RAIRO-Oper. Res. 52 (2018) 415–427
https://doi.org/10.1051/ro/2017063

RAIRO Operations Research
www.rairo-ro.org

ALGEBRAIC MODELLING OF A TWO LEVEL SUPPLY CHAIN WITH DEFECTIVE ITEMS

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Abstract. M. Khan and M.Y. Jaber, Optimal inventory cycle in a two-stage supply chain incorporating imperfect items from suppliers. *Int. J. Oper. Res.* **10** (2011) 442–457, have addressed a two level supply chain of defective items. They compared three coordination mechanisms, *i.e.* cycle time; K–multiplier cycle time; and 2^K–multiplier cycle time. This paper proposes a simpler algebraic solution for the K–multiplier cycle time mechanism without the use of differential calculus. The two level supply chain with defective items is illustrated with a numerical example. A sensitivity analysis is also provided.

Mathematics Subject Classification. 90B05.

Received January 10, 2017. Accepted September 7, 2017.

1. Introduction

The economic order quantity (EOQ) inventory model celebrated its 100th anniversary in 2013. The EOQ inventory model was proposed by Harris [16]. According to Cárdenas–Barrón *et al.* [11], Ford Whitman Harris can be considered as the Founding Father of the Inventory Theory.

Inventory and supply chain management are two important issues that researchers are studying in a holistic way recently. It is well known that the supply chain coordination is a centralized planning process that deals with production lot sizing, production scheduling, shipment quantities and inventory allocation. In the centralized production and replenishment decision policy, the global supply chain costs are optimized in an integrated manner. Whereas in the decentralized production and replenishment decision policy, each member within the supply chain considers optimizing their own costs individually. Several benefits of inventory coordination and information sharing among the supply chain members have been listed in the literature, see for instance the research works of Khan *et al.* [20] and Khan *et al.* [21].

Lately, the issue of inventory–distribution coordination in supply chain modeling has been dealt with in several research works. Mainly, these works have concentrated on the integrated vendor–buyer inventory and

Keywords. Supply chain management, integer multipliers, without derivatives, algebraic method.

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Article published by EDP Sciences

the joint economic lot–sizing problems. On the other hand, other researchers have proposed that the inventory–distribution coordination must be made by synchronizing the cycle time through all the supply chain stages. Moreover, there exit inventory supply chain models where the coordination is reached by implementing the integer multipliers mechanism. Here, the cycle time of a stage of the chain is an integer multiple of the cycle time of the adjacent downstream stage (*i.e.* Khouja [22]).

Salameh and Jaber [25] introduced a new direction of research to inventory modeling by adding screening process for defective items in an EOQ inventory model. Lately, this model has enjoyed a huge amount of interest from researchers and practitioners. The reader is referred to Khan *et al.* [18] for a review of the models that extend the work of Salameh and Jaber [25]. More recently, Sivashankari and Panayappan [34] considered defective items and studied how storage cost at a vendor's facility can be reduced by using two different production rates. Conversely, Darwish *et al.* [14] demonstrated the benefits of coordination in a two level supply chain where a fraction of vendor's lots are defective. Jauhari [17] studied the deterioration in vendor's production process in a two level supply chain where lead time varies linearly with lot size. Ben–Daya *et al.* [1] optimized the timings and quantities of inbound and outbound material in a three level supply chain. Cárdenas–Barrón *et al.* [7] revisited the model of Ben–Daya *et al.* [1] and proposed an improved algorithm that results in lesser total cost and lesser cycle time.

Kim and Sarkar [23] investigated the impact of investment to improve quality and reduce lead time in a multistage imperfect production process. On the one hand, Sarkar [29] illustrated the coordination in a supply chain with quantity discounts where the buyer uses a single setup multiple delivery strategy. On the other hand, Sarkar and Saren [31] studied inspection policies for a production process that randomly shifts to an out–of–control state. The reader may be referred to Cárdenas–Barrón *et al.* [9], Sarkar and Moon [28] and Tayyab and Sarkar [35] for more on these models.

