RAIRO Operations Research
RAIRO Oper. Res. **37** (2003) 157-177
DOI: 10.1051/ro:2003019

ANALYSIS OF OPERATING CHARACTERISTICS FOR THE HETEROGENEOUS BATCH ARRIVAL QUEUE WITH SERVER STARTUP AND BREAKDOWNS

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Communicated by Naoto Kaio

Abstract. In this paper we consider a like-queue production system in which server startup and breakdowns are possible. The server is turned on (*i.e.* begins startup) when N units are accumulated in the system and off when the system is empty. We model this system by an $M^{[x]}/M/1$ queue with server breakdowns and startup time under the N policy. The arrival rate varies according to the server's status: off, startup, busy, or breakdown. While the server is working, he is subject to breakdowns according to a Poisson process. When the server breaks down, he requires repair at a repair facility, where the repair time follows the negative exponential distribution. We study the steady-state behaviour of the system size distribution at stationary point of time as well as the queue size distribution at departure point of time and obtain some useful results. The total expected cost function per unit time is developed to determine the optimal operating policy at a minimum cost. This paper provides the minimum expected cost and the optimal operating policy based on assumed numerical values of the system parameters. Sensitivity analysis is also provided.

Keywords. Batch arrivals, breakdowns, control, sensitivity analysis, startup, stochastic decomposition.

Received December 4, 2002.

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1. Introduction

In this paper we consider the modeling of a production system in which the production does not start until some specified number of units, say N, are accumulated during the server off period. Units request for service usually arrive in batches with varying arrival rate. The server needs a startup time before providing the service; *i.e.*, he must perform certain pre-service work to gear up the machinery for operation. When the server is working, he may meet unpredictable breakdowns but is immediately repaired. This production system can be modeled by an $\mathrm{M}^{[x]}/\mathrm{M}/1$ queue with server breakdowns and startup time under the N policy.

In this system, units arrive following a compound Poisson process where the arrival size is a random variable and the arrival rate varies according to the server's status: off, startup, busy, or breakdown. The server is turned off as soon as the system becomes empty. When N units are accumulated in the system, the server is immediately turned on but is temporarily unavailable to the waiting units. He needs a startup time before starting his each service period. After the server finishes his startup, he starts to serve the waiting units until the system becomes empty. Whenever the server is working, it is assumed that the server can break down at any time. Whenever the server fails, it is immediately repaired at a repair facility.

The concept of N policy was first introduced by Yadin and Naor [23]. The so-called N policy means that the server does not start to provide service until there are N units waiting in the system. Batch arrival queues with N policy was first studied by Lee and Srinivasan [9]. Later, Lee *et al.* [10] concentrated on the interpretation of the system characteristics of the $\mathrm{M}^{[x]}/\mathrm{G}/1$ queueing system under the N policy.

Past work regarding queueing systems under N policy may be divided into two categories: (i) the case of server startup, and (ii) the case of server breakdowns. In the case of server startup, Baker [1] first proposed the N policy M/M/1 queueing system with exponential startup time. Borthakur *et al.* [2] extended Baker's results to the general startup time. The N policy M/G/1 queueing system with startup time was first studied by Minh [15] and was investigated by several researchers such as Medhi and Templeton [14], Takagi [18], Lee and Park [13], Hur and Paik [8], and so on. In the case of server breakdowns, Wang [20] first proposed the N policy Markovian queueing system with server breakdowns. Wang [21] and Wang *et al.* [22] extended Wang's model [20] to the N policy $\mathrm{M}/E_k/1$ and $\mathrm{M}/\mathrm{H}_2/1$ queueing system cases, respectively. They developed the analytic closed-form solutions and provided a sensitivity analysis.

The purpose of this paper is threefold. Firstly, the state equations are established to get the steady-state probability distribution as well as the departure point queue size distribution and some probability interpretations of the system characteristics are made. Secondly, we formulate the system's total expected cost in order to determine the optimal operating N policy numerically at the minimum cost for various values of system's parameters. Thirdly, we perform a sensitivity analysis.

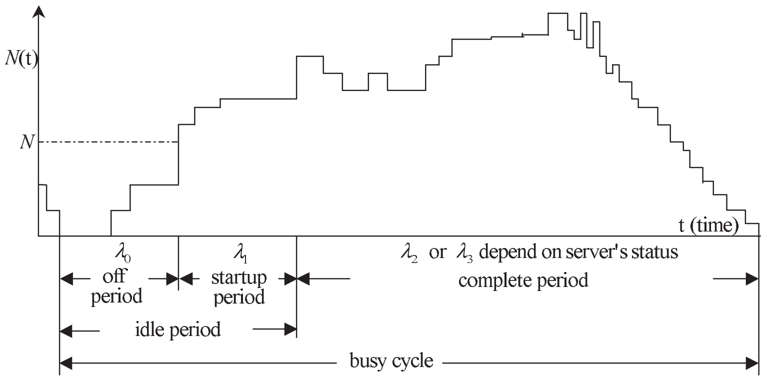

FIGURE 1. A queueing model with the busy cycle consisting of the idle period and complete period.

2. The system and assumptions

One *busy cycle* begins right when the system becomes empty and the server is off. The server remains off until there are N units in the system. We call this the *off period*. The startup period begins when the server performs startup as soon as the number of waiting units reach N and terminates when he starts providing the service. In the *startup period*, the units arriving while the server is performing startup as well as those arriving during the off period are not served yet. The *busy period* is initiated when the server completes his startup and starts serving the waiting units. During the *busy period*, the server may break down and starts his repair immediately. This is called the *breakdown period*. As soon as the server is repaired, he returns and provides service until there are no units in the system. Since the *complete period* starts when the *startup period* is over and terminates when there are no units in the system, the *complete period* is represented by the sum of the *busy period* and the *breakdown period*. We illustrate our model with a typical sample path shown in Figure 1. Furthermore, our model is considered under the following specifications:

1. The arrival process is a compound Poisson process with batch size X and various rates λ_i,$(i = 0, 1, 2, 3)$ where λ_0, λ_1, λ_2, and λ_3 denote the group arrival rates during the server off, startup, busy, and breakdown periods, respectively. Define that $b_k = \Pr(X = k)$ is the probability that the arrival size X is k $(k \geq 1)$. Arriving units within batches at the server form a single waiting line and are served in the order of their arrivals. The units within a batch are served one at a time by a single server. The service time for an individual unit is exponentially distributed with mean $1/\mu$.
2. The server is turned off when the system becomes empty. As soon as the number of units in the system reaches N, the server is immediately turned on but is temporarily unavailable to the waiting units. He needs to take

an exponential startup time with parameter r. As soon as the startup period is over, he serves the waiting units immediately.

