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A METHOD FOR SOLVING LINEAR PROGRAMMING WITH INTERVAL-VALUED TRAPEZOIDAL FUZZY VARIABLES

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Abstract. An efficient method to handle the uncertain parameters of a linear programming (LP) problem is to express the uncertain parameters by fuzzy numbers which are more realistic, and create a conceptual and theoretical framework for dealing with imprecision and vagueness. The fuzzy LP (FLP) models in the literature generally either incorporate the imprecisions related to the coefficients of the objective function, the values of the right-hand side, and/or the elements of the coefficient matrix. The aim of this article is to introduce a formulation of FLP problems involving interval-valued trapezoidal fuzzy numbers for the decision variables and the right-hand-side of the constraints. We propose a new method for solving this kind of FLP problems based on comparison of interval-valued fuzzy numbers by the help of signed distance ranking. To do this, we first define an auxiliary problem, having only interval-valued trapezoidal fuzzy cost coefficients, and then study the relationships between these problems leading to a solution for the primary problem. It is demonstrated that study of LP problems with interval-valued trapezoidal fuzzy variables gives rise to the same expected results as those obtained for LP with trapezoidal fuzzy variables.

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1. Introduction

The classical linear programming (LP) problem is to find the minimum or maximum values of a linear function under constraints represented by linear inequalities or equations. In many practical situations, it is not reasonable to require that the constraints or the objective function in LP problems be specified in precise, crisp terms. In such situations, it is desirable to use some type of fuzzy linear programming (FLP) problem. The FLP models in the literature generally either incorporate the imprecision related to the coefficients of the objective function, the values of the right-hand-side, and/or the elements of the coefficient matrix. In the past four decades, numerous researchers have studied various properties of the FLP problems and proposed different models for solving LP problems with fuzzy data. It would be almost impossible to cite all of them, and therefore in the following we will concentrate ourselves only on those contributions close to the topic considered in this paper. Tanaka and Asai [24] initially proposed a probabilistic linear programming formulation, where the coefficients of decision variables were crisp, while decision variables were obtained as fuzzy numbers. Verdegay [26] presented a concept of fuzzy objective based on the fuzzification principle and then in accordance with this concept solved

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the fuzzy linear mathematical programming problem. Ganesan and Veeramani [11] introduced a new method for solving linear programs with symmetric trapezoidal fuzzy numbers. Ebrahimnejad *et al.* [7] generalized their method for bounded linear programs with symmetric trapezoidal fuzzy numbers. Lotfi *et al.* [19] discussed full FLP (FFLP) problems in which all parameters and variables are triangular fuzzy numbers. Liang [17] applied the possibilistic linear programming approach as an application of fuzzy sets to solve fuzzy multi-objective project management decision problems. Moreover, Liang [18] developed a two-phase fuzzy goal programming method for solving the project management decision problems with multiple goals in uncertain environments. Hatami-Marbini and Tavana [12] proposed a new method for solving LP problems with fuzzy parameters. It provides an optimal solution that is not subject to specific restrictive conditions and supports the interactive participation of the decision maker in all steps of the decision-making process. Saati *et al.* [22] proposed a two-fold model which consists of two new methods for solving FLP problems in which the variables and the coefficients of the constraints are characterized by fuzzy numbers. Hatami-Marbini *et al.* [13] developed a new stepwise FLP model where fuzzy numbers are considered for the coefficients of the decision variables in the objective function, the coefficients of the decision variables in the constraints and the right-hand-side of the constraints. Ezzati *et al.* [9] based on a new lexicographic ordering on triangular fuzzy numbers, proposed a novel algorithm to solve FFLP problem by converting it into a multi-objective linear programming (MOLP) problem with three-objective functions. Kheirfam and Verdegay [14] studied the strictly sensitivity analysis for fuzzy quadratic programming when simultaneous and independently variations occur in both the right-hand-side of the constraints and the coefficients of the objective function. Ebrahimnejad and Verdegay [6] applied the fuzzy primal simplex algorithm and the fuzzy dual simplex algorithm for doing sensitivity analysis in linear programs with symmetric trapezoidal fuzzy numbers. Veeramani and Sumathi [25] proposed a solution procedure to solve fuzzy linear fractional programming (FLFP) problem where cost of the objective function, the resources and the technological coefficients are triangular fuzzy numbers. Kumar Das *et al.* [15] proposed a new approach for solving fully FLFP problems using the multi objective LP problem.

Although the FLP problem with all parameters as fuzzy numbers is the general case of the FLP, it may not be suitable for all FLP problems with different assumptions and sources of fuzziness. The FLP model proposed in this study belongs to those categories in which the decision variables and the right-hand-side of the constraints are represented by fuzzy numbers. This kind of fuzzy problems is known as linear programming with fuzzy variables (FVLP) and the most convenience method for solving it is based on the concept of comparison of fuzzy numbers by use of ranking functions which is desired in this paper. Maleki *et al.* [21] used the crisp solution of as an auxiliary problem, having only fuzzy cost coefficients, for finding the fuzzy solution of FVLP problems. Mahdavi-Amiri and Nasseri [20] showed that the auxiliary problem is indeed the dual of the FVLP problem. Then they stated and proved duality results obtained by a natural extension of the results in crisp linear programming. Using the obtained results, Mahdavi-Amiri and Nasseri [20] and Ebrahimnejad *et al.* [8] developed two new methods, namely the fuzzy dual simplex algorithm and fuzzy primal-dual simplex algorithm, respectively, for solving the FVLP problem directly, without any need of an auxiliary problem. After that, as a natural extension of the results in crisp linear programming problem, Ebrahimnejad [4] extended these algorithms for bounded FVLP problems.

It is acknowledged that triangular and trapezoidal membership functions are popularly circulated in the literature. However, there are only few papers dealing with the problems involving generalized interval-valued trapezoidal fuzzy numbers. Ye [30] solved the multiple attribute group decision-making problems with unknown weights, multiple attribute group decision-making methods with completely unknown weights of decision-makers and incompletely known weights of attributes in intuitionistic fuzzy setting and interval-valued intuitionistic fuzzy setting. Ebrahimnejad [5] introduced a formulation of transportation problem involving interval-valued trapezoidal fuzzy numbers for the transportation costs and values of supplies and demands and proposed an FLP approach for solving interval-valued trapezoidal fuzzy numbers transportation problem based on comparison of interval-valued fuzzy numbers by the help of signed distance ranking. Frahadinia [10] formulated the FLP problem where all parameters are considered as the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers meanwhile the decision variables are taken crisp values. However, to the best of our knowledge, till now there is no method in the literature to formulate FLP problems with the level (w^L, w^U)-interval-valued trapezoidal fuzzy variables. In this paper, we first introduce a formulation of FLP problems involving interval-valued trapezoidal fuzzy numbers for the decision variables and the right-hand-side of the constraints and then propose a new method for solving this kind of FLP problems based on comparison of interval-valued fuzzy numbers by the help of signed distance ranking. It is demonstrated that study of LP problems with interval-valued trapezoidal fuzzy variables gives rise to the same expected results as those obtained for linear programming with trapezoidal fuzzy variables in Maleki *et al.* [21], Mahdavi-Amiri *et al.* [20], Ebrahimnejad *et al.* [8] and Ebrahimnejad [3].

The remainder of this paper is organized as follows. In Section 2, we review some necessary concepts and backgrounds on fuzzy arithmetic. In Section 3, we first formulate the FLP problem with interval-valued fuzzy variables and then propose an effective approach for solving such a problem by introducing an auxiliary problem. Section 4 is devoted to illustration of the proposed method using an application example. Obtained results as well as the main advantages and disadvantages of the proposed method over the existing methods are discussed in Section 5. We present our conclusions in Section 6.

2. Preliminaries

This section is devoted to review some necessary background and notions of the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers which are applied throughout this paper.

Definition 2.1 ([2])**.** A w-level trapezoidal fuzzy number $\tilde{A}$, denoted by $\tilde{A} = (a_1, a_2, a_3, a_4; w)$,$0 < w \leq 1$, is a fuzzy set on $\mathbb{R}$ with the membership function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} w\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2, \\ w, & a_2 \leq x \leq a_3, \\ w\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $F_{TN}(w)$ be the family of all w-level trapezoidal fuzzy numbers, that is,

$$F_{TN}(w) = \left\{ \tilde{A} = (a_1, a_2, a_3, a_4; w), a_1 \leq a_2 \leq a_3 \leq a_4 \right\}, \quad 0 < w \leq 1. \tag{2.2}$$

Definition 2.2 ([29])**.** Let $\tilde{A}^L \in F_{TN}(w^L)$ and $\tilde{A}^U \in F_{TN}(w^U)$. A level (w^L, w^U)-interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$, denoted by $\tilde{\tilde{A}} = \left[\tilde{A}^L, \tilde{A}^U\right] = \left\langle (a_1^L, a_2^L, a_3^L, a_4^L; w^L), (a_1^U, a_2^U, a_3^U, a_4^U; w^U) \right\rangle$ is an interval-valued fuzzy set on $\mathbb{R}$ with the lower trapezoidal fuzzy number $\tilde{A}^L$ expressing by

$$\mu_{\tilde{A}^L}(x) = \begin{cases} w^L\frac{x-a_1^L}{a_2^L-a_1^L}, & a_1^L \leq x \leq a_2^L, \\ w^L, & a_2^L \leq x \leq a_3^L, \\ w^L\frac{a_4^L-x}{a_4^L-a_3^L}, & a_3^L \leq x \leq a_4^L, \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

and the upper trapezoidal fuzzy number $\tilde{A}^U$ expressing by

$$\mu_{\tilde{A}^U}(x) = \begin{cases} w^U\frac{x-a_1^U}{a_2^U-a_1^U}, & a_1^U \leq x \leq a_2^U, \\ w^U, & a_2^U \leq x \leq a_3^U, \\ w^U\frac{a_4^U-x}{a_4^U-a_3^U}, & a_3^U \leq x \leq a_4^U, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4}$$

where $a_1^L \leq a_2^L \leq a_3^L \leq a_4^L, a_1^U \leq a_2^U \leq a_3^U \leq a_4^U, 0 < w^L \leq w^U \leq 1, a_1^U \leq a_1^L$ and $a_4^L \leq a_4^U$. Moreover, $\tilde{A}^L \subseteq \tilde{A}^U$.

Let $F_{IVTN}(w^L, w^U)$ be the family of all level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers, that is,

$$F_{IVTN}(w^L, w^U) = \left\{ \tilde{\tilde{A}} = \left[\tilde{A}^L, \tilde{A}^U\right] = \left\langle (a_1^L, a_2^L, a_3^L, a_4^L; w^L), (a_1^U, a_2^U, a_3^U, a_4^U; w^U) \right\rangle : \tilde{A}^L \in F_{TN}(w^L), \right.$$
$$\left. \tilde{A}^U \in F_{TN}(w^U), a_1^U \leq a_1^L, a_4^L \leq a_4^U \right\}, \quad 0 < w^L \leq w^U \leq 1.$$

Definition 2.3. Let $\tilde{\tilde{A}} = \left[\tilde{A}^L, \tilde{A}^U\right] = \left\langle (a_1^L, a_2^L, a_3^L, a_4^L; w^L), (a_1^U, a_2^U, a_3^U, a_4^U; w^U) \right\rangle$ and $\tilde{\tilde{B}} = \left[\tilde{B}^L, \tilde{B}^U\right] = \left\langle (b_1^L, b_2^L, b_3^L, b_4^L; w^L), (b_1^U, b_2^U, b_3^U, b_4^U; w^U) \right\rangle$ belong to $F_{IVTN}(w^L, w^U)$ and k be a non-negative real number. Then the exact formulas for the extended addition and the scalar multiplication are defined as follows (see [27]):

$$\tilde{\tilde{A}} \oplus \tilde{\tilde{B}} = \left[\tilde{A}^L, \tilde{A}^U\right] = \left\langle (a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; w^L), (a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; w^U) \right\rangle$$

$$k\tilde{\tilde{A}} = \begin{cases} \left\langle (ka_1^L, ka_2^L, ka_3^L, ka_4^L; w^L), (ka_1^U, ka_2^U, ka_3^U, ka_4^U; w^U) \right\rangle, & k > 0, \\ \left\langle (ka_4^L, ka_3^L, ka_2^L, ka_1^L; w^L), (ka_4^U, ka_3^U, ka_2^U, ka_1^U; w^U) \right\rangle, & k < 0, \\ \left\langle (0,0,0,0; w^L), (0,0,0,0; w^U) \right\rangle = \tilde{\tilde{0}}, & k = 0. \end{cases}$$

Definition 2.4 ([2])**.** Let $r, 0 \in \mathbb{R}$. The signed distance from r to 0 is defined as $d(r, 0) = r$.