A common methodology in the above literature is to use differential calculus to optimize inventory systems. However, several researchers have developed and proposed easier solution approaches for the sake of optimization; for example Cárdenas–Barrón [3] and Wee and Chung [37]. Grubbström [15] was perhaps the first to propose the use of the algebraic optimization method to derive the famous EOQ inventory model without backorders. Since then, the algebraic method has received an extraordinary attention from several researchers around the world. Cárdenas–Barrón [2] applied an algebraic method to derive the EPQ inventory model considering shortages for the case where only one backlog cost is considered. Later, using the algebraic method Cárdenas–Barrón [3] formulated and solved an n–stage–multi–customer supply chain inventory model for the simple equal cycle time inventory coordination mechanism. Chung and Wee [13] developed an integrated three–stage inventory system taking into account planned backorders. They formulated and optimized the three–stage inventory system with four–decision–variables algebraically. Wee and Chung [37] also applied a simple algebraic method to derive the economic lot size of an integrated production–inventory system. Chiu [12] also presented a simple algebraic method to show that the optimal lot size and total production–inventory costs of an imperfect EMQ inventory model can be obtained without derivatives. Cárdenas–Barrón [4] considered the problem of optimal manufacturing batch size with rework process at a single–stage production system. He determined algebraically the optimal solution for two different inventory policies and established the range of real values for proportion of defective products for which there exists an optimal solution. Seliaman [33] revisited and extended the Chung and Wee [13] model to include a fourth stage. It is worth mentioning that with the algebraic method, researchers or practitioners, unexperienced with differential calculus, may also be capable to understand the optimization procedure easily.

The acceptance of the algebraic method as an optimization tool lies in the fact that it involves basic knowledge of mathematics. Cárdenas–Barrón [6] presented an in–depth literature review with regard to the use of algebraic optimization methods in the inventory field. The reader may be referred to the models in Cárdenas–Barrón [4], Cárdenas–Barrón [5], Cárdenas–Barrón *et al.* [7], Cárdenas–Barrón *et al.* [8] Cárdenas–Barrón *et al.* [9], Cárdenas–Barrón *et al.* [10], Sarkar [26], Sarkar *et al.* [30], Seliaman [32], Teng *et al.* [36], for more on these models with algebraic procedures. In comparison to the available literature, the contribution in this paper has been highlighted in Table 1.

TABLE 1. Contribution of the paper in comparison to other inventory models.

Author(s)	Supply Chain	Single Supplier	Multiple Suppliers	Defectives	Backorders/ Shortages	Algebraic Approach
Ben–Daya *et al.* [1]	√	√				√
Cárdenas–Barrón [3]	√		√			√
Cárdenas–Barrón [4]				√		√
Cárdenas–Barrón *et al.* [7]	√	√		√		√
Cárdenas–Barrón *et al.* [8]	√	√				√
Cárdenas–Barrón *et al.* [9]	√	√		√		√
Cárdenas–Barrón *et al.* [10]	√	√		√		√
Chiu [12]				√	√	√
Chung and Wee [13]	√	√			√	√
Darwish *et al.* [14]	√	√				
Jauhari [17]	√	√		√		
Khan and Jaber [19]	√	√		√		
Khan *et al.* [20]	√	√		√		
Khan *et al.* [21]	√	√		√		
Khouja [22]	√		√			
Kim and Sarkar [23]				√	√	
Papachristos and Konstantaras [24]				√		
Salameh and Jaber [25]				√		
Sarkar [26]	√	√				√
Sarkar *et al.* [27]				√	√	
Sarkar and Moon [28]				√	√	
Sarkar [29]	√	√		√	√	√
Sarkar *et al.* .[30]	√	√				√
Sarkar and Saren [31]				√		
Seliaman [32]	√		√			√
Seliaman [33]	√	√			√	√
Sivashankari and Panayappan [34]				√		
Tayyab and Sarkar [35]				√		
Teng *et al.* [36]	√	√			√	√
Wee and Chung [37]	√	√			√	√
Khan *et al.* [38]	√	√		√		
This Paper	√		√	√		√

In this paper, we consider the integer (K) multipliers inventory coordination mechanism in Khan and Jaber [19] and develop an algebraic solution for a two level supply chain with defective items. The algebraic optimization approach developed in this paper derives a simpler closed form solution for this type of supply chain systems. This approach depends on completing the perfect squares in the cost function which simplifies the solution procedures and avoids the need to establish optimality conditions needed in classical differential calculus methods.

The remainder of this paper is organized as follows. Section 2 presents the description the two stage supply chain model as presented in Khan and Jaber [19]. Section 3 describes the development of algebraic solution. Section 4 solves a numerical example. Section 5 provides a sensitivity analysis. Finally, Section 6 gives some concluding remarks and future research directions.

2. Two level supply chain model development

Khan and Jaber [19] optimized the cost of a supplier–vendor supply chain. In this chain, the suppliers would provide a known fraction of defectives in their supplies. To counter the impact of defective items, vendor institutes a complete screening of all the lots provided by the suppliers. The authors optimized the supply quantity and the number of supplies in each cycle by using three mechanisms. That is: (i) equal cycle time for suppliers and vendor, (ii) Supplier's cycle time is an integer multiplier of that of the vendor, and (iii) Supplier's cycle time is an integer power of two of that of the vendor. In this paper, a different approach for the optimization process is taken for the second (integer K–multipliers) mechanism with an assumption that all the suppliers adopt the same integer. The assumptions (Khan and Jaber [19]) are:

(1) The percentage of defective items is a continuous random variable with known probability density function.
(2) A 100% inspection of each lot is carried out.
(3) Demand occurs parallel to the inspection process and it is fulfilled by the items found to be perfect by the inspection process.
(4) There are no shortages.