3. When the server is working, the server can break down at any time with a Poisson breakdown rate α.
4. When the server fails, it is immediately repaired at a repair rate β, where the repair times are exponentially distributed.
5. If the server fails or one unit is in service, then the arriving units or waiting units have to wait in the queue until the server is free. The service is allowed to be interrupted if the server breaks down. Nevertheless, the server is immediately repaired. When the repair of a server is completed, the server immediately begins serving a unit.

2.1. Practical justifications of the model

A number of practical problems may be formulated as one in which the arrivals are in batches with arrival rate depending on system's state or server's state, and the server needs a startup time before providing the service and it may break down when working.

One particular problem where this model is applicable is in the study of a production line system. Consider a production line in manufacturing system of job-shop type, where the arrival stream of job orders follows a compound Poisson process with heterogeneous rates λ_i (i denotes the server's status, $i = 0, 1, 2, 3$) and each job order often requires the manufacture of a random number X ($X \geq 1$; that is, each job has more than one item). For economic efficiency, it is required that the production does not start until a specified number of orders, say N, is accumulated during an idle period. Each item within a job requires an exponentially distributed production time with mean $1/\mu$. As the size of job orders reaches N, the operator needs a startup time to operate machine before starting production and after the production is started, the production may be interrupted because of emergent events. But the production must immediately resume whenever the emergency is solved. The emergent event occurs according to a Poisson process with rate α and the time spent to recover an emergent event is exponentially distributed with mean $1/\beta$. We can interpret the emergent events as server breakdowns. The startup time is a random variable, which has an exponential distribution with mean $1/r$; it is corresponding to extra operations (for example, setup, warm up, etc.) before starting production.

In real-life situations it is not unusual to encounter that the arrivals join the queue in batches with different arrival rate, the server may perform some pre-service work to gear up the machinery, and the service may be interrupted. One may try to incorporate more realism in the model by considering that (i) batch arrivals occurs with different rate depending on the arrival time, system's state and server's state; (ii) the server performs the preparatory work before starting each service period; and (iii) the server is subject to breakdowns due to unpredictable or uncontrollable factors. Therefore, one may consider it necessary to investigate the heterogeneous arrival queue with server startup and breakdowns.

3. System steady probability

Let the state $i = 0$ represent the server is idle, while the state $i = 1$ represents the server is turned on and is in operation, and the state $i = 2$ represents the server is in operation but found to be broken down. In steady-state, the following notations are used.

$P_0(n) \equiv$ the probability that there are n units in the system when the server is idle. There are two situations when the server is idle: (i) the server is turned off if $n \leq N-1$, and (ii) it is turned on and performing startup if $n \geq N$;

$P_1(n) \equiv$ the probability that there are n units in the system when the server is turned on and is in operation, where $n = 1, 2, \ldots$; and

$P_2(n) \equiv$ the probability that there are n units in the system when the server is in operation but found to be broken down, where $n = 1, 2, \ldots$

It is easy to set up the following steady-state system equations:

$$\lambda_0 P_0(0) = \mu P_1(1), \tag{1}$$

$$\lambda_0 P_0(n) = \lambda_0 \sum_{k=0}^{n-1} b_{n-k} P_0(k), \qquad 1 \leq n \leq N-1 \tag{2}$$

$$(\lambda_1 + r) P_0(n) = \lambda_0 \sum_{k=0}^{N-1} b_{n-k} P_0(k) + \lambda_1 \sum_{k=N}^{n-1} b_{n-k} P_0(k), \qquad n \geq N \tag{3}$$

$$(\lambda_2 + \mu + \alpha) P_1(n) = \lambda_2 \sum_{k=1}^{n-1} b_{n-k} P_1(k) + \mu P_1(n+1) + \beta P_2(n), \quad 1 \leq n \leq N-1 \tag{4}$$

$$(\lambda_2 + \mu + \alpha) P_1(n) = \lambda_2 \sum_{k=1}^{n-1} b_{n-k} P_1(k) + \mu P_1(n+1) + \beta P_2(n) + r P_0(n), \quad n \geq N \tag{5}$$

$$(\lambda_3 + \beta) P_2(n) = \lambda_3 \sum_{k=1}^{n-1} b_{n-k} P_2(k) + \alpha P_1(n), \qquad n \geq 1 \tag{6}$$

where $a > b$ in the $\sum_{j=a}^{b}$ notation indicates that the term is zero.

The results for the N policy M/M/1 queueing system with an unreliable server are obtained by setting $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, and $r = \infty$ in (1)–(6). Equations (1)–(6) for $P_i(n)$ then correspond to the existing results in the literature (Wang [19]). The results for the ordinary $M^{[x]}$/M/1 queueing system with a reliable server are obtained by setting $N = 1$, $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $r = \infty$, $\alpha = 0$, and $\beta = \infty$ in (1)–(6). Equations (1)–(6) for $P_i(n)$ then correspond to the existing results in the literature (Gross and Harris [7], p. 157).

3.1. Derivations of $P_0(n)$

Solving (2)–(3) recursively, we finally get

$$P_0(n) = \begin{cases} P_0(0)\Psi(n), & n = 1, 2, \ldots N-1 \\ P_0(0)\Psi(N)\dfrac{\lambda_0}{\lambda_1 + r}, & n = N \\ P_0(0)\displaystyle\sum_{k=0}^{n-N-1}[\Lambda(n-k)\ \Theta(k)], & n = N+1, N+2, \ \ldots, \end{cases} \tag{7}$$

where

$$\Psi(n) = \begin{cases} 1, & n = 0 \\ \sum_{1\le k\le n}\sum_{\substack{\ell_1+\ell_2+\ldots+\ell_k=n \\ \ell_1,\ell_2,\ldots,\ell_k\in\{1,2,\cdots,n\}}} b_{\ell_1}b_{\ell_2}\ldots b_{\ell_k}, & n = 1, 2, \ \ldots \\ 0, & \text{otherwise}, \end{cases} \tag{8}$$

$$\Lambda(n) = \frac{\lambda_0}{\lambda_1 + r}\Big[\Psi(n) + \frac{\lambda_1 b_{n-N}\Psi(N)}{\lambda_1 + r} - \sum_{k=1}^{n-N} b_k\Psi(n-k)\Big], \quad n = N+1, N+2, \ \ldots,$$

and

$$\Theta(n) = \begin{cases} 1, & n = 0 \\ \displaystyle\sum_{1\le k\le n}\sum_{\substack{\ell_1+\ell_2+\ldots+\ell_k=n \\ \ell_1,\ell_2,\ldots,\ell_k\in\{1,2,\cdots,n\}}} h_{\ell_1}h_{\ell_2}\ldots h_{\ell_k}, & n = 1, 2, \ \ldots \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

with $h_j = \frac{\lambda_1 b_j}{\lambda_1 + r}$.