Definition 2.5 ([10])**.** Let $\tilde{\tilde{A}} \in F_{IVTN}(w^L, w^U)$. The α-cut set of $\tilde{\tilde{A}}$ denoted by $\tilde{\tilde{A}}(\alpha)$, is defined as follows (see Fig. 1):

$$\tilde{\tilde{A}}(\alpha) = \left[\tilde{A}^L(\alpha), \tilde{A}^U(\alpha)\right] = \begin{cases} \left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right] \bigcup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right], & 0 \leq \alpha \leq w^L \\ \left[\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha)\right], & w^L \leq \alpha \leq w^U, \end{cases}$$

where

$$\tilde{A}_l^L(\alpha) = a_1^L + (a_2^L - a_1^L)\frac{\alpha}{w^L}, \quad \tilde{A}_r^L(\alpha) = a_4^L + (a_4^L - a_3^L)\frac{\alpha}{w^L},$$
$$\tilde{A}_l^U(\alpha) = a_1^U + (a_2^U - a_1^U)\frac{\alpha}{w^U}, \quad \tilde{A}_r^U(\alpha) = a_4^U + (a_4^U - a_3^U)\frac{\alpha}{w^U},$$

Theorem 2.6 ([10])**.** *Let $\tilde{\tilde{A}} \in F_{IVTN}(w^L, w^U)$. The signed distance of $\tilde{\tilde{A}}$ from O_1 (y-axis) is given as follows:*

$$d(\tilde{A}, O_1) = \frac{1}{4}\left[a_1 + a_2 + a_3 + a_4\right], \quad \tilde{A}^L = \tilde{A}^U = \tilde{A}, \tag{2.5}$$

$$d(\tilde{\tilde{A}}, O_1) = \frac{1}{8}\left[a_1^L + a_2^L + a_3^L + a_4^L + a_1^U + a_2^U + a_3^U + a_4^U\right], \quad 0 < w^L = w^U \leq 1, \tag{2.6}$$

$$d(\tilde{\tilde{A}}, O_1) = \frac{1}{8}\left[a_1^L + a_2^L + a_3^L + a_4^L + 4a_1^U + 2a_2^U + 2a_3^U + 4a_4^U + 3(a_2^U + a_3^U - a_1^U - a_4^U)\frac{w^L}{w^U}\right],$$
$$0 < w^L < w^U \leq 1. \tag{2.7}$$

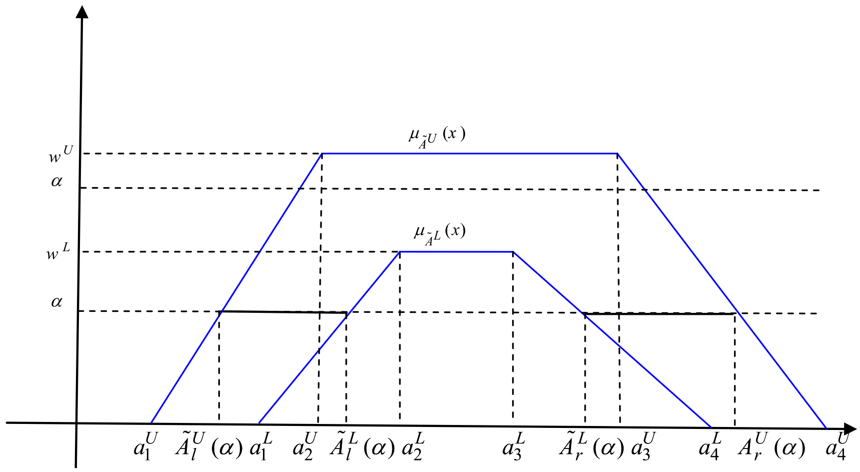

FIGURE 1. An α-cut of level (w^L, w^U)-interval-valued trapezoidal fuzzy number $\tilde{\tilde{A}}$.

Theorem 2.6 describes an efficient approach to order of level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers based on the concept of comparison of fuzzy numbers by the help of signed distance ranking.

Definition 2.7 ([10])**.** Let $\tilde{\tilde{A}}, \tilde{\tilde{B}} \in F_{IVTN}(w^L, w^U)$. Then the ranking of level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers in $F_{IVTN}(w^L, w^U)$ is defined on the basis of signed distance $d(., O_1)$ as follows:

$$\tilde{\tilde{A}} \preceq \tilde{\tilde{B}} \quad \text{iff } d(\tilde{\tilde{A}}, O_1) \leq d(\tilde{\tilde{B}}, O_1), \tag{2.8}$$
$$\tilde{\tilde{A}} \succ \tilde{\tilde{B}} \quad \text{iff } (\tilde{\tilde{A}}, O_1) > d(\tilde{\tilde{B}}, O_1) \tag{2.9}$$
$$\tilde{\tilde{A}} \approx \tilde{\tilde{B}} \quad \text{iff } (\tilde{\tilde{A}}, O_1) = d(\tilde{\tilde{B}}, O_1). \tag{2.10}$$

Notice that the signed distance $d(., O_1)$ provides us a linear ranking function, *i.e.* for any $\tilde{\tilde{A}}, \tilde{\tilde{B}} \in F_{IVTN}(w^L, w^U)$ and $k \in \mathbb{R}$ we have $d(k\tilde{\tilde{A}} \oplus \tilde{\tilde{B}}, O_1) = kd(\tilde{\tilde{A}}, O_1) + d(\tilde{\tilde{B}}, O_1)$.

In addition, $\big(F_{IVTN}(w^L, w^U), \approx, \prec\big)$ satisfies the law of trichotomy [2], that is, we have $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$ or $\tilde{\tilde{A}} \approx \tilde{\tilde{B}}$ or $\tilde{\tilde{B}} \prec \tilde{\tilde{A}}$.

3. Linear programming with interval-valued fuzzy variables

In this section, we consider the class of fuzzy linear programming problems where the decision variables and the right-hand-side of the constraints are represented as the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers and the rest of the parameters are represented by real numbers. Indeed, this kind of FLP problems is a generalized form of that considered by Maleki *et al.* [21], Mahdavi-Amiri and Nasseri [20], Ebrahimnejad *et al.* [8] and Ebrahimnejad [4].

Definition 3.1. A level (w^L, w^U)-interval-valued trapezoidal fuzzy variables linear programming (IVTFVLP) problem is defined as follows:

$$\begin{aligned} \text{Min} \quad & \tilde{u} \approx \tilde{\tilde{y}} b \\ \text{s.t.} \quad & \tilde{\tilde{y}} A \succeq \tilde{\tilde{c}} \\ & \tilde{\tilde{y}} \succeq \tilde{\tilde{0}}, \end{aligned} \tag{3.1}$$

where $b \in \mathbb{R}^m, \tilde{c} \in \left(F_{IVTN}(w^L, w^U)\right)^n, A \in \mathbb{R}^{m\times n}$ are given and $\tilde{y} \in \left(F_{IVTN}(w^L, w^U)\right)^m$ is to be determined.

Any vector $\tilde{y} \in \left(F_{IVTN}(w^L, w^U)\right)^m$ which satisfies all the constraints of (3.1) is said to be a level (w^L, w^U)-interval-valued trapezoidal fuzzy feasible solution. We denote the set of all these fuzzy feasible solutions by $\tilde{\tilde{S}}$, *i.e.* $\tilde{\tilde{S}} = \left\{\tilde{y} \in \left(F_{IVTN}(w^L, w^U)\right)^m : \tilde{y}A \succeq \tilde{c}, \tilde{y} \succeq \tilde{\tilde{0}}\right\}$.

In addition, $\tilde{y}^* \in \tilde{\tilde{S}}$ is called a level (w^L, w^U)-interval-valued trapezoidal fuzzy optimal solution if it holds $\tilde{y}^* b \preceq \tilde{y}b$ for all $\tilde{y} \in \tilde{\tilde{S}}$.

In the next subsection, we give an auxiliary problem, having only level (w^L, w^U)-interval-valued trapezoidal fuzzy cost coefficients, for an IVTFVLP problem and study the relationships between these problems. These relationships lead to a fuzzy optimal solution for the IVTFVLP problem.

3.1. Auxiliary problem in $F_{IVTN}(w^L, w^U)$ environment

In this subsection, we consider a class of FLP problem where the cost coefficients of objective function are represented as the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers and the rest of parameters are represented as real numbers. Indeed, this kind of FLP problem is a reduced form of level (w^L, w^U)-interval-valued trapezoidal fuzzy number linear programming (IVTFNLP) problem considered by Farhadinia [10], where all parameters expect of crisp decision variables were considered as the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers.

Definition 3.2. The problem

$$\begin{aligned} \text{Max} \quad & \tilde{\tilde{z}} \approx \tilde{\tilde{c}}x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, \end{aligned} \tag{3.2}$$

where $b \in \mathbb{R}^m, \tilde{\tilde{c}} \in \left(F_{IVTN}(w^L, w^U)\right)^n, A \in \mathbb{R}^{m\times n}$ is called the auxiliary problem of (3.1).

Consider the system of constraints (3.2) in its equality form where A is a matrix of order $(m \times n)$ and $rank(A) = m$. Therefore, A can be partitioned as $[B, N]$, where $B_{m\times m}$ is a non-singular matrix with $rank(B) = m$. Let y_j is the solutions of $By = a_j$. Moreover, the basic solution

$$x = (x_B, x_N) = (B^{-1}b, 0), \tag{3.3}$$

is a solution of$Ax = b$. This basic solution is feasible, whenever, $x_B \geq 0$, furthermore, the corresponding fuzzy objective function value is obtained as follows:

$$\tilde{\tilde{z}} \approx \tilde{\tilde{c}}_B x_B, \tilde{\tilde{c}}_B = (\tilde{\tilde{c}}_{B_1}, \tilde{\tilde{c}}_{B_2}, \ldots, \tilde{\tilde{c}}_{B_m}). \tag{3.4}$$

Suppose J_N be the set of indices associated with the current non-basic variables. For each non-basic variable $x_j, j \in J_N$ we define

$$\begin{aligned} \tilde{\tilde{z}}_j &\approx \tilde{\tilde{c}}_B B^{-1} a_j = \tilde{\tilde{c}}_B y_j = \sum_{i=1}^m \tilde{\tilde{c}}_{B_i} y_{ij} = \sum_{i=1}^m \left\langle (c^L_{1B_i}, c^L_{2B_i}, c^L_{3B_i}, c^L_{4B_i}; w^L), (c^U_{1B_i}, c^U_{2B_i}, c^U_{3B_i}, c^U_{4B_i}; w^U) \right\rangle y_{ij} \\ &= \sum_{\{i: y_{ij} \geq 0\}} \left\langle (c^L_{1B_i} y_{ij}, c^L_{2B_i} y_{ij}, c^L_{3B_i} y_{ij}, c^L_{4B_i} y_{ij}; w^L), (c^U_{1B_i}, c^U_{2B_i}, c^U_{3B_i}, c^U_{4B_i}; w^U) \right\rangle \\ &\quad + \sum_{\{i: y_{ij} < 0\}} \left\langle (c^L_{4B_i} y_{ij}, c^L_{3B_i} y_{ij}, c^L_{2B_i} y_{ij}, c^L_{1B_i} y_{ij}; w^L), (c^U_{4B_i} y_{ij}, c^U_{3B_i} y_{ij}, c^U_{2B_i} y_{ij}, c^U_{1B_i} y_{ij}; w^U) \right\rangle \\ &= \left\langle (z^L_{1j}, z^L_{2j}, z^L_{3j}, z^L_{4j}; w^L), (z^U_{1j}, z^U_{2j}, z^U_{3j}, z^U_{4j}; w^U) \right\rangle, \end{aligned} \tag{3.5}$$

where

$$
\begin{aligned}
z_{1j}^L &= \sum_{\{i:y_{ij}\geq 0\}} c_{1B_i}^L y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{4B_i}^L y_{ij}, z_{1j}^U = \sum_{\{i:y_{ij}\geq 0\}} c_{1B_i}^U y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{4B_i}^U y_{ij},\\
z_{2j}^L &= \sum_{\{i:y_{ij}\geq 0\}} c_{2B_i}^L y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{3B_i}^L y_{ij}, z_{2j}^U = \sum_{\{i:y_{ij}\geq 0\}} c_{2B_i}^U y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{3B_i}^U y_{ij},\\
z_{3j}^L &= \sum_{\{i:y_{ij}\geq 0\}} c_{3B_i}^L y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{2B_i}^L y_{ij}, z_{3j}^U = \sum_{\{i:y_{ij}\geq 0\}} c_{3B_i}^U y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{2B_i}^U y_{ij},\\
z_{4j}^L &= \sum_{\{i:y_{ij}\geq 0\}} c_{4B_i}^L y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{1B_i}^L y_{ij}, z_{4j}^U = \sum_{\{i:y_{ij}\geq 0\}} c_{4B_i}^U y_{ij} + \sum_{\{i:y_{ij}<0\}} c_{1B_i}^U y_{ij}.
\end{aligned}
\tag{3.6}
$$

Below, we state some important results concerning to improve a feasible solution, unbounded criteria and the optimality conditions for the auxiliary problem (3.2) with equality constraints.