The following notation is used through this paper.

Parameters:

J	=	Number of suppliers (an integer number)..
P_v	=	Vendor's production rate (units/time unit).
D_v	=	Vendor's demand rate (units/time unit).
D_s	=	Supplier's demand; *i.e.* D_s=$D_v w_s$ (units/time unit).
γ_s	=	Percentage of defective items supplied by supplier s(%).
d_s	=	Unit screening cost for the items provided by supplier s($/unit).
x_s	=	Screening rate for vendor for items provided by supplier s(units/time unit).
A_v	=	Vendor's fixed ordering or setup cost ($/order or setup).
$a_{v,s}$	=	Vendor's variable cost of ordering an item from supplier s($/unit).
A_s	=	Suppliers' setup cost ($/setup).
h_{v1}	=	Vendor's unit holding cost for raw materials ($/unit/time unit).
h_{v2}	=	Vendor's unit holding cost for finished products ($/unit/time unit).
h_s	=	Suppliers' unit holding cost ($/unit/time unit).
C_{vr}	=	Vendor's unit cost of the raw material ($/unit).
C_{vf}	=	Vendor's unit cost of the finished product ($/unit).
t_{ys}	=	Inspection time for items of type y from supplier s (time unit).
T_p	=	Vendor's cycle time for production (time unit).
T_d	=	Vendor's idle time in a cycle (time unit).
n_s	=	Number of types of parts provided by supplier s (an integer number).
u_{sy}	=	Number of parts of type y from supplier s needed for a product (units).
w_s	=	Number of items from supplier s, required for a product; $w_s = \sum_{y=1}^{y=n_s} u_{sy}$ (units).
y_s	=	Number of non–defective items supplied by supplier s; $y_s = D_v w_s\,(1-\gamma_s)$(units).
z	=	Minimum number of products that can be manufactured in a production cycle; here $z = Min\,(Int\,(y_s = D_v w_s\,(1-\gamma_s)))$ where $s = 1, 2, 3, \ldots, J$ (units).
l_s	=	Number of unused non–defective items of type $s$in a cycle; $l_s = y_s - z w_s$ (units).
Z_{1s}	=	Inventory level of parts from supplier s at the end of inspection process (units).
Z_{2s}	=	Inventory level of parts from supplier s after removing the defective items (units).
B_s	=	Products screening out; $B_s = Z_{1s} - Z_{2s}$(units).

Decision variables:

K = Integer multiplier for the coordination mechanism (an integer number).
T = Vendor's cycle time (time unit).

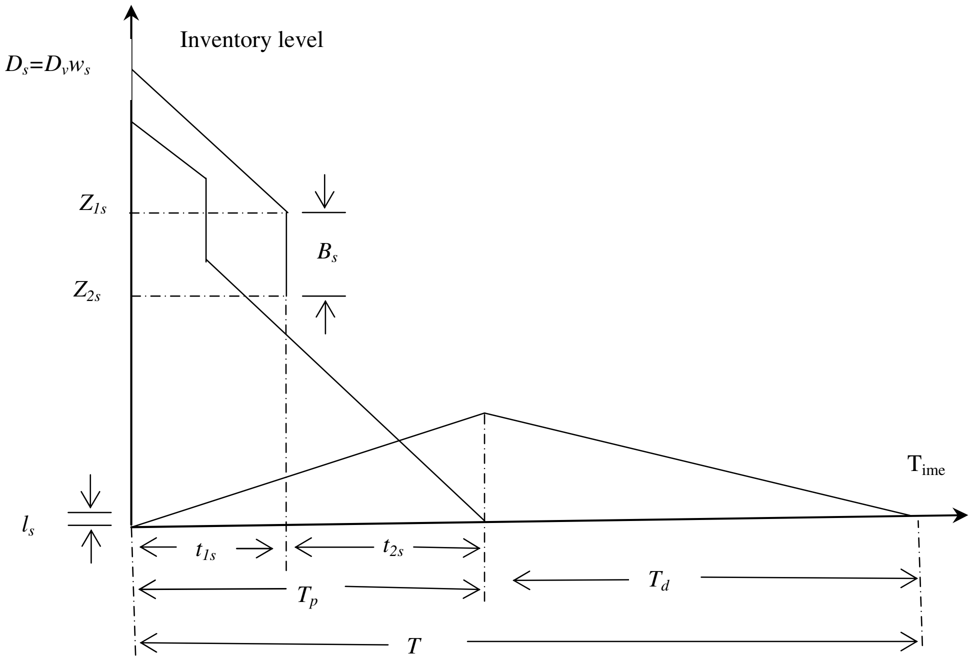

Figure 1. Inventory level for raw material and finished product.