Remark 1. As $b_1 = 1$ (single unit arrival), we have $\Psi(n) = 1$, for $n = 0, 1, 2\ldots$, $N-1$.

Remark 2. The meaning of (8) is sum up all possible products of k b_i's in which the total of subscript values of b equals n. As

$$\begin{aligned} \Psi(4) &= b_4 + b_3b_1 + b_2b_2 + b_1b_3 + b_1b_1b_2 + b_1b_2b_1 + b_2b_1b_1 + b_1b_1b_1b_1 \\ &= b_4 + 2b_3b_1 + b_2^2 + 3b_1^2b_2 + b_1^4. \end{aligned}$$

The meaning of (9) is the same as (8).

3.2. Probability generating function

Probability generating function (p.g.f.) technique may be used to obtain analytic solution $P_0(0)$ in neat closed-form expression as there is no way of solving (1)–(6) in a recursive manner. Define the respective probability generating functions (p.g.f.) of $P_0(n)$, $P_1(n)$, and $P_2(n)$ as follows:

$$H_0(z) = \sum_{n=0}^{N-1} z^n P_0(n),$$
$$H_1(z) = \sum_{n=N}^{\infty} z^n P_0(n),$$
$$Q(z) = \sum_{n=1}^{\infty} z^n P_1(n),$$

and

$$R(z) = \sum_{n=1}^{\infty} z^n P_2(n),$$

where $|z| \leq 1$.

Further, define the p.g.f. of the arrival size X as $X(z) = \sum_{k=1}^{\infty} z^k b_k$, It is found that $E(X) = X'(1)$ and $E[X(X-1)] = X''(1)$.

From (7), $H_0(z)$ is expressed in term of $P_0(0)$:

$$H_0(z) = P_0(0) \sum_{n=0}^{N-1} z^n \Psi(n) = I(z) P_0(0), \tag{10}$$

where $I(z) = \sum_{n=0}^{N-1} z^n \Psi(n)$ with $I(1) = \sum_{n=0}^{N-1} \Psi(n)$ and $I'(1) = \sum_{n=0}^{N-1} n\Psi(n)$.

In (2)–(3), (2)–(3) is multiplied by z^n $(n = 1, 2, \dots)$ and then the equations are added terms by terms (see Appendix 1). We finally obtain

$$H_1(z) = \frac{\lambda_0 P_0(0) + \lambda_0 [X(z) - 1] H_0(z)}{\lambda_1 + r - \lambda_1 X(z)}. \tag{11}$$

In (1), and (4)–(5), (1) is multiplied by z, (4)–(5) $(n = 1, 2, \dots)$ by z^{n+1}. Similarly, we get

$$\Big[\lambda_2 z X(z) - (\lambda_2 + \alpha + \mu) z + \mu\Big] Q(z) + \beta z R(z) = \lambda_0 z P_0(0) - r z H_1(z). \tag{12}$$

From (6), we use the same procedure as above to obtain

$$\alpha Q(z) + \big[\lambda_3 X(z) - \lambda_3 - \beta\big] R(z) = 0. \tag{13}$$

We solve $Q(z)$ and $R(z)$ from (12)–(13) and use (10)–(11) yielding

$$\begin{aligned} Q(z) = {} & z\lambda_0\Big(rI(z)+\lambda_1\Big)\Big(X(z)-1\Big)\Big(\lambda_3+\beta-\lambda_3 X(z)\Big) \\ & \times P_0(0)\Bigg/\Bigg\{\Big\{\lambda_1+r-\lambda_1 X(z)\Big\}\Big\{\Big[1-X(z)\Big]\Big[z\lambda_2\big[\lambda_3\big(1-X(z)\big)+\beta\big] \\ & +\mu\lambda_3\big(z-1\big)+z\alpha\lambda_3\Big]+\mu\beta\big(z-1\big)\Big\}\Bigg\}, \end{aligned} \tag{14}$$

$$\begin{aligned} R(z) = {} & z\alpha\lambda_0\Big(rI(z)+\lambda_1\Big)\Big(X(z)-1\Big) \\ & \times P_0(0)\Bigg/\Bigg\{\Big\{\lambda_1+r-\lambda_1 X(z)\Big\}\Big\{\Big[1-X(z)\Big]\Big[z\lambda_2\big[\lambda_3\big(1-X(z)\big)+\beta\big] \\ & +\mu\lambda_3\big(z-1\big)+z\alpha\lambda_3\Big]+\mu\beta\big(z-1\big)\Big\}\Bigg\}. \end{aligned} \tag{15}$$

Let $G(z)$ represent the p.g.f. of the number of units in the system; thus

$$G(z) = H_0(z)+H_1(z)+Q(z)+R(z). \tag{16}$$

Evaluating $H_0(1), H_1(1), Q(1)$, and $R(1)$ in (10)–(11) and (14)–(15), the numerator and denominator are both 0 in (14)–(15). We apply L'Hopital's rule once and finally obtain

$$H_0(1) = I(1)P_0(0), \tag{17}$$

$$H_1(1) = \frac{\lambda_0}{r}P_0(0), \tag{18}$$

$$Q(1) = \frac{\lambda_0\beta[rI(1)+\lambda_1]E(X)}{r\mu\beta-r[\lambda_2\beta+\alpha\lambda_3]E(X)}P_0(0), \tag{19}$$

$$R(1) = \frac{\lambda_0\alpha[rI(1)+\lambda_1]E(X)}{r\mu\beta-r[\lambda_2\beta+\alpha\lambda_3]E(X)}P_0(0). \tag{20}$$

To determine $P_0(0)$, using the normalizing condition finally yields

$$P_0(0) = \Big[I(1)+\frac{\lambda_0}{r}+\frac{\lambda_0(\alpha+\beta)[rI(1)+\lambda_1]E(X)}{r\mu\beta-r[\lambda_2\beta+\alpha\lambda_3]E(X)}\Big]^{-1}, \tag{21}$$

with $0 < P_0(0) < 1$ is sufficient for stationary.