Theorem 3.3. *The basic solution $x = (x_B, x_N) = (B^{-1}b, 0) = (\bar{b}, 0)$ is an optimal solution for the auxiliary problem* (3.2) *if $d(\tilde{\tilde{z}}_j, o_1) \geq d(\tilde{\tilde{c}}_j, o_1)$ for all $j \in J_N$.*

Proof. Suppose that basis B is the basic matrix associated with the basic solution $x = (x_B, x_N) = (B^{-1}b, 0)$. Both the basic variable and the objective function can be represented in terms of non-basic variables x_N as follows:

$$x_B = B^{-1}b - B^{-1}Nx_N, \tag{3.7}$$

$$\tilde{\tilde{z}} = \tilde{\tilde{c}}_B x_B + \tilde{\tilde{c}}_N x_N = \tilde{\tilde{c}}_B(B^{-1}b - B^{-1}Nx_N) + \tilde{\tilde{c}}_N x_N = \tilde{\tilde{c}}_B B^{-1}b - (\tilde{\tilde{c}}_B B^{-1}N - \tilde{\tilde{c}}_N)x_N. \tag{3.8}$$

Noting equation (3.8) and letting $\tilde{\tilde{z}}_0 = \tilde{\tilde{c}}_B B^{-1}b = \tilde{\tilde{c}}_B\bar{b}$ denote the fuzzy objective function value associated with the feasible basic solution $x = (B^{-1}b, 0) = (\bar{b}, 0)$, we get

$$\tilde{\tilde{z}} = \tilde{\tilde{z}}_\circ - \sum_{j\in J_N} (\tilde{\tilde{z}}_j - \tilde{\tilde{c}}_j)x_j. \tag{3.9}$$

Equivalently, equation (3.9) may be rewritten as follows with regard to sighed distance $d(., O_1)$:

$$d(\tilde{\tilde{z}}, O_1) = d(\tilde{\tilde{z}}_\circ, O_1) - \sum_{j\in J_N} \left(d(\tilde{\tilde{z}}_j, O_1) - d(\tilde{\tilde{c}}_j, O_1)\right) x_j. \tag{3.10}$$

Since $d(\tilde{\tilde{z}}_j, o_1) \geq d(\tilde{\tilde{c}}_j, o_1)$, examining equation (3.10) we conclude that $d(\tilde{\tilde{z}}, O_1) \leq d(\tilde{\tilde{z}}_\circ, O_1)$ or equivalently $\tilde{\tilde{z}} \preceq \tilde{\tilde{z}}_\circ$. This means that it is impossible to improve the fuzzy objective function value associated with the current solution $x = (x_B, x_N) = (B^{-1}b, 0) = (\bar{b}, 0)$. This indicates the current solution is optimal. □

Theorem 3.4. *If for a basic feasible solution with basis B and objective value $\tilde{\tilde{z}}$, it holds $d(\tilde{\tilde{z}}_k, O_1) < d(\tilde{\tilde{c}}_k, O_1)$ for some non-basic variable x_k while $y_k \not\leq 0$, then it might be obtained a new basic feasible solution with objective value $\tilde{\tilde{z}}_{new}$, such that $d(\tilde{\tilde{z}}, O_1) \leq d(\tilde{\tilde{z}}_{new}, O_1)$.*

Proof. Examining equation (3.7), we conclude that when x_k increases the value of some current basic variables decrease. In order to satisfy non-negativity, x_k is increased until the first point at which some basic variable

x_{B_r} drops to zero. Based on the results of crisp linear programming problems [1], we can increase x_k until

$$x_k = \frac{\bar{b}_r}{y_{rk}} = \min_{1\le i\le m}\left\{\frac{\bar{b}_r}{y_{ik}}\,|y_{ik}>0\right\}. \tag{3.11}$$

In this case, substituting a_k instead of a_{B_r} at basis $B=\left[a_{B_1},\ldots,a_{B_{r-1}},a_{B_r},a_{B_{r+1}},\ldots,a_{B_m}\right]$ gives a new basis as $B=\left[a_{B_1},\ldots,a_{B_{r-1}},a_k,a_{B_{r+1}},\ldots,a_{B_m}\right]$ leading to new feasible basic solution as follows:

$$\begin{aligned} x_{B_i} &= \bar{b}_i - y_{ik}\frac{\bar{b}_r}{y_{rk}}, \quad i=1,2,\ldots,m,\ i\ne r,\\ x_{B_r} &= x_k=\theta=\frac{\bar{b}_r}{y_{rk}}, \quad i=r,\\ x_j &= 0, \quad j\in J_N,\ j\ne k. \end{aligned} \tag{3.12}$$

The fuzzy objective function value associated with this new solution is obtained as follows:

$$\begin{aligned} \tilde{\tilde{z}}_{new} &\approx \tilde{\tilde{c}}_B x_B + \tilde{\tilde{c}}_N x_N = \sum_{\substack{i=1\\ i\ne r}}^{m}\left(\tilde{\tilde{c}}_{B_i}(\bar{b}_i - y_{ik}\frac{\bar{b}_r}{y_{rk}})\right) + \tilde{\tilde{c}}_k\frac{\bar{b}_r}{y_{rk}} = \sum_{\substack{i=1\\ i\ne r}}^{m}\left(\tilde{\tilde{c}}_{B_i}(\bar{b}_i - y_{ik}\theta)\right) + \tilde{\tilde{c}}_k\theta\\ &= \sum_{i=1}^{m}\left\langle(c^L_{1B_i},c^L_{2B_i},c^L_{3B_i},c^L_{4B_i};w^L),(c^U_{1B_i},c^U_{2B_i},c^U_{3B_i},c^U_{4B_i};w^U)\right\rangle(\bar{b}_i-y_{ik}\theta)\\ &\quad+\left\langle(c^L_{1k},c^L_{2k},c^L_{3k},c^L_{4k};w^L),(c^U_{1k},c^U_{2k},c^U_{3k},c^U_{4k};w^U)\right\rangle\theta \end{aligned} \tag{3.13}$$

Or

$$\tilde{\tilde{z}}_{new} \approx \left\langle z^L_{new}, z^U_{new}\right\rangle, \tag{3.14}$$

where

$$\begin{aligned} z^L_{new} &= (z^L_1-\theta {z'}^L_{1k}+\theta c^L_{1k}, z^L_2-\theta {z'}^L_{2k}+\theta c^L_{2k}, z^L_3-\theta {z'}^L_{3k}+\theta c^L_{3k}, z^L_4-\theta {z'}^L_{4k}+\theta c^L_{4k}; w^L),\\ z^U_{new} &= (z^U_1-\theta {z'}^U_{1k}+\theta c^U_{1k}, z^U_2-\theta {z'}^U_{2k}+\theta c^U_{2k}, z^U_3-\theta {z'}^U_{3k}+\theta c^U_{3k}, z^U_4-\theta {z'}^U_{4k}+\theta c^U_{4k}; w^U) \end{aligned} \tag{3.15}$$

and

$$z^L_1=\sum_{i=1}^m c^L_{1B_i}\bar{b}_i,\quad z^L_2=\sum_{i=1}^m c^L_{2B_i}\bar{b}_i,\quad z^L_3=\sum_{i=1}^m c^L_{3B_i}\bar{b}_i,\quad z^L_4=\sum_{i=1}^m c^L_{4B_i}\bar{b}_i, \tag{3.16}$$

$$z^U_1=\sum_{i=1}^m c^U_{1B_i}\bar{b}_i,\quad z^U_2=\sum_{i=1}^m c^U_{2B_i}\bar{b}_i,\quad z^U_3=\sum_{i=1}^m c^U_{3B_i}\bar{b}_i,\quad z^U_4=\sum_{i=1}^m c^U_{4B_i}\bar{b}_i, \tag{3.17}$$

$${z'}^L_{1k}=\sum_{i=1}^m c^L_{1B_i}y_{ik},\quad {z'}^L_{2k}=\sum_{i=1}^m c^L_{2B_i}y_{ik},\quad {z'}^L_{3k}=\sum_{i=1}^m c^L_{3B_i}y_{ik},\quad {z'}^L_{4k}=\sum_{i=1}^m c^L_{4B_i}y_{ik}, \tag{3.18}$$

$$z'^{U}_{1k} = \sum_{i=1}^{m} c^{U}_{1B_i} y_{ik}, \quad z'^{U}_{2k} = \sum_{i=1}^{m} c^{U}_{2B_i} y_{ik}, \quad z'^{U}_{3k} = \sum_{i=1}^{m} c^{U}_{3B_i} y_{ik}, \quad z'^{U}_{4k} = \sum_{i=1}^{m} c^{U}_{4B_i} y_{ik}, \tag{3.19}$$

Examining equations (3.14) and (3.15) it is clear that if $\theta = 0$, then

$$\tilde{\tilde{z}}_{new} \approx \left\langle (z^L_1, z^L_2, z^L_3, z^L_4; w^L), (z^U_1, z^U_2, z^U_3, z^U_4; w^U) \right\rangle \approx \tilde{\tilde{z}}.$$

This indicated that $d(\tilde{\tilde{z}}_{new}, O_1) = d(\tilde{\tilde{z}}, O_1)$.On the other hand, in the case of $\theta > 0$ we show that $d(\tilde{\tilde{z}}_{new}, O_1) \geq d(\tilde{\tilde{z}}, O_1)$.

$$\begin{aligned} z^L_{1k} + z^L_{4k} &= \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{1B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{4B_i} y_{ik} \right) + \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{4B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{1B_i} y_{ik} \right) \\ &= \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{1B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{1B_i} y_{ik} \right) + \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{4B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{4B_k} y_{ik} \right) \\ &= \left(\sum_{i=1}^{m} c^L_{1B_i} y_{ik} + \sum_{i=1}^{m} c^L_{4B_i} y_{ik} \right) = z'^{L}_{1k} + 4z'^{L}_{14}. \end{aligned} \tag{3.20}$$

In addition,

$$\begin{aligned} z^L_{2k} + z^L_{3k} &= \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{2B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{3B_i} y_{ik} \right) + \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{2B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{3B_i} y_{ik} \right) \\ &= \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{2B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{2B_i} y_{ik} \right) + \left(\sum_{\{i:y_{ik} \geq 0\}} c^L_{3B_i} y_{ik} + \sum_{\{i:y_{ik} < 0\}} c^L_{3B_k} y_{ik} \right) \\ &= \left(\sum_{i=1}^{m} c^L_{2B_i} y_{ik} + \sum_{i=1}^{m} c^L_{3B_i} y_{ik} \right) = z'^{L}_{2k} + 4z'^{L}_{3k}. \end{aligned} \tag{3.21}$$

In a similar way, we conclude that

$$z^U_{1k} + z^U_{4k} = z'^{U}_{1k} + 4z'^{U}_{4k}, \quad z^U_{2k} + z^U_{3k} = z'^{U}_{2k} + 4z'^{U}_{3k}. \tag{3.22}$$

Examining equations (3.20)–(3.22), we get

$$\begin{aligned} &\frac{1}{8} \left[z^L_{1k} + z^L_{2k} + z^L_{3k} + z^L_{4k} + 4z^U_{1k} + 2z^U_{2k} + 2z^U_{3k} + 4z^U_{4k} + 3(z^U_{2k} + z^U_{3k} - z^U_{1k} - z^U_{4k}) \frac{w^L}{w^U} \right] \\ &= \frac{1}{8} \left[z'^{L}_{1k} + z'^{L}_{2k} + z'^{L}_{3k} + z'^{L}_{4k} + 4z'^{U}_{1k} + 2z'^{U}_{2k} + 2z'^{U}_{3k} + 4z'^{U}_{4k} + 3(z'^{U}_{2k} + z'^{U}_{3k} - z'^{U}_{1k} - z'^{U}_{4k}) \frac{w^L}{w^U} \right]. \end{aligned} \tag{3.23}$$

This indicates that

$$d(\tilde{\tilde{z}}_k, O_1) = d(\tilde{\tilde{z}}'_k, O_1). \tag{3.24}$$

On the other hand, since $d(\tilde{\tilde{z}}_k, O_1) < d(\tilde{\tilde{c}}_k, O_1)$, with regard to equation (3.24) we conclude that

$$\frac{1}{8}\left[\begin{array}{l}(c_{1k}^L - {z'}_{1k}^L) + (c_{2k}^L - {z'}_{2k}^L) + (c_{3k}^L - {z'}_{3k}^L) + (c_{4k}^L - {z'}_{4k}^L)\\ +4(c_{1k}^U - {z'}_{1k}^U) + 2(c_{2k}^U - {z'}_{2k}^U) + 2(c_{3k}^U - {z'}_{3k}^U) + 4(c_{4k}^U - {z'}_{4k}^U)\\ +3\left[(c_{2k}^U - {z'}_{2k}^U) + (c_{3k}^U - {z'}_{3k}^U) - (c_{1k}^U - {z'}_{1k}^U) - (c_{4k}^U - {z'}_{4k}^U)\right]\frac{w^L}{w^U}\end{array}\right] > 0. \tag{3.25}$$

