RAIRO-Oper. Res. 52 (2018) 241–257
https://doi.org/10.1051/ro/2017081

RAIRO Operations Research
www.rairo-ro.org

INTUITIONISTIC FUZZY DEA/AR AND ITS APPLICATION TO FLEXIBLE MANUFACTURING SYSTEMS

Sanjeet Singh

Abstract. The concept of assurance region (AR) was proposed in Data Envelopment Analysis (DEA) literature to restrict the ratio of any two weights within a given lower and upper bounds so as to overcome the difficulty of ignoring or relying too much on any of the input or output while calculating the efficiency. Further, AR approach was extended to handle fuzzy input/output data. But, available information is not always sufficient to define the impreciseness in the input/output data using classical fuzzy sets. Intuitionistic Fuzzy Set (IFS) is a generalized fuzzy set to characterize the impreciseness by taking into account degree of hesitation also. In this paper, intuitionistic fuzzy DEA/AR approach has been proposed to evaluate the efficiency where input/output data are represented as intuitionistic fuzzy. Based on the expected value approach, classical cross efficiency has also been generalized to rank the DMUs for the case of intuitionistic fuzzy data. To the best of my knowledge, this is the first attempt to propose assurance region approach (DEA/AR) in DEA with intuitionistic fuzzy input/output data. This approach is useful for the experts and decision makers when they are hesitant about defining the degree of membership/non-membership of fuzzy data. Results have been illustrated and validated using a case of flexible manufacturing systems (FMS).

Mathematics Subject Classification. 90C05, 90C32, 90C70, 90C90.

Received September 28, 2016. Accepted October 30, 2017.

1. Introduction

Data Envelopment Analysis (DEA) was first introduced by Charnes *et al.* [1] to calculate the relative efficiency of firms or decision-making units (DMUs) producing multiple outputs by consuming multiple inputs. Based on Farrell's [2] work on productive efficiency, DEA relies on unknown input and output weights to measure the relative efficiency of DMUs. Over the past 39 years, DEA has been applied to a variety of application areas that include banks, hospitals, sports, manufacturing, finance, education, airlines, social welfare, crews, insurance, supply chain, *etc.* see [3–8]. As an extension to the classical CCR [1] and BCC models [9], researchers have proposed various models [3, 10] to handle diverse nature of practical problems. In DEA, DMUs select input and output weights that are most favorable to maximize their efficiency in the form of weighted output to input ratio. In this process, DMUs often rely too heavily only on some of the input and output parameters or ignore some of them altogether. However, in certain practical problems relative importance of pairwise input and output weights has to vary within the given bounds specified by the decision-maker, For example,

Keywords. DEA, fuzzy DEA, assurance region, intuitionistic fuzzy sets, fuzzy sets.

[1] Indian Institute of Management Calcutta, DH Road, Joka, Kolkata-700104.
[*] Corresponding author: sanjeet@iimcal.ac.in

in production systems, one has to have a certain minimum/maximum level of each raw material to produce a reliable and quality product. To overcome this shortcoming, Thompson *et al.* [11,12] proposed Assurance Region (AR) method. In AR method, relative importance bounds for pairwise input and output weights or multipliers (prices/costs) are imposed based on *a priori* information obtained through expert opinion, experience, common sense, and historical trends. Subsequently, DEA/AR method has been applied to many practical problems that include urban planning [13], small business development [14], Banks [15], Sports [16], hotels [17], among others.

One common limitation of the conventional DEA models is that they can handle crisp input and output data only. However, observed data in real-world problems are often imprecise or vague. The imprecise values for inputs and outputs may be the result of the incomplete, unquantifiable, and non-obtainable information. The uncertainty of the data may also be due to the error in the different stages of data collection. Fuzzy DEA models were proposed by various researchers to handle the uncertainty or vagueness in the data. Sengupta [18,19] was the first to incorporate fuzzy mathematical programming approach into DEA. In fuzzy DEA imprecise input and output values are represented by fuzzy numbers characterized by membership functions [20, 21]. Since then various researchers have developed fuzzy DEA models and methods to deal with imprecise input and output [22–30] values. The literature review by Emrouznejad *et al.* [31] may be referred for the taxonomy and summary of the current research in fuzzy DEA.

The main concept behind classical fuzzy sets is that the degree of membership of an element can take any value in the interval [0,1] unlike crisp set when it takes either 0 or 1 only. The degree of non-membership is automatically assumed to be one minus degree of membership. However, decision makers/experts/human beings, while expressing the degree of membership of an element in a fuzzy set, very often do not give any hint or do not express the corresponding degree of non-membership as the complement of the degree of membership. In most of the real life problems, evaluations are done by human beings who always have certain limitations of knowledge, lack of proper understanding of the problem context, uncontrollable external environment impacting input/output values. As a result, experts or decision–makers have some degree of hesitation in evaluation activities. This is where the concept of an intuitionistic fuzzy set (IFS) has the advantage over classical fuzzy set theory. The IFS was introduced by Atanassov [32]. The main advantage of IFS over the FS is that it separately defines the degree of membership and non-membership of an element in the set. Since its introduction in 1986, IFS has found applications to many practical problems such as medical diagnosis [33], pattern recognition [34], personnel selection [35], job-shop scheduling [36], consensus building [37], multi-attribute group decision-making [38], *etc.*

Although, the theory of IFS has been used extensively in decision-making problems, there are not many studies that have incorporated IFS to handle uncertainty or vagueness in DEA. Rouyendegh [39] was the first to use intuitionistic fuzzy TOPSIS method in a two-stage process to fully rank the DMUs. Gandotra *et al.* [40] proposed an algorithm to rank DMUs in the presence of intuitionistic fuzzy weighted entropy. Hajiagha *et al.* [41] developed a DEA model when input/output data was expressed in the form of IFS. They further extended the model to the case of weighted aggregated operator for IFS. Puri and Yadav [42] developed optimistic and pessimistic DEA models under intuitionistic fuzzy input data. They also presented the application of their proposed models through a case from the banking sector in India where some of the inputs were represented as Triangular Intuitionistic Fuzzy Numbers (TIFN).

This paper proposes a new intuitionistic fuzzy DEA/AR (IFDEA/AR) approach to handle intuitionistic fuzzy input/output data. To the best of my knowledge, this is the first study to analyze DEA/AR efficiency under intuitionistic fuzzy environment when input/output data and weights are considered as TIFN. Based on Expected Value Approach (EVA), a model is formulated to evaluate the relative efficiency of any DMU. A cross efficiency approach is also proposed for the complete ranking of DMUs in IFDEA/AR. To validate and illustrate the proposed approach and ranking method, a case of Flexible Manufacturing Systems (FMS) is considered where two inputs (capital and operating cost) and three outputs (work-in-process, number of tardy jobs, yield) are represented as TIFN.

The remainder of the paper proceeds as follows. Section 2 deals with related concepts of fuzzy sets, intuitionistic fuzzy sets. Section 3 describes the proposed approach of IFDEA/AR and ranking. In Section 4, a case of flexible manufacturing systems is presented to illustrate and validate the theoretical results. Section 5 concludes the paper with main findings of this research.

2. Preliminaries

In this section, basic concepts of fuzzy sets, intuitionistic fuzzy sets, intuitionistic fuzzy numbers, and triangular intuitionistic fuzzy numbers are introduced so as to facilitate further discussion.

2.1. Fuzzy set

Let Xbe the universe of discourse whose elements are denoted by x. A fuzzy set $\tilde{A}$ in X is defined by a set of ordered pairs

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\},$$

where fuzzy set $\tilde{A}$ is characterized by its membership function $\mu_{\tilde{A}} : X \to [0,1]$, which associates with each x in X, a real number $\mu_{\tilde{A}}(x)$ in [0,1]. The value $\mu_{\tilde{A}}(x)$ represents the degree of membership (belongingness) of x in $\tilde{A}$ and that of non-membership (non-belongingness) 1-$\mu_{\tilde{A}}(x)$.