From equations (10)–(11) and (14)–(16), we can see that the stochastic decomposition property by Fuhrmann and Cooper [6] doesn't holds for the heterogeneous arrival queues. As setting $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $G(z)$ can be simplified as

$$G(z) = \frac{[\lambda+rI(z)](1-z)\Big[\lambda(X(z)-1)-\beta\Big]P_0(0)}{\big[\lambda+r-\lambda X(z)\big]\Big\{\rho_0[1-X(z)]\{z[\lambda(1-X(z))+\beta]+\mu(z-1)+z\alpha\}+\beta(z-1)\Big\}}.$$

It is clear that the p.g.f. of the number of units in the $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with server breakdowns and startup can be decomposed into two independent terms as

$$G(z) = \zeta(z) \times G_o(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1), \tag{22}$$

where

$$\zeta(z) = \frac{[\lambda + rI(z)]P_0(0)}{(1-\rho_1)[\lambda + r - \lambda X(z)]} = \frac{r[\lambda + rI(z)]}{(\lambda + rI(1))[\lambda + r - \lambda X(z)]}, \tag{23}$$

and

$$G_o(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1) = \frac{(1-\rho_1)(1-z)\big[\lambda(X(z)-1)-\beta\big]}{\rho_0\Big[1-X(z)\Big]\Big[z\Big(\lambda\big(1-X(z)\big)+\beta\Big)+\mu\big(z-1\big)+z\alpha\Big]+\beta(z-1)}, \tag{24}$$

with $\rho_0 = \lambda/\mu$, $\rho = \rho_0 E[X]$ and $\rho_1 = \rho(1+\alpha/\beta)$.

From equation (22), we observe the p.g.f. of the number of units in the $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with server breakdowns and startup is the convolution of the p.g.f. of two independent random variables one of which is the number of units in the system corresponding to an ordinary $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with an unreliable server (second term) and the other is the number of arrivals during the residual life of the startup period (first term).

For vacation queues, Lee *et al.* [11,12] have shown that the stochastic decomposition property holds for the homogeneous arrival in the N policy M/G/1 queueing systems with a *reliable* server. Based on the earlier discussion, It is easily seen that the stochastic decomposition property holds for the homogeneous arrival in the N policy $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with an *unreliable* server and startup, too.

4. Departure point queue size distribution

In this section we derive the p.g.f. of the limiting queue size distribution at departure point of time. Following the argument of PASTA (see Chaudhry and Templeton [3]) we state that a departing customer will see $'i'$ units in the queue just after his departure if and only if there were $'i+1'$ units in the system just before the departure. Thus we may write

$$\pi_i = \theta P_1(i+1), \qquad i = 0, 1, 2, \ldots,$$

where $\pi_i = \Pr['i'$ units in the queue just after a departure], the definition of $P_1(i)$ was given in the Section 3, and θ is a constant to be evaluated.

Let $\Pi_q(z)$ be the p.g.f. of $\{\pi_i; i = 0, 1, 2, \ldots\}$, then

$$\Pi_q(z) = \frac{\theta}{z} Q(z),$$

where $Q(z)$ is given in (14).

Now using the normalizing condition; *i.e.*, limit of $\Pi_q(z)$ as $z \to 1$ is unity, we get

$$\theta = \left\{ \frac{\lambda_0 \beta [rI(1) + \lambda_1] E(X)}{r\mu\beta - r[\lambda_2\beta + \alpha\lambda_3] E(X)} P_0(0) \right\}^{-1}.$$

Thus the p.g.f. of the departure point queue size distribution is given by

$$\begin{aligned}\Pi_q(z) = {} & \theta\lambda_0 \Big(rI(z) + \lambda_1\Big)\Big(X(z) - 1\Big)\Big(\lambda_3 + \beta - \lambda_3 X(z)\Big) \\ & \times P_0(0) \Big/ \Bigg\{ \Big\{ \lambda_1 + r - \lambda_1 X(z) \Big\} \Big\{ \Big[1 - X(z)\Big] \Big[z\lambda_2 \big[\lambda_3\big(1 - X(z)\big) + \beta\big] \\ & + \mu\lambda_3\big(z - 1\big) + z\alpha\lambda_3 \Big] + \mu\beta\big(z - 1\big) \Big\} \Bigg\}. \end{aligned} \tag{25}$$

4.1. Some remarks

In particular, if we take $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\theta = 1/\rho$ and $\Pi_q(z)$ can be simplified as

$$\begin{aligned}\Pi_q(z) = {} & \rho_0 \Big(rI(z) + \lambda\Big)\Big(X(z) - 1\Big)\Big(\lambda + \beta - \lambda X(z)\Big) P_0(0) \Big/ \Bigg\{ \rho \Big\{ \lambda + r - \lambda X(z) \Big\} \\ & \times \Big\{ \rho_0 \Big[1 - X(z)\Big] \Big[z\big[\lambda\big(1 - X(z)\big) + \beta\big] + \mu\big(z - 1\big) + z\alpha \Big] + \beta\big(z - 1\big) \Big\} \Bigg\}. \end{aligned} \tag{26}$$

Now from equation (26) we observe that $\Pi_q(z)$ decomposes into three independent terms:

$$\Pi_q(z) = \zeta(z) T(z) \Pi_q(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1), \tag{27}$$

where $\zeta(z)$ is given by (23), and

$$T(z) = \frac{1 - X(z)}{E(X)(1 - z)},$$

and

$$\Pi_q(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1) = \frac{(1-\rho_1)(z-1)\big[\lambda(1 - X(z)) + \beta\big]}{\rho_0\Big[1 - X(z)\Big]\Big[z\Big(\lambda\big(1 - X(z)\big) + \beta\Big) + \mu\big(z-1\big) + z\alpha\Big] + \beta(z-1)}.$$

It is important to be noted that the departure point queue size distribution given by (27) decomposes into three independent random variables: one (the first term) is that the number of units arrive during the residual life of the startup period. Particularly, we may call it queue size distribution due to residual startup period. Another (the second term) is the number of units placed before an arbitrary test unit (tagged unit) in a batch in which the tagged unit arrives (see Takagi [17], p. 46), and the last one (the third term) is the departure point queue size of the ordinary $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with an unreliable server.

The above discussions tell us that, the stochastic decomposition property of the departure point queue size holds for the homogeneous arrival in the N policy $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with an unreliable server and startup.

If $\alpha = 0$ and $\beta = \infty$, equation (26) can be simplified as

$$\Pi_q(z) = \frac{\lambda(\lambda + rI(z))(X(z) - 1)P_0(0)}{\rho(\lambda + r - \lambda X(z))\Big[z\lambda(1 - X(z)) + \mu(z-1)\Big]},$$

which verifies the p.g.f. of the departure point queue size distribution for the N policy $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with a reliable server and startup.

Suppose that we have $\alpha = 0$ and $\beta = \infty$; then if we put $N = 1$ and $\Pr[X = 1]$=1, $X(z) = z$ and $E(X) = 1$ and therefore $T(z) = 1$. Hence (26) becomes

$$\Pi_q(z) = \frac{(\lambda + r)P_0(0)}{(1 - \rho_0 z)(\lambda + r - \lambda z)},$$

which furnishes the p.g.f. of the departure point queue size distribution for an ordinary M/M/1 queueing system with a reliable server and startup (see Choudhury [4]).