Now based on equation (3.15), we have

$$\begin{aligned}&\frac{1}{8}\left[\begin{array}{l}(z_1^L - \theta {z'}_{1k}^L + \theta c_{1k}^L) + (z_2^L - \theta {z'}_{2k}^L + \theta c_{2k}^L) + (z_3^L - \theta {z'}_{3k}^L + \theta c_{3k}^L) + ({z'}_4^L - \theta {z'}_{4k}^L + \theta c_{4k}^L)\\ +4(z_1^U - \theta {z'}_{1k}^U + \theta c_{1k}^U) + 2(z_2^U - \theta {z'}_{2k}^U + \theta c_{2k}^U) + 2(z_3^U - \theta {z'}_{3k}^U + \theta c_{3k}^U) +\\ 4(z_4^U - \theta {z'}_{4k}^U + \theta c_{4k}^U)\\ +3\left[(z_2^U - \theta {z'}_{2k}^U + \theta c_{2k}^U) + (z_3^U - \theta {z'}_{3k}^U + \theta c_{3k}^U) - (z_1^U - \theta {z'}_{1k}^U + \theta c_{1k}^U) - (z_4^U - \theta {z'}_{4k}^U + \theta c_{4k}^U)\frac{w^L}{w^U}\right]\end{array}\right]\\ &= \frac{1}{8}\left[z_1^L + z_2^L + z_3^L z_4^L + 4z_1^U + 2z_2^U + 2z_3^U + 4z_4^U + 3(z_2^U + z_3^U - z_1^U - z_4^U)\frac{w^L}{w^U}\right]\\ &\quad +\theta\frac{1}{8}\left[\begin{array}{l}(c_{1k}^L - {z'}_{1k}^L) + (c_{2k}^L - {z'}_{2k}^L) + (c_{3k}^L - {z'}_{3k}^L) + (c_{4k}^L - {z'}_{4k}^L)\\ +4(c_{1k}^U - {z'}_{1k}^U) + 2(c_{2k}^U - {z'}_{2k}^U) + 2(c_{3k}^U - {z'}_{3k}^U) + 4(c_{4k}^U - {z'}_{4k}^U)\\ +3\left[(c_{2k}^U - {z'}_{2k}^U) + (c_{3k}^U - {z'}_{3k}^U) - (c_{1k}^U - {z'}_{1k}^U) - (c_{4k}^U - {z'}_{4k}^U)\right]\frac{w^L}{w^U}\end{array}\right].\end{aligned} \tag{3.26}$$

In this case, with regard to equation (3.25), we conclude that

$$\begin{aligned}&\frac{1}{8}\left[\begin{array}{l}(z_1^L - \theta {z'}_{1k}^L + \theta c_{1k}^L) + (z_2^L - \theta {z'}_{2k}^L + \theta c_{2k}^L) + (z_3^L - \theta {z'}_{3k}^L + \theta c_{3k}^L) + (z_4^L - \theta {z'}_{4k}^L + \theta c_{4k}^L)\\ +4(z_1^U - \theta {z'}_{1k}^U + \theta c_{1k}^U) + 2(z_2^U - \theta {z'}_{2k}^U + \theta c_{2k}^U) + 2(z_3^U - \theta {z'}_{3k}^U + \theta c_{3k}^U) + 4(z_4^U - \theta {z'}_{4k}^U + \theta c_{4k}^U)\\ +3\left[(z_2^U - \theta {z'}_{2k}^U + \theta c_{2k}^U) + (z_3^U - \theta {z'}_{3k}^U + \theta c_{3k}^U) - (z_1^U - \theta {z'}_{1k}^U + \theta c_{1k}^U) - (z_4^U - \theta {z'}_{4k}^U + \theta c_{4k}^U)\frac{w^L}{w^U}\right]\end{array}\right]\\ &\geq \frac{1}{8}\left[z_1^L + z_2^L + z_3^L z_4^L + 4z_1^U + 2z_2^U + 2z_3^U + 4z_4^U + 3(z_2^U + z_3^U - z_1^U - z_4^U)\frac{w^L}{w^U}\right].\end{aligned} \tag{3.27}$$

This means that $d(\tilde{\tilde{z}}_{new}, O_1) \geq d(\tilde{\tilde{z}}, O_1)$. ☐

Theorem 3.5. *If for a basic feasible solution with basis B and objective value $\tilde{\tilde{z}}$, it holds $d(\tilde{\tilde{z}}_k, O_1) < d(\tilde{\tilde{c}}_k, O_1)$ for some non-basic variable x_k while $y_k \nleq 0$, then the optimal solution of the auxiliary problem (3.2) is unbounded.*

Proof. Examining equation (3.26), we conclude that

$$d(\tilde{\tilde{z}}_{new}, O_1) = d(\tilde{z}', O_1) + \theta d(\tilde{\tilde{c}}_k - \tilde{\tilde{z}}_k, O_1). \tag{3.28}$$

Since $d(\tilde{\tilde{z}}_k, O_1) < d(\tilde{\tilde{c}}_k, O_1)$, we have $d(\tilde{\tilde{c}}_k - \tilde{\tilde{z}}_k, O_1) > 0$. This indicates that if $\theta > 0$, $d(\tilde{\tilde{z}}_{new}, O_1)$ can be made arbitrarily large and hence we have unbounded solutions. ☐

Now we are in a position to apply the fuzzy primal simplex algorithm [10] in $F_{IVTN}(w^L, w^U)$ environment for solving the auxiliary problem (3.2).

Algorithm 3.6. Primal simplex algorithm in fuzzy sense (Maximization Problem).

Initialization step
Choose a starting feasible basic solution with basis B. The initial feasible basic solution is given by $x_B = B^{-1}b = \bar{b}, x_N = 0$ and the fuzzy objective value is $\tilde{\tilde{z}} \approx \tilde{\tilde{c}}_B x_B$. Define $\tilde{\tilde{z}}_j = \tilde{\tilde{c}}_B B^{-1} a_j$ and $y_j = B^{-1}a_j$.
Let Table 1 be the initial simplex tableau corresponding to basis B.

TABLE 1. The initial IVTFVLP simplex tableau.

Basis	x_B	x_N	RHS
$d(\tilde{\tilde{z}}, O_1)$	0	$d(\tilde{\tilde{z}}_N - \tilde{\tilde{c}}_N, O_1)$	$d(\tilde{\tilde{z}}, O_1)$
$\tilde{\tilde{z}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{z}}_N - \tilde{\tilde{c}}_N = \tilde{\tilde{c}}_B Y_N - \tilde{\tilde{c}}_N$	$\tilde{\tilde{z}} = \tilde{\tilde{c}}_B \bar{b}$
x_B	I	Y_N	$\bar{b} = B^{-1}b$

Main step
(1) Calculate $d(\tilde{\tilde{z}}_j, O_1) - d(\tilde{\tilde{c}}_j, O_1) = d(\tilde{\tilde{c}}_B B^{-1} a_j, O_1) - d(\tilde{\tilde{c}}_j, O_1)$ for all $j \in J_N$ and let $d(\tilde{\tilde{z}}_k, O_1) - d(\tilde{\tilde{c}}_k, O_1) = \max_{j \in J_N} \left\{ d(\tilde{\tilde{z}}_j, O_1) - d(\tilde{\tilde{c}}_j, O_1) \right\}$. If $d(\tilde{\tilde{z}}_k, O_1) - d(\tilde{\tilde{c}}_k, O_1) \leq 0$, then stop with the current basic solution as an optimal solution.
(2) If $y_k \leq 0$, then stop with the conclusion that the problem is unbounded.
(3) If $y_k \not\leq 0$, then x_k enters the basis and x_{Br} leaves the basis providing that $\frac{\bar{b}_r}{y_{rk}} = \min_{1 \leq i \leq m} \left\{ \frac{\bar{b}_i}{y_{ik}} | y_{rk} > 0 \right\}$.
(4) Update the tableau by pivoting at y_{rk}. Update the basic and nabasic variable where x_k enters the basic and x_{Br} leaves the basic and go to (1).

3.2. Relationships between the IVTFVLP problem and the auxiliary problem

In this subsection, we shall discuss the relationship between the primary and the auxilirary problem.

Theorem 3.7. *If $\tilde{\tilde{y}}_\circ$ is any level (w^L, w^U)-interval-valued trapezoidal fuzzy feasible solution for (3.1) and $x_\circ$ is any feasible solution to (3.2), then we have $d(\tilde{\tilde{y}}_\circ b, O_1) \geq d(\tilde{\tilde{c}} x_\circ, O_1)$.*

Proof. Since $\tilde{\tilde{y}}_\circ$ is a fuzzy feasible solution for (3.1), we have $\tilde{\tilde{y}}_\circ A \succeq \tilde{\tilde{c}}$ and $\tilde{\tilde{y}}_\circ \succeq \tilde{\tilde{0}}$. This means that $d(\tilde{\tilde{y}}_\circ A, O_1) \geq d(\tilde{\tilde{c}}, O_1)$ and $d(\tilde{\tilde{y}}_\circ, O_1) \geq 0$. Multiplying $d(\tilde{\tilde{y}}_\circ A, O_1) \geq d(\tilde{\tilde{c}}, O_1)$ on the left by $x_\circ \geq 0$, we get

$$d(\tilde{\tilde{y}}_\circ A x_\circ, O_1) \geq d(\tilde{\tilde{c}} x_\circ, O_1). \tag{3.29}$$

On other hands, since $x_\circ$ is a fuzzy feasible solution for (3.2), we have $Ax_\circ \leq b$ and $x_\circ \geq 0$. Multiplying $Ax_\circ \leq b$ on the right by $d(\tilde{\tilde{y}}_\circ, O_1) \geq 0$, we get

$$d(\tilde{\tilde{y}}_\circ A x_\circ, O_1) \leq d(\tilde{\tilde{y}}_\circ b, O_1). \tag{3.30}$$

Examining equations (3.29) and (3.30), we conclude that $d(\tilde{\tilde{y}}_\circ b, O_1) \geq d(\tilde{\tilde{c}} x_\circ, O_1)$. □

Theorem 3.8. *If $\tilde{\tilde{y}}_\circ$ is any level (w^L, w^U)-interval-valued trapezoidal fuzzy feasible solution for (3.1) and $x_\circ$ is any feasible solution to (3.2) such that $d(\tilde{\tilde{y}}_\circ b, O_1) = d(\tilde{\tilde{c}} x_\circ, O_1)$, then $\tilde{\tilde{y}}_\circ$ is a level (w^L, w^U)-interval-valued trapezoidal fuzzy optimal solution of (3.1) and $x_\circ$ is an optimal solution of (3.2).*

Proof. It is straightforward, using Theorem 3.8. □

Theorem 3.9. *If the auxiliary problem (3.2) has an optimal solution, then the IVTFVLP problem (3.1) has a level (w^L, w^U)-interval-valued trapezoidal fuzzy optimal solution.*

Proof. We first transform (3.2) into the standard form, *i.e.*

$$\begin{aligned} \text{Max} \quad & \tilde{\tilde{z}} \approx \tilde{\tilde{c}}x + \tilde{\tilde{0}}x_s \\ \text{s.t.} \quad & Ax + x_s = b \\ & x, x_s \geq 0. \end{aligned} \tag{3.31}$$

Let $x_\circ = (x_B, x_N) = (B^{-1}b, 0)$ is an optimal solution for the auxiliary problem (3.31). From Theorem 3.3 we have

$$\tilde{\tilde{z}}_j = \tilde{\tilde{c}}_B B^{-1} a_j \succeq \tilde{\tilde{c}}_j, \quad j = 1, 2, \ldots, n, n+1, \ldots, n+m.$$

Or equivalently

$$\begin{aligned} \tilde{\tilde{c}}_B B^{-1} a_j \succeq \tilde{\tilde{c}}_j, \quad & j = 1, 2, \ldots, n, \\ \tilde{\tilde{c}}_B B^{-1} e_j \succeq \tilde{\tilde{0}}, \quad & j = n+1, \ldots, n+m. \end{aligned}$$

Hence, we must have

$$\begin{aligned} \tilde{\tilde{c}}_B B^{-1} A &\succeq \tilde{\tilde{c}}, \\ \tilde{\tilde{c}}_B B^{-1} &\succeq \tilde{\tilde{0}}. \end{aligned}$$

Now, let $\tilde{\tilde{y}}_\circ \approx \tilde{\tilde{c}}_B B^{-1}$. Using the above inequalities we can write

$$\begin{aligned} \tilde{\tilde{y}}_\circ A &\succeq \tilde{\tilde{c}}, \\ \tilde{\tilde{y}}_\circ &\succeq \tilde{\tilde{0}}. \end{aligned}$$

Then $\tilde{\tilde{y}}_\circ$ is a level (w^L, w^U)-interval-valued trapezoidal fuzzy feasible solution for IVTFVLP problem (3.1) such that

$$\tilde{\tilde{y}}_\circ b \approx \tilde{\tilde{c}}_B B^{-1} b \approx \tilde{\tilde{c}}_B x_B = \tilde{\tilde{c}} x_\circ.$$

Hence is $\tilde{\tilde{y}}_\circ$ a level (w^L, w^U)-interval-valued trapezoidal fuzzy feasible optimal for IVTFVLP problem (3.1) based upon Theorem 3.8. □

Corollary 3.10. *If $x_\circ = (x_B, x_N) = (B^{-1}b, 0)$ with basis B as the optimal basis, is an optimal solution for the auxiliary problem, then $\tilde{\tilde{y}}_\circ \approx \tilde{\tilde{c}}_B B^{-1}$ is the fuzzy optimal solution of the IVTFVLP problem.*

Theorem 3.11. *If the auxiliary problem (3.2) has an unbounded solution, then the IVTFVLP problem (3.1) has no fuzzy feasible solution.*

Proof. It is straightforward, using Theorem 3.9. □

We conclude that in order to solve a linear programming problem with the interval-valued trapezoidal fuzzy variables it is sufficient to solve the auxiliary problem by the method discussed in Section 3.2. We can then obtain the interval-valued trapezoidal fuzzy optimal solution of our problem using the theorems of this subsection.