The $\mu_{\tilde{A}}(x)$ indicates evidence only for the degree of membership of x in A, human beings or decision makers do not provide any evidence or preference for degree of non-membership, and it is by default assumed to be *1*-$\mu_{\tilde{A}}(x)$. Atanassov [32] introduced the concept of Intuitionistic Fuzzy Set (IFS) which is characterized by both membership and non-membership function.

2.2. Intuitionistic fuzzy set (IFS) and intuitionistic fuzzy number (IFN)

Let X be a universe of discourse. An IFS [32] $\tilde{A}^I$ in X is a set of ordered triples

$$\tilde{A}^I = \{(x, \mu_{\tilde{A}^I}(x), \nu_{\tilde{A}^I}(x)) : x \in X\},$$

where IFS $\tilde{A}^I$ is characterized by its membership function $\mu_{\tilde{A}^I} : X \to [0,1]$and non-membership function $\nu_{\tilde{A}^I} : X \to [0,1]$such that

$$0 \leqslant \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leqslant 1 \forall x \in X.$$

For each $x \in X, \mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ represent the degree of membership and non-membership respectively, of x in $\tilde{A}^I_.$ Degree of hesitation (or intuitionistic fuzzy index) $\pi_{\tilde{A}^I}(x)$ of x in $\tilde{A}^I$ for each $x \in X$ is defined as follows:

$$\pi_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x) - \nu_{\tilde{A}^I}(x).$$

Here, it may be seen that fuzzy set is a particular case of IFS when $\nu_{\tilde{A}^I}(x) = 1 - \mu_{\tilde{A}^I}(x)$.

Let $\tilde{A}^I$ be an IFS with its membership function , $\mu_{\tilde{A}^I}$and non-membership function $\nu_{\tilde{A}^I}$. Then $\tilde{A}^I$ is said to be an IFN [42] if:

(i) $\tilde{A}^I$ is intuitionistic fuzzy normal, *i.e.*, there exist at least two points $x_0, x_1 \in X$ such that $\mu_{\tilde{A}^I}(x_0) = 1$ and $\nu_{\tilde{A}^I}(x_1) = 1$.
(ii) $\tilde{A}^I$ is intuitionistic fuzzy convex for $\mu_{\tilde{A}^I}(x)$, *i.e.*, $\mu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \geqslant \min(\mu_{\tilde{A}^I}(x_1), \mu_{\tilde{A}^I}(x_2))$, $\forall x_1, x_2 \in \Re, 0 \leqslant \lambda \leqslant 1$.
(iii) $\tilde{A}^I$ is intuitionistic fuzzy concave for $\nu_{\tilde{A}^I}(x)$, *i.e.*, $\nu_{\tilde{A}^I}(\lambda x_1 + (1-\lambda)x_2) \leqslant \max(\nu_{\tilde{A}^I}(x_1), \nu_{\tilde{A}^I}(x_2)), \forall x_1, x_2 \in \Re, 0 \leqslant \lambda \leqslant 1$.

From this, an IFN can be represented mathematically as $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4)$ by eight numbers $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \Re$ such that $b_1 \leqslant a_1 \leqslant b_2 \leqslant a_2 \leqslant a_3 \leqslant b_4$ with membership function $\mu_{\tilde{A}^I}$ and

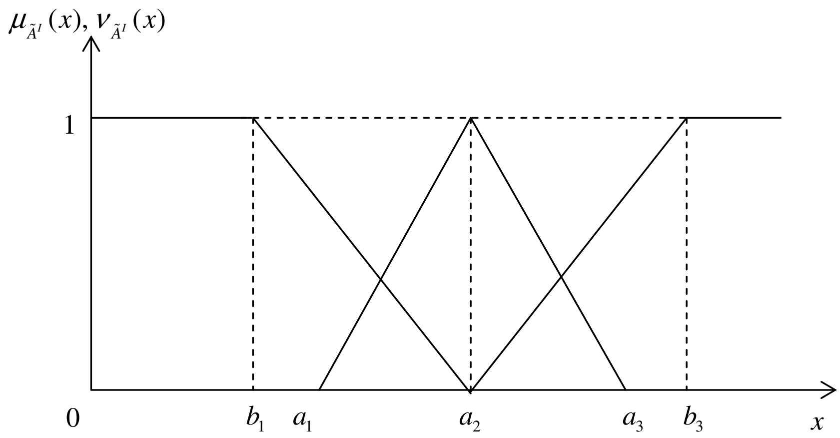

FIGURE 1. Triangular intuitionistic fuzzy number.

non-membership function$\nu_{\tilde{A}^I}$ given by

$$\mu_{\tilde{A}^I}(x) = \begin{cases} f_{\tilde{A}^I}(x), & a_1 \leqslant x < a_2, \\ 1, & a_2 \leqslant x \leqslant a_3, \\ g_{\tilde{A}^I}(x), & a_3 < x \leqslant a_4, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} h_{\tilde{A}^I}(x), & b_1 \leqslant x < b_2, \\ 1, & b_2 \leqslant x \leqslant b_3, \\ k_{\tilde{A}^I}(x), & b_3 < x \leqslant b_4, \\ 0, & \text{otherwise} \end{cases}$$

where, $f_{\tilde{A}^I}$, $g_{\tilde{A}^I}$, $h_{\tilde{A}^I}, k_{\tilde{A}^I} : \Re \to [0,1]$ are piecewise continuous functions and $\leqslant \mu_{\tilde{A}^I}(x) + \nu_{\tilde{A}^I}(x) \leqslant 1$. The functions $f_{\tilde{A}}, k_{\tilde{A}}$ are strictly increasing in $[a_1, a_2)$ and $(b_3, b_4]$ respectively and functions $g_{\tilde{A}}, h_{\tilde{A}}$ are strictly decreasing in $(a_3, a_4]$ and $[b_1, b_2)$ respectively.

2.3. Triangular intuitionistic fuzzy number (TIFN)

A TIFN [43], denoted by $\tilde{A}^I = (a_1, a_2, a_3; b_1, a_2, b_3)$, is a subset of IFS in R with the following membership and non-membership functions respectively:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leqslant x < a_2, \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leqslant x \leqslant a_3, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_{\tilde{A}^I}(x) = \begin{cases} \dfrac{x - a_2}{b_1 - a_2}, & b_1 \leqslant x \leqslant a_2, \\ \dfrac{x - a_2}{b_3 - a_2}, & a_2 \leqslant x \leqslant b_3, \\ 1, & \text{otherwise} \end{cases}$$

where $b_1 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant b_3$ and TIFN is graphically represented as in Figure 1.

Here, it may be noted that $\mu_{\tilde{A}^I}(x)$ and $\nu_{\tilde{A}^I}(x)$ are increasing functions with the constant rate in $[a_1, a_2]$ and $[a_2, b_3]$ respectively, and decreasing functions with the constant rate in $[a_2, a_3]$ and $[b_1, a_2]$ respectively. In this paper, for simplicity, triangular intuitionistic fuzzy numbers are used to represent the uncertainty. Data for the case study in this work has been randomly generated from the crisp data and fuzzy data in [44, 45] where crisp values have been kept as modal values of intuitionistic triangular fuzzy data.