The vendor's inventory level of items obtained from one supplier and the finished products is depicted in Figure 1 which is adopted from Khan and Jaber [19]. Because of the inspection process that runs for time t_{1s}, any incoming lot of raw items will be divided into two sub–lots. The first sub–lot represents the non–defective items provided by suppliers: $Dvws$ (1− γ_s). The second is defective sub–lot, B_s.

Since the inspection process is instituted at the rate of x, then the rate of delivering inspected non–defective items supplied by supplier s is $x_s\,(1-\gamma_s)$. Based on the third assumption above, the rate of delivering non–defective items by supplier s is more than vendor's demand. This condition can be expressed as:

$$x_s\,(1-\gamma_s) \geqslant D_v w_s$$

Papachristos and Konstantaras [24] pointed out that the above inequality is not meaningful without assuming that the screening speed is always greater than or equal to the demand rate. Therefore, the following assumption is also added:

$$x_s \geqslant D_v$$

The inventory holding cost for the vendor consists of the carrying cost for the items as they are being assembled into finished products during the production portion of the cycle; and the carrying cost of the finished products during the non–production portion of the cycle. Therefore, the vendor's cost of raw material for ordering, holding and screening as in Khan and Jaber [19] is

$$C_{vr} = D_v \sum_{s=1}^{J} a_{vs} w_s + \sum_{s=1}^{J} \left[h_{v1} \left\{ \frac{(D_v w_s T - l_s)(1-\gamma_s)}{2} + \frac{\gamma_s (D_v w_s T - l_s)^2}{T x_s} + l_s \right\} + \frac{d_s (D_v w_s T - l_s)}{T} \right]$$

and vendor's total cost for the product is given by

$$C_{vf} = \frac{A_v}{T} + \frac{T}{2} h_{v2} D_v \left(1 - \frac{D_v}{P_v}\right)$$

So, vendor's total cost is as follow

$$\begin{aligned} TC_v = D_v \sum_{s=1}^{J} a_{vs} w_s + \sum_{s=1}^{J} \left[h_{v1} \left\{ \frac{(D_v w_s T - l_s)(1-\gamma_s)}{2} + \frac{\gamma_s (D_v w_s T - l_s)^2}{T x_s} + l_s \right\} + \frac{d_s (D_v w_s T - l_s)}{T} \right] \\ + \frac{A_v}{T} + \frac{T}{2} h_{v2} D_v \left(1 - \frac{D_v}{P_v}\right) \end{aligned}$$

The supplier's inventory in vendor's non–production time drops in steps and then the total cost is

$$TC_s = \frac{1}{T} \sum_{s=1}^{J} \frac{A_s}{K} + + \frac{T}{2} \sum_{s=1}^{J} (K-1) h_s D_s$$

Consequently, the total cost of the supply chain in Khan and Jaber [19] is

$$\begin{aligned} TC = \frac{A_v}{T} + \frac{1}{T} \sum_{s=1}^{J} \frac{A_s}{K} + D_v \sum_{s=1}^{J} a_{vs} w_s + \frac{T}{2} h_{v2} D_v \left(1 - \frac{D_v}{P_v}\right) + \frac{T}{2} \sum_{s=1}^{J} (K-1) h_s D_s \\ + \sum_{s=1}^{J} \left[h_{v1} \left\{ \frac{(D_v w_s T - l_s)(1-\gamma_s)}{2} + \frac{\gamma_s (D_v w_s T - l_s)^2}{T x_s} + l_s \right\} + \frac{d_s (D_v w_s T - l_s)}{T} \right] \end{aligned} \tag{2.1}$$

3. Algebraic optimization

Notice that suppliers share the same integer multiplier K. Now, equation (2.1) can be expressed as

$$\begin{aligned} TC = \frac{1}{T} \sum_{s=1}^{J} \frac{A_s}{K} + \frac{A_v}{T} + D_v \sum_{s=1}^{J} a_{vs} w_s \\ + \sum_{s=1}^{J} \left[h_{v1} \left\{ \frac{T(1-\gamma_s) D_v w_s}{2} - \frac{l_s (1-\gamma_s)}{2} + \frac{\gamma_s \left(D_v^2 w_s^2 T^2 - 2 D_v w_s T l_s + l_s^2\right)}{T x_s} + l_s \right\} + \frac{d_s (D_v w_s T - l_s)}{T} \right] \\ + \frac{T}{2} h_{v2} D_v \left(1 - \frac{D_v}{P_v}\right) + \frac{T}{2} \sum_{s=1}^{J} (K-1) h_s D_s \end{aligned} \tag{3.1}$$