Remark 3.12. For an IVTFVLP problem with maximization objective function, the auxiliary problem is defined as a minimization problem having only interval-valued trapezoidal fuzzy cost coefficients. In this case, the primal simplex algorithm in fuzzy sense for a minimization problem can also be handled directly as follows. Let $d(\tilde{\tilde{z}}_k, O_1) - d(\tilde{\tilde{c}}_k, O_1)$ instead be the minimum $d(\tilde{\tilde{z}}_j, O_1) - d(\tilde{\tilde{c}}_j, O_1)$ for $j \in J_N$; the stopping criterion is that $d(\tilde{\tilde{z}}_k, O_1) - d(\tilde{\tilde{c}}_k, O_1) \geq 0$. Otherwise, the steps are as previously presented in Algorithm 3.6.

4. Application examples

To illustrate the efficiency of the proposed method, we examine two realistic real-world examples.

Example 4.1. A farmer who raises chickens would like to determine the amounts of the available ingredients that would meet certain nutritional requirements. The available ingredients and their cost per serving, along with the units of nutrients per serving in the ingredients are summarized in Table 2.

The minimum daily requirements generally are imprecise numbers with the level (w^L, w^U)-interval-valued trapezoidal possibility distributions over the planning horizon due to incomplete or unobtainable information. For example, the minimum daily requirement of the protein and carbohydrates are $\left\langle(40,45,65,70;\frac{2}{3}),(35,40,70,75;1)\right\rangle$ and $\left\langle(60,65,85,90;\frac{2}{3}),(55,60,90,95;1)\right\rangle$, respectively. The objective is to determine which mix will meet certain nutritional requirements at a minimum cost.

This problem is evidently an uncertain optimization problem due to variations in minimum daily requirements. So the amount of each unit of ingredients will be uncertain. Hence, we will model the problem as a level (w^L, w^U)-interval-valued trapezoidal fuzzy variables linear programming problem. Let $\tilde{\tilde{x}}_1$ and $\tilde{\tilde{x}}_2$ are the uncertain daily amount of protein and carbohydrates to determine the optimal combination, respectively. In this case, the problem is formulated as follows:

$$\begin{aligned}
\text{Min}\ \ & \tilde{\tilde{u}} \approx 80\tilde{\tilde{y}}_1 + 60\tilde{\tilde{y}}_2 \\
\text{s.t.}\ \ & 4\tilde{\tilde{y}}_1 + 1\tilde{\tilde{y}}_1 \succeq \left\langle(40,45,65,70;\frac{2}{3}),(35,40,70,75;1)\right\rangle \\
& 2\tilde{\tilde{y}}_1 + 3\tilde{\tilde{y}}_1 \succeq \left\langle(60,65,85,90;\frac{2}{3}),(55,60,90,95;1)\right\rangle \\
& \tilde{\tilde{y}}_1, \tilde{\tilde{y}}_2 \succeq \tilde{\tilde{0}}.
\end{aligned} \tag{4.1}$$

The auxiliary problem associated with this interval-valued trapezoidal fuzzy variable linear programming problem is given as follows:

$$\begin{aligned}
\text{Max}\ \ & \tilde{\tilde{z}} \approx \left\langle(40,45,65,70;\frac{2}{3}),(35,40,70,75;1)\right\rangle x_1 + \left\langle(60,65,85,90;\frac{2}{3}),(55,60,90,95;1)\right\rangle x_2 \\
\text{s.t.}\ \ & 4x_1 + 2x_2 \leq 80 \\
& x_1 + 3x_2 \leq 60 \\
& x_1, x_2 \geq 0.
\end{aligned} \tag{4.2}$$

Table 2. The data of Example 4.1.

Nutrient	Ingredient	
	Corn	Lime
Protein	4	1
Carbohydrates	2	3
Cost	80	60

For solving this auxiliary problem we rewrite it in the following form with introducing the slack variables x_3 and x_4:

$$\begin{aligned} \text{Max} \quad & \tilde{\tilde{z}} \approx \left\langle (40,45,65,70;\frac{2}{3}),(35,40,70,75;1)\right\rangle x_1 + \left\langle (60,65,85,90;\frac{2}{3}),(55,60,90,95;1)\right\rangle x_2 \\ \text{s.t.} \quad & 4x_1+2x_2+x_3=80 \\ & x_1+3x_2+x_4=60 \\ & x_1,x_2,x_3,x_4 \geq 0. \end{aligned} \tag{4.3}$$

The auxiliary problem (4.3) can be solved by Algorithm 3.6. The initial primal simplex tableau in fuzzy sense is given in Table 3.

From Table 3, we find that x_2 enters and x_4 leaves the basis. Pivoting in the x_2 column and the x_4 row, we get a new tableau that is given in Table 4.

From Table 3, we find that x_1 enters and x_3 leaves the basis. Pivoting in the x_1 column and the x_3 row, we get a new tableau that is given in Table 5.

By Algorithm 3.6 and the fact $d(\tilde{\tilde{z}}_j,O_1) \leq d(\tilde{\tilde{c}}_j,O_1)$ for all $j \in J_N$, the optimal solution to the auxiliary problem (4.2) is clearly $x_\circ = (x_B=(x_1,x_2),x_N=(x_3,x_4)) = (12,16,0,0)$ with the optimal basis as $B=[a_1,a_2]=$

TABLE 3. The first iteration.

Basis	x_1	x_2	x_3	x_4	RHS
$d(\tilde{\tilde{z}},O_1)$	-110	-150	0	0	0
$\tilde{\tilde{z}}$	$\langle(-70,-65,-45,-40;2/3),$ $(-75,-70,-40,-35;1)\rangle$	$\langle(-90,-85,-65,-60;2/3),$ $(-95,-90,-60,-55;1)\rangle$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$
x_3	4	2	1	0	80
x_4	1	3	0	1	60

TABLE 4. The second iteration.

Basis	x_1	x_2	x_3	x_4	RHS
$d(\tilde{\tilde{z}},O_1)$	-60	0	0	50	3000
$\tilde{\tilde{z}}$	$\langle(-50,-130/3,-50/3,$ $-10;2/3),$ $(-170/3,-50,-10,$ $-10/3;1)\rangle$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\langle(20,65/3,85/3,$ $30;2/3),$ $(55/3,20,30,$ $95/3;1)\rangle$	$\langle(1200,1300,1700,1800;2/3),$ $(1100,1200,1800,1900;1)\rangle$
x_3	$10/3$	0	1	$-2/3$ 40	
x_2	$1/3$	1	0	$1/3$	20

TABLE 5. The optimal iteration.

Basis	x_1	x_2	x_3	x_4	RHS
$d(\tilde{\tilde{z}},O_1)$	0	0	18	38	3720
$\tilde{\tilde{z}}$	$\tilde{\tilde{0}}$	$\tilde{\tilde{0}}$	$\langle(3,5,13,15;2/3)$ $(1,3,15,17;1)\rangle$	$\langle(10,13,25,28;2/3),$ $(7,10,28,31;1)\rangle$	$\langle(1320,1500,2220,2400;2/3),$ $(1140,1320,2400,2580;1)\rangle$
x_1	1	0	$3/10$	$-1/5$	12
x_2	0	1	$-1/10$	$1/5$	16

$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$. Based on the results of Section 3, especially Corollary 3.10, the level (w^L, w^U)-interval-valued trapezoidal fuzzy optimal solution to problem (4.1) is given as follows:

$$\begin{aligned}\tilde{\tilde{y}}_\circ = (\tilde{\tilde{y}}_1, \tilde{\tilde{y}}_2) \approx \tilde{\tilde{c}}_B B^{-1} &\approx \left(\left\langle \left(40, 45, 65, 70; \frac{2}{3}\right), (35, 40, 70, 75; 1)\right\rangle, \left\langle \left(60, 65, 85, 90; \frac{2}{3}\right), (55, 60, 90, 95; 1)\right\rangle\right) \begin{bmatrix} \frac{3}{10} & \frac{-1}{5} \\ \frac{-1}{10} & \frac{2}{5} \end{bmatrix} \\ &\approx \left(\left\langle \left(3, 5, 13, 15; \frac{2}{3}\right), (1, 3, 13, 17; 1)\right\rangle, \left\langle \left(10, 13, 25, 28; \frac{2}{3}\right), (7, 10, 28, 31; 1)\right\rangle\right).\end{aligned}$$

The optimal value of problem (4.1) is achieved by putting $\tilde{\tilde{y}}_\circ = (\tilde{\tilde{y}}_1, \tilde{\tilde{y}}_2)$ in $\tilde{\tilde{u}} \approx 80\tilde{\tilde{y}}_1 + 60\tilde{\tilde{y}}_2$ as follows:

$$\begin{aligned}\tilde{\tilde{u}} &\approx 80 \left\langle \left(3, 5, 13, 15; \frac{2}{3}\right), (1, 3, 15, 17; 1)\right\rangle + 60 \left\langle \left(10, 13, 25, 28; \frac{2}{3}\right), (7, 10, 28, 31; 1)\right\rangle \\ &\approx \left\langle \left(840, 1180, 2540, 2880; \frac{2}{3}\right), (500, 840, 2880, 3220; 1)\right\rangle.\end{aligned}$$

In addition, the membership function of the constraints and the objective function can be formulated as

$$\mu_{\tilde{y}_1^L}(y) = \begin{cases} \frac{2y-6}{6}, & 3 \le y \le 5, \\ \frac{2}{3}, & 5 \le y \le 13, \\ \frac{30-2y}{6}, & 13 \le y \le 15, \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{\tilde{y}_1^U}(y) = \begin{cases} \frac{y-1}{2}, & 1 \le y \le 3, \\ 1, & 3 \le y \le 15, \\ \frac{17-y}{2}, & 15 \le y \le 17, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{y}_2^L}(y) = \begin{cases} \frac{2y-20}{9}, & 10 \le y \le 13, \\ \frac{2}{3}, & 13 \le y \le 25, \\ \frac{56-2y}{6}, & 25 \le y \le 28, \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{\tilde{y}_2^U}(y) = \begin{cases} \frac{y-7}{3}, & 7 \le y \le 10, \\ 1, & 10 \le y \le 28, \\ \frac{31-y}{3}, & 28 \le y \le 31, \\ 0, & \text{otherwise.} \end{cases}$$

$$\mu_{\tilde{u}_1^L}(y) = \begin{cases} \frac{2y-1680}{680}, & 840 \le y \le 1180, \\ \frac{2}{3}, & 1180 \le y \le 2540, \\ \frac{5760-2y}{680}, & 2540 \le y \le 2880, \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{\tilde{u}_2^U}(y) = \begin{cases} \frac{y-500}{340}, & 500 \le y \le 840, \\ 1, & 840 \le y \le 2880, \\ \frac{3220-y}{340}, & 2880 \le y \le 3220, \\ 0, & \text{otherwise.} \end{cases}$$

Now we use a real life interval-valued fuzzy transportation problem in order to present the applicability of the proposed approach.

Example 4.2. One of the most important factors for the development of each country is the existence of an efficient and appropriate network in that country to meet transportation needs. Each year, a huge amount of oil products are transported to the Iran's transportation network to meet demand of different parts of Iran for these products. The issue is how to transport these products into the internal transport network to meet the demands of destinations using the available supply at refineries. Transportation of oil products with tankers and ships has problems that increase the importance of the pipeline. Here we formulate a petrol transportation problem in an uncertain environment. The problem can be summarized as follows. There are three petrol refineries at cities Shiraz (S1), Tabriz (S2) and Kermanshah (S3) to transport the petrol to different warehouses located in six cities Tehran (D1), Esfahan (D2), Arak (D3), Abadan (D4), Semnan (D5) and Yazd (D6) by pipelines. The task for the decision-maker is to make the transportation plan for the next year. The decision maker is

Table 6. Interval-valued fuzzy petrol transportation problem.

Refinery	Warehouse					
	Warehouse 1	Warehouse 2	Warehouse 3	Warehouse 4	Warehouse 5	Warehouse 6
Refinery 1	2	4	6	8	4	6
Refinery 2	3	5	7	5	3	9
Refinery 3	2	3	4	6	5	3

Table 7. Interval-valued fuzzy supply from the three refineries.

Refinery	Supply (in million liters)
Refinery 1	$\langle(450, 500, 700, 750; 1), (400, 450, 700, 800; 1)\rangle$
Refinery 2	$\langle(350, 400, 600, 650; 1), (300, 350, 600, 700; 1)\rangle$
Refinery 3	$\langle(500, 550, 750, 800; 1), (450, 500, 750, 850; 1)\rangle$

Table 8. Interval-valued fuzzy demand from the six warehouses.

