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GENERALIZED CHARACTERIZATION OF THE CONVEX ENVELOPE OF A FUNCTION *

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Abstract. We investigate the minima of functionals of the form

$$\int_{[a,b]} g(\dot{u}(s))\mathrm{d}s$$

where g is strictly convex. The admissible functions $u : [a,b] \longrightarrow \mathbb{R}$ are not necessarily convex and satisfy $u \leq f$ on $[a,b]$, $u(a) = f(a)$, $u(b) = f(b)$, f is a fixed function on $[a,b]$. We show that the minimum is attained by $\bar{f}$, the convex envelope of f.

Keywords: Convex envelope, optimization, strict convexity, cost function.

1. Introduction

Consider a real valued function $f : [a,b] \longrightarrow \mathbb{R}$, the convex envelope of f is the largest convex function majorized by f on $[a,b]$, see [6]. It will be denoted by $\bar{f}$. Assume that $f \in W^{1,1}[a,b]$. We have proved [4], that $\bar{f}$ is the unique solution of the following optimization problem:

$$(\mathrm{P}) \quad \min L(u)$$
$$u \in F(f)$$

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where

$$L(u) = \int_{[a,b]} \sqrt{1+\dot{u}^2(s)}\mathrm{d}s.$$

and

$$F(f) = \left\{u \in W^{1,1}[a,b];\ u \le f \text{ on } [a,b], u(a) = f(a), u(b) = f(b)\right\} \cdot \qquad (1)$$

Since $L(u)$ is the length of u, then the previous result is due to the fact that $\bar{f}$ is the shortest function majorized by f which coincides with f on a and b. In this paper we consider problems of the form:

$$\text{(Q)} \qquad \min_{u \in F(f)} \int_{[a,b]} g(\dot{u}(s))\mathrm{d}s$$

where g is strictly convex. Denote by G the cost function of (Q). We show that $\bar{f}$ is the unique solution of program (Q).

The organization of this paper is as follows:

In Section 2, we discuss some preliminary results concerning the properties of the Sobolev space $W^{1,1}[a,b]$ and the convex envelope of a given function. In Section 3, we study the problem (Q), we show that the minimum is attained by $\bar{f}$, therefore, we obtain a general characterization of the convex envelope. Finally, Section 4 is concerned with some concluding remarks.

2. Preliminary results

Consider a continuous function $u : [a,b] \longrightarrow \mathbb{R}$. Denote

$$J = \{x \in [a,b] : \bar{u}(x) \neq u(x)\} \qquad (2)$$

and

$$K = \{x \in [a,b] : \bar{u}(x) = u(x)\} \cdot \qquad (3)$$

Since $\bar{u}$ is convex then $v = \bar{u} - u$ is continuous on $]a,b[$. It is clear that $J = v^{-1}(]-\infty,0[) \cap]a,b[$, then J and K are measurable.

Assume that J is not empty, the following lemma holds.

Lemma 2.1. *Let $x_0 \in J$ then there exists $]x_1,x_2[\subset J$ such that $x_0 \in]x_1,x_2[$ and*

$$\bar{u}(x) = [(u(x_2) - u(x_1))/(x_2 - x_1)](x - x_1) + u(x_1),\ \forall x \in]x_1,x_2[.$$

Proof. Let

$$x_1 = \max[a, x_0] \cap K$$

and

$$x_2 = \min[x_0, b] \cap K.$$

Remark that x_1 and x_2 are well defined since u and $\bar{u}$ are continuous. Let D be the affine function defined by $D(x_1) = \bar{u}(x_1)$ and $D(x_2) = \bar{u}(x_2)$. Since $\bar{u}$ is convex, then $\bar{u}(x) \leq D(x)$ for all $x \in]x_1, x_2[$. Consider the convex function $\tilde{u}$ defined by

$$\tilde{u}(x) = D(x), \forall x \in]x_1, x_2[$$

and

$$\tilde{u}(x) = \bar{u}(x), \forall x \in [a, b] \backslash]x_1, x_2[,$$

then $\bar{u} \leq \tilde{u} \leq u$. Since $\bar{u}$ is the largest convex function majorized by u then $\tilde{u} = \bar{u}$. It follows that $\bar{u}(x) = D(x)$, $\forall x \in]x_1, x_2[$. Since $x_1, x_2 \in K$ then $\bar{u}(x_1) = u(x_1)$ and $\bar{u}(x_2) = u(x_2)$. It follows that

$$\bar{u}(x) = [(u(x_2) - u(x_1))/(x_2 - x_1)](x - x_1) + u(x_1),\ \forall x \in]x_1, x_2[.$$

□

Assume that $]a, b[\cap K$ is not empty, then the following lemma holds, see [1].

Lemma 2.2. *Let $x_0 \in]a, b[\cap K$. If u is differentiable at x_0, then $\bar{u}$ is differentiable at x_0. Moreover $\bar{u}'(x_0) = u'(x_0)$.*

Now, let us discuss some properties of $W^{1,1}[a, b]$.

The Sobolev space $W^{1,1}[a, b]$ is defined by

$$\begin{aligned} W^{1,1}[a, b] &= \{u \in L^1; \exists g \in L^1 \text{ such that} \\ &\qquad \int_{[a,b]} u\varphi' = -\int_{[a,b]} g\varphi,\ \forall \varphi \in C_c^1([a, b])\}. \end{aligned}$$

By [2], $u \in W^{1,1}[a, b]$ if and only if u is absolutely continuous (AC) on $[a, b]$. In the sequel, we need the following result [7] and [3].

