RAIRO-Oper. Res. 52 (2018) 1411–1428
https://doi.org/10.1051/ro/2018027

RAIRO Operations Research
www.rairo-ro.org

A WEAK PERTURBATION THEORY FOR APPROXIMATIONS OF INVARIANT MEASURES IN M/G/1 MODEL

Badredine Issaadi*, Karim Abbas and Djamil Aïssani

Abstract. The calculation of the stationary distribution for a stochastic infinite matrix is generally difficult and does not have closed form solutions, it is desirable to have simple approximations converging rapidly to this distribution. In this paper, we use the weak perturbation theory to establish analytic error bounds for the M/G/1 model. Numerical examples are carried out to illustrate the quality of the obtained error bounds.

Mathematics Subject Classification. 68M20, 60K25, 60J10.

Received February 2, 2017. Accepted April 27, 2018.

1. Introduction

Queueing models are more suitable for representing many practical problems. Specifically, the queueing models are used for the evaluation of performance measures of communication, computer, and manufacturing systems [26]. The evaluation of performance measures of queueing models has been widely studied in the literature. Particularly, computing the stationary distribution associated with the queueing model is a challenging problem, because all other important performance measures of the model can be obtained from the stationary distribution. A variety of approaches have been proposed in the literature for approximately or indirectly solving this stationary distribution. The predominant approach is to obtain either the generating function of the stationary distribution or an analytical expression for the stationary distribution containing a Laplace-Stieltjes transform; see, for example [1]. Moreover, when there an infinite buffers for waiting customers in the system, the evaluation of performance measures of such systems becomes more challenging and sometimes even problematic, because the memory requirements and computation time of the used approaches grow exponentially with the number of buffers in the system.

There are many methods for approximating stationary distribution of Markov chain with denumerable-state space. Especially, the well-known technique applicable for limiting model sizes is state truncation [22]. Indeed, approximating the performance measures of Markov chain with denumerable-state space by those corresponding to Markov chains with a finite-state space is an interesting problem. Computationally, when we solve for the stationary distribution of a Markov chain with denumerable-state space, the transition probability matrix of the Markov chain has to be truncated in some way into a finite matrix. Then we will use the stationary distribution of the truncated Markov chain as an approximation to that of the finite-state space, where we expect that as the truncation size increases to infinity, the solution for the truncated Markov chain converges to that of

Keywords and phrases: Truncation, queueing system, weak stability, algorithm.

LaMOS, University of Bejaia, Targua Ouzemour, Bejaia 06000, Algeria.

* Corresponding author: issaadi_badredine@yahoo.fr

Article published by EDP Sciences

the Markov chain with denumerable-state space. While for many application problems the justification of the convergence could be made by the physical meanings of the finite and the countable state Markov chains, it is not always easy to formally justify this claim. In this paper, we develop an algorithmic approach based on the weak perturbation theory for constructing reliable approximations for performance measures of queueing models. Specifically, we investigate if the perturbation analysis and the truncation estimation of the state-space of Markov chains can be integrated into one framework.

Let P, a stochastic infinite matrix, irreducible and positive recurrent, then it admits a unique stationary distribution π, the calculation of this distribution is generally difficult if not impossible. Truncation of the state space then becomes necessary, *i.e.* one solution is to approach P by a finite stochastic matrix P_k, where we assume that P_k has unique stationary distribution, denoted by π_k. The advantage of using finite state space Markov chains lies in the simplicity of the computation of stationary distributions, so that computational procedures such as direct methods based on Gaussian elimination, iterative methods such as Gauss-Seidel and decompositional methods are to be used. The question about the effect of switching from P to P_k on the stationary behavior is expressed by $\pi - \pi_k$, the difference between the stationary distributions. More specifically, let $\|.\|_{TV}$ denote the total variation norm, then the above problem can be phrased as follows: How to choose the threshold k such that $\|\pi - \pi_k\|_{TV}$ (the distance between stationary probabilities of the truncated model and that of the main model) be less than ε, for arbitrarily small ε? Finding a such bounds is of practicable importance, which enables one to determine the needed threshold size.

Convergence proofs when the truncation size tends to infinity have been investigated by Sarymsakov [19] and are crystallized most notably by Seneta [20, 21]. The existence of the stationary distribution π_k has been studied for various types of matrices P_k, in particular by Seneta [22, 23]. Most of their results are included in a paper by Gibson and Seneta [7]. Wolf [29] was focused in particular on the approximation of stationary distribution of an infinite matrix, irreducible, recurrent positive, by the stationary distributions of finite matrices, he examines four types of finite matrices. For instance, Heyman [10] provided a probabilistic treatment of the problem. Kalashnikov and Rachev [12] have also studied the problem of the approximation of an infinite Markov chain, the main part of their work is oriented towards the uniform approximation of the initial chain by finite chains constructed by augmentation of the first column. Simonot [25] was examine the case of an infinite irreducible stochastic matrix, dominated by stochastically monotone chains, the rate of convergence of π_k to π was derived in terms of Foster-Lyapounov condition. Tweedie [27] provided simple error bound in the case of geometrically ergodic chains, stochastically monotone chains, and those dominated by stochastically monotone chains. Moreover, a computable bounds are obtained for polynomially and geometrically ergodic chains by Liu [15]. Recently, Hervé and Ledoux [9] have provided explicit connections between the V-geometric ergodicity of P and that of P_k approximating P. A special attention is paid to obtain an efficient way to specify the convergence rate for P from that of P_k and conversely. Moreover, explicit bounds are obtained for the total variation distance between π_k and π.

The classical model M/G/1 queue is used to represent a large number of real-life computer and networking applications. For example, M/G/1 queues have been applied to evaluate the performance of devices such as volumes in a storage subsystem [3], web servers [18], or nodes in an optical ring network [2]. The contribution of our paper is as follows. We are interested in computing the error bound of the stationary queue length distribution of M/G/1 queueing model through finite truncation of some buffers, provided their stability holds. It is then natural to approximate the stationary distribution of M/G/1 queueing model through truncating some buffers. We may expect that such a truncation well approximates the original model as the truncation level (or size) becomes large. Therefore, we use the weak perturbation theory [9]. More precisely, we are interested in the perturbation of the structure of the transition probability matrix (which is not a parametric perturbation). And we analyzed the effect of the perturbation of the size buffer (*i.e.* k which is the size of the transition probability matrix) on the stationary characteristics of the M/G/1 model. Note that the change in the size buffer (k) is not small enough. The approach presented in this paper is a transform-free approach as it avoids the use Laplace transforms and/or numerical inversion techniques, which are predominantly used in the literature.

Indeed the stationary distribution of the M/G/1 queueing model can be found by using the well-known Pollaczek-Khinchine formula [8]. This formula involves the use of Laplace transforms. However, if we assume the service time distribution is a heavy-tailed distribution, then its stationary distribution cannot be easily computed, because Laplace transforms of such distributions oftentimes do not have an analytic closed form. This is, in particular, the case for the Pareto and Weibull distributions. Therefore, the current analytical theory of these systems, which use the Laplace transform of the heavy-tailed distributions, is of limited range and difficult or even impossible to use in such cases [1]. When the stationary distribution of the M/G/1 queue cannot be computed exactly it needs to be approximated.

The paper is organized as follows. Section 2 presents the necessary definitions and notation. Section 3 is devoted to establish the bounds on the perturbation. Numerical examples are presented in Section 4.

2. Notations and preliminaires

In this section, we introduce necessary notations. For the basic theorems of the weak stability method, we refer to [9]. Specifically, we take P to be the transition probability matrix on $\mathcal{S} = \mathbb{N}$, and π as an invariant or stationary distribution for P. Let $\mathcal{S}_k = \{0, \ldots, k\}$, we are interested in procedures for approximating π using P_k, where P_k is derived from the linear augmentation (in the last column) of the $\mathcal{S}_k \times \mathcal{S}_k$ "northwest truncation" of P. Let $\widehat{P}_k$, $k \geq 1$ be defined the associated (extended) sub-stochastic matrix of P_k on $\mathcal{S}$, a typical instance of sequence $\widehat{P}_k$, $k \geq 1$ is given by considering the extended by zeros of P_k on $\mathcal{S}$ (*e.g.* see [27]).