Warehouse	Demand (in million liters)
Warehouse 1	$\langle(100, 125, 225, 250; 1), (75, 100, 225, 275; 1)\rangle$
Warehouse 2	$\langle(175, 200, 300, 325; 1), (150, 175, 300, 350; 1)\rangle$
Warehouse 3	$\langle(125, 150, 250, 275; 1), (100, 125, 250, 300; 1)\rangle$
Warehouse 4	$\langle(275, 300, 400, 425; 1), (250, 275, 400, 450; 1)\rangle$
Warehouse 5	$\langle(325, 350, 450, 475; 1), (300, 325, 450, 500; 1)\rangle$
Warehouse 6	$\langle(300, 325, 425, 450; 1), (275, 300, 425, 475; 1)\rangle$

certain about the transportation cost of the petrol from the different refineries to the different warehouses so these parameters are represented by real numbers in Table 6. However, since the transportation plan is made in advance, he/she generally cannot estimate this data precisely. In this kind of situation, the usual way is to obtain the interval-valued fuzzy data based upon past experience or expert advice. Tables 6 and 7 summarize the potential supply available from the three refineries and the forecast demand from the six warehouses for the upcoming year, respectively. The problem is to determine a feasible "shipping pattern" from refineries to warehouses that minimizes the total transportation costs.

To formulate this problem as an Interval-valued FVLP problem, we first define an interval-valued fuzzy variable for each decision that the decision maker must make. Because the decision maker must determine how much coal is sent from each refinery to each warehouse (see Tab. 8), we define $\tilde{\tilde{y}}_{ij}$ as the interval-valued fuzzy amount of petrol in liters transported from refinery i to warehouse j (for $i = 1, 2, 3$ and $j = 1, 2, 3, 4, 5, 6$).

In terms of these interval-valued fuzzy variables, the total cost of supplying the petrol demands to the warehouses may be written as:

$$
\begin{aligned}
&2\tilde{\tilde{y}}_{11} + 4\tilde{\tilde{y}}_{12} + 6\tilde{\tilde{y}}_{13} + 8\tilde{\tilde{y}}_{14} + 4\tilde{\tilde{y}}_{15} + 6\tilde{\tilde{y}}_{16}\\
&+3\tilde{\tilde{y}}_{21} + 5\tilde{\tilde{y}}_{22} + 7\tilde{\tilde{y}}_{23} + 5\tilde{\tilde{y}}_{24} + 3\tilde{\tilde{y}}_{25} + 9\tilde{\tilde{y}}_{26}\\
&+2\tilde{\tilde{y}}_{31} + 3\tilde{\tilde{y}}_{32} + 4\tilde{\tilde{y}}_{33} + 6\tilde{\tilde{y}}_{34} + 5\tilde{\tilde{y}}_{35} + 3\tilde{\tilde{y}}_{36}
\end{aligned}
$$

There are two types of constraints. First, the total petrol supplied by each refinery cannot exceed the refinery's capacity. For example, the total amount of petrol sent from refinery 1 to the six warehouses is $\langle(450, 500, 700, 750; 1), (400, 450, 700, 800; 1)\rangle$ million liters. Each variable with first subscript 1 represents a shipment of petrol from refinery 1, so this restriction is expressed by the following constraint

$$\tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{12} + \tilde{\tilde{y}}_{13} + \tilde{\tilde{y}}_{14} + \tilde{\tilde{y}}_{15} + \tilde{\tilde{y}}_{16} \approx \langle (450, 500, 700, 750; 1), (400, 450, 700, 800; 1) \rangle$$

In a similar fashion, we can find constraints that reflect other refineries' capacities. In this case, this problem contains the following three supply constraints:

$$\begin{array}{l}
\tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{12} + \tilde{\tilde{y}}_{13} + \tilde{\tilde{y}}_{14} + \tilde{\tilde{y}}_{15} + \tilde{\tilde{y}}_{16} \approx \langle (450, 500, 700, 750; 1), (400, 450, 700, 800; 1) \rangle \text{ (Refinery 1 supply constraint)} \\
\tilde{\tilde{y}}_{21} + \tilde{\tilde{y}}_{22} + \tilde{\tilde{y}}_{23} + \tilde{\tilde{y}}_{24} + \tilde{\tilde{y}}_{25} + \tilde{\tilde{y}}_{26} \approx \langle (350, 400, 600, 650; 1), (300, 350, 600, 700; 1) \rangle \text{ (Refinery 2 supply constraint)} \\
\tilde{\tilde{y}}_{31} + \tilde{\tilde{y}}_{32} + \tilde{\tilde{y}}_{33} + \tilde{\tilde{y}}_{34} + \tilde{\tilde{y}}_{35} + \tilde{\tilde{y}}_{36} \approx \langle (500, 550, 750, 800; 1), (450, 500, 750, 850; 1) \rangle \text{ (Refinery 3 supply constraint)}
\end{array}$$

Second, we need constraints that ensure that each warehouse will receive sufficient petrol to meet its peak demand. For example, warehouse 1 must receive $\langle (100, 125, 225, 250; 1), (75, 100, 225, 275; 1) \rangle$ million liters. Each variable with second subscript 1 represents a shipment of petrol to warehouse 1, so we obtain the following constraint:

$$\tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{21} + \tilde{\tilde{y}}_{31} \approx \langle (100, 125, 225, 250; 1), (75, 100, 225, 275; 1) \rangle$$

Similarly, we obtain a constraint for each of warehouses 2, 3, 4, 5 and 6. A constraint that ensures a location receives its demand is a demand constraint. The company must satisfy the following six demand constraints:

$$\begin{array}{l}
\tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{21} + \tilde{\tilde{y}}_{31} \approx \langle (100, 125, 225, 250; 1), (75, 100, 225, 275; 1) \rangle \text{ (Warehouse 1 demand constraint)} \\
\tilde{\tilde{y}}_{12} + \tilde{\tilde{y}}_{22} + \tilde{\tilde{y}}_{32} \approx \langle (175, 200, 300, 325; 1), (150, 175, 300, 350; 1) \rangle \text{ (Warehouse 2 demand constraint)} \\
\tilde{\tilde{y}}_{13} + \tilde{\tilde{y}}_{23} + \tilde{\tilde{y}}_{33} \approx \langle (125, 150, 250, 275; 1), (100, 125, 250, 300; 1) \rangle \text{ (Warehouse 3 demand constraint)} \\
\tilde{\tilde{y}}_{14} + \tilde{\tilde{y}}_{24} + \tilde{\tilde{y}}_{34} \approx \langle (275, 300, 400, 425; 1), (250, 275, 400, 450; 1) \rangle \text{ (Warehouse 4 demand constraint)} \\
\tilde{\tilde{y}}_{15} + \tilde{\tilde{y}}_{25} + \tilde{\tilde{y}}_{35} \approx \langle (325, 350, 450, 475; 1), (300, 325, 450, 500; 1) \rangle \text{ (Warehouse 5 demand constraint)} \\
\tilde{\tilde{y}}_{16} + \tilde{\tilde{y}}_{26} + \tilde{\tilde{y}}_{36} \approx \langle (300, 325, 425, 450; 1), (275, 300, 425, 475; 1) \rangle \text{ (Warehouse 6 demand constraint)}
\end{array}$$

Because all the $\tilde{\tilde{y}}_{ij}$s must be non-negative, we add the sign restrictions $\tilde{\tilde{y}}_{ij} \succeq \tilde{\tilde{0}}$, $(i = 1, 2, 3,\ j = 1, 2, 3, 4, 5, 6)$.

Combining the objective function, supply constraints, demand constraints, and sign restrictions yields the following interval-valued FVLP formulation of the regional petrol transportation problem:

$$\begin{array}{ll}
\text{Min} & \tilde{\tilde{u}} \approx 2\tilde{\tilde{y}}_{11} + 4\tilde{\tilde{y}}_{12} + 6\tilde{\tilde{y}}_{13} + 8\tilde{\tilde{y}}_{14} + 4\tilde{\tilde{y}}_{15} + 6\tilde{\tilde{y}}_{16} + 3\tilde{\tilde{y}}_{21} + 5\tilde{\tilde{y}}_{22} + 7\tilde{\tilde{y}}_{23} + 5\tilde{\tilde{y}}_{24} + 3\tilde{\tilde{y}}_{25} + 9\tilde{\tilde{y}}_{26} \\
& +2\tilde{\tilde{y}}_{31} + 3\tilde{\tilde{y}}_{32} + 4\tilde{\tilde{y}}_{33} + 6\tilde{\tilde{y}}_{34} + 5\tilde{\tilde{y}}_{35} + 3\tilde{\tilde{y}}_{36} \\
\text{s.t.} & \tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{12} + \tilde{\tilde{y}}_{13} + \tilde{\tilde{y}}_{14} + \tilde{\tilde{y}}_{15} + \tilde{\tilde{y}}_{16} \approx \langle (450, 500, 700, 750; 1), (400, 450, 700, 800; 1) \rangle \\
& \tilde{\tilde{y}}_{21} + \tilde{\tilde{y}}_{22} + \tilde{\tilde{y}}_{23} + \tilde{\tilde{y}}_{24} + \tilde{\tilde{y}}_{25} + \tilde{\tilde{y}}_{26} \approx \langle (350, 400, 600, 650; 1), (300, 350, 600, 700; 1) \rangle \\
& \tilde{\tilde{y}}_{31} + \tilde{\tilde{y}}_{32} + \tilde{\tilde{y}}_{33} + \tilde{\tilde{y}}_{34} + \tilde{\tilde{y}}_{35} + \tilde{\tilde{y}}_{36} \approx \langle (500, 550, 750, 800; 1), (450, 500, 750, 850; 1) \rangle \\
& \tilde{\tilde{y}}_{11} + \tilde{\tilde{y}}_{21} + \tilde{\tilde{y}}_{31} \approx \langle (100, 125, 225, 250; 1), (75, 100, 225, 275; 1) \rangle \\
& \tilde{\tilde{y}}_{12} + \tilde{\tilde{y}}_{22} + \tilde{\tilde{y}}_{32} \approx \langle (175, 200, 300, 325; 1), (150, 175, 300, 350; 1) \rangle \\
& \tilde{\tilde{y}}_{13} + \tilde{\tilde{y}}_{23} + \tilde{\tilde{y}}_{33} \approx \langle (125, 150, 250, 275; 1), (100, 125, 250, 300; 1) \rangle \\
& \tilde{\tilde{y}}_{14} + \tilde{\tilde{y}}_{24} + \tilde{\tilde{y}}_{34} \approx \langle (275, 300, 400, 425; 1), (250, 275, 400, 450; 1) \rangle \\
& \tilde{\tilde{y}}_{15} + \tilde{\tilde{y}}_{25} + \tilde{\tilde{y}}_{35} \approx \langle (325, 350, 450, 475; 1), (300, 325, 450, 500; 1) \rangle \\
& \tilde{\tilde{y}}_{16} + \tilde{\tilde{y}}_{26} + \tilde{\tilde{y}}_{36} \approx \langle (300, 325, 425, 450; 1), (275, 300, 425, 475; 1) \rangle \\
& \tilde{\tilde{y}}_{ij} \succeq \tilde{\tilde{0}}, \quad i = 1, 2, 3,\ \ j = 1, 2, 3, 4, 5, 6.
\end{array} \qquad (4.4)$$

Figure 2 is a graphical representation of the interval-valued fuzzy petrol transportation problem.

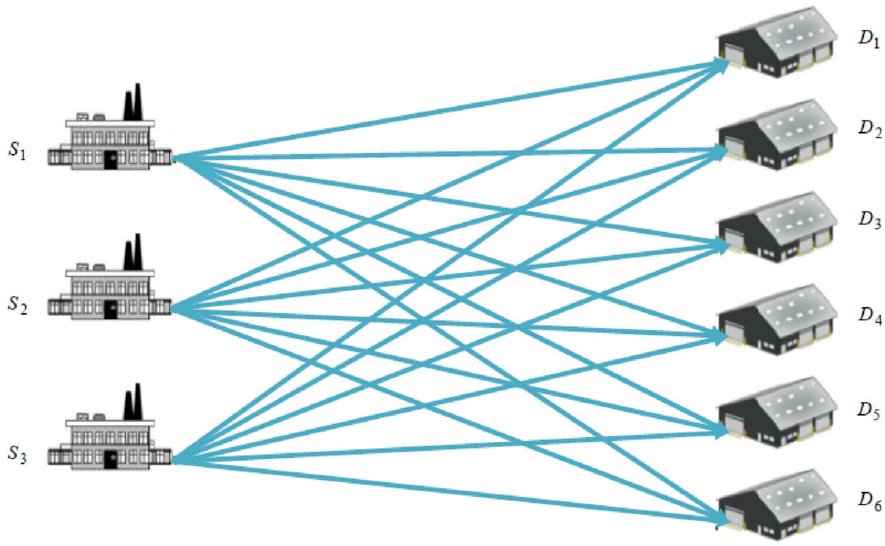

FIGURE 2. Petrol transportation problem.