Proposition 2.1. *u is (AC) on $[a, b]$ if and only if u is differentiable a.e. on $[a, b]$, $u' \in L^1$ and*

$$u(x) = u(a) + \int_{[a,x]} u'(t)\mathrm{d}t \ \ (a \leq x \leq b).$$

It follows from Lemma 2.2 and Proposition 2.1 that if $u \in W^{1,1}[a, b]$ then $\bar{u} \in W^{1,1}[a, b]$, see [4].

3. General characterization of the convex envelope

Consider the problem (Q) introduced in Section 1. The cost function and the feasible set associated with (Q) are denoted respectively by G and $F(f)$. In order to show that $\bar{f}$ is the unique solution of (Q), we need to prove the following lemmas:

Lemma 3.1. *Assume that g is strictly convex. Let u_1 and $v_1 \in F(f)$ such that u_1 and v_1 are convex and $u_1 \leq v_1$. Then, $G(v_1) \leq G(u_1)$.*

Proof. Let u_1 and $v_1 \in F(f)$ such that u_1 and v_1 are convex and $u_1 \leq v_1$. Consider the auxiliary problem

$$\text{(Pv1)} \quad \begin{aligned} &\min G(u) \\ &u \in F(f) \\ &u_1 \leq u \leq v_1 \text{ on } [a,b] \\ &u \text{ is convex.} \end{aligned}$$

Remark that (Pv1) is a problem of minimizing functionals of the form

$$\int_{[a,b]} g(\dot{u}(s))\mathrm{d}s$$

on a set of convex functions. Moreover, u_1, $v_1 \in F(f)$, then $u_1 = v_1$ on $\partial[a,b]$. It is clear that program (Pv1) has the same form as problems considered in Theorem 1 of [5]. Since u_1 is convex and g is strictly convex, then v_1 is the unique solution of (Pv1). In the other hand, u_1 is a feasible function, it follows that

$$G(v_1) \leq G(u_1).$$

□

Lemma 3.2. *Let $u \in W^{1,1}[a,b]$. If g is convex then*

$$\int_J g(\dot{\bar{u}}(s))\mathrm{d}s \leq \int_J g(\dot{u}(s))\mathrm{d}s. \tag{4}$$

Proof. Assume that J is not empty. Let $x_0 \in J$. By Lemma 2.1 there exists $]x_1, x_2[\subset J$ such that $x_0 \in]x_1, x_2[$ and

$$\bar{u}(x) = [(u(x_2) - u(x_1))/(x_2 - x_1)](x - x_1) + u(x_1), \ \forall x \in]x_1, x_2[. \tag{5}$$

It follows that

$$\dot{\bar{u}}(s) = (u(x_2) - u(x_1))/(x_2 - x_1), \ \ \forall s \in]x_1, x_2[. \tag{6}$$

From the Jensen's inequality, see [7], we deduce that for all $v \in L^1$,

$$g\left([1/(x_2 - x_1)] \int_{[x_1,x_2]} v(s)\mathrm{d}s\right) \leq [1/(x_2 - x_1)] \int_{[x_1,x_2]} g(v(s))\mathrm{d}s. \qquad (7)$$

For $v = \dot{u}$, we obtain

$$g(u(x_2) - u(x_1)/x_2 - x_1) \leq [1/(x_2 - x_1)] \int_{[x_1,x_2]} g(\dot{u}(s))\mathrm{d}s. \qquad (8)$$

It follows from (6) that

$$\int_{[x_1,x_2]} g(\dot{\bar{u}}(s))\mathrm{d}s \leq \int_{[x_1,x_2]} g(\dot{u}(s))\mathrm{d}s. \qquad (9)$$

By the same way as in Lemma 2.4 of [4], we obtain

$$\int_J g(\dot{\bar{u}}(s))\mathrm{d}s \leq \int_J g(\dot{u}(s))\mathrm{d}s. \qquad (10)$$

□

Now, it is easy to show the following theorem:

Theorem 3.1. *Assume that g is strictly convex. Then, $\bar{f}$ is the unique solution of (Q).*

Proof. Let $u \in F(f)$, writing $[a, b] = J \cup K$ and using Lemma 3.2, we show that

$$G(\bar{u}) \leq G(u). \qquad (11)$$

By Lemma 3.1,

$$G(\bar{f}) \leq G(\bar{u}).$$

Then

$$G(\bar{f}) \leq G(u), \ \forall u \in F(f). \qquad (12)$$

The uniqueness of the solution follows from the strict convexity of G. For details, we refer the reader to the proof of Theorem 2.1 of [4]. □

4. Concluding remarks

We end this paper by two remarks:

First, the function

$$g : x \longmapsto \sqrt{1+x^2}$$

is strictly convex, then Theorem 2.1 of [4] is a particular case of Theorem 3.1. The cost function in [4] is geometrically interpreted as the length of the considered function. The difficulty, in the general case, is the fact that we do not know a geometric interpretation of the cost function. The second remark is concerned with Theorem 1 of [5] where the admissible functions are assumed to be convex. We remark, by Theorem 3.1, that, in the case of one dimension, the minimum is attained by the convex envelope of f without requiring the admissible functions to be convex.

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