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ON CONSTRAINT QUALIFICATIONS IN DIRECTIONALLY DIFFERENTIABLE MULTIOBJECTIVE OPTIMIZATION PROBLEMS *

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Abstract. We consider a multiobjective optimization problem with a feasible set defined by inequality and equality constraints such that all functions are, at least, Dini differentiable (in some cases, Hadamard differentiable and sometimes, quasiconvex). Several constraint qualifications are given in such a way that generalize both the qualifications introduced by Maeda and the classical ones, when the functions are differentiable. The relationships between them are analyzed. Finally, we give several Kuhn-Tucker type necessary conditions for a point to be Pareto minimum under the weaker constraint qualifications here proposed.

Keywords. Multiobjective optimization problems, constraint qualification, necessary conditions for Pareto minimum, Lagrange multipliers, tangent cone, Dini differentiable functions, Hadamard differentiable functions, quasiconvex functions.

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1. Introduction

Constraint qualifications have a significant role in optimization problems, since they allow us to guarantee the effective intervention of the objective function in the Fritz John type necessary conditions for a point to be an optimum. Since the first decade of the 50's, the study of these qualifications has been the aim of several researchers with different approaches, proposing various regularity conditions.

Maeda [10] studies multiobjective optimization problems with differentiable functions between finite-dimensional spaces and gives a Kuhn-Tucker type necessary condition for a Pareto optimum of a function over a feasible set defined by inequality constraints, assuring that the multipliers of the objective function are all positive under a regularity condition, called generalized Guignard constraint qualification. He also studies other qualifications, showing that this one is the weakest.

Preda and Chitescu [13] develop, at first, results similar to those obtained by Maeda, considering Dini-quasiconvex and directionally differentiable functions. But owing to the requirement on the objective functions to be Dini-quasiconvex and Dini-quasiconcave with convex and concave Dini derivatives, their necessary optimality conditions (Ths. 3.1 and 3.2) are very restrictive. On the other hand, the necessary condition expressed in Theorem 3.2, assuring the existence of positive multipliers for the objective functions, has a mistake that will be corrected in this paper.

Jiménez and Novo [7] extend the results obtained by Maeda for differentiable functions, by considering equality constraints, not considered by Maeda nor by Preda and Chitescu. They also introduced new qualifications that are sufficient conditions for what the afore mentioned papers called generalized Guignard constraint qualification.

In the present paper, the results obtained by Maeda, Preda and Chitescu and Jiménez and Novo are extended, by considering Dini or Hadamard differentiable functions and equality constraints. Furthermore, new qualifications are also introduced and the relationships between them are studied, thus obtaining a scheme which generalizes the ones of Bazaraa and Shetty [2], Figure 6.4, Maeda [10], Figure 1, Preda and Chitescu [13], Figure 1, and Jiménez and Novo [7], Figure 1.

This paper is structured as follows: Section 2 contains the definitions and notations we use and some previous results. In Section 3 several constraint qualifications are proposed and the relationships between them are studied. Finally, in Section 4, several necessary optimality conditions of the Kuhn-Tucker type are obtained, *i.e.* such that they assure the positivity of the multipliers under the weaker qualifications proposed.

2. Notations and preliminaries

Let x and y be two points of $\mathbb{R}^n$. Throughout this paper, we use the following notations.

$$x \leq y \text{ if } x_i \leq y_i,\ i = 1, \ldots, n. \ x < y \text{ if } x_i < y_i,\ i = 1, \ldots,\ n.$$

Let S be a subset of $\mathbb{R}^n$. As usual, $\operatorname{cl} S$, $\operatorname{co} S$, $\operatorname{cone} S$ and $\operatorname{lin} S$ will denote the closure, convex hull, generated cone and generated subspace by S, respectively. $B(x_0, \delta)$ is the open ball of center x_0 and radius $\delta > 0$.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered

$$\text{(MOP)}\,\text{Min}\{f(x) : x \in S\}\cdot$$

It is said that the point $x_0 \in S$ is a local Pareto minimum, denoted $x_0 \in \text{LMin}(f, S)$, if there exists a neighborhood of x_0, $B(x_0, \delta)$, such that

$$S_f \cap S \cap B(x_0, \delta) = \emptyset, \tag{1}$$

where $S_f = \{x \in \mathbb{R}^n :\ f(x) \leq f(x_0),\ f(x) \neq f(x_0)\}$.

The usual concepts of Pareto minimum, weak Pareto minimum and local weak Pareto minimum are also used. They will be denoted by $\text{Min}(f, S)$, $\text{WMin}(f, S)$ and $\text{LWMin}(f, S)$, respectively.