The auxiliary problem associated with this interval-valued fuzzy transportation problem is given as follows:

$$\begin{aligned}
\text{Max}\ \ & \tilde{\tilde{z}} \approx \langle (450,500,700,750;1),(400,450,700,800;1)\rangle\, u_1 + \langle (350,400,600,650;1),(300,350,600,700;1)\rangle\, u_2 \\
& + \langle (500,550,750,800;1),(450,500,750,850;1)\rangle\, u_3 + \langle (100,125,225,250;1),(75,100,225,275;1)\rangle\, v_1 \\
& + \langle (175,200,300,325;1),(150,175,300,350;1)\rangle\, v_2 + \langle (125,150,250,275;1),(100,125,250,300;1)\rangle\, v_3 \\
& + \langle (275,300,400,425;1),(250,275,400,450;1)\rangle\, v_4 + \langle (325,350,450,475;1),(300,325,450,500;1)\rangle\, v_5 \\
& + \langle (300,325,425,450;1),(275,300,425,475;1)\rangle\, v_6 \\
\text{s.t.}\ \ & u_1+v_1 \le 2, u_1+v_2 \le 4, u_1+v_3 \le 6, u_1+v_4 \le 8, u_1+v_5 \le 4, u_1+v_6 \le 6, \\
& u_2+v_1 \le 3, u_2+v_2 \le 5, u_2+v_3 \le 7, u_2+v_4 \le 5, u_2+v_5 \le 3, u_2+v_6 \le 9, \\
& u_3+v_1 \le 2, u_3+v_2 \le 3, u_3+v_3 \le 4, u_3+v_4 \le 6, u_3+v_5 \le 5, u_3+v_6 \le 3, \\
& u_1,u_2,u_3 \le 0, v_1,v_2,v_3,v_4,v_5,v_6 \ge 0. \qquad (4.5)
\end{aligned}$$

Using Algorithm 3.6, the optimal solution of the auxiliary problem (4.5) is given as follows:

$$u_1=2, u_2=1, u_3=1, v_1=0, v_2=2, v_3=3, v_4=4, v_5=2, v_6=2. \qquad (4.6)$$

Now, according to Corollary 3.10, the interval-valued fuzzy optimal solution of the petrol transportation problem (4.4) is given as follows:

$$\begin{aligned}
&\tilde{\tilde{y}}_{11} = \tilde{\tilde{y}}_{12} \approx \langle (100,125,225,250;1),(75,100,225,275;1)\rangle\,, \\
&\tilde{\tilde{y}}_{14} \approx \langle (250,250,250,250;1),(250,250,250,250;1)\rangle\,, \\
&\tilde{\tilde{y}}_{24} \approx \langle (275,300,400,425;1),(250,275,400,450;1)\rangle\,, \\
&\tilde{\tilde{y}}_{25} \approx \langle (75,100,200,225;1),(50,75,200,300;1)\rangle\,, \\
&\tilde{\tilde{y}}_{32} \approx \langle (75,75,75,75;1),(75,75,75,75;1)\rangle\,, \\
&\tilde{\tilde{y}}_{33} \approx \langle (125,150,250,275;1),(100,125,250,300;1)\rangle\,, \\
&\tilde{\tilde{y}}_{36} \approx \langle (300,325,425,450;1),(275,300,425,475;1)\rangle\,,
\end{aligned}$$

$$\tilde{\tilde{y}}_{13} = \tilde{\tilde{y}}_{15} = \tilde{\tilde{y}}_{16} = \tilde{\tilde{y}}_{21} = \tilde{\tilde{y}}_{22} = \tilde{\tilde{y}}_{23} = \tilde{\tilde{y}}_{26} = \tilde{\tilde{y}}_{31} = \tilde{\tilde{y}}_{34} = \tilde{\tilde{y}}_{35} \approx \tilde{\tilde{0}}. \tag{4.7}$$

Therefore, the optimal transportation cost is given as follows:

$$\tilde{\tilde{u}} \approx \langle (5825, 6350, 8450, 8975; 1), (5300, 5825, 8450, 9650; 1) \rangle . \tag{4.8}$$

Finally, the membership function of the objective function can be formulated as

$$\mu_{\tilde{u}_1^L}(y) = \begin{cases} \frac{y-5825}{525}, & 5825 \leq y \leq 6350, \\ 1, & 6350 \leq y \leq 8450, \\ \frac{8975-y}{525}, & 8450 \leq y \leq 8975, \\ 0, & \text{otherwise.} \end{cases} \qquad \mu_{\tilde{u}_2^U}(y) = \begin{cases} \frac{y-5300}{525}, & 5300 \leq y \leq 5825, \\ 1, & 5825 \leq y \leq 8450, \\ \frac{9650-y}{1200}, & 8450 \leq y \leq 9650, \\ 0, & \text{otherwise.} \end{cases}$$

5. Results and discussions

In this section, the main advantages of the proposed method over the existing methods are explored. Here, we shall point out that the level (w^L, w^U)-interval-valued trapezoidal fuzzy variables linear programming problem studied in this paper is not in the form of a problem whose model also involves fuzzy cost coefficients and fuzzy constraint matrix. This kind of interval-valued FLP problem is called fully interval-valued FLP problem. Although the fully interval-valued FLP problem group is the general case of interval-valued FLP problems, it may not be suitable for all interval-valued FLP problems with different assumptions and sources of fuzziness. Moreover, it needs to point out that for some real application problems such as transportation problem, the constraint matrix cannot be in the form of interval-valued fuzzy matrix. We have reported the significant results of solving interval-valued fuzzy transportation problems as an application of the formulated interval-valued trapezoidal FVLP problem in the previous section. However, the development of the proposed method for solving fully interval-valued FLP problems; is left to the next study.

Now, the proposed method in this study is compared with the existing methods [2, 5, 10, 23] for solving interval-valued FLP problems. Then, the advantages of the proposed approach over the existing methods are explored.

Chiang [3] considered the FLP problems with statistical data and proposed an approach for solving the following FLP problem in which the right-hand side of the constraints and the constraint matrix are interval-valued triangular fuzzy numbers:

$$\begin{aligned} \text{Min} \quad & yb \\ \text{s.t.} \quad & y\tilde{\tilde{A}} \succeq \tilde{\tilde{c}} \\ & y \geq 0. \end{aligned} \tag{5.1}$$

Chiang [2] considered statistical confidence intervals to derive the interval-valued triangular fuzzy numbers given in model (5.1). Then, he used the signed distance ranking for the interval-valued triangular fuzzy numbers to obtain the linear programming in the fuzzy sense. Hence, Chiang [2] proposed the following crisp LP problem in order to obtain the optimal solution of interval-valued FLP problem (5.1):

$$\begin{aligned} \text{Min} \quad & yb \\ \text{s.t.} \quad & yd(\tilde{\tilde{A}}, O_1) \geq d(\tilde{\tilde{c}}, O_1) \\ & y \geq 0. \end{aligned} \tag{5.2}$$

The main defects of this approach can be classified as follows. First, in this approach, the signed distance ranking is used to defuzzify LP in the fuzzy sense. Using simplex method one can get the optimal solution of crisp LP problem (5.2). Such a solution method cannot communicate all fuzzy information into the optimization process without neglecting valuable uncertain information. Second, according to this approach, the optimal solutions of the interval-valued FLP problem (5.1) are real numbers. This represents a compromise in the case of interval-valued fuzzy data. Third, the objective function of interval-valued FLP problem (5.1) should be interval-valued fuzzy number because of interval-valued fuzzy constraints matrix and interval-valued fuzzy resource vector. However, this approach gives a crisp objective function value. The proposed approach in this study overcomes the first defect by solving an auxiliary problem, having only interval-valued fuzzy cost coefficients and without using a signed distance ranking to defuzzify interval-valued FLP problem. In the proposed model (3.1), the decision variables are interval-valued fuzzy numbers. Hence, it is possible to overcome the second and third defects by solving the interval-valued FLP problem (3.1) and by obtaining interval-valued fuzzy optimal solutions.

Su [23] first used interval-valued fuzzy numbers to fuzzify the crisp LP to three cases and then applied a signed distance ranking to defuzzify the resulting interval-valued FLP problems. To describe this approach, consider the following crisp LP problem:

$$\begin{aligned} \text{Min} \ \ & yb \\ \text{s.t.} \ \ & yA \geq c \\ & y \geq 0. \end{aligned} \tag{5.3}$$

In the first case, Su [23] used the level $(w^L, 1)$-interval-valued triangular fuzzy numbers to fuzzify the coefficients of decision variables in the objective function. In the second case, the level $(w^L, 1)$-interval-valued triangular fuzzy numbers have been used to fuzzify the right-hand side of the constraints (resources vector) and the coefficients of decision variable in the constraints (constraints matrix). For the third case, Su [23] combined the first and second cases. Without loss of the generality, we explore the third case. Su [23] fuzzified the crisp parameter t to the following level $(w^L, 1)$-interval-valued triangular fuzzy number:

$$\tilde{\tilde{t}} = \left\langle (t-\delta_2, t, t+\delta_3; w^L), (t-\delta_1, t, t+\delta_4; 1) \right\rangle, \quad 0 < \delta_2 < \delta_1 < t, \quad 0 < \delta_3 < \delta_4. \tag{5.4}$$

Now, if the parameters of the crisp LP problem (5.3) are fuzzified to the $(w^L, 1)$-interval-valued triangular fuzzy numbers regarding the equation (5.4), the following interval-valued FLP problem is obtained:

$$\begin{aligned} Min \ & y\tilde{\tilde{b}} \\ s.t. \ \ & y\tilde{\tilde{A}} \succeq \tilde{\tilde{c}} \\ & y \geq 0. \end{aligned} \tag{5.5}$$

In this case, Su [23] used the signed distance ranking for the interval-valued triangular fuzzy numbers to obtain the following LP in the fuzzy sense:

$$\begin{aligned} \text{Min} \ \ & yd(\tilde{\tilde{b}}, O_1) \\ \text{s.t.} \ \ & yd(\tilde{\tilde{A}}, O_1) \geq d(\tilde{\tilde{c}}, O_1) \\ & y \geq 0. \end{aligned} \tag{5.6}$$

The crisp optimal solution of the problem (5.6) obtained by simplex method is considered as the optimal solution of the interval-valued FLP problem (5.5). The main defects of this approach can be classified as follows. First, similar to the approach proposed by Chiang (2001), such a solution method neglects the valuable uncertain information in the optimization process because of using the signed distance ranking for defuzzification LP in the fuzzy sense. Second, similar to the approach proposed by Chiang [2], this approach only provide precise solutions for the interval-valued FLP problem (5.5), which represent a compromise in cases of interval-valued fuzzy data.

Third, such approach is only effective in reflecting ambiguity and vagueness in parameters by representing uncertainties as the level $(w^L, 1)$-interval-valued triangular fuzzy numbers, while the level (w^L, w^h)-interval-valued trapezoidal fuzzy numbers are most often used for representing uncertainties in real-life applications. The interval-valued trapezoidal fuzzy decision variables in model (3.1) ensure that the optimal solution of interval-valued FLP under consideration is an interval-valued trapezoidal fuzzy number. Hence, it is possible to overcome the second and the third defects by solving the corresponding interval-valued FLP problem.

Farhadinia [10] proposed an approach for solving the following FLP problem in which the right-hand side of the constraints, the constraint matrix and the coefficients of decision variables in the objective function are represented in terms of interval-valued trapezoidal fuzzy numbers:

$$\begin{aligned} \text{Min} \quad & y\tilde{\tilde{b}} \\ \text{s.t.} \quad & y\tilde{\tilde{A}} \succeq \tilde{\tilde{c}} \\ & y \geq 0. \end{aligned} \tag{5.7}$$

Using the signed distance ranking, Farhadinia [10], first converted the interval-valued trapezoidal fuzzy resource vector and interval-valued trapezoidal fuzzy constraints matrix into crisp ones and then proposed a method for solving the following FLP problem with interval-valued trapezoidal fuzzy cost coefficients:

$$\begin{aligned} \text{Min} \quad & y\tilde{\tilde{b}} \\ \text{s.t.} \quad & yd(\tilde{\tilde{A}}, O_1) \geq d(\tilde{\tilde{c}}, O_1) \\ & y \geq 0. \end{aligned} \tag{5.8}$$

The main defects of this approach can be classified as follows. First, this approach converts the interval-valued fuzzy feasible space into a crisp one, leading to potential losses of valuable uncertain information in the feasible region. Second, solving the model (5.8) leads to the crisp optimal solution for the interval-valued FLP problem (5.7). This represents a compromise in the case of interval-valued fuzzy data. The proposed approach in this research can overcome the first and the second defects, by not converting the interval-valued fuzzy feasible space into a crisp one and by considering the decision variables as interval-valued fuzzy ones, respectively.