For any positive measure μ on $\mathcal{S}$, and for any space of bounded measurable functions $f = \{f(x)\}_{x\in\mathcal{S}} \in \mathbb{C}^{\mathcal{S}}$, we associate with each transition operator $\forall k \in \mathbb{N}^* \cup \{\infty\}$, $\widehat{P}_k$ the linear mappings:

$$(\mu\widehat{P}_k)(k) = \sum_{x\in\mathcal{S}} \mu(x)\widehat{P}_k(x,k), \tag{2.1}$$

$$(\widehat{P}_k f)(k) = \sum_{x\in\mathcal{S}} f(x)\widehat{P}_k(k,x), \tag{2.2}$$

where by convention $\widehat{P}_\infty := P$.
Consider any unbounded increasing sequence $V = \{V(x)\}_{x\in\mathcal{S}} \in [1, +\infty)^{\mathcal{S}}$ with $V(0) = 1$, and the associated weighted space $(\mathcal{B}_1, \|\cdot\|_1)$ given by

$$\mathcal{B}_1 := \left\{ f \in \mathbb{C}^{\mathcal{S}} \,:\, \|f\|_1 = \sup_{x\in\mathcal{S}} |f(x)| V(x)^{-1} < \infty \right\}.$$

In the following, PV/V and each $\widehat{P}_k V/V$ are supposed to be bounded on $\mathcal{S}$. Besides that, P (resp. every $\widehat{P}_k$) is supposed to have a stationary distribution π (resp. an invariant bounded positive measure $\widehat{\pi}_k$) on $\mathcal{S}$ such that $\pi(V) < \infty$ (resp. $\widehat{\pi}_k(V) < \infty$), $\widehat{\pi}_k$ is the (extended) stationary distribution on $\mathcal{S}$ derived from π_k the stationary distribution of P_k on $\mathcal{S}_k$. We consider the vectors $\mathbb{1}_{\mathcal{S}_k}$ and $\mathbb{1}_{\mathcal{S}}$ defined as follows:

$$\forall x \in \mathcal{S}, \quad \mathbb{1}_{\mathcal{S}_k} = (\mathbb{1}_{\mathcal{S}_k}(x))_{x\in\mathcal{S}} := \begin{cases} 1 & \text{if } x \in \mathcal{S}_k, \\ 0 & \text{if } x \notin \mathcal{S}_k, \end{cases}$$

and,

$$\forall x \in \mathcal{S}, \quad \mathbb{1}_{\mathcal{S}}(x) := 1.$$

So, $\widehat{P}_k \mathbb{1}_{\mathcal{S}_k} = \mathbb{1}_{\mathcal{S}_k}$ and $\widehat{\pi}_k(\mathbb{1}_{\mathcal{S}_k}) = 1$. Finally we assume that

$$\lim_{k\to+\infty} \pi(\mathbb{1}_{\mathcal{S}_k}) = 1 \quad \text{and} \quad \lim_{k\to+\infty} \widehat{\pi}_k(\mathbb{1}_{\mathcal{S}}) = 1. \tag{2.3}$$

A necessary condition so that, the sequence $\{\widehat{\pi}_k\}_{k\geq 1}$ converges in total variation to π is given by: $\pi(\mathbb{1}_{\mathcal{S}_k}) - 1 = (\pi - \widehat{\pi}_k)(\mathbb{1}_{\mathcal{S}_k})$ and $\widehat{\pi}_k(\mathbb{1}_{\mathcal{S}}) - 1 = (\widehat{\pi}_k - \pi)(\mathbb{1}_{\mathcal{S}})$.

Next we introduce the concept of V-geometric ergodicity, see [17], for details.

Lemma 2.1. *Suppose that P and $\widehat{P}_k$ are V-geometrically ergodic. Then the following conditions are holds:*

1. *there exist some rate $\rho \in (0,1)$ and constant $C > 0$ such that*

$$\forall n \geq 0, \quad \sup_{f\in\mathcal{B}_1, \|f\|_1\leq 1} \|P^n f - \pi(f)\mathbb{1}_{\mathcal{S}}\|_1 \leq C\,\rho^n, \tag{V}$$

2. *there exist some rate $\rho_k \in (0,1)$ and constant $C_k > 0$ such that*

$$\forall n \geq 0, \quad \sup_{f\in\mathcal{B}_1, \|f\|_1\leq 1} \|\widehat{P}_k{}^n f - \widehat{\pi}_k(f)\mathbb{1}_{\mathcal{S}_k}\|_1 \leq C_k\,\rho_k{}^n, \tag{V_k}$$

The property (V) is a consequence of drift inequality and conditions of irreducibility and aperiodicity of P. Otherwise, the property (V_k) hold since the finite matrix $\widehat{P}_k$ is stochastic and 1 is the only eigenvalue of modulus 1 for $\widehat{P}_k$, it is therefore a direct consequence of the irreducibility and aperiodicity of $\widehat{P}_k$.

Now let define by $(\mathcal{L}(\mathcal{B}_0,\mathcal{B}_1), \|\cdot\|_{0,1})$ the space of all the bounded linear maps from $\mathcal{B}_0$ to $\mathcal{B}_1$, equipped with its usual norm:

$$\|T\|_{0,1} = \sup\big\{\|Tf\|_1,\ f \in \mathcal{B}_0,\ \|f\|_0 \leq 1\big\}, \tag{2.4}$$

where $(\mathcal{B}_0, \|\cdot\|_0)$ is the Banach space of bounded measurable functions on $\mathcal{S}$ equipped with its usual norm:

$$\mathcal{B}_0 := \big\{f \in \mathbb{C}^{\mathcal{S}}\ : \|f\|_0 = \sup_{x\in\mathcal{S}} |f(x)| \leq 1\big\}.$$

The strong perturbation theory [13], requires that $\{\widehat{P}_k\}_{k\geq 1}$ converges to P in operator norm on $\mathcal{B}_1$. Unfortunately the convergence of $\|P - \widehat{P}_k\|_1$ to 0 is a condition quite demanding, so it is not always satisfied, even in simple cases [5, 24, 25]. This is why we use the weak perturbation theory due to Keller and Liverani [14, 16] which invokes the weakened convergence property

$$\|\widehat{P}_k - P\|_{0,1} := \sup_{f\in\mathcal{B}_0, \|f\|_0\leq 1} \|\widehat{P}_k f - Pf\|_1 \xrightarrow[k\to+\infty]{} 0. \tag{$C_{0,1}$}$$

Definition 2.2. For all P and $\widehat{P}_k$ defined previously, we have the following Uniform Weak Drift condition:

$$\exists\,\delta \in (0,1),\ \exists L > 0,\ \forall k \in \mathbb{N}^* \cup \{\infty\}, \quad \widehat{P}_k V \leq \delta V + L\,\mathbb{1}_{\mathcal{S}}, \tag{UWD}$$

where by convention $\widehat{P}_\infty := P$.

Hervé and Ledoux [9] have provided explicit connections between the V-geometric ergodicity of P and that of $\widehat{P}_k$ approximating P. A special attention is paid to obtain an efficient way to specify the convergence rate for P from that of $\widehat{P}_k$ and conversely. Moreover, explicit bounds are obtained for the total variation distance between $\widehat{\pi}_k$ and π.

Reminder that the total variation distance between $\widehat{\pi}_k$ and π is given by

$$\|\widehat{\pi}_k - \pi\|_{TV} := \sup_{\|f\|_0\leq 1} |\widehat{\pi}_k(f) - \pi(f)|.$$

The next theorem is relevant to estimate $\|\widehat{\pi}_k - \pi\|_{TV}$.