Because of the difficulties in verifying condition (1), different approximations at x_0 of the sets S and S_f are normally used, which have a simpler structure and are easier to obtain. The tangent cones are the approximations more usually used.

Definition 2.1. Let $S \subset \mathbb{R}^n,\ x_0 \in \operatorname{cl} S$.
(a) The tangent cone to S at the point x_0 is
$T(S, x_0) = \{v \in \mathbb{R}^n :\ \exists t_k \to 0^+,\ \exists x_k \in S \text{ such that } (x_k - x_0)/t_k \to v\}$.
(b) The cone of attainable directions is
$A(S, x_0) = \{v \in \mathbb{R}^n :\ \forall t_k \to 0^+,\ \exists x_k \in S \text{ such that } (x_k - x_0)/t_k \to v\}$.
(c) The cone of linear directions is
$Z(S, x_0) = \{v \in \mathbb{R}^n :\ \exists \delta > 0 \text{ such that } x_0 + tv \in S\ \forall t \in (0, \delta]\}\cdot$

For these cones, we have the following inclusions

$$Z(S, x_0) \subset A(S, x_0) \subset T(S, x_0). \tag{2}$$

A complete and rigorous analysis of these cones in a greater detail can be found in Bazaraa and Shetty [2] and in Aubin and Frankowska [1].

Let $D \subset \mathbb{R}^n$. Then the polar cone to D is $D^* = \{\xi \in \mathbb{R}^n :\ \langle \xi, d\rangle \leq 0\ \forall d \in D\}$.

The normal cone to S at x_0 is the polar to the tangent cone, *i.e.*, $N(S, x_0) = T(S, x_0)^*$.

Note that if the sets are defined through function constraints, their approximation is realized through the cones defined by the directional derivatives of the functions.

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^p,\ x_0, v \in \mathbb{R}^n$.
(a) The Dini derivative (or directional derivative) of f at x_0 in the direction v is

$$Df(x_0, v) = \lim_{t\to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

(b) The Hadamard derivative of f at x_0 in the direction v is

$$df(x_0, v) = \lim_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(c) f is Dini differentiable (respectively Hadamard differentiable) at x_0 if its Dini derivative (resp. Hadamard derivative) exists for all the directions.

The following properties are well-known:
– if f is Fréchet differentiable at x_0 with Fréchet differential $\nabla f(x_0)$, then $df(x_0, v) = \nabla f(x_0)v$;
– if $df(x_0, v)$ exists, then also $Df(x_0, v)$ exists and they are equal.

Definition 2.3. The Dini subdifferential of a Dini differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is

$$\partial_D f(x_0) = \{\xi \in \mathbb{R}^n :\ \langle \xi, v\rangle \le Df(x_0, v)\ \forall v \in \mathbb{R}^n\}.$$

If $Df(x_0, v)$ is a convex function in v, then there exists the subdifferential (in the Convex Analysis sense) of this function at $v = 0$: $\partial Df(x_0, \cdot)(0)$. This set is nonempty, compact and convex and $\partial_D f(x_0) = \partial Df(x_0, \cdot)(0)$.

In this paper, the following generalized convexity notions will be used.

Definition 2.4. Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $f : \Gamma \to \mathbb{R}$, and $x_0 \in \Gamma$.

(a) f is quasiconvex at x_0 if $\forall x \in \Gamma,\ f(x) \le f(x_0)\ \Rightarrow\ f(\lambda x + (1-\lambda)x_0) \le f(x_0)\ \forall \lambda \in (0, 1)$;
(b) f is quasiconcave at x_0 if $-f$ is quasiconvex at x_0;
(c) f is quasilinear at x_0 if f is quasiconvex and quasiconcave at x_0;
(d) f is pseudoconvex at x_0 if $\forall x \in \Gamma$, $f(x) < f(x_0) \Rightarrow Df(x_0, x - x_0) < 0$;
(e) f is pseudoconcave at x_0 if $-f$ is pseudoconvex at x_0. f is pseudolinear at x_0 if f is pseudoconvex and pseudoconcave at x_0;
(f) f is linearlike at x_0 if $f(x) = f(x_0) + Df(x_0, x - x_0)\ \forall x \in \Gamma$;
(g) f is Dini-quasiconvex at x_0 if $\forall x \in \Gamma,\ f(x) \le f(x_0)\ \Rightarrow\ Df(x_0, x-x_0) \le 0$;
(h) f is Dini-quasilinear at x_0 if f and $-f$ are Dini-quasiconvex at x_0;
(i) f is quasiconvex on Γ if f is quasiconvex at each point of Γ. The other concepts here introduced can be defined on a set in a similar way.