which can be further rewritten as

$$
\begin{aligned}
TC = {} & \frac{1}{T}\sum_{s=1}^{J}\frac{A_s}{K} + \frac{A_v}{T} + D_v\sum_{s=1}^{J}a_{vs}w_s \\
& + \sum_{s=1}^{J}\left[h_{v1}\left\{\frac{T\left(1-\gamma_s\right)D_v w_s}{2} - \frac{l_s\left(1-\gamma_s\right)}{2} + \frac{T\gamma_s D_v^2 w_s^2}{x_s} - \frac{2D_v\gamma_s w_s l_s}{x_s} + \frac{\gamma_s l_s^2}{Tx_s} + l_s\right\} + d_s D_v w_s - \frac{d_s l_s}{T}\right] \\
& + \frac{T}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) + \frac{T}{2}\sum_{s=1}^{J}\left(K-1\right)h_s D_s
\end{aligned}
\tag{3.2}
$$

Now, grouping similar terms (with regard to T and $1/T$)

$$
\begin{aligned}
TC = {} & \frac{1}{T}\sum_{s=1}^{J}\frac{A_s}{K} + \frac{A_v}{T} + \frac{h_{v1}}{T}\sum_{s=1}^{J}\frac{\gamma_s l_s^2}{x_s} - \frac{1}{T}\sum_{s=1}^{J}d_s l_s \\
& + T\sum_{s=1}^{J}\frac{h_{v1}\left(1-\gamma_s\right)D_v w_s}{2} + T\sum_{s=1}^{J}\frac{h_{v1}\gamma_s D_v^2 w_s^2}{x_s} + \frac{T}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) + \frac{T}{2}\sum_{s=1}^{J}\left(K-1\right)h_s D_s \\
& + D_v\sum_{s=1}^{J}a_{vs}w_s + \sum_{s=1}^{J}d_s D_v w_s + h_{v1}\sum_{s=1}^{J}l_s - h_{v1}\sum_{s=1}^{J}\frac{2D_v\gamma_s w_s l_s}{x_s} - \sum_{s=1}^{J}\frac{h_{v1}l_s\left(1-\gamma_s\right)}{2}
\end{aligned}
\tag{3.3}
$$

Again we can rewrite the total cost as

$$
\begin{aligned}
TC = {} & \frac{1}{T}\left(\sum_{s=1}^{J}\frac{A_s}{K} + A_v + h_{v1}\sum_{s=1}^{J}\frac{\gamma_s l_s^2}{x_s} - \sum_{s=1}^{J}d_s l_s\right) \\
& + T\left\{\sum_{s=1}^{J}\frac{h_{v1}\left(1-\gamma_s\right)D_v w_s}{2} + \sum_{s=1}^{J}\frac{h_{v1}\gamma_s D_v^2 w_s^2}{x_s} + \frac{1}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) + \frac{1}{2}\sum_{s=1}^{J}\left(K-1\right)h_s D_s\right\} \\
& + D_v\sum_{s=1}^{J}a_{vs}w_s + \sum_{s=1}^{J}d_s D_v w_s + h_{v1}\sum_{s=1}^{J}l_s - h_{v1}\sum_{s=1}^{J}\frac{2D_v\gamma_s w_s l_s}{x_s} - \sum_{s=1}^{J}\frac{h_{v1}l_s\left(1-\gamma_s\right)}{2}
\end{aligned}
\tag{3.4}
$$

which can be represented in a compact form as:

$$
TC = \frac{W}{T} + TY + X \tag{3.5}
$$

where

$$
\begin{aligned}
Y &= \sum_{s=1}^{J}\frac{h_{v1}\left(1-\gamma_s\right)D_v w_s}{2} + \sum_{s=1}^{J}\frac{h_{v1}\gamma_s D_v^2 w_s^2}{x_s} + \frac{1}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) + \frac{1}{2}\sum_{s=1}^{J}\left(K-1\right)h_s D_s \\
W &= \sum_{s=1}^{J}\frac{A_s}{K} + A_v + h_{v1}\sum_{s=1}^{J}\frac{\gamma_s l_s^2}{x_s} - \sum_{s=1}^{J}d_s l_s \\
X &= D_v\sum_{s=1}^{J}a_{vs}w_s + \sum_{s=1}^{J}d_s D_v w_s + h_{v1}\sum_{s=1}^{J}l_s - h_{v1}\sum_{s=1}^{J}\frac{2D_v\gamma_s w_s l_s}{x_s} - \sum_{s=1}^{J}\frac{h_{v1}l_s\left(1-\gamma_s\right)}{2}
\end{aligned}
$$

Now, using the simple algebraic steps proposed by Cárdenas–Barrón [3]. By factorizing the term $\frac{1}{T}$ and completing the perfect square then the annual total cost for the entire supply chain in equation (3.5) can be expressed as

$$TC = \frac{1}{T}\left(T^2Y - 2T\sqrt{YW} + W + 2T\sqrt{YW}\right) + X \tag{3.6}$$

It is important to remark that X is a constant and it does not depend on any of the decision variables.