2.4. Arithmetic operations on TIFNs

Let $\tilde{A}^I_1 = (a_1, a_2, a_3; b_1, a_2, b_3)$ and $\tilde{A}^I_2 = (a_1^{'}, a_2^{'}, a_3^{'}; b_1^{'}, a_2^{'}, b_3^{'})$ be two TIFNs

Addition: $\tilde{A}_1^I \oplus \tilde{A}_2^I = (a_1 + a_1^{'}, a_2 + a_2^{'}, a_3 + a_3^{'}; b_1 + b_1^{'}, a_2 + a_2^{'}, b_3 + b_3^{'})$. Subtraction: $\tilde{A}_1^I \Theta \tilde{A}_2^I = (a_1 - a_3^{'}, a_2 - a_2^{'}, a_3 - a_1^{'}; b_1 - b_3^{'}, a_2 - a_2^{'}, b_3 - b_1^{'})$. Multiplication: $\tilde{A}_1^I \otimes \tilde{A}_2^I = (p_1, p_2, p_3; p_1^{'}, p_2, p_3^{'})$, where

$$p_1 = \min\left\{a_1 a_1^{'}, a_1 a_3^{'}, a_3 a_1^{'}, a_3 a_3^{'}\right\},\ p_3 = \max\left\{a_1 a_1^{'}, a_1 a_3^{'}, a_3 a_1^{'}, a_3 a_3^{'}\right\}$$
$$p_1^{'} = \min\left\{b_1 b_1^{'}, b_1 b_3^{'}, b_3 b_1^{'}, b_3 b_3^{'}\right\},\ p_3^{'} = \max\left\{b_1 b_1^{'}, b_1 b_3^{'}, b_3 b_1^{'}, b_3 b_3^{'}\right\},\ p_2 = a_2 a_2^{'}$$

For $\tilde{A}_1^I, \tilde{A}_2^I > 0$, we have

$$\tilde{A}_1^I \otimes \tilde{A}_2^I = (a_1 a_1^{'}, a_2 a_2^{'}, a_3 a_3^{'}; b_1 b_1^{'}, a_2 a_2^{'}, b_3 b_3^{'})$$

Scalar multiplication: $k\tilde{A}^I = \{ka_1, ka_2, ka_3; kb_1, ka_2, kb_3\}$ for $k > 0$ and

$$k\tilde{A}^I = \{ka_3, ka_2, ka_1; kb_3, ka_2, kb_1\} \quad \text{for} \quad k < 0.$$

2.5. Expected value and ordering of TIFNs

The expected value [46] of a TIFN is given by (proof is omitted)

$$EV(\tilde{A}^I) = (b_1 + a_1 + 4a_2 + a_3 + b_3)/8. \tag{2.1}$$

An accuracy function $H(\tilde{A}^I) = ((a_1 + 2a_2 + a_3) + (b_1 + 2a_2 + b_3))/8$ for TIFN was defined by Nagoogani and Ponnalagu [47] which is essentially the same as the expected value $EV(A)$. Based on the accuracy function following order has been established:

$$\tilde{A}^I \geqslant \tilde{B}^I \Leftrightarrow H(\tilde{A}^I) \geqslant H(\tilde{B}^I) \tag{2.2}$$

where $H(\tilde{A}^I)$ and $H(\tilde{B}^I)$ are the accuracy functions for the TIFN $\tilde{A}^I$ and $\tilde{B}^I$ respectively.

3. Methodology

Consider n DMUs each having m inputs and s outputs. Let x_{ij}, and y_{rj} be the amount of $i^{th}(i = 1, 2, \ldots, m)$ input and rth ($r = 1, 2, \ldots, s$) output, respectively, of jth ($j = 1, 2, \ldots, n$) DMU. Let u_i and v_r denote the weights for the ith input and rth output, respectively, under the efficiency evaluation of the kth DMU. Following BCC DEA model was first formulated by Banker *et al.* [9] to calculate the efficiency of the kth DMU under crisp inputs and outputs:

$$\text{Max } E_k = \frac{\sum\limits_{r=1}^{s} v_r y_{rk} + v_0}{\sum\limits_{i=1}^{m} u_i x_{ik}}$$

subject to

$$\frac{\sum\limits_{r=1}^{s} v_r y_{rj} + v_0}{\sum\limits_{i=1}^{m} u_i x_{ij}} \leqslant 1, \quad j = 1,\ 2, \ldots, n,$$

$$u_i \geqslant \varepsilon > 0, \quad i = 1, 2, \ldots, m,$$

$$v_r \geqslant \varepsilon > 0, \quad r = 1, 2, \ldots, s,$$

v_0 is unrestricted in sign.

Here, ε is a small non-Archimedean number [48]. Above model can be converted to an equivalent multiplier form (linear programming) given below by applying Charnes-Cooper transformation [49]. (BCC–DEA)

$$\begin{aligned}
&\text{Max } E_k = \sum_{r=1}^{s} v_r y_{rk} + v_0 \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{m} u_i x_{ij} = 1, \\
&\qquad \sum_{r=1}^{s} v_r y_{rj} - \sum_{i=1}^{m} u_i x_{ij} + v_0 \leqslant 0, \quad j = 1,\ 2, \ldots, n, \\
&\qquad u_i \geqslant \varepsilon > 0, \quad i = 1, 2, \ldots, m, \\
&\qquad v_r \geqslant \varepsilon > 0, \quad r = 1, 2, \ldots, s, \\
&\qquad v_0 \text{ is unrestricted in sign.}
\end{aligned}$$

In conventional DEA models, input and output weights are assumed to be non-negative. As a result, weights for inputs and outputs with very high and very low values tend to be zero, and therefore such inputs and outputs often ignored in the efficiency evaluation. To overcome this problem, Charnes and Cooper [48] introduced a small non-Archimedean number ε as in above models. However, subsequently, issues were faced regarding the fixed value selection of ε which is required by the models to actually calculate the numeric value of efficiency Thompson *et al.* [12] suggested the Assurance Region (AR) approach in the form of pairwise relative importance of input and output weights. AR approach proved to be a robust strategy to address the difficulty in assigning a fixed value to ε as well as to deal with the situations when decision makers assign relative importance to various input and output parameters. In AR, the relative importance of weights is expressed in the form of following mathematical inequalities:

$$\frac{l_p^o}{u_q^o} \leqslant \frac{v_p}{v_q} \leqslant \frac{u_p^o}{l_q^o}, \quad p < q = 2, 3, \ldots, s, \tag{3.1}$$

$$\frac{l_p^I}{u_q^I} \leqslant \frac{u_p}{u_q} \leqslant \frac{u_p^I}{l_q^I}, \quad p < q = 2, 3, \ldots, m. \tag{3.2}$$

For the ease of algebraic simplification,

$$\text{let } g_{pq}^l = \frac{l_p^o}{u_q^o}, g_{pq}^u = \frac{u_p^o}{l_q^o}, h_{pq}^l = \frac{l_p^I}{u_q^I}, \quad \text{and} \quad h_{pq}^u = \frac{u_p^I}{l_q^I}. \tag{3.3}$$

Adding these AR constraint in (BCC-DEA), we get the following DEA/AR model.

(DEA/AR)

$$\text{Max } E_k = \sum_{r=1}^{s} v_r y_{rk} + v_0$$

subject to

$$\sum_{i=1}^{m} u_i x_{ij} = 1,$$

$$\sum_{r=1}^{s} v_r y_{rj} - \sum_{i=1}^{m} u_i x_{ij} + v_0 \leqslant 0, \quad j = 1,\ 2, \ldots, n,$$

$$-v_p + g_{pg}^l v_q \leqslant 0,\ \ v_p - g_{pg}^u v_q \leqslant 0, \quad \forall p < q = 2, 3, \ldots, s,$$

$$-u_p + h_{pg}^l u_q \leqslant 0,\ \ u_p - h_{pg}^u u_q \leqslant 0, \quad \forall p < q = 2, 3, \ldots, m,$$

$$u_i \geqslant \varepsilon > 0, \quad i = 1, 2, \ldots, m,$$

$$v_r \geqslant \varepsilon > 0, \quad r = 1, 2, \ldots, s,$$

v_0 is unrestricted in sign.