From earlier inferences we have

$$G_0(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1) = \Pi_q(z; \mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1),$$

which verifies the results by Cooper [5].

Further, the result by Choudhury [4] as

$$G_q(z;\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1)=\Pi_q(z;\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1)[1+\rho_0(1-X(z))],$$

where $G_q(z;\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1)$ is the p.g.f. of the number of units in the queue at stationary point in time equilibrium state for the ordinary $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with an unreliable server.

5. Expected number of arrivals in the system

Using (10)–(11), and (14)–(16), we compute the mean queue length

$$\begin{aligned}
L_N &= \frac{\mathrm{d}G(z)}{\mathrm{d}z}|_{z=1}\\
&= \Bigg\{ I'(1)+\frac{\lambda_0 E(X)}{r^2}\Big[I(1)+\lambda_1+\frac{\lambda_1[\alpha+\beta][rI(1)+\lambda_1][E(X)]}{\mu\beta-(\lambda_2\beta+\alpha\lambda_3)E(X)}\Big]\\
&\quad+\frac{\lambda_0}{2r[\mu\beta-(\lambda_2\beta+\alpha\lambda_3)E(X)]}\Big[\Big[rI(1)+\lambda_1\Big]\Big[\alpha+\beta\Big]E[X(X+1)]\\
&\quad+2rI'(1)(\alpha+\beta)E(X)-2\lambda_3\big[rI(1)+\lambda_1\big][E(X)]^2\Big]\\
&\quad+\frac{\lambda_0}{2r[\mu\beta-(\lambda_2\beta+\alpha\lambda_3)E(X)]^2}\Big[\Big[rI(1)+\lambda_1\Big]\Big[\alpha+\beta\Big]E(X)\\
&\quad\times\Big[\big(\lambda_2\beta+\alpha\lambda_3\big)E[X(X+1)]+2\lambda_3\big[\mu-\lambda_2E(X)\big]E(X)\Big]\Big]\Bigg\}P_0(0), \quad (28)
\end{aligned}$$

where $P_0(0)$ is given is (21).

5.1. Special cases

In this section, we present some existing results in the literature which are special cases of our model.

Case A: If $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $b_1=1$, $r=\infty$, $\alpha=0$, and $\beta=\infty$, the ordinary M/M/1 queueing system with a reliable server case. When $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $b_1=1$, $r=\infty$, $\alpha=0$, and $\beta=\infty$, expression (14) for $Q(z)$ reduces to a special case of expression (2.14) of Gross and Harris ([7], p. 67).

Case B: If $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $r=\infty$, $\alpha=0$, and $\beta=\infty$, the ordinary $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with a reliable server case. When $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $r=\infty$, $\alpha=0$, and $\beta=\infty$, expression (14) for $Q(z)$ reduces to a special case of expression (3.3) of Gross and Harris [7].

Case C: If $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $b_1=1$, and $r=\infty$, the ordinary M/M/1 queueing system with an unreliable server case. When $N=1$, $\lambda_0=\lambda_1=\lambda_2=\lambda_3$, $b_1=1$, and $r=\infty$, expressions (14) for $Q(z)$ and (15) for $R(z)$ reduce to a special case of expressions (16) and (17) of Wang [19], respectively.

Case D: If $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, $r = \infty$, $\alpha = 0$, and $\beta = \infty$, the N policy M/M/1 queueing system with a reliable server case. When $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, $r = \infty$, $\alpha = 0$, and $\beta = \infty$, expression (14) for $Q(z)$ reduces to a special case of expression (4.62) of Sivazlian and Stanfel ([16], p. 255).

Case E: If $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, and $r = \infty$, the N policy M/M/1 queueing system with an unreliable server case. When $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, and $r = \infty$, expressions (14) for $Q(z)$ and (15) for $R(z)$ reduce to a special case of expressions (18) and (19) of Wang [20], respectively.

Case F: If $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, $\alpha = 0$, and $\beta = \infty$, the N policy M/M/1 queueing system with a reliable server case. When $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3$, $b_1 = 1$, $\alpha = 0$, and $\beta = \infty$, expression (28) for L_N reduces to a special case of expression $L(n)$ of Baker ([1], p. 72).

6. Optimal design of the N policy

Let O, S, B, and D denote the lengths of the server off, startup, busy, and breakdown periods, respectively. Applying the memoryless property of the Poisson process, we find that the mean length of the off period is

$$E[O] = N/\lambda_0. \tag{29}$$

The expected length of the off period, the startup period, the busy period, and the breakdown period, are denoted by $E[O]$, $E[S]$, $E[B]$ and $E[D]$, respectively. The expected length of a busy cycle is given by

$$E[C] = E[O] + E[S] + E[B] + E[D].$$

From (17)–(20), we obtain the following long-run fraction of time the server is off, startup, busy, and broken down, respectively:

$$\frac{E[O]}{E[C]} = H_0(1) = I(1)P_0(0), \tag{30}$$

$$\frac{E[S]}{E[C]} = H_1(1) = \frac{\lambda_0}{r}P_0(0), \tag{31}$$

$$\frac{E[B]}{E[C]} = Q(1) = \frac{\lambda_0\beta[rI(1)+\lambda_1]E(X)}{r\mu\beta - r[\lambda_2\beta + \alpha\lambda_3]E(X)}P_0(0), \tag{32}$$

$$\frac{E[D]}{E[C]} = R(1) = \frac{\lambda_0\alpha[rI(1)+\lambda_1]E(X)}{r\mu\beta - r[\lambda_2\beta + \alpha\lambda_3]E(X)}P_0(0). \tag{33}$$

Thus, we have the expected number of busy cycles per unit time

$$\frac{1}{E[C]} = \frac{\lambda_0 I(1)P_0(0)}{N}. \tag{34}$$

6.1. Determining the optimal policy

We develop the total expected cost function per unit time for the $M^{[x]}/M/1$ queue under the N policy with server breakdowns and startup time, in which N is a decision variable. Following the cost structure is constructed, our objective is to determine the optimal operating N policy so as to minimize this function. Let

$C_h \equiv$ holding cost per unit time for each unit present in the system;
$C_f \equiv$ setup cost per busy cycle;
$C_o \equiv$ cost per unit time for keeping the server off;
$C_s \equiv$ startup cost per unit time for the preparatory work of the server before starting the service;
$C_b \equiv$ cost per unit time for keeping the server on and in operation;
$C_d \equiv$ breakdown cost per unit time for a broken server.