In the next proposition we summarize some properties of the generalized convex functions previously introduced.

Proposition 2.1. *Let $\Gamma \subset \mathbb{R}^n$ be a convex set, $f : \Gamma \to \mathbb{R}$, and $x_0 \in \Gamma$.*
(a) [3] Th. 3.5.2) f is quasiconvex on Γ if and only if the level sets $\Gamma_\alpha = \{x \in \Gamma :\ f(x) \le \alpha\}$ are convex for all $\alpha \in \mathbb{R}$.

(b) Let f be Dini differentiable at x_0. If f is quasiconvex at x_0, then f is Dini-quasiconvex at x_0.
(c) ([5] Th. 3.5) if f is pseudoconvex at x_0 and continuous on Γ, then f is quasi-convex at x_0.
(d) ([5] Th. 3.2) if f is continuous and Dini-quasiconvex on Γ, then f is quasi-convex on Γ.

Remark 2.1. The following implications can easily be proved for a linear type function:

(i) if f is linearlike at x_0, then f is pseudolinear at x_0 and quasilinear at x_0;
(ii) if f is quasilinear at x_0 and Dini differentiable at x_0, then f is Dini-quasilinear at x_0.

The second implication follows from Proposition 2.1(b). The converse of (i) does not hold. It can be proved, for instance, with the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x| + x^2$ and the point $x_0 = 0$.

Moreover, the reverse of Proposition 2.1(b) does not hold (*i.e.*, if f is Dini-quasiconvex at x_0, then f is quasiconvex at x_0), even if f is differentiable at x_0. The function $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$, is an evident counterexample for $x_0 = 0$.

Also, there is no in general implication relation between the concepts of pseudoconvexity at a point and quasiconvexity at a point. In fact, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \|(x,y)\|\varphi(y/x^2) & \text{if } x > 0,\ 1 \leq y/x^2 \leq 3 \\ \|(x,y)\| & \text{otherwise} \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is defined by $\varphi(\alpha) = |\alpha - 2|$ if $1 \leq \alpha \leq 3$ and $\varphi(\alpha) = 1$ otherwise. Its Dini-derivative at $x_0 = (0,0)$ is $Df(x_0, v) = \|v\|\ \forall v \in \mathbb{R}^2$. It follows that f is pseudoconvex at x_0, since $\forall x \in \mathbb{R}^2\ f(x) \geq f(x_0)$, and it is not Dini-quasiconvex at x_0 over any neighborhood of x_0 (it is sufficient to consider the points $x_n = (\delta_n, \delta_n^2)$ with $\delta_n \to 0^+$, then $f(x_n) = f(x_0)$ and $Df(x_0, x_n - x_0) > 0$). Hence, thanks to Proposition 2.1(b), f is not quasiconvex at x_0. Furthermore, this function is pseudolinear at x_0 and, consequently, it is not true that a pseudolinear function at x_0 is quasilinear at x_0 (or Dini-quasilinear at x_0).

The function $f(x) = x^3$ is quasiconvex at $x_0 = 0$ but it is not pseudoconvex.

Finally, none of the linear types guarantees by itself the continuity of the derivative. Example 3.1 in [4] Chapter 1 shows this fact.

We say that the convex sets B_j, $j \in J = \{1, \ldots, m\}$, of $\mathbb{R}^n$ are positively linearly independent (p.l.i.) if

$$0 \in \sum_{j \in J} \lambda_j B_j,\ \lambda \geq 0 \quad \Rightarrow \quad \lambda = 0,$$

i.e., if $0 \notin \operatorname{co}(\cup_{j \in J} B_j)$.

In Section 4 necessary conditions for a local Pareto minimum with positive multipliers for the objective functions will be obtained. It is however necessary to establish first a Tucker type alternative theorem [11] Theorem 3, Chapter 2.4. We choose the version obtained by Ishizuka [6] Proposition 2.2 in a simplified form, and we give it in a suitable form for our purposes.