3.1. Proposed Intuitionistic fuzzy DEA/AR

Consider again n homogeneous DMUs, and each DMU has m intuitionistic fuzzy inputs and s intuitionistic fuzzy outputs. Let $\tilde{x}_{ij}^I = (x_{ij}^1, x_{ij}^2, x_{ij}^3; x_{ij}^{1'}, x_{ij}^2, x_{ij}^{3'})$ and $\tilde{y}_{rj}^I = (y_{rj}^1, y_{rj}^2, y_{rj}^3; y_{rj}^{1'}, y_{rj}^2, y_{rj}^{3'})$ be the $i^{th}\,(i = 1, 2, \ldots, m)$ and rth $(r = 1, 2, \ldots, s)$ intuitionistic fuzzy input and output, respectively, of the jth $(j = 1, 2, \ldots, n)$ DMU. Then intuitionistic fuzzy DEA/AR (IFDEA/AR) to measure the efficiency of the kth DMU is given by

(IFDEA/AR)

$$\text{Max } \tilde{E}_k^I = \sum_{r=1}^{s} \tilde{v}_r^I \otimes \tilde{y}_{rk}^I + v_0$$

subject to

$$\sum_{i=1}^{m} \tilde{u}_i^I \otimes \tilde{x}_{ij}^I = \tilde{1},$$

$$\sum_{r=1}^{s} \tilde{v}_r^I \otimes y_{rj}^I \varTheta \sum_{i=1}^{m} \tilde{u}_i^I \otimes \tilde{x}_{ij}^I + v_0 \leqslant \tilde{0}, \quad j = 1,\ 2, \ldots, n,$$

$$g_{pg}^l \tilde{v}_q^I \varTheta \tilde{v}_p^I \leqslant \tilde{0},\ \ \tilde{v}_p^I \varTheta g_{pg}^u \tilde{v}_q^I \leqslant \tilde{0}, \quad \forall p < q = 2, 3, \ldots, s,$$

$$h_{pg}^l \tilde{u}_q^I \varTheta \tilde{u}_p^I \leqslant \tilde{0},\ \ \tilde{u}_p^I \varTheta h_{pg}^u \tilde{u}_q^I \leqslant \tilde{0}, \quad \forall p < q = 2, 3, \ldots, m,$$

$$\tilde{u}_i^I \geqslant \varepsilon > 0, \quad i = 1, 2, \ldots, m,$$

$$\tilde{v}_r^I \geqslant \varepsilon > 0, \quad r = 1, 2, \ldots, s,$$

v_0 is unrestricted in sign.

Here, $\tilde{u}_i^I = (u_i^1, u_i^2, u_i^3; u_i^{1'}, u_i^2, u_i^{3'})$ and $\tilde{v}_r^I = (v_r^1, v_r^2, v_r^3; v_r^{1'}, v_r^2, v_r^{3'})$ are the intuitionistic fuzzy weights of the ith intuitionistic fuzzy input and rth intuitionistic fuzzy output, respectively.

Substituting the values of intuitionistic fuzzy inputs, outputs, and weights, IFDEA/AR model can be rewritten as

$$
\begin{aligned}
&\text{Max } \tilde{E}_k^I = \sum_{r=1}^{s} (v_r^1, v_r^2, v_r^3; v_r^{1'}, v_r^2, v_r^{3'}) \otimes (y_{rk}^1, y_{rk}^2, y_{rk}^3; y_{rk}^{1'}, y_{rk}^2, y_{rk}^{3'}) + v_0 \\
&\text{subject to} \\
&\qquad \sum_{i=1}^{m} (u_i^1, u_i^2, u_i^3; u_i^{1'}, u_i^2, u_i^{3'}) \otimes (x_{ik}^1, x_{ik}^2, x_{ik}^3; x_{ik}^{1'}, x_{ik}^2, x_{ik}^{3'}) = (1,\ 1,\ 1; 1,\ 1,\ 1), \\
&\qquad \sum_{r=1}^{s} (v_r^1, v_r^2, v_r^3; v_r^{1'}, v_r^2, v_r^{3'}) \otimes (y_{rj}^1, y_{rj}^2, y_{rj}^3; y_{rj}^{1'}, y_{rj}^2, y_{rj}^{3'}) \\
&\qquad \Theta \sum_{i=1}^{m} (u_i^1, u_i^2, u_i^3; u_i^{1'}, u_i^2, u_i^{3'}) \otimes (x_{ij}^1, x_{ij}^2, x_{ij}^3; x_{ij}^{1'}, x_{ij}^2, x_{ij}^{3'}) + v_0 \\
&\qquad \leqslant (0,0,0;0,0,0), \quad j = 1,\ 2, \ldots, n, \\
&\qquad g_{pq}^l \left(v_q^1, v_q^2, v_q^3; v_q^{1'}, v_q^2, v_q^{3'} \right) \Theta \left(v_p^1, v_p^2, v_p^3; v_p^{1'}, v_p^2, v_p^{3'} \right) \leqslant (0,0,0;0,0,0), \quad \forall p < q = 2,3,\ldots,s, \\
&\qquad \left(v_p^1, v_p^2, v_p^3; v_p^{1'}, v_p, v_q^{3'} \right) \Theta g_{pq}^u \left(v_q^1, v_q^2, v_q^3; v_q^{1'}, v_q^2, v_q^{3'} \right) \leqslant (0,0,0;0,0,0), \quad \forall p < q = 2,3,\ldots,s, \\
&\qquad h_{pq}^l \left(u_q^1, u_q^2, u_q^3; u_q^{1'}, u_q^2, u_q^{3'} \right) \Theta \left(u_p^1, u_p^2, u_p^3; u_p^{1'}, u_p^2, u_p^{3'} \right) \leqslant (0,0,0;0,0,0), \quad \forall p < q = 2,3,\ldots,m, \\
&\qquad \left(u_p^1, u_p^2, u_p^3; u_p^{1'}, u_p^2, u_p^{3'} \right) \Theta h_{pq}^u \left(u_q^1, u_q^2, u_q^3; u_q^{1'}, u_q^2, u_q^{3'} \right) \leqslant (0,0,0;0,0,0), \quad \forall p < q = 2,3,\ldots,m, \\
&\qquad \left(u_i^1, u_i^2, u_i^3; u_i^{1'}, u_i^2, u_i^{3'} \right) \geqslant \varepsilon > 0, \quad i = 1,2,\ldots,m, \\
&\qquad \left(v_r^1, v_r^2, v_r^3; v_r^{1'}, v_r^2, v_r^{3'} \right) \geqslant \varepsilon > 0, \quad r = 1,2,\ldots,s, \\
&\qquad v_0 \text{ is unrestricted in sign.}
\end{aligned}
$$