Theorem 2.3 ([9]). *Assume that Condition (UWD) holds. Set $\Delta_k := \|\widehat{P}_k - P\|_{0,1}$ and $A := 1 + L/(1-\delta)$. If, for some $k \geq 1$, $\widehat{P}_k$ is V-geometrically ergodic with rate and constant (ρ_k, C_k) in (V_k), then*

$$\|\widehat{\pi}_k - \pi\|_{TV} \ \leq \ |1 - \pi(\mathbb{1}_{\mathcal{S}_k})| + \frac{L}{1-\delta}\left(\frac{2C_k}{\rho_k} + \frac{A}{\ln({\rho_k}^{-1})}\,|\ln \Delta_k|\right)\Delta_k. \tag{2.5}$$

If in addition (V_k) is satisfied for all k, then the constants C_k and ρ_k of (V_k) can be chosen such as, for all $k \geq 1$:

- the sequence (C_k) is bounded;
- $\limsup \rho_k < 1$.

These two non-trivial assertions derive from Keller-Liverani's theorem (see [9]). Thus, under the assumptions (UWD), (V_k) and (V), then $\widehat{\pi}_k$ tends to π in total variation distance with the convergence speed $|1 - \pi(\mathbb{1}_{\mathcal{S}_k})| + O(|\ln \Delta_k|\Delta_k)$.

The advantage of inequality (2.5) is that it is true for every integer $k \geq 1$. We have just to know C_k and ρ_k (see Lem. 2.4).

Lemma 2.4 ([9]). *The matrix $\widehat{P}_k$ is assumed to be V-geometrically ergodic, that is 1 is a simple eigenvalue of the stochastic matrix P_k and is the unique eigenvalue of modulus one. Suppose that an explicit upper bound $\widetilde{\rho}_k \in (0,1)$ of the second eigenvalue of P_k is known. Let any ρ_k be such that*

$$\widetilde{\rho}_k < \rho_k < 1 \tag{2.6}$$

and define $s \equiv s(\rho_k) \in \mathbb{N}^*$ as the smallest integer such that

$$\|{P_k}^s - \pi_k(\cdot)\mathbb{1}_{\mathcal{S}_k}\|_1 \leq {\rho_k}^s. \tag{2.7}$$

Then, we obtain the following estimate

$$\begin{aligned} &\forall n \geq 0,\ \|\widehat{P}_k^n - \widehat{\pi}_k(\cdot)\mathbb{1}_{\mathcal{S}_k}\|_1 \leq C_k\,{\rho_k}^n \leq \overline{C}_k\,{\rho_k}^n \\ &\qquad \textit{with}\ \ C_k := \frac{\max_{0\leq r\leq s-1}\|{P_k}^r - \pi_k(\cdot)\mathbb{1}_{\mathcal{S}_k}\|_1}{{\rho_k}^{s-1}} \quad \overline{C}_k := \frac{1-\delta+2L}{(1-\delta)\rho_k^{s-1}}, \end{aligned} \tag{2.8}$$

where the eigenvalue $\widetilde{\rho}_k$ of P_k can be computed using any standard techniques. For example, Jacobi's techniques are still relevant and have led to popular and powerful algorithms, to compute the eigenvalues of symmetric matrices. Another longstanding method that is of great significance and serves as the basis for many algorithms is the Power iteration, *e.g.*, Krylov methods, inverse iteration, QR-method (*e.g.* see [4, 6, 28]).

3. Analysis of the model

3.1. Model description

Consider an M/G/1 (FIFO,∞) queue. Customers arrive in a Poisson process with parameter λ and are served by a single server. Let the service times of these customers be independent and identically distributed random variables $\{Z_n,\ n = 1, 2, 3, \ldots\}$ with $P(Z_n \leq t) = H(t)$, $t \geq 0$; $\mathbf{E}(Z_n) = 1/\mu$; $\mathbf{V}(Z_n) = \sigma_z^2$. We assume that Z_n is the service time of the nth customer. Let $X(t)$ be the number of customers in the system at time t and identify $t_0 = 0, t_1, t_2, \ldots$ as the departure epochs of customers. As described above, at these points the remaining service times of customers are zero. Let $X_n = X(t_n + 0)$ be the number of customers in the system soon after the nth departure. We can show that $\{X_n,\ n = 0, 1, 2, \ldots\}$ is a Markov chain as follows.

Let A_n be the number of customers arriving during Z_n. With the Poisson assumption for the arrival process we have

$$a_j = P(A_n = j) = \int_0^\infty P(X_n = j | Z_n) P(t < Z_n \le t + \mathrm{d}t) \tag{3.1}$$

$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!} \mathrm{d}H(t), \quad j = 0,\, 1,\, 2 \ldots. \tag{3.2}$$

Hence $\{X_n,\, n = 0,\, 1,\, 2,\, \ldots\}$ is a Markov chain. Its parameter space is made up of departure points, and the state space S is the number of customers in the system; $\mathcal{S} = \{0,\, 1,\, 2,\, \ldots\}$. Because of the imbedded nature of the parameter space, it is known as an *imbedded Markov chain.*

Let

$$P(i,j) = \mathbf{P}(X_{n+1} = j | X_n = i) = \begin{cases} a_j & \text{if } j \ge 0,\, i = 0, \\ a_{j-i+1} & \text{if } 1 \le i \le j+1, \\ 0 & \text{else.} \end{cases} \tag{3.3}$$

The transition probability matrix P for the Markov chain is

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \ddots \end{array} \!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ & a_0 & a_1 & \cdots \\ & & a_0 & \cdots \\ & & & \cdots \end{pmatrix} \end{array}.$$

Let $\mathcal{S}_k = \{0, \ldots, k\}$ for any $k \ge 1$. Consider the northwest corner of the order k of the matrix P: $T_k = (p(i,j))_{(i,j)\in\mathcal{S}_k^2}$. P being irreducible, there exists at least one line i for which $\sum_{j=1}^k p(i,j) < 1$ so that the truncated matrix T_k is not stochastic.

From T_k we construct a stochastic matrix $P_k = (p_k(i,j))_{(i,j)\in\mathcal{S}_k^2}$ verifying $P_k \ge T_k$, that is $p_k(i,j) \ge p(i,j)$ for $(i,j) \in \mathcal{S}_k^2$; this can be done as follows: The lost probability mass during the truncation of P is redistributed on the last column of T_k, more precisely,

$$p_k(i,j) = p(i,j) + \mathbb{1}_k(j) \sum_{\ell > k} p(i,\ell) \quad \text{for } (i,j) \in \mathcal{S}_k^2,$$

where

$$\mathbb{1}_k(j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Using the probability distribution $\{a_j,\, j = 0,\, 1,\, 2 \ldots\}$ defined in (3.2), we get the transition probability matrix

$$P_k = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ k \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \cdots & k \end{array} \\ \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & 1 - \sum_0^{k-1} a_j \\ a_0 & a_1 & a_2 & \cdots & 1 - \sum_0^{k-1} a_j \\ & a_0 & a_1 & \cdots & 1 - \sum_0^{k-2} a_j \\ & & & & \vdots \\ & & & & 1 - a_0 \end{pmatrix} \end{array}.$$

Such a matrix is generally called a linear augmentation (in the last column here) of the northwest corner truncation of P. For other kinds of augmentations and related literature, see Seneta [22]. This system is the same that M/G/1 queue described earlier, with the restriction that the capacity for the number of customers in the system is $k+1$. Since the state space for the imbedded Markov chain is the number in the system soon after departure, $k+1$ will not be included in the state space; $\mathcal{S}_k = \{0, 1, 2, \ldots, k\}$.