Proposition 2.2 (generalized Tucker alternative theorem). *Let $f_1, \dots, f_p, g_1, \dots, g_m$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $h_1, \dots, h_r$ linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ given by $h_k(v) = \langle c_k, v\rangle$, $k \in K = \{1, \dots, r\}$. Suppose that for each $i \in \{1, \dots, p\}$ the cone*

$$D_i = \operatorname{cone}\operatorname{co}(\cup_{j\neq i}\partial f_j(0)) + \operatorname{cone}\operatorname{co}(\cup_{j=1}^m \partial g_j(0)) + \operatorname{lin}\{c_k : \ k \in K\}$$

is closed. Then, the following statements are equivalent:
(a) The system

$$f(v) \leq 0, \ f(v) \neq 0, \ g(v) \leq 0, \ h(v) = 0$$

has no solution $v \in \mathbb{R}^n$.
(b) There exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that $\lambda > 0$, $\mu \geq 0$ and

$$0 \in \sum_{i=1}^p \lambda_i \partial f_i(0) + \sum_{j=1}^m \mu_j \partial g_j(0) + \sum_{k=1}^r \nu_k c_k.$$

Proof. To transform this proposition into the one of Ishizuka, it is enough to consider $A_i = \partial f_i(0)$, $i = 1, \dots, p$, $B_j = \partial g_j(0)$, $j = 1, \dots, m$, $B_{m+k} = \operatorname{co}\{-c_k, c_k\} = [-c_k, c_k]$, $k = 1, \dots, r$, which implies $f_i(v) = \operatorname{Max}_{a\in A_i}\langle a, v\rangle$, $i = 1, \dots, p$, $g_j(v) = \operatorname{Max}_{b\in B_j}\langle b, v\rangle$, $j = 1, \dots, m$. Let

$$g_{m+k}(v) = \operatorname*{Max}_{c\in B_{m+k}} \langle c, v\rangle = \operatorname{Max}\{-\langle c_k, v\rangle, \langle c_k, v\rangle\} = |h_k(v)|, \ k = 1, \dots, r.$$

We have $\partial g_{m+k}(0) = B_{m+k}$ and the equation $h_k(v) = 0$ is equivalent to $g_{m+k}(v) \leq 0$. By means of this notation, as the cones D_i are closed, according to Ishizuka's Proposition 2.2 [6], (a) is equivalent to

(c) There exist $(\lambda, \mu, \alpha) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that $\lambda > 0$, $(\mu, \alpha) \geq 0$ and $0 \in \sum_{i=1}^p \lambda_i \partial f_i(0) + \sum_{j=1}^m \mu_j \partial g_j(0) + \sum_{k=1}^r \alpha_k[-c_k, c_k]$.

Taking into account that $c \in \alpha_k[-c_k, c_k]$ for some $\alpha_k \geq 0$ if and only if there exists $\nu_k \in \mathbb{R}$ such that $c = \nu_k c_k$, proposition (c) is equivalent to (b). □

In order to decide if the cones D_i are closed, we have the following criterium.

Remark 2.2. Note that if $0 \notin \operatorname{co}(\cup_{j\neq i} A_j \cup \cup_{j=1}^m B_j) + \operatorname{lin}\{c_k : \ k \in K\}$, then D_i is closed. This follows from Proposition 3.6 in [8].

Note that if $0 \notin C = \operatorname{co}(\cup_{i=1}^p A_i \cup \cup_{j=1}^m B_j) + \operatorname{lin}\{c_k : \ k \in K\}$, then the p cones D_i are closed. But this condition is incompatible with Proposition 2.2(b) and, consequently, with Proposition (a). As a matter of fact, if $0 \notin C$ and $u = Proy_C(0)$, then the vector $v = -u$ is a solution of the system in (a).

Now we consider a set S defined by equality and inequality constraints and a point of S at which we need to obtain the tangent cone. This is done in Proposition 2.6.

From now on, we shall assume that the feasible set of problem (MOP) is defined by

$$S = \{x \in \mathbb{R}^n : \ g(x) \leq 0, \ h(x) = 0\}, \tag{3}$$

where $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^r$, whose component functions are, respectively, $g_j, \ j \in J = \{1, \ldots, m\}$, $h_k, \ k \in K = \{1, \ldots, r\}$. We shall adopt the following notation. Given $x_0 \in S$, the active index set at x_0 is $J_0 = \{j \in J : \ g_j(x_0) = 0\}$. The sets defined by the constraints g and h are denoted, respectively, by $G = \{x \in \mathbb{R}^n : \ g(x) \leq 0\}$, $H = \{x \in \mathbb{R}^n : \ h(x) = 0\}$, so $S = G \cap H$.

We suppose that all functions considered are continuous at x_0 and that the active constraints are Dini differentiable at x_0. The cones that we shall use in order to approximate S at x_0 are (linearized cones):

$$C_0(S) = \{v \in \mathbb{R}^n : \ Dg_j(x_0, v) < 0 \ \forall j \in J_0, \ Dh_k(x_0, v) = 0 \ \forall k \in K\},$$
$$C(S) = \{v \in \mathbb{R}^n : \ Dg_j(x_0, v) \leq 0 \ \forall j \in J_0, \ Dh_k(x_0, v) = 0 \ \forall k \in K\}\cdot$$

$C_0(G)$ and $C(G)$ are defined in an analogous way and we denote $K(H) = \operatorname{Ker} Dh(x_0, \cdot)$. Consequently, $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$.