By applying arithmetic operations on intuitionistic TIFN fuzzy numbers (as given in Sect. 2.4), above model can be transformed into the following model:

$$
\begin{aligned}
&\text{Max } \tilde{E}_k^I = \left(\sum_{r=1}^{s} v_r^1 y_{rk}^1, \sum_{r=1}^{s} v_r^2 y_{rk}^2, \sum_{r=1}^{s} v_r^3 y_{rk}^3; \sum_{r=1}^{s} v_r^{1'} y_{rk}^{1'}, \sum_{r=1}^{s} v_r^2 y_{rk}^2, \sum_{r=1}^{s} v_r^{3'} y_{rk}^{3'} \right) + v_0 \\
&\text{subject to} \\
&\qquad \left(\sum_{i=1}^{m} u_i^1 x_{ik}^1, \sum_{i=1}^{m} u_i^2 x_{ik}^2, \sum_{i=1}^{m} u_i^3 x_{ik}^3; \sum_{i=1}^{m} u_i^{1'} x_{ik}^{1'}, \sum_{i=1}^{m} u_i^2 x_{ik}^2, \sum_{i=1}^{m} u_i^{3'} x_{ik}^{3'} \right) = (1,\ 1,\ 1; 1,\ 1,\ 1), \\
&\qquad \begin{pmatrix} \sum\limits_{r=1}^{s} v_r^1 y_{rj}^1 - \sum\limits_{i=1}^{m} u_i^3 x_{ij}^3, \sum\limits_{r=1}^{s} v_r^2 y_{rj}^2 - \sum\limits_{i=1}^{m} u_i^2 x_{ij}^2, \sum\limits_{r=1}^{s} v_r^3 y_{rj}^3 - \sum\limits_{i=1}^{m} u_i^1 x_{ij}^1; \\ \sum\limits_{r=1}^{s} v_r^{1'} y_{rj}^{1'} - \sum\limits_{i=1}^{m} u_i^{3'} x_{ij}^{3'}, \sum\limits_{r=1}^{s} v_r^2 y_{rj}^2 - \sum\limits_{i=1}^{m} u_i^2 x_{ij}^2, \sum\limits_{r=1}^{s} v_r^{3'} y_{rj}^{3'} - \sum\limits_{i=1}^{m} u_i^{1'} x_{ij}^{1'} \end{pmatrix} \\
&\qquad\qquad + v_0 \leqslant (0,0,0;0,0,0), j = 1,\ 2, \ldots, n,
\end{aligned}
$$

$$
\begin{aligned}
&\left(g_{pq}^{l}v_{q}^{1}-v_{p}^{3},g_{pq}^{l}v_{q}^{2}-v_{p}^{2},g_{pq}^{l}v_{q}^{3}-v_{p}^{1};g_{pq}^{l}v_{q}^{1'}-v_{p}^{3'},g_{pq}^{l}v_{q}^{2}-v_{p}^{2},g_{pq}^{l}v_{q}^{3'}-v_{p}^{1'}\right)\\
&\qquad\qquad \leqslant (0,0,0;0,0,0), \forall p<q=2,3,\ldots,s,\\
&\left(v_{p}^{1}-g_{pq}^{u}v_{q}^{3},v_{p}^{2}-g_{pq}^{u}v_{q}^{2},v_{p}^{3}-g_{pq}^{u}v_{q}^{1};v_{p}^{1'}-g_{pq}^{u}v_{q}^{3'},v_{p}^{2}-g_{pq}^{u}v_{q}^{2},v_{q}^{3'}-g_{pq}^{u}v_{q}^{1'}\right)\\
&\qquad\qquad \leqslant (0,0,0;0,0,0), \forall p<q=2,3,\ldots,s,\\
&\left(h_{pq}^{l}u_{q}^{1}-u_{p}^{3},h_{pq}^{l}u_{q}^{2}-u_{p}^{2},h_{pq}^{l}u_{q}^{3}-u_{p}^{1};h_{pq}^{l}u_{q}^{1'}-u_{p}^{3'},h_{pq}^{l}u_{q}^{2}-u_{p}^{2},h_{pq}^{l}u_{q}^{3'}-u_{p}^{1'}\right)\\
&\qquad\qquad \leqslant (0,0,0;0,0,0), \forall p<q=2,3,\ldots,m,\\
&\left(u_{p}^{1}-h_{pq}^{u}u_{q}^{3},u_{p}^{2}-h_{pq}^{u}u_{q}^{2},u_{p}^{3}-h_{pq}^{u}u_{q}^{1};u_{p}^{1'}-h_{pq}^{u}u_{q}^{3'},u_{p}^{2}-h_{pq}^{u}u_{q}^{2},u_{p}^{3'}-h_{pq}^{u}u_{q}^{1'}\right)\\
&\qquad\qquad \leqslant (0,0,0;0,0,0), \forall p<q=2,3,\ldots,m,\\
&\left(u_{i}^{1},u_{i}^{2},u_{i}^{3};u_{i}^{1'},u_{i}^{2},u_{i}^{3'}\right)\geqslant \varepsilon>0, \quad i=1,2,\ldots,m,\\
&\left(v_{r}^{1},v_{r}^{2},v_{r}^{3};v_{r}^{1'},v_{r}^{2},v_{r}^{3'}\right)\geqslant \varepsilon>0, \quad r=1,2,\ldots,s,\\
&v_0 \text{ is unrestricted in sign.}
\end{aligned}
$$

Above model is intuitionistic fuzzy linear programming problem (IFLPP) which can easily be transformed into crisp linear programming problem by applying ordering method based on expected value approach described in Section 2.5. The crisp LPP is given as follows:

$$
\text{Max } EV(\tilde{E}_k^I)=EV\left(\left(\sum_{r=1}^{s}v_r^1y_{rk}^1,\sum_{r=1}^{s}v_r^2y_{rk}^2,\sum_{r=1}^{s}v_r^3y_{rk}^3;\sum_{r=1}^{s}v_r^{1'}y_{rk}^{1'},\sum_{r=1}^{s}v_r^2y_{rk}^2,\sum_{r=1}^{s}v_r^{3'}y_{rk}^{3'}\right)+v_0\right)
$$

subject to

$$
\begin{aligned}
&EV\left(\left(\sum_{i=1}^{m}u_i^1x_{ik}^1,\sum_{i=1}^{m}u_i^2x_{ik}^2,\sum_{i=1}^{m}u_i^3x_{ik}^3;\sum_{i=1}^{m}u_i^{1'}x_{ik}^{1'},\sum_{i=1}^{m}u_i^2x_{ik}^2,\sum_{i=1}^{m}u_i^{3'}x_{ik}^{3'}\right)\right)=EV((1,1,1;1,1,1)),\\
&EV\left(\begin{pmatrix}\sum\limits_{r=1}^{s}v_r^1y_{rj}^1-\sum\limits_{i=1}^{m}u_i^3x_{ij}^3,\ \sum\limits_{r=1}^{s}v_r^2y_{rj}^2-\sum\limits_{i=1}^{m}u_i^2x_{ij}^2,\ \sum\limits_{r=1}^{s}v_r^3y_{rj}^3-\sum\limits_{i=1}^{m}u_i^1x_{ij}^1;\\ \sum\limits_{r=1}^{s}v_r^{1'}y_{rj}^{1'}-\sum\limits_{i=1}^{m}u_i^{3'}x_{ij}^{3'},\ \sum\limits_{r=1}^{s}v_r^2y_{rj}^2-\sum\limits_{i=1}^{m}u_i^2x_{ij}^2,\ \sum\limits_{r=1}^{s}v_r^{3'}y_{rj}^{3'}-\sum\limits_{i=1}^{m}u_i^{1'}x_{ij}^{1'}\end{pmatrix}+v_0\right)\\
&\qquad\qquad \leqslant EV((0,0,0;0,0,0)), \quad j=1,\ 2,\ldots,n,\\
&EV\left(\left(g_{pq}^{l}v_{q}^{1}-v_{p}^{3},g_{pq}^{l}v_{q}^{2}-v_{p}^{2},g_{pq}^{l}v_{q}^{3}-v_{p}^{1};g_{pq}^{l}v_{q}^{1'}-v_{p}^{3'},g_{pq}^{l}v_{q}^{2}-v_{p}^{2},g_{pq}^{l}v_{q}^{3'}-v_{p}^{1'}\right)\right)\\
&\qquad\qquad \leqslant EV\left((0,0,0;0,0,0)\right), \quad \forall p<q=2,3,\ldots,s,\\
&EV\left(\left(v_{p}^{1}-g_{pq}^{u}v_{q}^{3},v_{p}^{2}-g_{pq}^{u}v_{q}^{2},v_{p}^{3}-g_{pq}^{u}v_{q}^{1};v_{p}^{1'}-g_{pq}^{u}v_{q}^{3'},v_{p}^{2}-g_{pq}^{u}v_{q}^{2},v_{q}^{3'}-g_{pq}^{u}v_{q}^{1'}\right)\right)\\
&\qquad\qquad \leqslant EV\left((0,0,0;0,0,0)\right), \quad \forall p<q=2,3,\ldots,s,\\
&EV\left(\left(h_{pq}^{l}u_{q}^{1}-u_{p}^{3},h_{pq}^{l}u_{q}^{2}-u_{p}^{2},h_{pq}^{l}u_{q}^{3}-u_{p}^{1};h_{pq}^{l}u_{q}^{1'}-u_{p}^{3'},h_{pq}^{l}u_{q}^{2}-u_{p}^{2},h_{pq}^{l}u_{q}^{3'}-u_{p}^{1'}\right)\right)\\
&\qquad\qquad \leqslant EV\left((0,0,0;0,0,0)\right), \quad \forall p<q=2,3,\ldots,m,
\end{aligned}
$$