The associated (extended) sub-Markov kernel $\widehat{P}_k$ on $\mathcal{S}$ is defined by:

$$\forall (i,j) \in \mathcal{S}^2, \quad \widehat{P}_k(i,j) := \begin{cases} P_k(i,j) & \text{if } (i,j) \in \mathcal{S}_k^2, \\ 0 & \text{if } (i,j) \notin \mathcal{S}_k^2. \end{cases}$$

Assume that stochastic matrix P_k has an invariant probability measure π_k, then $\widehat{\pi}_k$ is the probability measure on $\mathcal{S}$ defined by

$$\forall x \in \mathcal{S}, \quad \widehat{\pi}_k(x) := \begin{cases} \pi_k(x) & \text{if } x \in \mathcal{S}_k, \\ 0 & \text{if } x \notin \mathcal{S}_k. \end{cases}$$

3.2. Weak perturbation bounds

Consider any unbounded increasing sequence $V := \{V(x)\}_{x \in \mathcal{S}} \in [1, +\infty)^{\mathcal{S}}$ with $V(0) = 1$, and the associated weighted space $(\mathcal{B}_1, \|\cdot\|_1)$ given by

$$\mathcal{B}_1 := \left\{ f \in \mathbb{C}^{\mathcal{S}} \, : \|f\|_1 = \sup_{x \in \mathcal{S}} |f(x)| V(x)^{-1} < \infty \right\}.$$

Specifically, for $\beta > 1$, we will choose,

$$V(x) = \beta^x, \quad \text{for } x \in \mathcal{S}.$$

Lemma 3.1. *Suppose that in the M/G/1 queueing system the following conditions holds:*

(a) $\lambda \mathbf{E}(\xi) < 1$, *where ξ is the service time (geometric ergodicity),*
(b) $\exists\, \theta > 0, \mathbf{E}(e^{\theta \xi}) = \displaystyle\int_0^\infty e^{\theta t} \, \mathrm{d}H(t) < \infty$ *(Cramér condition).*

then, $\exists\, \beta_0 > 1$ such that

$$\beta_0 = \sup \left\{ \beta \, : \, \psi\left(\lambda\left(1 - \beta\right)\right) < \beta \right\}, \tag{3.4}$$

where,

$$\psi\left(\lambda\left(1-\beta\right)\right) = \int_0^\infty e^{-(\lambda(1-\beta))x} \, \mathrm{d}H(x).$$

Proof. Let $\Psi(\beta) = \psi\left(\lambda\left(1-\beta\right)\right)$. Ψ is continuous differentiable in $[1, \theta]$, so

$$\Psi'(\beta) = \lambda \int_0^\infty x \; e^{-(\lambda(1-\beta))x} \, \mathrm{d}H(x),$$
$$\Psi''(\beta) = \lambda^2 \int_0^\infty x^2 \; e^{-(\lambda(1-\beta))x} \, \mathrm{d}H(x).$$

According to the relation between monotony and the sign of the derivative, then the function Ψ is strictly convex in $[1,\theta]$. Let us define the function:

$$\Phi(\beta) = \frac{\Psi(\beta)}{\beta} = \frac{1}{\beta}\int_0^{\infty} e^{-(\lambda(1-\beta))x}\,\mathrm{d}H(x).$$

For $\beta = 1$ we have $\Phi(1) = 1$. Furthermore, $\Phi'(\beta) = \left(\beta\Psi'(\beta) - \Psi(\beta)\right)/\beta^2$. Then

$$\Phi'(1) = \Psi'(1) - \Psi(1) \;=\; \lambda\int_0^{\infty} x\,\mathrm{d}H(x) - \int_0^{\infty}\mathrm{d}H(x) \;=\; \lambda\mathbf{E}(\xi) - 1 \;<\; 0$$

where ξ is the service time. So in the neighborhood of 1, Φ is decreasing. Then, $\exists\,\beta > 1$ such that $\Phi(\beta) < \Phi(1)$ (*i.e.* $\Psi(\beta)/\beta < 1$). Moreover, from the convexity of the function $\Psi(\beta)$, $\exists\,\beta_0 > 0$ such that

$$\forall\,\beta \in (1,\beta_0):\;\; \Phi(\beta) < 1 \;\Rightarrow\; \psi\left(\lambda\left(1-\beta\right)\right) < \beta.$$

$\square$

Remark 3.2. Consider the function $\beta = \psi\left(\lambda\left(1-\beta\right)\right)$. Then ψ has only one fixed-point and can then be used to calculate β_0 iteratively using successive substitution as follows:

$$\beta_{i+1} = \psi\left(\lambda\left(1-\beta_i\right)\right).$$

This method of solution is also called fixed-point iteration. The steps of this iterative process, are:
Step 1: Let β_1 be such that $\beta_1 > 1$,
Step 2: $\beta_{i+1} = \int_0^{\infty} e^{-(\lambda(1-\beta_i))x}\,\mathrm{d}H(x)$,
Step 3: The stopping criterion is given by the following formula: $\left|\frac{\beta_{i+1}-\beta_i}{\beta_{i+1}}\right| < \varepsilon$, where $\varepsilon > 0$ is a fixed tolerance.

Lemma 3.3. *For all β such that $1 < \beta < \beta_0$, we have*

$$\forall k \in \mathbb{N}^* \cup \{\infty\}, \quad \widehat{P}_k V \leq \delta V + L\,1_{\mathcal{S}} \tag{3.5}$$

with,

$$\delta := \delta(\beta) \;=\; \frac{1}{\beta}\psi\left(\lambda\left(1-\beta\right)\right) < 1, \tag{3.6}$$

$$L := L(\beta) \;=\; (\beta - 1)\delta(\beta) < \infty, \tag{3.7}$$

where by convention $\widehat{P}_\infty := P$.

Proof. In a first step, we prove that

$$\exists\,\delta \in (0,1),\; \exists\,L > 0, \quad PV \leq \delta V + L\,1_{\mathcal{S}}. \tag{3.8}$$

According to equation (2.2), we have:

$$\forall\,i \in \mathcal{S}, \quad PV(i) = \sum_{j\in\mathcal{S}} V(j)P(i,j)$$

(a) For $i = 0$:

$$PV(0) \;=\; \int_0^\infty e^{-(1-\beta)\lambda t}\,\mathrm{d}H(t) \;=\; \psi\left(\lambda\left(1-\beta\right)\right), \tag{3.9}$$

(b) For $i \geq 1$:

$$PV(i) \;=\; \beta^{i-1}\int_0^\infty e^{-(1-\beta)\lambda t}\,\mathrm{d}H(t) \;=\; \frac{1}{\beta}\psi\left(\lambda\left(1-\beta\right)\right)\,V(i). \tag{3.10}$$

From (3.10), we take

$$\delta := \delta(\beta) \;=\; \frac{1}{\beta}\psi\left(\lambda\left(1-\beta\right)\right). \tag{3.11}$$

And, from (3.9), we have

$$PV(0) = \delta V(0) + L = \delta + L = \psi\left(\lambda\left(1-\beta\right)\right),$$

Then, we define L as,

$$L := L(\beta) \;=\; \left(1 - \frac{1}{\beta}\right)\psi\left(\lambda\left(1-\beta\right)\right) \;=\; (\beta-1)\delta(\beta). \tag{3.12}$$

The proof of,

$$\exists\,\delta \in (0,1),\ \exists\,L > 0,\ \forall k \in \mathbb{N}^*,\quad \widehat{P}_k V \leq \delta V + L\,1_{\mathcal{S}} \tag{3.13}$$

is similar by exchanging the role of P by $\widehat{P}_k$.

Hence, according to equation (2.2), we have:

$$\begin{aligned}
&\forall\, i \in \mathcal{S}_k, \quad \widehat{P}_k V(i) = \sum_{j\in\mathcal{S}} V(j)\widehat{P}_k(i,j) \;=\; \sum_{j\in\mathcal{S}_k} V(j)P_k(i,j),\\
&\forall\, i \notin \mathcal{S}_k, \quad \widehat{P}_k V(i) = 0.
\end{aligned}$$

(a) For $i = 0$:

$$\begin{aligned}
\widehat{P}_k V(0) &= \sum_{j<k}\beta^j P(0,j) + \beta^k \sum_{j\geq k} P(0,j)\\
&\leq \sum_{j\in\mathcal{S}}\beta^j P(0,j) \;=\; \psi\left(\lambda\left(1-\beta\right)\right),
\end{aligned} \tag{3.14}$$

(b) For $1 \leq i \leq k$:

$$\begin{aligned}
\widehat{P}_k V(i) &= \sum_{j\in\mathcal{S}_{k-1}}\beta^j P(i,j) + \beta^k \sum_{j\geq k} P(i,j)\\
&\leq \sum_{j\in\mathcal{S}}\beta^j P(i,j) \;=\; \frac{1}{\beta}\psi\left(\lambda\left(1-\beta\right)\right)\,V(i).
\end{aligned} \tag{3.15}$$

□

Remark 3.4. We can notice that the relation (3.13) can be given in a simpler way, we have for all $i \in \mathcal{S}_k$,

$$(\widehat{P}_k V)(i) = \sum_{j=0}^{k-1} P(i,j)V(j) + V(k) \sum_{j \geq k} P(i,j) \leq PV(i), \tag{3.16}$$

so $(\widehat{P}_k V)(i) \leq \delta V(i) + L$.