Our aim is to obtain the inclusions

$$C_0(S) \subset T(S, x_0) \subset C(S). \tag{4}$$

This is done in the following propositions.

Proposition 2.3 ([12], (Prop. 3.1)). *Let $Dg_j(x_0, \cdot)$, $j \in J_0$ be convex, $Dh(x_0, \cdot)$ linear and $C_0(S) \neq \emptyset$. Then $\operatorname{cl} C_0(S) = C(S)$.*

Proposition 2.4. *Suppose that for each $j \in J_0$, either g_j is Hadamard differentiable at x_0 or g_j is Dini-quasiconvex at x_0 and $Dg_j(x_0, \cdot)$ is continuous on $\mathbb{R}^n$, and for each $k \in K$, either h_k is Hadamard differentiable at x_0 or Dini-quasilinear at x_0 with $Dh_k(x_0, \cdot)$ continuous. Then*

$$T(S, x_0) \subset C(S).$$

The proof of the previous proposition is similar to that of Lemma 3.2 in [9].

Proposition 2.5. *If there is no equality constraints, $S = G$, and the functions g_j, $j \in J_0$, are Dini differentiable at x_0, then*

$$C_0(G) \subset Z(G, x_0) \subset \begin{cases} A(G, x_0) \subset T(G, x_0) \\ C(G). \end{cases}$$

Proposition 2.6 [9] (Cor. 3.5). *Let us suppose the following:*
(a) h is continuous on a neighborhood of x_0, Fréchet differentiable at x_0 and $\{\nabla h_k(x_0):\ k \in K\}$ is linearly independent;
(b) for each $j \in J_0$, g_j is either Dini-quasiconvex and continuous on a neighborhood of x_0 or Hadamard differentiable at x_0, in both cases with convex derivative at x_0;
(c) $C_0(S) \neq \emptyset$.
Then

$$\operatorname{cl} C_0(S) = A(S, x_0) = T(S, x_0) = C(S).$$

Note that, by [8], Theorem 3.9, we have that $\{\nabla h_k(x_0) :\ k \in K\}$ is linearly independent and $C_0(S) \neq \emptyset$ if and only if the following implication is true:

$$0 \in \sum_{j\in J_0} \mu_j \partial_D g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0),\ \mu \geq 0\ \Rightarrow\ \mu = 0,\ \nu = 0,$$

which is constraint qualification (CQ2) in [9]. By Proposition 2.1(d), if g_j is Dini-quasiconvex and continuous on a neighborhood of x_0, then g_j is quasiconvex on such a neighborhood.

3. Constraint qualifications in multiobjective optimization

Let us consider the multiobjective optimization problem

$$\text{(MOP)}\ \operatorname{Min}\{f(x) :\ x \in S\},$$

where the feasible set S is given by (3) and $f : \mathbb{R}^n \to \mathbb{R}^p$ has component functions f_i, $i \in I = \{1, \ldots, p\}$.

By keeping the notation of Section 2, given $x_0 \in S$, the following sets are considered: $F = \{x :\ f(x) \leq f(x_0)\}$, $S^0 = F \cap S$ and for each $i \in I$, $F^i = \{x :\ f_j(x) \leq f_j(x_0)\ \forall j \in I \setminus \{i\}\}$ and $S^i = F^i \cap S$. Obviously $F = \cap_{i=1}^{p} F^i$ and $S^0 = \cap_{i=1}^{p} S^i$. Since the sets given above are defined by function constraints, the corresponding linearized cones can be defined. Let us remark that for the set F all functions f_i, $i \in I$, are active at x_0 and for the set F^i the same is true for the functions f_j, $j \in I \setminus \{i\}$. We have $C_0(S^i) = C_0(F^i) \cap C_0(G) \cap K(H)$, $C(S^i) = C(F^i) \cap C(G) \cap K(H)$ and similar expressions for $C_0(S^0)$ and $C(S^0)$.

It is a known result that x_0 is a local Pareto minimum to problem (MOP) if and only if for each $i = 1, \ldots, p$, x_0 is a local minimum of the scalar problem

$$(\mathrm{P}_i)\ \operatorname{Min}\{f_i(x) :\ x \in S^i\}\cdot$$

We consider now different qualifications for problem (MOP) in the approaches of Maeda [10], Preda and Chitescu [13] and Jiménez and Novo [7]. The implications between the various qualifications are also analyzed.

We suppose that all functions are Dini differentiable at x_0, unless we specify another thing.