$$
\begin{aligned}
&EV\left(\left(u_p^1-h_{pq}^u u_q^3, u_p^2-h_{pq}^u u_q^2, u_p^3-h_{pq}^u u_q^1; u_p^{1'}-h_{pq}^u u_q^{3'}, u_p^2-h_{pq}^u u_q^2, u_p^{3'}-h_{pq}^u u_q^{1'}\right)\right)\\
&\qquad\qquad \leqslant EV\left((0,0,0;0,0,0)\right), \quad \forall p<q=2,3,\ldots,m,\\
&EV\left(\left(u_i^1, u_i^2, u_i^3; u_i^{1'}, u_i^2, u_i^{3'}\right)\right)\\
&\qquad\qquad \geqslant \varepsilon>0, \quad i=1,2,\ldots,m,\\
&EV\left(\left(v_r^1, v_r^2, v_r^3; v_r^{1'}, v_r^2, v_r^{3'}\right)\right)\geqslant \varepsilon>0, \quad r=1,2,\ldots,s,\\
&v_0 \ \text{ is unrestricted in sign.}
\end{aligned}
$$

Applying equations (2.1) and (2.2) on the above model, we obtain (IFAR)

$$
\text{Max } E_k^I=\frac{1}{8}\left(\sum_{r=1}^{s}\left(v_r^{1'}y_{rk}^{1'}+v_r^1y_{rk}^1+4v_r^2y_{rk}^2+v_r^3y_{rk}^3+v_r^{3'}y_{rk}^{3'}\right)\right)+v_0
$$

subject to

$$
\begin{aligned}
&\sum_{i=1}^{m}\left(u_i^{1'}x_{ik}^{1'}+u_i^1x_{ik}^1+4u_i^2x_{ik}^2+u_i^3x_{ik}^3+u_i^{3'}x_{ik}^{3'}\right)=8,\\
&\sum_{r=1}^{s}\left(v_r^{1'}y_{rj}^{1'}+v_r^1y_{rj}^1+4v_r^2y_{rj}^2+v_r^3y_{rj}^3+v_r^{3'}y_{rj}^{3'}\right)\\
&-\sum_{i=1}^{m}\left(u_i^{1'}x_{ij}^{1'}+u_i^1x_{ij}^1+4u_i^2x_{ij}^2+u_i^3x_{ij}^3+u_i^{3'}x_{ij}^{3'}\right)+8v_0\leqslant 0, \quad j=1,\ 2,\ldots,n,\\
&g_{pq}^l\left(v_q^{1'}+v_q^1+4v_q^2+v_q^3+v_q^{3'}\right)-v_p^{3'}-v_p^3-4v_p^2-v_p^1-v_p^{1'}\leqslant 0, \quad \forall p<q=2,3,\ldots,s,\\
&v_p^{1'}+v_p^1+4v_p^2+v_p^3+v_q^{3'}-g_{pq}^u\left(v_q^{3'}+v_q^3+4v_q^2+v_q^1+v_q^{1'}\right)\leqslant 0, \quad \forall p<q=2,3,\ldots,s,\\
&h_{pq}^l\left(u_q^{1'}+u_q^1+4u_q^2+u_q^3+u_q^{3'}\right)-u_p^{3'}-u_p^3-4u_p^2-u_p^2-u_p^{1'}\leqslant 0, \quad \forall p<q=2,3,\ldots,m,\\
&u_p^{1'}+u_p^1+4u_p^2+u_p^3+u_p^{3'}-h_{pq}^u\left(u_q^{3'}+u_q^3+4u_q^2+u_q^1+u_q^{1'}\right)\leqslant 0, \quad \forall p<q=2,3,\ldots,m,\\
&u_i^{1'}+u_i^1+4u_i^2+u_i^3+u_i^{3'}\geqslant \varepsilon>0, \quad i=1,2,\ldots,m,\\
&v_r^{1'}+v_r^1+4v_r^2+v_r^3+v_r^3\geqslant \varepsilon>0, \quad r=1,2,\ldots,s\\
&u_i^{3'}\geqslant u_i^3\geqslant u_i^2\geqslant u_i^1\geqslant u_i^{1'}\geqslant \varepsilon>0, \quad i=1,2,\ldots,m,\\
&v_r^{3'}\geqslant v_r^3\geqslant v_r^2\geqslant v_r^1\geqslant v_r^{1'}\geqslant \varepsilon>0, \quad r=1,2,\ldots,s.\\
&v_0 \ \text{ is unrestricted in sign.}
\end{aligned}
$$

The model (IFAR) is a crisp LPP model and E_k^{I*} is known as IFAR efficiency of the kth DMU.

Definition 3.1. The kth DMU is IFAR efficient if E_k^{I*} =1 otherwise it is IFAR inefficient.

Definition 3.2. The intuitionistic fuzzy efficiency of kth DMU under assurance region is defined as $\tilde{E}_k^{I*} = \sum_{r=1}^{s} \tilde{v}_r^{I*} \otimes \tilde{y}_{rk}^I + v_0$, where $\tilde{v}_r^{I*} = (v_r^{1*}, v_r^{2*}, v_r^{3*}; v_r^{1'*}, v_r^{2*}, v_r^{3'*})$ is the rth optimal weight in crisp IFAR model.

3.2. Complete Ranking of DMUs under Intuitionistic Fuzzy Environment

DEA has been proved a very effective nonparametric technique in identifying efficient and inefficient DMUs. Using the idea of best self–evaluation, DEA calculates a most favorable set of weights for input and output parameters to maximize the efficiency of the observed DMU. However, its inability to distinguish among efficient DMUs and nature of self-evaluation has been criticized. DEA provides the efficiency score as 1 (or 100%) to all efficient DMUs despite the fact that these DMUs may differ in actual performance when evaluated w.r.t. a different set of input and output weights. Thus, the different methods of ranking the DMUs proposed in DEA literature (Refs. [50] for a review of these methods). Sexton *et al.* [51] developed the concept of cross efficiency to rank DMUs using DEA. It was later studied by Doyle and Green [52]. In cross efficiency, DMUs are ranked based on the idea of peer evaluation. Cross evaluation of each DMU is done not only using self-evaluation but also using peer–evaluation. Cross evaluation overcomes the problem of unrealistic weight selection without needing prior weight restriction information from the experts. Ranking or ordering of DMUs is another main advantage of the cross-evaluation. Doyle and Green [52] observed that the case of alternate optima (non–unique set of weights) reduces the utility of the cross–efficiency. Cross–efficiency scores of the remaining DMUs will depend on which particular set of DEA weights is used to calculate the efficiency of the observed DMU. To address the difficulty of non–uniqueness of weights, Sexton *et al.* [51] and Doyle and Green [52] formulated the aggressive (benevolent) models in order to select the optimal weights so that these weights not only maximize the efficiency of the observed DMU but also minimize (maximize) the efficiency of the remaining DMUs. Cross–efficiency has found a wide variety of applications, for example, preference voting [53], Railways [54], social welfare [55], electricity distribution [56], Sports [57], and others.