Factorizing the perfect squared trinomial in a squared binomial one obtains:

$$TC = \frac{1}{T}\left(T\sqrt{Y} - \sqrt{W}\right)^2 + 2\sqrt{YW} + X \tag{3.7}$$

It should be noted that equation (3.7) reaches its minimum with respect to T when setting

$$\left(T\sqrt{Y} - \sqrt{W}\right)^2 = 0$$

Hence, the optimal basic cycle time T^* is

$$T^* = \sqrt{\frac{W}{Y}} \tag{3.8}$$

which reduces to the same cycle length as in Khan and Jaber [19].

The corresponding minimum cost is given by

$$TC = 2\sqrt{YW} + X \tag{3.9}$$

Now, it is required to derive the optimal value of the integer multiplier K and we will do as follow. Considering the terms Y and W again

$$Y = \sum_{s=1}^{J}\frac{h_{v1}\left(1-\gamma_s\right)D_v w_s}{2} + \sum_{s=1}^{J}\frac{h_{v1}\gamma_s D_v^2 w_s^2}{x_s} + \frac{1}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) + \frac{1}{2}\sum_{s=1}^{J}\left(K-1\right)h_s D_s$$

$$W = \sum_{s=1}^{J}\frac{A_s}{K} + A_v + h_{v1}\sum_{s=1}^{J}\frac{\gamma_s l_s^2}{x_s} - \sum_{s=1}^{J} d_s l_s$$

Where Y can be rewritten as

$$Y = \frac{h_{v1}}{2}\sum_{s=1}^{J}\left(1-\gamma_s\right)D_v w_s + h_{v1}\sum_{s=1}^{J}\frac{\gamma_s D_v^2 w_s^2}{x_s} + \frac{1}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) - \frac{1}{2}\sum_{s=1}^{J}h_s D_s + \frac{K}{2}\sum_{s=1}^{J}h_s D_s$$

Let it be represented as

$$Y = K\alpha + \beta \tag{3.10}$$

with

$$\alpha = \frac{1}{2}\sum_{s=1}^{J}h_s D_s$$

$$\beta = \frac{h_{v1}}{2}\sum_{s=1}^{J}\left(1-\gamma_s\right)D_v w_s + h_{v1}\sum_{s=1}^{J}\frac{\gamma_s D_v^2 w_s^2}{x_s} + \frac{1}{2}h_{v2}D_v\left(1-\frac{D_v}{P_v}\right) - \frac{1}{2}\sum_{s=1}^{J}h_s D_s$$

Similarly W can be expressed as

$$W = \frac{\delta}{K} + \varphi \tag{3.11}$$

Where

$$\delta = \sum_{s=1}^{J} A_s$$

$$\varphi = A_v + h_{v1} \sum_{s=1}^{J} \frac{\gamma_s l_s^2}{x_s} - \sum_{s=1}^{J} d_s l_s$$

Now, substituting both Y and W into equation (3.9) one gets

$$TC = 2 \left\{ (K\alpha + \beta) \left(\frac{\delta}{K} + \varphi \right) \right\}^{\frac{1}{2}} + X$$

or

$$TC = 2 \left\{ \frac{1}{K} \left(K\sqrt{\alpha\varphi} - \sqrt{\beta\delta} \right)^2 + \left(\sqrt{\alpha\delta} + \sqrt{\beta\varphi} \right)^2 \right\}^{\frac{1}{2}} + X$$

One can easily see that this reaches its minimum when letting:

$$K\sqrt{\alpha\varphi} - \sqrt{\beta\delta} = 0$$

Hence, the optimal value of integer multiplier is derived as

$$K^* = \sqrt{\frac{\beta\delta}{\alpha\varphi}} \tag{3.12}$$

Since the value of K is a positive integer, the following condition must be satisfied:

$$K^* (K^* - 1) \leqslant (K^*)^2 \leqslant K^* (K^* + 1)$$

Now, one can substitute $K*$ from equation (3.12) into equation (3.8) to find the optimal basic cycle time $T*$. Also, substituting $K*$ and $T*$ into equation (3.9) derives the optimal annual total cost in a closed form.