Let us consider the following hypotheses:

(H0) h is continuous on a neighborhood of x_0, Fréchet differentiable at x_0 and $\{\nabla h_k(x_0) : k \in K\}$ is linearly independent.

(H1) Each function of the set $\{f_i,\ g_j :\ i \in I,\ j \in J_0\}$ is either Dini-quasiconvex at x_0 with continuous derivative on $\mathbb{R}^n$ or Hadamard differentiable at x_0.

(H2) Each function of the set $\{f_i,\ g_j :\ i \in I,\ j \in J_0\}$ is either Dini-quasiconvex and continuous on a neighborhood of x_0 or Hadamard differentiable at x_0.

(H3) For each $i = 1, \ldots, p$, $T(S^i, x_0) \subset C(S^i)$ holds true.

(H4) Each function in the set $\{Df_i(x_0, \cdot),\ Dg_j(x_0, \cdot) :\ i \in I,\ j \in J_0\}$ is convex.

Definition 3.1. The next constraint qualifications are considered:

1. *Generalized Guignard (GGCQ):* $C(S^0) = \cap_{i=1}^{p} \operatorname{cl} \operatorname{co} T(S^i, x_0)$.
2. *Abadie (ACQ):* $C(S^0) = T(S^0, x_0)$ and (H3).
3. *Generalized Abadie (GACQ):* $C(S^0) = \cap_{i=1}^{p} T(S^i, x_0)$ and (H3).
4. *Global Cottle (GCCQ):* $C_0(F) \cap C_0(S) \neq \emptyset$, (H0) and (H4).
5. *Cottle (CCQ):* for each $i = 1 \ldots p$, $C_0(S^i) \neq \emptyset$ (H0) and (H4).
6. Slater type.
   a) *Slater (SCQ):* $f_i,\ i \in I,\ g_j,\ j \in J_0$, are pseudoconvex at x_0 ; $h_k,\ k \in K$, are Dini-quasilinear at x_0, (H0), (H4) and for each $i = 1, \ldots, p$ there exists $x_i \in \mathbb{R}^n$ such that

$$f_j(x_i) < f_j(x_0)\ \forall j \neq i,\ g_j(x_i) < 0\ \forall j \in J_0,\ \text{and}\ h_k(x_i) = 0\ \forall k \in K. \tag{5}$$

   b) *Differentiable Slater (DSCQ):* $f_i,\ i \in I,\ g_j,\ j \in J_0$, are pseudoconvex at x_0, (H0), (H4) and for each $i = 1, \ldots, p$ there exists $x_i \in \mathbb{R}^n$ such that

$$f_j(x_i) < f_j(x_0)\ \forall j \neq i,\ g_j(x_i) < 0\ \forall j \in J_0,\ \text{and}\ x_i - x_0 \in K(H). \tag{6}$$

7. *Linearlike (LLCQ):* $f_i,\ g_j,\ h_k,\ i \in I,\ j \in J_0,\ k \in K$, are all linearlike at x_0 with continuous derivative.
8. *Linearlike objectives (LLO):* $f_i,\ i \in I$, are linearlike at x_0 with convex derivative, each $g_j,\ j \in J_0$, has convex derivative and is either Hadamard differentiable or Dini-quasiconvex at x_0, $h_k,\ k \in K$, are affine and $C(F) \cap C_0(G) \cap K(H) \neq \emptyset$.
9. *Mangasarian-Fromovitz.* Each qualification in this group must verify (H0) and (H4):
   a) With positively linearly independent objectives *(PIOMF):* $C(F) \cap C_0(S) \neq \emptyset$ and $C_0(F) \cap K(H) \neq \emptyset$.
   b) With quasiindependent objectives *(QIOMF):* $C(F) \cap C_0(S) \neq \emptyset$ and for each $i = 1, \ldots, p$ we have that $C_0(F^i) \cap K(H) \neq \emptyset$.
   c) With positively linearly independent constraints *(PICMF):* $C_0(F) \cap C(S) \neq \emptyset$ and $C_0(G) \cap K(H) \neq \emptyset$.
   d) With $C(F) \cap C_0(G) \cap K(H) \neq \emptyset$ and for each $i = 1, \ldots, p$, $C(F) \cap C_0(G) \cap K(H) \not\subset \operatorname{Ker} Df_i(x_0, \cdot)$. We speak in this case of *Preda-Chitescu Mangasarian-Fromovitz qualification (PCMF).*

10. *Zangwill (ZCQ):* $\operatorname{cl} Z(S^0, x_0) = C(S^0)$ and (H3).
11. *Kuhn-Tucker (KTCQ):* $A(S^0, x_0) = C(S^0)$ and (H3).
12. *Reverse (RCQ):* $f_i,\ g_j,\ i \in I,\ j \in J_0$, are pseudoconcave at x_0, (H1) and $h_k,\ k \in K$, are linearlike at x_0 with continuous derivative.