3.2.1. Proposed Cross-efficiency approach in IFDEA/AR

Considering the intuitionistic fuzzy relative efficiency of a given DMUs as described in definition 2, We can extend the usual concept of cross efficiency [51,52] to IFAR environment.

Definition 3.3. If $\tilde{v}_d^{I*} = (v_d^{1*}, v_d^{2*}, v_d^{3*}; v_d^{1'*}, v_d^{2*}, v_d^{3'*})$ is an optimal intuitionistic weight (optimal solution to (IFAR)) for the d^{th} DMU, then the intuitionistic fuzzy cross efficiency of the jth $(j = 1, 2, \ldots., n)$ DMU is defined as an intuitionistic fuzzy number $\tilde{E}_{dj}^I$ where

$$\tilde{E}_{dj}^I = \frac{\sum_{r=1}^{s} \tilde{v}_d^{I*} \otimes \tilde{y}_{rj}^I + v_0^*}{\sum_{i=1}^{m} \tilde{u}_d^{I*} \otimes \tilde{x}_{ij}^I}.$$

Next, as it is usually done in crisp cross efficiency, IFAR intuitionistic fuzzy efficiency score $\bar{\tilde{E}}_j^I$ of the jth $(j = 1, 2, \ldots, n)$ DMU is the average of all $\tilde{E}_{dj}^I$ for all DMUs $j = 1, 2, \ldots, n$, *i.e.*

$$\bar{\tilde{E}}_j^I = \frac{1}{n} \sum_{d=1}^{n} \tilde{E}_{dj}^I. \tag{3.4}$$

By applying intuitionistic fuzzy arithmetic provided in Section 2.4, intuitionistic fuzzy cross efficiency (IFCE) score of the jth DMU is an intuitionistic fuzzy number of the following form

$$\bar{\tilde{E}}_j^I = \left(E_j^1, E_j^2, E_j^2; E_j^{1'}, E_j^2, E_j^{3'} \right)$$

Finally, the complete ranking of DMU can be made by using the ordering method in equation (2.2) for IFN as follows:

Let $\tilde{\bar{E}}_j^I$ and $\tilde{\bar{E}}_k^I$ IFCE scores of the jth and kth DMU respectively, then

$$\tilde{\bar{E}}_j^I \geqslant \tilde{\bar{E}}_j^I \Leftrightarrow H\left(\tilde{\bar{E}}_j^I\right) \geqslant H\left(\tilde{\bar{E}}_k^I\right). \tag{3.5}$$

4. A case of flexible manufacturing systems

In this section, a case of flexible manufacturing system selection has been considered to illustrate the applicability of the proposed IFDEA/AR approach. The data for this empirical analysis has been taken from the study of Shiang-Tai Liu [45] which was originally taken from the work of Shang and Sueyoshi [44]. Twelve FMS alternative (DMUs) are considered each having two input parameters; 1) Capital and Operating costs 2) Floor space requirement and three output parameters including improvements in; 1) Qualitative factor 2) Work in process (WIP) 3) Number of tardy jobs 4)Yield. The yield has been taken as throughput minus scrape and rework. Analytic hierarchy process has been used to assess the improvements in qualitative factors which include flexibility, learning, exposure to labor union unrest among others. Improvements in WIP, Number of tardy jobs, Yield has been calculated using simulation. For a detailed discussion on inputs and outputs, work by Shang and Sueyoshi [44] may be referred. We have modified the data (as shown in Tab. 1) from the study of Shiang–Tai Liu [45] by taking fuzzy inputs/outputs as intuitionistic fuzzy in the form of TIFN. Here, remaining crisp input/output can be taken as degenerated TIFN.

In the study of Shiang-Tai Liu [45], relative importance (in a scale of 1.0) for different input and output parameters has been defined in the form of intervals with lower and upper bounds as follows:

$$\begin{aligned}
v_1 &= [0.7500, 0.8333]: && \text{relative importance of capital and operating costs.}\\
v_2 &= [0.1667, 0.2500]: && \text{relative importance of floor space requirement.}\\
u_1 &= [0.4023, 0.4667]: && \text{relative importance of qualitative factor.}\\
u_2 &= [0.0795, 0.1361]: && \text{relative importance of WIP.}\\
u_3 &= [0.1392, 0.1850]: && \text{relative importance of no. of tardy jobs.}\\
u_4 &= [0.2766, 0.3146]: && \text{relative importance of yield.}
\end{aligned}$$

Now, AR for the relative importance of input and output parameters can be calculated (using Eqs. (3.1)–(3.3)) as below

$$\begin{aligned}
&h_{12}^l = 3, \quad h_{12}^u = 4.998, \quad g_{12}^l = 2.9559, g_{13}^l = 2.1746, \quad g_{14}^l = 1.2788\\
&g_{12}^u = 5.8704, \quad g_{13}^u = 3.3527, \quad g_{14}^u = 3.4291, \quad g_{23}^l = 0.4297, \quad g_{24}^l = 0.2527\\
&g_{23}^u = 0.9777, \quad g_{24}^u = 0.4920, \quad g_{34}^l = 0.4425, \quad g_{34}^u = 0.6686
\end{aligned}$$

i.e.

$$3 \leqslant \frac{u_1}{u_2} \leqslant 4.998,\ 2.9559 \leqslant \frac{v_1}{v_2} \leqslant 5.8704,\ 2.1746 \leqslant \frac{v_1}{v_3} \leqslant 3.3527,\ 1.2778 \leqslant \frac{v_1}{v_4} \leqslant 3.4291,$$
$$0.4297 \leqslant \frac{v_2}{v_3} \leqslant 0.9777,\ 0.2527 \leqslant \frac{v_2}{v_4} \leqslant 0.4920,\ 0.4425 \leqslant \frac{v_3}{v_4} \leqslant 0.6686.$$

We have applied the IFDEA/AR models developed to calculate IDEA/AR efficiency of FMS alternatives and same has been shown in Table 2.

IFCE score ($\tilde{\bar{E}}_j^I$) of any jth FMS alternative has been calculated using equations (3.4). Based on the IFCE score, the performance of the FMS alternatives has been ranked (using Eq. (3.5)) as shown in Table 3. It can be seen that alternative 4 is the best, followed by alternatives 5, 1, and so on. The above IFCE score associated ranking can help the manufacturing firm to choose the preferred alternatives when the inputs and outputs are represented as crisp and intuitionistic fuzzy data.

TABLE 1. Intuitionistic fuzzy input and output data for the FMS alternatives.