Ebrahimnejad [5] formulated an uncertain transportation problem (TP), as a special case of LP problem, where all the parameters (cost, supply and demand) as well as the amounts of commodity are represented as the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers. Then, TP in $F_{IVTN}(w^L, w^U)$ environment is:

$$\begin{aligned} \text{Min} \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{\tilde{c}}_{ij} \otimes \tilde{\tilde{x}}_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} \tilde{\tilde{x}}_{ij} = \tilde{\tilde{a}}_i, \quad i = 1, 2, \dots, m, \\ & \sum_{i=1}^{m} \tilde{\tilde{x}}_{ij} = \tilde{\tilde{b}}_j, \quad j = 1, 2, \dots, n, \\ & \tilde{\tilde{x}}_{ij} \succeq \tilde{\tilde{0}}, \quad i = 1, 2, \dots, m, \ j = 1, 2, \dots, n. \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \tilde{\tilde{a}}_i &= \big\langle (a_{i1}^L, a_{i2}^L, a_{i3}^L, a_{i4}^L; w^L), (a_{i1}^U, a_{i2}^U, a_{i3}^U, a_{i4}^U; w^U) \big\rangle \in F_{IVTN}^{+}(w^L, w^U), \quad i = 1, 2, \dots, m, \\ \tilde{\tilde{b}}_j &= \big\langle (b_{j1}^L, b_{j2}^L, b_{j3}^L, b_{j4}^L; w^L), (b_{j1}^U, b_{j2}^U, b_{j3}^U, b_{j4}^U; w^U) \big\rangle \in F_{IVTN}^{+}(w^L, w^U), \quad j = 1, 2, \dots, n, \end{aligned}$$

$$\tilde{\tilde{c}}_{ij} = \left\langle (c_{ij1}^L, c_{ij2}^L, c_{ij3}^L, c_{ij4}^L; w^L), (c_{ij1}^U, c_{ij2}^U, c_{ij3}^U, c_{ij4}^U; w^U) \right\rangle \in F_{IVTN}^{+}(w^L, w^U), \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n,$$
$$\tilde{\tilde{x}}_{ij} = \left\langle (x_{ij1}^L, x_{ij2}^L, x_{ij3}^L, x_{ij4}^L; w^L), (x_{ij1}^U, x_{ij2}^U, x_{ij3}^U, x_{ij4}^U; w^U) \right\rangle \in F_{IVTN}^{+}(w^L, w^U), \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.$$

Ebrahimnejad [5] proposed the following crisp LP problem in order to obtain the interval-valued trapezoidal fuzzy optimal solution of interval-valued trapezoidal fuzzy TP (5.9):

$$\begin{aligned}
\text{Min} \ & d\left(\sum_{i=1}^{m}\sum_{j=1}^{n} \tilde{\tilde{c}}_{ij} \otimes \tilde{\tilde{x}}_{ij}, O_1\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} d\left(\tilde{\tilde{c}}_{ij} \otimes \tilde{\tilde{x}}_{ij}, O_1\right) \\
\text{s.t.} \ & \sum_{j=1}^{n} x_{ij1}^L = a_{i1}^L, \sum_{j=1}^{n} x_{ij2}^L = a_{i2}^L, \sum_{j=1}^{n} x_{ij3}^L = a_{i3}^L, \sum_{j=1}^{n} x_{ij4}^L = a_{i4}^L, \quad i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} x_{ij1}^U = a_{i1}^U, \sum_{j=1}^{n} x_{ij2}^U = a_{i2}^U, \sum_{j=1}^{n} x_{ij3}^U = a_{i3}^L, \sum_{j=1}^{n} x_{ij4}^U = a_{i4}^U, \quad i = 1, 2, \ldots, m, \\
& \sum_{i=1}^{m} x_{ij1}^L = b_{j1}^L, \sum_{i=1}^{m} x_{ij2}^L = b_{j2}^L, \sum_{i=1}^{m} x_{ij3}^L = b_{j3}^L, \sum_{i=1}^{m} x_{ij4}^L = b_{j4}^L, \quad j = 1, 2, \ldots, n, \\
& \sum_{i=1}^{m} x_{ij1}^U = b_{j1}^U, \sum_{i=1}^{m} x_{ij2}^U = b_{j2}^U, \sum_{i=1}^{m} x_{ij3}^U = b_{j3}^U, \sum_{i=1}^{m} x_{ij4}^U = b_{j4}^U, \quad j = 1, 2, \ldots, n, \\
& x_{ij1}^U \geq 0, x_{ij1}^L - x_{ij1}^U \geq 0, x_{ij2}^L - x_{ij1}^L \geq 0, x_{ij3}^L - x_{ij2}^L \geq 0, x_{ij4}^L - x_{ij3}^L \geq 0, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n, \\
& x_{ij2}^U - x_{ij1}^U \geq 0, x_{ij3}^U - x_{ij2}^U \geq 0, x_{ij4}^U - x_{ij3}^U \geq 0, x_{ij4}^U - x_{ij4}^L \geq 0, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n.
\end{aligned} \tag{5.10}$$

Although both the proposed method in this research and the proposed method by Ebrahimnejad [5] gives interval interval-valued fuzzy optimal solution, there are an important reason for using our proposed method. According to the proposed approach by Ebrahimnejad [5], each interval-valued fuzzy supply constraint $\sum_{j=1}^{n} \tilde{\tilde{x}}_{ij} = \tilde{\tilde{a}}_i$ is converted into eight crisp supply constraints. Also, each interval-valued fuzzy demand constraint $\sum_{i=1}^{m} \tilde{\tilde{x}}_{ij} = \tilde{\tilde{b}}_j$ is converted into eight crisp demand constraints. Moreover, each non-negative constraint $\tilde{\tilde{x}}_{ij} \succeq \tilde{\tilde{0}}$ is converted eight crisp boundary constraints. The primary interval-valued fuzzy TP (5.9) has $m+n$ constraints and mn decision variables, whereas the converted problem (5.10) has $8(m+n+mn)$ constraints and $8mn$ variables. It should be noted that an LP problem with p constraints and q decision variables can be solved using Khachian's ellipsoid algorithm and Karmarkar's projective algorithm within an effort of $O\left[q^4 L\right]$ and $O\left[(pq)^6 L\right]$. Here L is the number of binary bits required to record all the data of the problem and is known as the input length of the LP problem [1]. Thus, there is a direct relationship between the number of constraints and variables in LP problems and the computational complexity of the algorithms. Thus, problem (5.10) has a complexity of $O\left[(8pq)^4 L\right]$ and $O\left[(8(m+n+mn)(8mn))^6 L\right]$ in Khachichian's and Karmarkar's algorithms, respectively, whereas the auxiliary problem for solving model (5.9) has the total of complexities $O\left[(m+n)^4 L\right]$ and $O\left[((m+n)(mn))^6 L\right]$, respectively. This shows that the new proposed approach in this research is highly economical compared with problem (5.10) from a computational viewpoint. Hence, from a computation point of view the proposed method is preferable to the Ebrahimnejad's [5] method for solving the interval-valued fuzzy TP.

In sum, the main advantages of the proposed method over the existing methods are summarized as follows:

(1) By introducing the level (w^L, w^U)-interval-valued trapezoidal fuzzy variables linear programming problem, we indeed generalized the linear programming problem with fuzzy variables [8, 20, 21] and proposed an algorithm in a more general setting.

(2) We point out that the level (w^L, w^U)-interval-valued trapezoidal fuzzy number linear programming problem studied by Frahadinia [10] is not in the form of a problem whose model involves fuzzy decision variables. The proposed algorithm in this contribution overcomes this shortcoming.

(3) Ebrahimnejad [3] employed the linear ranking function $\Re(\tilde{\tilde{A}}) = \frac{1}{4}\left[a_1 + a_2 + a_3 + a_4\right]$, which was first proposed by Yager cite28, to define orders on trapezoidal fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4)$.This linear ranking function is a special version of $d(., O_1)$ if a level (w^L, w^U)-interval valued trapezoidal fuzzy number is reduce to a trapezoidal fuzzy number. With this in mind, if the proposed method is applied to Example 6.1 in Mahdavi Amiri and Nasseri [20] and Example 4.1 in Ebrahimnejad *et al.* [8], it leads to the exact same results as obtained before. This demonstrates that study of LP problems with interval-valued trapezoidal fuzzy variables gives rise to the same expected results as those obtained for linear programming with trapezoidal fuzzy variables.

(4) Lai and Hwang [16] assumed that the parameters have a triangular possibility distribution. They used an auxiliary problem which is solved by multi-objective linear programming methods, while we solved the auxiliary problem with the help of generalization simplex algorithm in fuzzy sense without any multi-objective solution method.

(5) An extension of this approach for solving bounded interval-valued trapezoidal FVLP problems in which the interval-valued fuzzy decision variables are restricted within interval-valued fuzzy lower and upper bounds is immediate. To do this, assume that the constraint $\tilde{\tilde{l}} \preceq \tilde{\tilde{y}} \preceq \tilde{\tilde{u}}$ is added to constraints of the problem (3.1). By changing this inequality into two inequalities of the form $\tilde{\tilde{y}} \succeq \tilde{\tilde{l}}$ and $-\tilde{\tilde{y}} \succeq -\tilde{\tilde{u}}$, the constraints $\tilde{\tilde{y}}A \succeq \tilde{\tilde{c}}, \tilde{\tilde{l}} \preceq \tilde{\tilde{y}} \preceq \tilde{\tilde{u}}$ would simplify to $\tilde{\tilde{y}} \begin{pmatrix} A \\ I \\ -I \end{pmatrix} \succeq \begin{pmatrix} \tilde{\tilde{c}} \\ \tilde{\tilde{l}} \\ -\tilde{\tilde{u}} \end{pmatrix}$. Now let $A' = \begin{pmatrix} A \\ I \\ -I \end{pmatrix}$ and $\tilde{\tilde{c'}} = \begin{pmatrix} \tilde{\tilde{c}} \\ \tilde{\tilde{l}} \\ -\tilde{\tilde{u}} \end{pmatrix}$. Thus we get $\tilde{\tilde{y}}A' \succeq \tilde{\tilde{c}}'$ which is equivalent to constraint of the problem (3.1). This means that the proposed approach can be used for solving bounded interval-valued FVLP problems.

(6) According to the proposed approach, it is possible to solve two different problems by use of one algorithm at a same time. In fact, by solving an auxiliary problem, having only interval-valued trapezoidal fuzzy cost coefficients, an interval-valued fuzzy optimal solution is obtained for the problem involving interval-valued trapezoidal fuzzy numbers for the decision variables and the right-hand-side of the constraints.

(7) The proposed algorithm for solving the auxiliary problem not only can be used for solving interval-valued FVLP problems, but also can be used for solving that real-life LP models such shortest path problems, in which just the cost coefficients are represented in terms of interval-valued fuzzy numbers.

(8) There is a lack of solution methodologies in the literature for fuzzy mathematical programs including interval-valued fuzzy parameters [2, 10, 23]. The existing solution approaches for interval-valued FLP problems considered uncertainty only on the parameter values, *i.e.*, cost coefficients, resources vector and technological matrix. However, decision variables can also be considered as interval-valued fuzzy numbers. In detail, obtaining interval-valued fuzzy solutions instead of crisp values which provide ranges of flexibility to decision maker seems more impressive. To best of our knowledge, no work has been studied on interval-valued fuzzy linear programming with interval-valued fuzzy decision variables and interval-valued fuzzy resources. Briefly, the interval-valued fuzzy solutions for the FLP problems can be comparatively better than the currently available crisp solutions.

(9) Majority of existing solution methodologies [2, 5, 10, 23] for FLP problems are transformation methods in which the fuzzy mathematical program is first converted into its crisp equivalent form and then solved by a classical optimization technique. In other words, these methods transform the FLP problem into a crisp LP problem and generally utilize optimization techniques for classical mathematical programming to obtain a solution. Actually, these methods are indirect solution techniques and may cause loss of information while the transformation process. In addition, the generated crisp solutions by these methods may present a very less flexibility to the decision maker. Furthermore, the model size (number of constraints and decision variables) may generally increase due to the transformation process. Therefore, the required computational

effort will also increase in the solution phase. However, the proposed method in this study does not increase the number of constraints and variables directly. The main advantage of proposed method is that utilizing the auxiliary problem for solving interval-valued fuzzy FVLP problem under consideration is highly economical compared with indirect solution techniques from a computational viewpoint, regarding the number of constraints and variables.

(10) The proposed method in this study can be extended to solve interval-valued fuzzy fractional LP problems. That is why this paper can be a good starting point for future research.

6. Concluding remarks

These days a number of researchers have shown interest in the area of fuzzy linear programming problems and various attempts have been made to study the solution of these problems. In this paper we studied a class of fuzzy linear programming problem known as the level (w^L, w^U)-interval-valued trapezoidal fuzzy variable linear programming problem in which the right-hand-side vectors and the decision variables are represented by the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers while the rest of the parameters are represented by real numbers. We proposed a very effective method for solving this problem with the help of an auxiliary problem. The introduced auxiliary problem has been solved based on comparison of the level (w^L, w^U)-interval-valued trapezoidal fuzzy numbers with the help of singed distance ranking.

On the basis of the present study, it can be concluded that the fuzzy variable linear programming problem which can be solved by the existing methods [8, 20, 21], can be solved by the proposed method in this study too. However, the level (w^L, w^U)-interval-valued trapezoidal fuzzy variable linear programming problem which can be solved by the proposed approach, cannot be solved by any of the existing methods [8, 20, 21].

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