Lemma 3.1.
(i) If h is linearlike at x_0 with continuous Dini derivative, then h is Hadamard differentiable at x_0.
(ii) If h is pseudolinear and Dini-quasilinear at x_0, then
(a) $Z(H, x_0) = T(H, x_0) = K(H)$;
(b) $H = x_0 + K(H)$.
(iii) If $g_j,\ j \in J_0$, are pseudoconcave at x_0, then $Z(G, x_0) = C(G)$.

Proof.
(i) It is an elementary exercise.
(ii) (a) $Z(H, x_0) \subset T(H, x_0)$ is true for all sets H and $T(H, x_0) \subset K(H)$ by Proposition 2.4. We now prove that $K(H) \subset Z(H, x_0)$. Let $v \in K(H)$, then $Dh(x_0, v) = 0$ and therefore $Dh(x_0, (x_0 + tv) - x_0) \geq 0\ \forall t > 0$. Since h is pseudoconvex, $h(x_0 + tv) \geq h(x_0) = 0\ \forall t > 0$, and analogously, due to the pseudoconcavity, $h(x_0 + tv) \leq h(x_0) = 0$. Consequently $h(x_0 + tv) = 0$, *i.e.*, $x_0 + tv \in H\ \forall t > 0$, which implies $v \in Z(H, x_0)$.
(b) We have just proved that $x_0 + tv \in H\ \forall t > 0$. Taking $t = 1$, we have $v \in H - x_0$ and thus $K(H) \subset H - x_0$. Now we prove the reverse inclusion. Let $x \in H$, hence $h(x) - h(x_0) \leq 0$. Since h is Dini-quasiconvex at x_0, it follows $Dh(x_0, x - x_0) \leq 0$. Likewise with $-h$, we get $-Dh(x_0, x - x_0) \leq 0$. Consequently, $Dh(x_0, x - x_0) = 0$, which means, $x - x_0 \in K(H)$.
(iii) From Proposition 2.5, we only have to prove that $C(G) \subset Z(G, x_0)$. Let $v \in C(G)$, then $Dg_j(x_0, v) \leq 0\ \forall j \in J_0$. By pseudoconcavity, $g_j(x_0 + tv) \leq g_j(x_0) = 0\ \forall t > 0$. If $j \in J \setminus J_0$, by the continuity of g_j we have $g_j(x_0 + tv) < 0$ for all t small enough. Therefore, $x_0 + tv \in G$, and consequently $v \in Z(G, x_0)$, thus completing the proof. □

We remark that just by using the definition, we get that if h is linearlike at x_0 with linear Dini derivative, then h is affine. If h is linearlike at x_0 with continuous Dini derivative, then part (ii) of Lemma 3.1 holds true (according to Rem. 2.1, h is pseudolinear and Dini-quasilinear at x_0).

In Theorem 3.1 below, the relationship between the different constraint qualifications are established. In order to prove the theorem we need a previous lemma. The inclusion relationships in the lemma are obvious and the proof of the second part is similar to that of Proposition 2.3.

Lemma 3.2. *If $Dh(x_0, \cdot)$ is linear and $Df_i(x_0, \cdot),\ Dg_j(x_0, \cdot),\ i \in I,\ j \in J_0$ are convex, then*

$$C_0(S^0) = C_0(F) \cap C_0(G) \cap K(H) \subset \left\{ \begin{array}{c} C(F) \cap C_0(G) \cap K(H) \\ C_0(F) \cap C(G) \cap K(H) \end{array} \right\}$$
$$\subset C(F) \cap C(G) \cap K(H) = C(S^0),$$

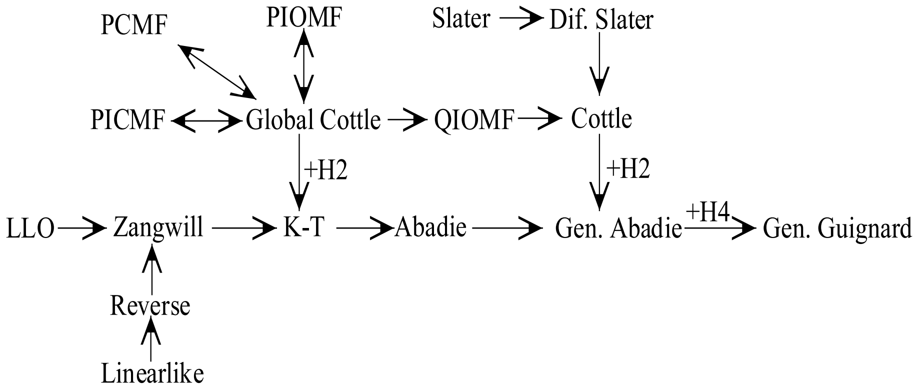

FIGURE 1. Relationship between the constraint qualifications.