	Inputs	1			Improvements in outputs	
FMS Alternative	Capital and Operating cost ($00000)	Floor space needed (OOOsqft)	Qualitative (%)	WIP(IO)	No. of Tardy (%)	Yield (00)
1	(16.17,17.02,17.87,15.17,17.02,18.87)	5	42	(43.0,45.3,47.6;42,45.3,48.6)	(13.5,14.2,14.9;12.85,14.2,15.55)	(28.6,30.1,31.6;27.6,30.1,32.6)
2	(15.64,16.46,17.28;14.50,16.46,18.42)	4.5	39	(38.1,40.1,42.1;3 6.9,40.1,43.3)	(12.4,13.0,13.7;12.0,13.0,14.0)	(28.3,29.8,31.3;27.0,29.8,32.6)
3	(11.17,11.76,12.35; 10.67,11.76,12.85)	6	26	(37.6,39.6,41.6;37.0,39.6,42.2)	(13.1,13.8,14.5;12.73,13.8,14.87)	(23.3,24.5,25.7;22.5,25.5,26.5)
4	(9.99,10.52,11.05;9.54,10.52,11.5)	4	22	(34.2,36.0,37.8;33.5,36.0,38.5)	(10.7,11.3,11.9;10.4,11.3,12.2)	(23.8,25.0,26.3;22.8,25,27.2)
5	(9.03,9.50,9.97;8.73,9.50,10.27)	3.8	21	(32.5,34.2,35.9;31.5,34.2,3 6.9)	(11.4,12.0,12.6;10.8,12.0,13.2)	(19.4,20.4,21.4,-18.88,20.4,21.92)
6	(4.55,4.79,5.03 ;4.05,4.79,5.53)	5.4	10	(19.1,20.1,21.1; 18.2,20.1,22.0)	(4.8,5.0,5.3;4.6,5.0,5.4)	(15.7,16.5,17.3,-15.0,16.5,18.0)
7	(5.90,6.21,6.52;5.68,6.21,6.74)	6.2	14	(25.2,26.5,27.8;24.2,26.5,28.8)	(6.7,7.0,7.3;6.3,7.0,7.7)	(18.7,19.7,20.7,-17.8,19.7,21.6)
8	(10.56,11.12,11.68;9.96,11.12,12.28)	6	25	(34.1,35.9,37.7;33.7,35.9,38.1)	(8.6,9.0,9.5;8.4,9.0,9.6)	(23.5,24.7,25.9;22.5,24.7,26.9)
9	(3.49,3.67,3.85;3.30,3.67,4.04)	8	4	(16.5,17.4,18.3,'15.9,17.4,18.9)	(0.1,0.1,0.1;0.1,0.1;0.1)	(17.2,18.1,19.0;16.3,18.1,19.9)
10	(8.48,8.93,9.38;8.15,8.93,9.71)	7	16	(32.6,34.3,36.0,'31.6,34.3,37.0)	(6.2,6.5,6.8;6.0,6.5,7.0)	(19.6,20.6,21.6,18.5,20.6,22.7)
11	(16.85,17.74,18.63;15.85,17.74,19.63)	7.1	43	(43.3,45.6,47.9;42.0,45.6,49.2)	(13.3,14.0,14.7,-13.0,14.0,15.0)	(29.5,31.1,32.7;29.0,31.1,33.2)
12	(14.11,14.85,15.59;13.28,14.85,16.42)	6.2	27	(36.8,38.7,40.6;35.8,38.7,41.6)	(13.1,13.8,14.5,-12.6,13.8,15.0)	(24.1,25.4,26.7;23.8,25.4,27.0)

TABLE 2. Intuitionistic fuzzy efficiency and IFAR efficiency of FMS alternatives.

FMS Alternative (DMU)	Intuitionistic Fuzzy Efficiency	IFAR Efficiency
1	(0.8613, 0.8773, 1.379; 0.8497, 0.8773, 1.4007)	1
2	(0.5419, 0.5447, 0.5448; 0.5401, 0.5447, 4.1365)	0.9931
3	(0.4680, 0.4782, 0.4867; 0.4612, 0.4782, 4.6517)	0.9976
4	(0.9639, 0.9998, 1.0373; 0.9411, 0.9998, 1.0585)	1
5	(0.8451, 0.8593, 1.4283; 0.8358, 0.8593, 1.4539)	1
6	(0.5047, 0.5301, 0.5567; 0.4820, 0.5301, 4.0332)	1
7	(0.9622, 0.9652, 0.9684; 0.9595, 0.9652, 1.2492)	1
8	(0.6335, 0.6439, 0.6549; 0.6305, 0.6439, 3.2345)	0.9661
9	(1, 1, 1; 1, 1, 1)	1
10	(0.1837, 0.1862, 0.1866; 0.1821, 0.1862, 5.4492)	0.8435
11	(0.5979, 0.6310, 0.6641; 0.5869, 0.6310, 3.6270)	1
12	(0.4835, 0.4978, 0.5120; 0.4802, 0.4978, 3.0853)	0.819

TABLE 3. Ranking of the FMS alternatives.

FMS Alternative	IFCE	Ranking
1	0.8729	3
2	0.8557	6
3	0.8659	4
4	0.8824	1
5	0.8811	2
6	0.7609	9
7	0.8618	5
8	0.8396	8
9	0.5275	12
10	0.7239	10
11	0.8479	7
12	0.7119	11

4.1. Results validation

For results validation, we will compare our findings with those of Liu [45]. Liu [45] developed a fuzzy DEA/AR method to evaluate the efficiency of FMS alternatives when inputs and outputs are represented as fuzzy and crisp data. His work was based on the α-cuts approach for fuzzy numbers to calculate the lower and upper bounds of the fuzzy efficiency scores of the FMS alternatives. In his work, ranking of the alternatives was done based on the self–efficiency scores whereas in this study, ranking is based on the proposed cross efficiency approach for intuitionistic fuzzy data. A comparison of rankings of FMS alternatives, using the proposed approach with those of Liu (2014), has been shown in the Table 4. We notice that our approach is in agreement with Liu [45] for the ranking of alternatives of 2, 5, and 8. Furthermore, we notice that alternative 4 is at the top in our approach and ranked third in Liu [45]. Rankings of most of the other alternatives in the proposed approach differ by one or two positions with the work of Liu [45]. This difference in ranking is due to the fact that in the proposed approach ranking of alternatives are ranked based on the peer evaluation (cross efficiency) whereas they are ranked using self–evaluation in Liu [45].

5. CONCLUSIONS

This paper has proposed a new idea of assurance region method to handle the practical situations when some vague or imprecise input and output data are defined as intuitionistic fuzzy sets instead of fuzzy sets. The concept of intuitionistic fuzzy set allows defining the degree of non–membership which cannot simply be taken as

TABLE 4. Ranking of FMS alternatives using the proposed approach.

FMS alternative (DMU)	Proposed Model (IFDEA/AR)		Liu (2014)	
	Cross-efficiency (IFCE)	Ranking	Efficiency	Ranking
1	0.8729	3	0.8445	4
2	0.8557	6	0.8022	6
3	0.8659	4	0.7951	7
4	0.8824	1	0.8524	3
5	0.8811	2	0.8985	2
6	0.7609	9	0.8289	5
7	0.8618	5	0.9166	1
8	0.8396	8	0.7741	8
9	0.5275	12	0.5954	10
10	0.7239	10	0.5887	11
11	0.8479	7	0.7668	9
12	0.7119	11	0.4852	12

the complement of the degree of membership. This is useful in practical situations when experts/decision makers are not confident enough in assigning the degree of membership/ non–membership for the imprecise data. We have also proposed the generalization of the classical cross efficiency approach to the case of intuitionistic fuzzy DEA/AR. The case of twelve flexible manufacturing alternatives is discussed to illustrate the theoretical results developed in this paper. Using the expected value approach for intuitionistic fuzzy numbers, the ranking of the FMS alternatives have been provided. Results have been compared with the existing fuzzy DEA/AR approach. The difference in the ranking is mainly attributed to the fact that in our approach alternatives are ranked based on the peer evaluation (cross efficiency) whereas, in the existing work on fuzzy DEA/AR, they are ranked using self-evaluation. The results help the firms to select the "best" FMS alternative when inputs/outputs are represented as intuitionistic fuzzy data. Our approach of efficiency evaluation under assurance region is particularly useful when available information is not sufficient to define the impreciseness in input/output data using classical fuzzy set.

Acknowledgements. The author would like to express his gratitude to the esteemed Associate Editor and anonymous reviewers for their valuable comments and suggestions which have significantly improved the quality of the paper.

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