Using the definitions of each cost element and its corresponding system characteristics, the total expected cost function per unit time is given by

$$T_{\text{cost}}(N) = C_h L_N + \frac{C_f}{E[C]} + C_o\frac{E[O]}{E[C]} + C_s\frac{E[S]}{E[C]} + C_b\frac{E[B]}{E[C]} + C_d\frac{E[D]}{E[C]}. \qquad (35)$$

Using (28), (30)–(34), the results of (35) can be explicitly expressed which is a very long and complex formula for $T_{\text{cost}}(N)$. This is due to the fact that there are many parameters (*e.g.*, X, λ_0, λ_1, λ_2, λ_3, r, μ, α, and β) involved in our model. We obtain the optimal value N which minimizes the cost function, $T_{\text{cost}}(N)$, by differentiating it with respect to N and setting the result to be zero, *i.e.*,

$$\frac{\partial T_{\text{cost}}(N)}{\partial N} = 0. \qquad (36)$$

The solution N to (36) may not be integer, and the optimal positive integer value of N is one of the integers surrounding N^* which gives a smaller cost T_{cost}. Here, it should be pointed out explicitly that the solution really gives the minimum value, and the $\frac{\partial^2 T_{\text{cost}}(N)}{\partial^2 N}|_{N=N^*}$ is greater than 0 when the values of system parameters satisfy suitable conditions. However, it is quite tedious to present the explicitly expression. Therefore, we will perform the numerical experiments to demonstrate that the function is really convex and the solution gives a minimum.

6.2. Numerical studies

We now perform a sensitivity analysis on the optimum value N^* based on changes in specific values of the system parameters. Let the batch size X be a geometric distribution with parameter p, and employ the following cost elements:

Case 1: $C_h = 5, C_o = 10, C_b = 100, C_d = 200, C_s = 125, C_f = 500$.
Case 2: $C_h = 5, C_o = 20, C_b = 200, C_d = 400, C_s = 250, C_f = 500$.
Case 3: $C_h = 5, C_o = 20, C_b = 200, C_d = 400, C_s = 500, C_f = 500$.
Case 4: $C_h = 5, C_o = 20, C_b = 200, C_d = 400, C_s = 500, C_f = 1000$.
Case 5: $C_h = 10, C_o = 20, C_b = 200, C_d = 400, C_s = 500, C_f = 1000$.

TABLE 1. The optimal value of N and its minimum expected cost for geometric batch size ($X \sim$ Geo(p)) and (p, λ_0, λ_1, λ_2, λ_3, μ, β) = (0.55, 0.3, 0.4, 0.5, 0.2, 1.2, 3.0).

	(r, α)	(0.3, 0.05)	(0.5, 0.05)	(0.7, 0.05)	(0.5, 0.1)	(0.5, 0.2)	(0.5, 0.3)
Case 1	N^*	5	5	5	5	5	5
	$T_{\text{cost}}(N^*)$	137.064	126.850	122.909	129.199	133.976	138.878
Case 2	N^*	7	6	6	6	6	6
	$T_{\text{cost}}(N^*)$	216.592	203.671	198.293	207.772	216.012	224.323
Case 3	N^*	10	8	8	8	8	8
	$T_{\text{cost}}(N^*)$	230.151	213.355	206.096	217.315	225.279	233.321
Case 4	N^*	11	9	9	9	9	9
	$T_{\text{cost}}(N^*)$	234.343	218.423	211.721	222.311	230.132	238.033
Case 5	N^*	7	6	6	6	6	6
	$T_{\text{cost}}(N^*)$	290.849	265.334	254.811	269.839	279.014	288.447

TABLE 2. The optimal value of N and its minimum expected cost for geometric batch size ($X \sim$ Geo(p)) and (p, λ_0, λ_1, λ_2, λ_3, r, α) = (0.55, 0.3, 0.4, 0.5, 0.2, 0.2, 0.05).

	(μ, β)	(1.0, 3.0)	(1.2, 3.0)	(1.4, 3.0)	(1.2, 2.0)	(1.2, 4.0)	(1.2, 5.0)
Case 1	N^*	3	5	6	5	5	5
	$T_{\text{cost}}(N^*)$	218.097	151.310	139.592	152.317	150.811	150.514
Case 2	N^*	4	7	9	7	7	7
	$T_{\text{cost}}(N^*)$	314.058	234.154	213.320	235.991	233.239	232.691
Case 3	N^*	6	11	13	11	11	11
	$T_{\text{cost}}(N^*)$	323.686	251.901	234.812	253.609	251.050	250.542
Case 4	N^*	6	11	14	11	11	11
	$T_{\text{cost}}(N^*)$	325.875	255.511	239.096	257.174	254.684	254.189
Case 5	N^*	4	7	9	7	7	7
	$T_{\text{cost}}(N^*)$	447.726	325.534	307.626	327.344	324.640	324.107

The optimal value of N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ for the above five cases are shown in Table 1 for $(p, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \mu, \beta) = (0.55, 0.3, 0.4, 0.5, 0.2, 1.2, 3.0)$ and for various values of (r, α). One observes from Table 1 that (i) $T_{\text{cost}}(N^*)$ increases as r decreases or α increases for any cases; (ii) for a large value r, N^* does not change at all when r changes from 0.5 to 0.7 for any cases; and (iii) N^* does not change at all when α changes from 0.1 to 0.3 for any cases. Intuitively, N^* is insensitive to changes in α.

The optimal value of N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ for five cost cases are shown in Table 2 for $(p, \lambda_0, \lambda_1, \lambda_2, \lambda_3, r, \alpha) = (0.55, 0.3, 0.4, 0.5, 0.2, 0.2, 0.05)$ and for different values of (μ, β). From Table 2, we observe that (i) $T_{\text{cost}}(N^*)$ increases as μ and β decrease for any cases; (ii) N^* increases as μ increases for any cases; and (iii) N^* does not change at all when β changes from 2.0 to 5.0 for any cases. Intuitively, N^* is insensitive to changes in β.

TABLE 3. The optimal value of N and its minimum expected cost for geometric batch size ($X \sim$ Geo(p)) and $(p, \lambda_2, \lambda_3, r, \mu, \alpha, \beta) = (0.45, 0.6, 0.2, 0.2, 2.0, 0.05, 3.0)$.

	(λ_0, λ_1)	(0.2, 0.5)	(0.3, 0.5)	(0.5, 0.5)	(0.3, 0.3)	(0.3, 0.4)	(0.3, 0.6)
Case 1	N^*	7	6	5	6	6	6
	$T_{\text{cost}}(N^*)$	136.445	154.447	172.548	146.156	150.230	158.787
Case 2	N^*	10	9	8	9	9	10
	$T_{\text{cost}}(N^*)$	200.510	228.795	258.891	218.824	223.787	233.729
Case 3	N^*	14	14	13	14	14	14
	$T_{\text{cost}}(N^*)$	219.954	252.381	288.762	245.011	248.641	256.222
Case 4	N^*	15	15	14	15	15	15
	$T_{\text{cost}}(N^*)$	223.307	256.363	293.649	249.311	252.780	260.050
Case 5	N^*	9	9	8	9	9	9
	$T_{\text{cost}}(N^*)$	298.267	339.986	384.818	327.165	333.388	346.918

TABLE 4. The optimal value of N and its minimum expected cost for geometric batch size ($X \sim$ Geo(p)) and $(p, \lambda_0, \lambda_1, r, \mu, \alpha, \beta) = (0.45, 0.3, 0.4, 0.2, 2.0, 0.05, 3.0)$.