Lemma 3.5. *For all β such that $1 < \beta < \beta_0$, we have:*

$$\|\widehat{P}_k - P\|_{0,1} \leq \Delta_k = \max\left\{2\left(1 - \sum_{j \in \mathcal{S}_k} a_j\right), \frac{2}{\beta^k}\left(1 - \sum_{j \in \mathcal{S}_1} a_j\right), \frac{1}{\beta^{k+1}}\right\}. \tag{3.17}$$

Proof. Using the definition of the weakened convergence property ($C_{0,1}$), we have

$$\|\widehat{P}_k - P\|_{0,1} \leq \sup_{i \in \mathcal{S}} \frac{1}{V(i)} \sum_{j \in \mathcal{S}} |P(i,j) - \widehat{P}_k(i,j)|.$$

Let,

$$\forall\ i \in \mathcal{S}: \quad \Delta(i) = \frac{\sum_{j \in \mathcal{S}} |P(i,j) - \widehat{P}_k(i,j)|}{V(i)}.$$

(a) For $i = 0$:

$$\Delta(0) \ = \ 2\sum_{j>k} a_j \ = \ 2\left(1 - \sum_{j \in \mathcal{S}_k} a_j\right), \tag{3.18}$$

(b) For $1 \leq i \leq k$:

$$\Delta(i) \ = \ \frac{2}{\beta^i} \sum_{j > k-i+1} a_j \ = \ \frac{2}{\beta^i}\left(1 - \sum_{j \in \mathcal{S}_{k-i+1}} a_j\right), \tag{3.19}$$

(c) For $i > k$:

$$\Delta(i) = \frac{1}{V(i)} \sum_{j \in \mathcal{S}} P(i,j) \ = \ \frac{1}{\beta^i} \ \leq \ \frac{1}{\beta^{k+1}}, \tag{3.20}$$

finally, we have

$$\begin{aligned} \|\widehat{P}_k - P\|_{0,1} \leq \Delta_k &= \max\left\{2\left(1 - \sum_{j \in \mathcal{S}_k} a_j\right), \frac{2}{\beta^k}\left(1 - \sum_{j \in \mathcal{S}_1} a_j\right), \frac{1}{\beta^{k+1}}\right\}. \\ &\xrightarrow[k \to +\infty]{} 0 \end{aligned} \tag{3.21}$$

$\square$

The following theorem provides the total variation distance between stationary distributions $\widehat{\pi}_k$ and π. This can be done by using Theorem 2.3.

Theorem 3.6. *Assume that $\lambda < \mu$, and for all β such that $1 < \beta < \beta_0$. Set*

$$A := 1 + \frac{L}{1-\delta}, \tag{3.22}$$

then, we have the following estimation

$$\|\widehat{\pi}_k - \pi\|_{TV} \;\leq\; \left(\frac{1}{\beta^{k+1}} + \left(\frac{2C_k}{\rho_k} + \frac{A|\ln \Delta_k|}{\ln(\rho_k^{-1})}\right)\Delta_k\right)(A-1). \tag{3.23}$$

Proof. From the Theorem 2.3, we have

$$\|\widehat{\pi}_k - \pi\|_{TV} \;\leq\; |1 - \pi(\mathbb{1}_{\mathcal{S}_k})| + \frac{L}{1-\delta}\left(\frac{2C_k}{\rho_k} + \frac{A}{\ln({\rho_k}^{-1})}\,|\ln \Delta_k|\right)\Delta_k,$$

where, $|1-\pi(\mathbb{1}_{\mathcal{S}_k})| = \sum_{j\geq k+1}\pi(j)$ can be estimated as follows. From Lemma 3.3, for all β such that $1 < \beta < \beta_0$ the condition (3.5) holds. Then $\pi(V) \leq \delta\pi(V) + L$, hence $\pi(V) \leq L/(1-\delta)$, furthermore

$$\begin{aligned}\sum_{j\geq k+1}\pi(j) &\leq \frac{1}{V(k+1)}\sum_{j\geq k+1}\pi(j)V(j)\\ &\leq \frac{1}{\beta^{k+1}}\pi(V)\\ &= \frac{L}{(1-\delta)\beta^{k+1}}\\ &= \frac{A-1}{\beta^{k+1}}.\end{aligned}$$

Hence, we obtain the following result

$$\|\widehat{\pi}_k - \pi\|_{TV} \;\leq\; \mathbf{WSB}(\beta,\rho_k) = \left(\frac{1}{\beta^{k+1}} + \left(\frac{2C_k}{\rho_k} + \frac{A|\ln \Delta_k|}{\ln(\rho_k^{-1})}\right)\Delta_k\right)(A-1)$$

with Δ_k defined in (3.17), and the constants ρ_k and C_k can be calculated using the Lemma 2.4. $\square$

Note that the bound in Theorem 3.6 has β and ρ_k as a free parameters. This gives the opportunity to minimize $\mathbf{WSB}(\beta,\rho_k)$ with respect to β and ρ_k. This leads to the following optimization problem.

$$(P)\qquad \begin{array}{cc}\displaystyle\min_{\beta,\rho_k} & \mathbf{WSB}(\beta,\rho_k)\\ \text{s.t.} & 1 < \beta < \beta_0,\\ & \widetilde{\rho}_k < \rho_k < 1,\end{array}$$

where, we have supposed that $\widetilde{\rho}_k \in (0,1)$ the second eigenvalue of P_k is known, and β_0, δ, L, C_k, A and Δ_k are already defined. By inserting $\varepsilon > 0$ small, all inequalities can be made strict and in the above optimization problem can be solved using any standard technique. In the sequel, we assume that $(\beta, \rho_k) \in \Theta = [1+\varepsilon, \beta_0 - \varepsilon] \times [\widetilde{\rho}_k + \varepsilon, 1-\varepsilon]$, so the optimization problem can be defined as follows

$$(P) \qquad \min_{\Theta} \mathbf{WSB}(\beta, \rho_k).$$

WSB being continuous on Θ, so the function **WSB** is bounded and reaches its bounds, *i.e.* **WSB** has a global minimum on Θ. *A priori*, these minimums may not be unique (can be reached several times on Θ).

4. Numerical example

In this section we will apply our bound put forward in Theorem 3.6.

4.1. Approximation algorithm

In this subsection we elaborate the algorithms which allows us to get the domain of the approximation and to determine the error on the stationary distribution due to the approximation. Solve the optimization problem (P) is of great importance, this will allow us to define the best choice of the couple (β, ρ_k) on Θ which gives the minimal value of the bound $\mathbf{WSB}(\beta, \rho_k)$ for error of the approximation $\|\widehat{\pi}_k - \pi\|_{TV}$, such that $\|\widehat{\pi}_k - \pi\|_{TV} \leq \mathbf{WSB}(\beta, \rho_k)$. The idea is then to go through the stability domain Θ with a fixed step size ε and save the smallest value found in the sense of the objective function **WSB**.