4. Numerical example

In this section a numerical example of a two–stage supply chain is solved. This example is taken from Khan and Jaber [19] and its data is shown in Table 2. The parameters in this example satisfy the following two conditions necessary to avoid shortages:

(1) $x_s (1 - \gamma_s) \geqslant D_v w_s$
(2) $x_s \geqslant D_v$

By applying the developed solution procedure one obtains the optimal cycle time length of 0.146 for the vendor. The optimal integer multiplier for the two suppliers is given as 10. Hence, the optimal cycle length at the suppliers' stage is 1.460. Finally, the total cost under this solution is \$ 7142.09. For the same example and under the equal cycle coordination time mechanism, the optimal basic cycle time is 0.42 years, and the total cost is $TC = \$\ 10326.38$. As shown in Table 3, using the integer multipliers coordination mechanism will make about 30.84% costs saving for the entire supply chain as compared to the traditional equal cycle coordination time mechanism.

TABLE 2. Data for the example in appropriate units according to notation.

Av	A_1	A_2	a_{v1}	a_{v2}	h_1	h_2	u_1	u_2	d_s	D_v	P_v
200	800	800	1	1	0.5	1	1	1	0.1	1000	3000

h_{v1}	h_{v2}	γ_1	γ_2	x_s
2	25	0.07	0.08	175 200

TABLE 3. The example results with comparison to the equal cycle coordination time mechanism.

	T	K	Suppliers' cost	Vendor's cost	Entire Chain cost
Equal time cycle	0.420	–	3807.03	6519.35	10326.38
Integer multipliers	0.146	10	2081.70	5060.39	7142.09
Saving	–	–	1725.33	1458.96	3184.29
Saving%	–	–	45.32	22.38	30.84

TABLE 4. Effects of the screening rate on the on the optimal solution and associated costs.

%Saving over the equal cycle	Saving over the equal cycle	Total cost	Vendor cost	Screening cost	Suppliers cost	K	T	x_s
35.99	4014.98	7142.094	3939.31	193.13	3202.783	10	0.145598	175 200
35.99	4015.00	7142.107	3939.33	193.13	3202.781	10	0.145597	166 440
35.99	4015.02	7142.121	3939.34	193.13	3202.778	10	0.145597	157 680
35.99	4015.05	7142.136	3939.36	193.13	3202.775	10	0.145597	148 920
35.99	4015.08	7142.154	3939.38	193.13	3202.772	10	0.145596	140 160
35.99	4015.11	7142.174	3939.41	193.13	3202.769	10	0.145595	131 400
35.99	4015.15	7142.197	3939.43	193.13	3202.764	10	0.145595	122 640
35.99	4015.19	7142.224	3939.46	193.13	3202.76	10	0.145594	113 880
35.99	4015.38	7142.335	3939.60	193.13	3202.74	10	0.145591	87 600
35.99	4015.47	7142.389	3939.66	193.13	3202.73	10	0.145589	78 840
35.99	4015.58	7142.456	3939.74	193.13	3202.718	10	0.145587	70 080

5. SENSITIVITY ANALYSIS

Sensitivity analysis is conducted to examine the impact of changing some of the inventory model parameters on the optimal solution and its associated costs. Effects of changing the screening rate on the model results and the cost saving from using integer multipliers mechanism over the equal time cycle mechanism are shown in Table 4. It is observed that a variation in the range of (70 080–175 200) has no drastic impact on the optimal solution or the gained cost saving over the equal time cycle mechanism. Table 5 shows the impact of changing the percentage of defective items provided by suppliers on the optimal solution and associated costs. It can observed that increasing the percentage of defective items increases the cycle time at the suppliers stage by increasing the integer multiplier. In turns, the total cost at the suppliers also increases. Additionally, the gained cost saving over the equal time cycle mechanism increases also.

Table 6 shows how the optimal solution responds to changes in the setup costs at the suppliers and inventory holding costs and setup costs at the vendor. It can be observed that increasing A_v decreases the cost saving over the equal time cycle mechanism. However, increase of A_s will increase this cost saving. In addition, increasing inventory holding costs at the vendor will also increase the cost saving over the equal time cycle mechanism.

TABLE 5. Effects of percentage of defective items supplied by suppliers on the optimal solution and associated costs.