	(λ_2, λ_3)	(0.4, 0.1)	(0.6, 0.1)	(0.8, 0.1)	(0.6, 0.4)	(0.6, 0.5)	(0.6, 0.6)
Case 1	N^*	7	6	4	6	6	6
	$T_{\text{cost}}(N^*)$	140.846	150.082	209.148	150.537	150.698	150.863
Case 2	N^*	10	9	6	9	9	9
	$T_{\text{cost}}(N^*)$	205.135	223.541	299.140	224.293	224.554	224.819
Case 3	N^*	16	14	9	14	14	14
	$T_{\text{cost}}(N^*)$	233.780	248.438	314.849	249.060	249.276	249.498
Case 4	N^*	17	15	10	15	15	15
	$T_{\text{cost}}(N^*)$	238.461	252.583	317.663	253.188	253.398	253.614
Case 5	N^*	10	9	6	9	9	9
	$T_{\text{cost}}(N^*)$	320.243	333.159	437.816	333.871	334.125	334.387

The optimal value of N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ for five cases are shown in Table 3 for $(p, \lambda_2, \lambda_3, r, \mu, \alpha, \beta) = (0.45, 0.6, 0.2, 0.2, 2.0, 0.05, 3.0)$ and for various values of (λ_0, λ_1). One observes from Table 3 that (i) $T_{\text{cost}}(N^*)$ increases as λ_0 and λ_1 increase for any cases; (ii) N^* slightly changes when λ_0 changes from 0.2 to 0.5 for any cases; and (iii) N^* does not change at all when λ_1 changes from 0.3 to 0.6 for any cases. Intuitively, N^* is insensitive to changes in λ_1.

The optimal value of N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ are shown in Table 4 for $(p, \lambda_0, \lambda_1, r, \mu, \alpha, \beta) = (0.45, 0.3, 0.4, 0.2, 2.0, 0.05, 3.0)$ and for different values of (λ_2, λ_3). From Table 4, we find that (i) $T_{\text{cost}}(N^*)$ increases as λ_2 and λ_3 increase for any cases; (ii) N^* increases as λ_2 decreases for any cases; and (iii) N^* and $T_{\text{cost}}(N^*)$ do not change at all when λ_3 changes from 0.1 to 0.6 for any cases. Intuitively, N^* is insensitive to changes in λ_3.

Table 5. The optimal value of N and its minimum expected cost for geometric batch size ($X \sim$ Geo(p)) and $(p, \lambda_0, \lambda_3, r, \mu, \alpha, \beta) = (0.35, 0.3, 0.2, 0.2, 2.0, 0.05, 3.0)$.

	(λ_1, λ_2)	(0.4, 0.4)	(0.4, 0.5)	(0.4, 0.6)	(0.3, 0.6)	(0.5, 0.6)	(0.6, 0.6)
Case 1	N^*	6	5	4	4	4	5
	$T_{\text{cost}}(N^*)$	161.506	173.806	219.092	211.378	226.704	234.165
Case 2	N^*	9	8	6	5	6	7
	$T_{\text{cost}}(N^*)$	237.573	256.366	310.400	302.102	318.490	326.327
Case 3	N^*	14	12	9	9	9	10
	$T_{\text{cost}}(N^*)$	266.163	280.927	327.880	321.434	334.412	340.982
Case 4	N^*	15	13	10	10	10	10
	$T_{\text{cost}}(N^*)$	269.987	284.291	330.458	324.249	336.777	343.189
Case 5	N^*	9	8	6	6	6	6
	$T_{\text{cost}}(N^*)$	360.776	379.384	459.573	446.811	472.565	485.740

Table 6. The optimal value of N and its minimum expected cost for uniform batch size ($X \sim U(1,4)$) and $(\lambda_0, \lambda_3, r, \mu, \alpha, \beta) = (0.3, 0.2, 0.2, 2.0, 0.05, 3.0)$.

	(λ_1, λ_2)	(0.4, 0.4)	(0.4, 0.5)	(0.4, 0.6)	(0.3, 0.6)	(0.5, 0.6)	(0.6, 0.6)
Case 1	N^*	5	5	4	4	5	5
	$T_{\text{cost}}(N^*)$	147.944	152.052	162.036	156.705	167.367	172.774
Case 2	N^*	9	8	7	7	8	8
	$T_{\text{cost}}(N^*)$	217.720	227.281	244.349	238.449	250.257	255.937
Case 3	N^*	13	12	11	11	11	11
	$T_{\text{cost}}(N^*)$	237.603	246.785	262.955	258.645	267.367	271.875
Case 4	N^*	13	12	11	11	11	11
	$T_{\text{cost}}(N^*)$	243.437	252.142	267.433	263.250	271.726	276.121
Case 5	N^*	9	8	7	7	7	7
	$T_{\text{cost}}(N^*)$	335.049	340.409	355.476	347.682	363.753	372.442

The optimal value of N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ are shown in Table 5 for $(p, \lambda_0, \lambda_3, r, \mu, \alpha, \beta) = (0.35, 0.3, 0.2, 0.2, 2.0, 0.05, 3.0)$ and for different values of (λ_1, λ_2). Table 5 depicts that (i) $T_{\text{cost}}(N^*)$ increases as λ_1 and λ_2 increase for any cases; (ii) N^* increases as λ_2 decreases for any cases; and (iii) N^* slightly changes when λ_1 changes from 0.3 to 0.6 for any cases.

Furthermore, we choose the uniform batch size (set $X \equiv U(1,4)$). The numerical results for the optimal value N, N^*, and its minimum expected cost $T_{\text{cost}}(N^*)$ are shown in Table 6 for $(\lambda_0, \lambda_3, r, \mu, \alpha, \beta) = (0.3, 0.2, 0.2, 2.0, 0.05, 3.0)$ and for different values of (λ_1, λ_2). Table 6 shows that (i) $T_{\text{cost}}(N^*)$ increases as λ_1 and λ_2 increase for any cases; (ii) N^* increases as λ_2 decreases for any cases; and (iii) N^* slightly changes when λ_1 changes from 0.3 to 0.6 for any cases.