Algorithm 1: Computing the perturbation bound $\mathbf{WSB}(\beta_{\text{opt}}, \rho_{\text{opt}})$.

```
    INITIALISATION: Definition of the inputs ;
     λ, μ, k and ε > 0 small;
     Fix ρ_opt = ∞, β_opt = ∞ and WSB_opt = ∞;
BEGIN
  if The condition (λ < μ) hold then
      Calculate P_k, π_k (the invariant measure of P_k) and ρ̃_k ∈ (0,1) (the second eigenvalue of P_k);
      Calculate β_0 (from Rem. (3.2)) ;
      ρ_k = ρ̃_k;
      while ρ_k < 1 do
          ρ_k = ρ_k + ε;
          β = 1;
          while β < β_0 do
              β = β + ε;
              Calculate δ(β) (from (3.6));
              Calculate L(β) (from (3.7)), and A (from (3.22));
              Calculate s and C_k (from Lem. 2.4);
              Calculate WSB(β, ρ_k) (from Thm. 3.6);
              if WSB(β, ρ_k) < WSB_opt then
                  WSB_opt = WSB(β, ρ_k);
                  β_opt = β;
                  ρ_opt = ρ_k;
```

The optimization problem (P) can be solved also using Acceptance-Rejection algorithm, we refer to [11]. Let θ denote the couple (β, ρ_k), then to introduce the idea implemented in this algorithm, we simulate from bivariate uniform distribution on Θ, $\theta_1, \ldots, \theta_N \sim U_\Theta$, and we use the approximation $\mathbf{WSB}_{\text{opt}} = \min(\mathbf{WSB}(\theta_1), \ldots, \mathbf{WSB}(\theta_N))$.

The all steps of this algorithm are given as follows:

Algorithm 2: Computation of the weak stability bound $\mathbf{WSB}(\beta, \rho_k)$.

INITIALISATION: Fix N. (must be very large);
Fix $\mathbf{WSB}_{\text{opt}} = \infty$;

BEGIN

For $i = 1$ **to** N **do**

Generate $(u_1, u_2) \sim U_\Theta$;

if $\boldsymbol{WSB}(u_1, u_2) < \boldsymbol{WSB}_{\text{opt}}$ **then**

$\mathbf{WSB}_{\text{opt}} = \mathbf{WSB}(u_1, u_2)$;

$\beta_{\text{opt}} = u_1$;

$\rho_{\text{opt}} = u_2$;

where U is the bivariate uniform distribution on Θ and β_0, $\widetilde{\rho}_k$, $\mathbf{WSB}$ and δ are already defined previously;

4.2. Numerical validation

In order to examine the effectiveness of Algorithm 1, and to explore the stationary performance of the M/G/1 model compared to its truncated model, we provide a series of numerical results for different performance of the model. The results obtained are shown in tables and figures. We consider four types of service time distributions: Deterministic (D), Exponential (M), Hyperexponential (H_2) and Erlang (E_2). Meanwhile, for the specified distributions of the service time, we obtain different coefficients of variation (CV).

4.2.1. A deterministic or fixed service time

We consider the chain embedded at departure times of an M/D/1 queue (*i.e.* with Poisson input rate, and service times deterministic of unit length). The deterministic arrival process has CV = 0. The probability (3.2) is given by:

$$a_j = \frac{\lambda^j}{j!} e^{-\lambda}, \quad j = 0, 1, \ldots$$

Let us choose, for example: $\lambda = 0.6$ and $k = 10$. The approximation domain has been determined using the Algorithm 1:

$$\begin{aligned} \beta \in (1, \beta_0) &= (1, 2.57), \\ \rho_k \in (\widetilde{\rho}_k, 1) &= (0.8545, 1). \end{aligned}$$

So, we can give an idea about the error due to the approximation on the stationary distribution, by showing its curve as function of β and ρ_k (Fig. 1). For that, we display the strong stability bound $\mathbf{WSB}(\beta, \rho_{\mathbf{k}})$ as function of the norm parameter β and the parameter ρ_k. From the Figure 1, we see that this curve has a minimum with respect to the parameter β and ρ_k. This is clearly shown in the Figure 2 where we have represented the curve $\mathbf{WSB}(\beta, \rho_{\text{opt}})$ in function of β and where we have fixed the parameter ρ_k to $\rho_{\text{opt}} = 0.8645$ (the optimal value of ρ_k that minimise $\mathbf{WSB}(\beta, \rho_k)$ with respect to ρ_k). From this Figure we can notice that the curve has an unique minimum at a certain point, denoted by β_{opt}, this is well shown on the right part of Figure 2, where we have zoomed it. Thereby, inducing that it has an unique minimum at a certain point, denoted by β_{opt}. By applying the above algorithms, one can obtain the smallest error at $\beta = \beta_{\text{opt}} = 2.08$, and the corresponding error value due to the approximation on the stationary distribution of the number of customers is:

$$||\pi - \widetilde{\pi}||_{TV} \leq \mathbf{WSB}(2.08, 0.8645) = 2.8885.$$

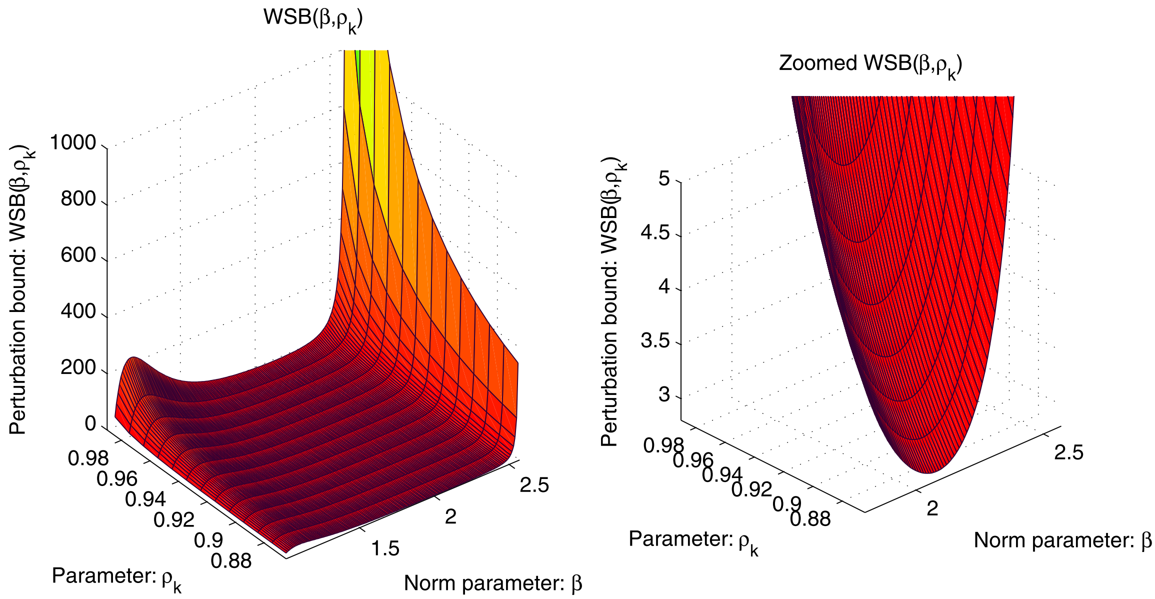

FIGURE 1. Weak stability bound $\mathbf{WSB}(\beta, \rho_k)$ as function of β and ρ_k.

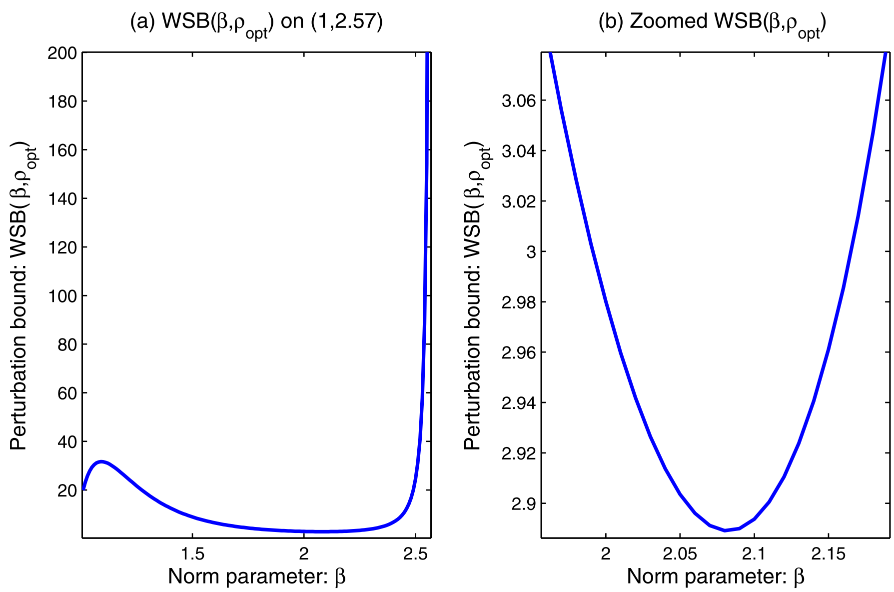

FIGURE 2. Weak stability bound $\mathbf{WSB}(\beta, \rho_{\text{opt}})$ as function of β.