%Saving over the equal cycle	Saving over the equal cycle	Total cost	Vendor cost	Screening cost	Suppliers cost	K	T	γ_s
36.13	4055.081	7169.647	3962.58	151.43	3207.069	10	0.144	0.01
36.10	4048.414	7165.539	3959.19	158.44	3206.344	10	0.144	0.02
36.08	4041.739	7161.237	3955.61	165.43	3205.624	10	0.145	0.03
36.05	4035.059	7156.742	3951.83	172.39	3204.907	10	0.145	0.04
36.03	4028.372	7152.053	3947.86	179.33	3204.195	10	0.145	0.05
35.96	4008.276	7136.824	3934.74	200.00	3202.084	10	0.146	0.08
37.10	4138.369	7015.771	3605.60	185.27	3410.168	11	0.136	0.10
40.29	4503.661	6673.091	2987.40	141.56	3685.686	12	0.120	0.15
44.21	4951.485	6247.89	1652.50	69.50	4595.393	16	0.092	0.20

TABLE 6. Effect of holding and setup costs on the optimal solution and associated costs.

%Saving	Saving	Total cost	Vendor cost	Screening cost	Suppliers cost	K	T	Parameter	
38.60	4258.362	6773.512	3160.70	191.92	3612.81	12	0.124	150	
38.03	4205.547	6851.508	3431.43	192.38	3420.077	11	0.131	160	
37.49	4155.058	6927.108	3518.24	192.50	3408.869	11	0.133	170	
36.95	4104.444	7002.761	3779.75	192.93	3223.007	10	0.141	180	
36.46	4059.248	7072.926	3860.37	193.03	3212.56	10	0.144	190	A_v
35.99	4014.98	7142.094	3939.31	193.13	3202.783	10	0.146	200	
35.52	3971.6	7210.304	4016.68	193.23	3193.629	10	0.148	210	
35.06	3929.069	7277.597	4092.54	193.32	3185.057	10	0.150	220	
34.21	3850.85	7405.137	4406.51	193.77	2998.63	9	0.160	240	
33.81	3813.462	7467.084	4476.38	193.84	2990.708	9	0.162	250	
34.23	3643.404	7001.724	4187.64	193.24	2814.081	9	0.148	700	
35.14	3831.944	7072.926	3983.48	193.03	3089.447	10	0.144	750	
35.57	3924.241	7107.632	3961.14	193.08	3146.491	10	0.145	775	
35.99	4014.98	7142.094	3939.31	193.13	3202.783	10	0.146	800	
36.38	4104.23	7176.316	3917.97	193.18	3258.35	10	0.147	825	A_s
36.76	4192.056	7210.304	3897.09	193.23	3313.218	10	0.148	850	
37.13	4278.518	7244.063	3876.65	193.27	3367.411	10	0.149	875	
37.50	4365.702	7275.567	3668.69	193.02	3606.882	11	0.143	900	
37.86	4451.271	7307.211	3645.25	193.06	3661.96	11	0.144	925	
38.20	4535.613	7338.66	3622.25	193.10	3716.409	11	0.145	950	
35.99	4014.98	7142.094	3939.31	193.13	3202.783	10	0.146	2	
37.48	4442.374	7409.651	4085.37	192.47	3324.281	11	0.133	4	
38.78	4849.491	7656.968	4221.37	191.84	3435.598	12	0.123	6	h_{v1}
39.90	5238.091	7888.391	4349.40	191.24	3538.988	13	0.114	8	
40.90	5610.297	8106.552	4617.42	190.91	3489.134	13	0.110	10	
32.42	3252.497	6779.725	3539.51	193.94	3240.214	9	0.165	18	
33.57	3480.956	6888.384	3719.65	193.79	3168.737	9	0.161	20	
34.59	3698.589	6994.576	3885.89	193.65	3108.687	9	0.158	22	h_{v2}
35.55	3911.823	7093.32	3862.48	193.20	3230.838	10	0.147	24	
36.40	4116.09	7190.39	4013.72	193.06	3176.668	10	0.144	26	

6. Conclusions

This paper contributes to the supply chain modeling literature proposing an algebraic approach to determine the cycle length and integer multiplier in the coordination scheme of a two level supply chain model given by Khan and Jaber [19]. This approach is more convenient for students and practitioners who are not familiar with differential calculus. The developed solution method is illustrated by solving a numerical example. The sensitivity analysis indicates that increasing the percentage of defective items increases the cycle time at the suppliers' stage by increasing the integer multiplier. In turns, the total cost at the suppliers also increases. This results agree with the results reported by Khan *et al.* [38]. The proposed model can be extended in a number of ways. example, one could use this approach to optimize a multiple tier supply chain. The possibility of shortages could also be explored in the suggested inventory model considering the portion of defective items as random variable. Besides, the impact of errors in screening can also be studied, using the same approach.

Acknowledgements. The first author would like to acknowledge the support for this research provided by Deanship of Scientific Research: King Faisal University under research grant (150174). The third author was supported by the Tecnológico de Monterrey Research Group in Industrial Engineering and Numerical Methods 0822B01006.

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