It can be easily see from Table 1 through 6 that (i) N^* increases as C_s increases or C_h decreases (see Case 2–3 and Case 4–5); and (ii) C_h and C_s have a larger effect on N^* than C_f (see Case 3–4). Tables 5 and 6 indicate that X affects N^*.

From our numerical investigations, it appears that (i) α, β, and λ_3 do not affect N^*; (ii) r, λ_0 and λ_1 rarely affect N^*; and (iii) λ_2 and μ affect N^* significantly. It is interested that C_h and C_s have much stronger effect on N^* than X, $\lambda_0, \lambda_1, \lambda_2, \lambda_3, r, \mu, \alpha$, and β.

7. Conclusions

In this paper, we have developed the analytic closed-form solutions for the $\mathrm{M}^{[\mathrm{x}]}/\mathrm{M}/1$ queueing system with server breakdowns and startup time under the N policy. More especially, the stochastic decomposition property of state-steady probabilities and departure point queue size distribution has been investigated and some important remarks have been given. The model is very useful for real systems since the behavior of arriving units is considered. Usually, the problems related to the Markovian queueing systems are treated as special cases of this model. We also have performed a sensitivity analysis among the optimal value of N, specific values of system parameters, and the cost elements. Through the numerical results, we were able to analyze the complex but exact solutions for a practical and general queueing system, make an intelligent decision based on quantitative measures.

Appendix 1. The derivations of $H_1(z)$

Equation (2) is multiplied by $z^n (n = 1, 2, \ldots, N-1)$ and are expanded as follows:

$$z\lambda_0 P_0(1) = z\lambda_0 b_1 P_0(0),$$

$$z^2\lambda_0 P_0(2) = z^2\lambda_0 b_2 P_0(0) + z^2\lambda_0 b_1 P_0(1),$$

$$z^3\lambda_0 P_0(3) = z^3\lambda_0 b_3 P_0(0) + z^3\lambda_0 b_2 P_0(1) + z^3\lambda_0 b_1 P_0(2),$$

$$\vdots$$

$$z^{N-1}\lambda_0 P_0(N-1) = z^{N-1}\lambda_0 b_{N-1} P_0(0) + z^{N-1}\lambda_0 b_{N-2} P_0(2) + \ldots + z^{N-1}\lambda_0 b_1 P_0(N-2).$$

The equations listed above are added terms by terms. Thus we have

$$\lambda_0 \sum_{n=1}^{N-1} z^n P_0(n) = \lambda_0 P_0(0) \sum_{n=1}^{N-1} z^n b_n + \lambda_0 z P_0(1) \sum_{n=1}^{N-2} z^n b_n + \lambda_0 z^2 P_0(2) \sum_{n=1}^{N-3} z^n b_n + \cdots + \lambda_0 z^{N-2} P_0(N-2) \sum_{n=1}^{1} z^n b_n,$$

or equivalently

$$\lambda_0 H_0(z) - \lambda_0 P_0(0) = \lambda_0 P_0(0) \sum_{n=1}^{N-1} z^n b_n + \lambda_0 z P_0(1) \sum_{n=1}^{N-2} z^n b_n + \lambda_0 z^2 P_0(2) \sum_{n=1}^{N-3} z^n b_n + \cdots + \lambda_0 z^{N-2} P_0(N-2) \sum_{n=1}^{1} z^n b_n. \quad \text{(A.1)}$$

Similarly, equation (3) is multiplied by $z^n (n = N, N+1, \dots)$ and are expanded as follows:

$$z^N(\lambda_1 + r)P_0(N) = z^N \lambda_0 b_N P_0(0) + z^N \lambda_0 b_{N-1} P_0(1) + z^N \lambda_0 b_{N-2} P_0(2) + \cdots + z^N \lambda_0 b_1 P_0(N-1),$$

$$z^{N+1}(\lambda_1+r)P_0(N+1) = z^{N+1}\lambda_0 b_{N+1} P_0(0) + z^{N+1}\lambda_0 b_N P_0(1) + z^{N+1}\lambda_0 b_{N-1} P_0(2) + \cdots + z^{N+1}\lambda_0 b_2 P_0(N-1) + z^{N+1}\lambda_1 b_1 P_0(N),$$

$$z^{N+2}(\lambda_1+r)P_0(N+2) = z^{N+2}\lambda_0 b_{N+2} P_0(0) + z^{N+2}\lambda_0 b_{N+1} P_0(1) + z^{N+2}\lambda_0 b_N P_0(2) + \cdots + z^{N+2}\lambda_0 b_3 P_0(N-1) + z^{N+2}\lambda_1 b_2 P_0(N) + z^{N+2}\lambda_1 b_1 P_0(N+1),$$

$$\vdots$$

The equations listed above are added terms by terms. Thus we have

$$(\lambda_1 + r) \sum_{n=N}^{\infty} z^n P_0(n) = \lambda_0 P_0(0) \sum_{n=N}^{\infty} z^n b_n + \lambda_0 z P_0(1) \sum_{n=N-1}^{\infty} z^n b_n + \lambda_0 z^2 P_0(2) \sum_{n=N-2}^{\infty} z^n b_n + \cdots + \lambda_0 z^{N-2} P_0(N-2) \sum_{n=2}^{\infty} z^n b_n + \lambda_0 z^{N-1} P_0(N-1) \sum_{n=1}^{\infty} z^n b_n + \lambda_1 z^N P_0(N) \sum_{n=1}^{\infty} z^n b_n + \lambda_1 z^{N+1} P_0(N+1) \sum_{n=1}^{\infty} z^n b_n + \dots,$$

or equivalently

$$
\begin{aligned}
(\lambda_1 + r)H_1(z) = \lambda_0 P_0(0) \sum_{n=N}^{\infty} z^n b_n + \lambda_0 z P_0(1) \sum_{n=N-1}^{\infty} z^n b_n \\
+ \lambda_0 z^2 P_0(2) \sum_{n=N-2}^{\infty} z^n b_n + \cdots + \lambda_0 z^{N-2} P_0(N-2) \sum_{n=2}^{\infty} z^n b_n \\
+ \lambda_0 z^{N-1} P_0(N-1) \sum_{n=1}^{\infty} z^n b_n + \lambda_1 z^N P_0(N) \sum_{n=1}^{\infty} z^n b_n \\
+ \lambda_1 z^{N+1} P_0(N+1) \sum_{n=1}^{\infty} z^n b_n + \ldots \quad \text{(A.2)}
\end{aligned}
$$

Adding (A.1) to (A.2) and rearranging some terms, it finally yields

$$[(\lambda_1 + r) - \lambda_1 X(z)]H_1(z) = \lambda_0[X(z) - 1]H_0(z) + \lambda_0 P_0(0),$$

which gets (11).

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