Similarly, the obtained numerical results for the weak stability bounds in D/M/1 queue are given in the following Table 1, where we set $\lambda = 0.6$.

4.2.2. Exponential service times

We consider the M/M/1 queue. The stationary distributions of the number of customers in the system at departure points can be obtained in a similar way for the following fixed parameters: $\lambda = 1$ and $\mu = 2$, then

TABLE 1. Weak stability bounds for M/D/1 queue.

$\lambda = 0.6$, $M/D_1/1$									
k	β_0	β	$\widetilde{\rho}_k$	ρ_k	δ	L	s	C_k	$\mathbf{WSB}(\beta, \rho_k)$
5	2.5700	1.0100	0.7503	0.8701	0.9961	0.0100	2	2.1645	17.8228
10	2.5700	2.0800	0.8539	0.8645	0.9191	0.9926	2	3.1641	2.8885
15	2.5700	2.2400	0.8762	0.8865	0.9394	1.1649	2	5.6046	0.1044
20	2.5700	2.3300	0.8841	0.8945	0.9533	1.2678	2	8.3979	0.0024
25	2.5700	2.3800	0.8884	0.8982	0.9617	1.3271	2	11.1340	4.2030e-005
30	2.5700	2.4100	0.8905	0.9003	0.9669	1.3634	2	13.6706	6.5151e-007
35	2.5700	2.4300	0.8918	0.9016	0.9706	1.3879	2	16.0267	9.2071e-009
40	2.5700	2.4500	0.8921	0.9024	0.9742	1.4127	2	18.8869	1.2161e-010
45	2.5700	2.4700	0.8930	0.9029	0.9780	1.4377	2	22.4949	1.5309e-012
50	2.5700	2.4800	0.8932	0.9034	0.9799	1.4503	2	25.1450	1.8487e-014

TABLE 2. Weak stability bounds for M/M/1 queue.

$\lambda = 1$, $\mu = 2$, M/M/1									
k	β_0	β	$\widetilde{\rho}_k$	ρ_k	δ	L	s	C_k	$\mathbf{WSB}(\beta, \rho_k)$
5	2.0000	1.4000	0.6664	0.6767	0.8929	0.3571	2	1.5432	12.3777
10	2.0000	1.6600	0.8181	0.8283	0.8991	0.5934	2	1.6965	4.6573
15	2.0000	1.7700	0.8549	0.8651	0.9187	0.7074	2	2.8203	0.5796
20	2.0000	1.8200	0.8694	0.8791	0.9313	0.7636	2	3.8985	0.0458
25	2.0000	1.8500	0.8758	0.8860	0.9401	0.7991	2	4.9467	0.0029
30	2.0000	1.8800	0.8799	0.8898	0.9498	0.8359	2	6.3576	1.5725e-004
35	2.0000	1.8900	0.8823	0.8921	0.9533	0.8485	2	7.2144	7.8445e-006
40	2.0000	1.9100	0.8835	0.8937	0.9607	0.8742	2	8.8706	3.6723e-007
45	2.0000	1.9200	0.8844	0.8947	0.9645	0.8873	2	10.1435	1.6353e-008
50	2.0000	1.9200	0.8858	0.8955	0.9645	0.8873	2	10.6948	7.0014e-010

the associated traffic intensity is $\sigma = \lambda/\mu = 0.5$ and the associated coefficient of variation is CV $= 1$. The computational results for this example are shown in Table 2.

4.2.3. Erlang service times

The Erlang E_n is the simplest phase-type distribution and it represents the distribution of the time taken by a Markov process to traverse n phases of exponential service. We may use this representation to provide a Markov model for the number of customers in the system in queue $M/E_n/1$.

Consider the queue $M/E_n/1$. Suppose that the inter-arrival times are distributed according to the exponential distribution with parameter λ, let the service time distribution be given as

$$f(t) = e^{-n\mu t} \frac{(n\mu t)^{n-1} n\mu}{(n-1)!}, \quad t > 0.$$

Its coefficient of variation is defined as the ratio between the standard deviation and the average, then CV$=1/\sqrt{n} < 1$. In our numerical computation, we take $n = 2$ with CV$= 1/\sqrt{2}$, and we use Table 3 to present the error on the stationary distributions of the number of customers in the system at departure points under the fixed parameters: $\lambda = 1$ and $\mu = 2$, we obtain:

TABLE 3. Weak stability bounds for M/E$_2$/1 queue.

$\lambda = 1$, $\mu = 2$, $\ell = 2$, M/E$_2$/1									
k	β_0	β	$\widetilde{\rho}_k$	ρ_k	δ	L	s	C_k	**WSB**(β, ρ_k)
5	2.4300	1.6600	0.6811	0.6914	0.8640	0.5702	2	1.4463	9.6148
10	2.4300	2.0100	0.8089	0.8191	0.8904	0.8993	2	2.5058	1.3707
15	2.4300	2.1400	0.8385	0.8482	0.9141	1.0420	2	4.2221	0.0615
20	2.4300	2.2100	0.8489	0.8591	0.9301	1.1254	2	6.0477	0.0018
25	2.4300	2.2500	0.8540	0.8643	0.9403	1.1754	2	7.7962	4.1335e-005
30	2.4300	2.2800	0.8575	0.8672	0.9485	1.2141	2	9.6679	8.3809e-007
35	2.4300	2.3000	0.8593	0.8690	0.9543	1.2405	2	11.4307	1.5541e-008
40	2.4300	2.3200	0.8602	0.8701	0.9602	1.2675	2	13.3504	2.6964e-010
45	2.4300	2.3300	0.8607	0.8709	0.9633	1.2811	2	34.9371	4.4853e-012
50	2.4000	2.3000	0.9000	0.9094	0.9900	1.3000	2	1.5006e+003	9.4280e-014

TABLE 4. Weak stability bounds for M/H$_2$/1 queue.

$\lambda = 1$,$\mu_1 = 2$, $\mu_2 = 2.5$, $p = 0.4$, M/H$_2$/1									
k	β_0	β	$\widetilde{\rho}_k$	ρ_k	δ	L	s	C_k	**WSB**(β, ρ_k)
5	2.2300	1.6200	0.6360	0.6559	0.8504	0.5272	2	1.5245	7.1528
10	2.2300	1.8800	0.7822	0.8019	0.8725	0.7678	2	1.8960	1.3147
15	2.2300	1.9800	0.8175	0.8374	0.8945	0.8766	2	2.9964	0.0866
20	2.2300	2.0400	0.8309	0.8511	0.9121	0.9486	2	4.2311	0.0038
25	2.2300	2.0800	0.8379	0.8577	0.9259	1.0000	2	5.5802	1.3153e-004
30	2.2300	2.1000	0.8417	0.8615	0.9335	1.0268	2	6.6871	4.0423e-006
35	2.2300	2.1200	0.8436	0.8638	0.9415	1.0545	2	8.0385	1.1388e-007
40	2.2300	2.1400	0.8451	0.8653	0.9501	1.0831	2	9.7699	3.0269e-009
45	2.2300	2.1400	0.8465	0.8663	0.9501	1.0831	2	10.2046	7.6274e-011
50	2.2300	2.1600	0.8469	0.8671	0.9592	1.1126	2	12.6322	1.8644e-012

4.2.4. Hyperexponential inter-arrival times

The hyperexponential process has CV ≥ 1. Thereby, we assume that the density function of the H$_2$ distribution having a balanced means is defined as follows:

$$f(t) = q\mu_1 e^{-\mu_1 t} + (1-q)\mu_2 e^{-\mu_2 t}, \quad t \geq 0,$$

where $0 \leq q \leq 1$ and $\mu^{-1} = q\mu_1^{-1} + (1-q)\mu_2^{-1}$. By altering q, one can obtain different CVs. The corresponding CV of this distribution is given by:

$$CV = \sqrt{\frac{1 + (2q-1)^2}{1 - (2q-1)^2}}.$$

In Table 4 we show the error on the stationary distributions of the number of customers in the system at departure points for M/H$_2$/1 queue. For this numerical example, we have fixed the parameters of the queueing model as follows: $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 2.5$ and $q = 0.4$, and the associated coefficient of variation is CV = 1.0408.

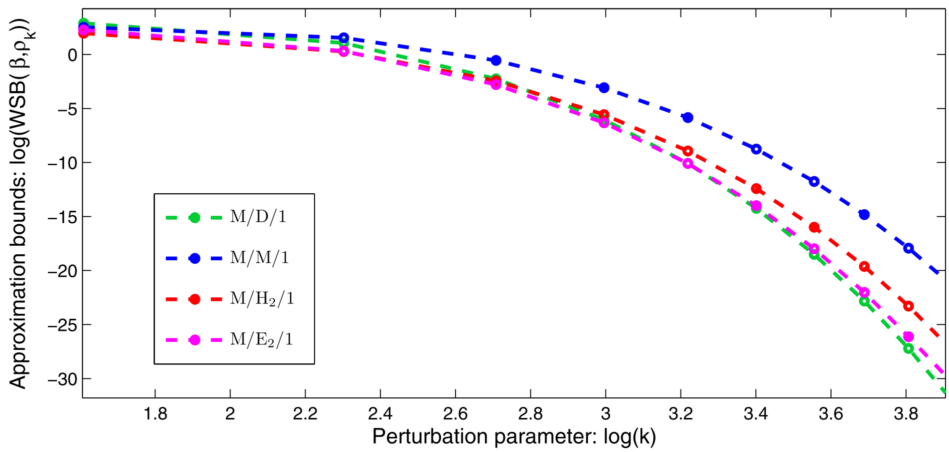

FIGURE 3. Bound of $\log(\mathbf{WSB}(\beta, \rho_{\text{opt}}))$ in function of $\log(k)$.

Figure 3, illustrates the behavior of the weak stability bound by varying the parameter k which is the rank of the truncation, based on the results obtained in Tables 1–4 the approximation error is very small, which does not allow to represent on the same scale the four curves corresponding to the four types of the service times distributions considered previously, it's for this reason that we represent the curves $\log(\mathbf{WSB}(\beta, \rho_{\text{opt}}))$ in function of $\log(k)$.

From Figure 3, we observe that for the four types of service times distributions: Deterministic (D), Exponential (M), Hyperexponential (H_2) and Erlang (E_2) the approximation error decreases by increasing the rank of the truncation, and we can expect that when we increase the size of the truncation ($k > 50$) this error will tend to 0 and becomes negligible, ie $\pi(j) \simeq 0, \ \forall j > k$, And therefore the stationary probabilities of the model with infinite state space can be computed based only on the truncated model $\pi(j) \simeq \pi_k(j), \ \forall j \leq k$.

5. CONCLUSION

In this paper, we used the weak stability method, which is an analytical method that leads to qualitative bounds, to estimate the error of approximation of the stationary distribution of the M/G/1 queue by those of the troncated model. We proved the stability conditions and next obtained stability inequalities with exactly computing of the constants. A series of numerical examples were used to illustrate the potential use of weak stability bounds.

Acknowledgements. The authors would like to thank the area editor and the anonymous reviewers for their valuable remarks and constructive comments that greatly contributed to improving the final version of the paper. We are so grateful to Prof. Loïc Hervé for his useful discussions and for his advices and encouragements.

REFERENCES

[1] J. Abate, G.L. Choudhury and W. Whitt, An introduction to numerical transform inversion and its application to probability models, in Computational Probability, edited by W.K. Grassmann. Kluwer Academic Publishers (2000) 257–323.

[2] N. Bouabdallah, A.L. Beylot, E. Dotaro and G. Pujolle, Resolving the fairness issues in bus-based optical access networks. *IEEE J. Sel. Areas Commun.* **23** (2005) 1444–1457.

[3] A. Brandwajn, Models of DASD subsystems with multiple access paths: a throughput-driven approach. *IEEE Trans. Comput. C* **32** (1983) 451–463.

[4] R. Butt, Introduction to Numerical Analysis Using MATLAB. Inc., United States (2009).

[5] D. Ferré, L. Hervé and J. Ledoux, Regular perturbation of V-geometrically ergodic Markov chains. *J. Appl. Probab.* **50** (2013) 184–194.

[6] W. Ford, Numerical Linear Algebra With Applications: Using MATLAB, 1st edn. Academic Press (2015).

[7] D. Gibson and E. Seneta, Monotone infinite stochastic matrices and their augmented truncations. *Stoch. Process. Appl.* **24** (1987) 287–292.

[8] D. Gross and C. Harris, Fundamentals of Queueing Theory. Wiley (1985).

[9] L. Hervé and J. Ledoux. Approximating Markov chains and V-geometric ergodicity via weak perturbation theory. *Stoch. Process. Appl.* **124** (2014) 613–638.

[10] D.P. Heyman, Approximating the stationary distribution of an infinite stochastic matrix. *J. Appl. Probab.* **28** (1991) 96–103.

[11] B. Issaadi, K. Abbas and D. Aïssani, Perturbation Analysis of the GI/M/s Queue. *Methodol. Comput. Appl. Probab.* **19** (2017) 819–841.

[12] V.V. Kalashnikov and S.T. Rachev, Mathematical Methods for Construction of Queueing Models. Wadsworth & Brooks Cole (1990).

[13] N.V. Kartashov, Strong Stable Markov Chains. Edition VSP, Utrecht, The Netherlands (1996).

[14] G. Keller and C. Liverani, Stability of the spectrum for transfer operators. *Ann. Scuola Norm. Sci.* **4** (1999) 141–152.

[15] Y. Liu, Augmented truncation approximations of discrete-time Markov chains *J. Oper. Res. Lett.* **38** (2010) 218–222.

[16] C. Liverani, Rigorous numerical investigation of the statistical properties of piecewise expanding maps. A feasibility study. *Nonlinearity* **14** (2001) 463–490.

[17] S. Meyn and R.L. Tweedie, Markov chains and stochastic stability, 2nd edition. Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo (2009).

[18] M. Molina, P. Castelli and G. Foddis, Web traffic modeling exploiting. TCP connections' temporal clustering through HTML-REDUCE. *IEEE Netw.* **14** (2000) 46–55.

[19] T. A. Sarymsakov, Sur les Chaines de Markoff a une Infinite Dénombrable d'États Possibles. *Doklady Akad. Sci. U.R.S.S* **48** (1945) 159–161.

[20] E. Seneta, Finite Approximations to Infinite Non-Negative Matrices. *Proc. Cambridge Phil. Soc.* **63** (1967) 983–992.

[21] E. Seneta, The Principles of Truncations in Applied Probability. *Comm. Math. Univ. North Carolina* **9** (1968) 533–539.

[22] E. Seneta, Non-negative Matrices and Markov Chains. Springer-Verlag, New York (1980).

[23] E. Seneta, Computing the stationary distributon for infinite Markov chains. *Linear Algebra Appl.* **34** (1980) 259–267.

[24] T. Shardlow and A.M. Stuart, A perturbation theory for ergodic Markov chains and application to numerical approximations. *SIAM J. Numer. Anal.* **37** (2000) 1120–1137.

[25] F. Simonot, Sur l'approximation de la distribution stationnaire d'une chaine de Markov stochastiquement monotone. *Stoch. Process. Appl.* **56** (1995) 133–149.

[26] H.C. Tijms, Stochastic modelling and analysis, in A Computational Approach. John Wiley, New York (1986).

[27] R.L. Tweedie, Truncation approximations of invariant measures for markov chains *J. Appl. Probab.* **35** (1998) 517–536.

[28] D.S. Watkins, The Matrix Eigenvalue Problem: GR and Krylov Subspace Methods. SIAM (2007).

[29] D. Wolf, Approximation of the invariant probability distribution of an infinite stochastic matrix. *Adv. Appl. Probab.* **12** (1980) 710–726.