and if some of the sets $C_0(S^0)$, $C(F) \cap C_0(G) \cap K(H)$ *and* $C_0(F) \cap C(G) \cap K(H)$ *is nonempty then its closure is* $C(S^0)$.

Theorem 3.1. *The following implications are verified:*
1. Linearlike ⇒ Reverse ⇒ Zangwill.
2. Linearlike objectives ⇒ Zangwill.
3. Slater ⇒ Differentiable Slater ⇒ Cottle.
4. PICMF ⇔ Global Cottle ⇔ PIOMF ⇔ PCMF.
5. Global Cottle ⇒ QIOMF ⇒ Cottle.
6. Cottle and (H2) ⇒ Generalized Abadie.
7. Global Cottle and (H2) ⇒ Kuhn-Tucker.
8. a) Zangwill ⇒ Kuhn-Tucker ⇒ Abadie ⇒ Generalized Abadie.
b) Generalized Abadie and (H4) ⇒ Generalized Guignard.

The results above are summarized in Figure 1, which generalizes the similar figures in [2], Figure 6.4, [10], Figure 1, [13], Figure 1 and [7], Figure 1.

Proof.
1. a) Linearlike ⇒ Reverse. It suffices to observe that if a function is linearlike, then it is pseudoconcave and, since it has continuous derivative, by Lemma 3.1(i), it follows that it is Hadamard differentiable, which implies (H1).
b) Reverse ⇒ Zangwill. Lemma 3.1(iii) shows that $Z(F \cap G, x_0) = C(F \cap G)$, and by Lemma 3.1(ii), $Z(H, x_0) = K(H)$. Thus

$$\begin{aligned} Z(S^0) &= Z(F \cap G \cap H, x_0) = Z(F \cap G, x_0) \cap Z(H, x_0) \\ &= C(F \cap G) \cap K(H) = C(S^0). \end{aligned}$$

Since (H1) is true, (H3) follows from Proposition 2.4.
2. Linearlike objectives ⇒ Zangwill. From Proposition 2.5 it follows $C_0(G) \subset Z(G, x_0)$. Since f is linearlike at x_0, by means of Lemma 3.1(iii), we get $Z(F, x_0) =$

$C(F)$. As h is affine, $Z(H, x_0) = K(H)$. Hence we have

$$C(F) \cap C_0(G) \cap K(H) \subset Z(F, x_0) \cap Z(G, x_0) \cap Z(H, x_0) = Z(S^0, x_0) \subset T(S^0, x_0). \tag{7}$$

As f is linearlike with continuous derivative, from Lemma 3.1(i), f is Hadamard differentiable. Since g_j, $j \in J_0$, is Hadamard differentiable or Dini-quasiconvex, by Proposition 2.4, $T(S^i, x_0) \subset C(S^i)$ $\forall i \in I$ and $T(S^0, x_0) \subset C(S^0)$. Taking this last inclusion and (7) into account and using Lemma 3.2, we can conclude that $\operatorname{cl} Z(S^0, x_0) = C(S^0)$.

3. a) Slater $\Rightarrow$ Differentiable Slater. For each problem (P_i) there exists x_i verifying (5). In particular, $h_k(x_i) = h_k(x_0)$, and by the Dini-quasiconvexity of h_k and $-h_k$, $\nabla h_k(x_0)(x_i - x_0) = 0$, $\forall k \in K$, which means that $x_i - x_0 \in K(H)$.
b) Differentiable Slater $\Rightarrow$ Cottle. For each $i = 1, \ldots, p$, there exists x_i verifying (6). Because of the pseudoconvexity of f_j and g_j, we have

$$Df_j(x_0, x_i - x_0) < 0 \ \forall j \neq i, \ \ Dg_j(x_0, x_i - x_0) < 0 \ \forall j \in J_0.$$

By hypothesis, $x_i - x_0 \in K(H)$, and consequently $x_i - x_0 \in C_0(S^i)$.

4. a) PICMF $\Rightarrow$ Global Cottle. By assumption we have $C_0(G) \cap K(H) \neq \emptyset$, and hence from Proposition 2.3

$$\operatorname{cl}[C_0(G) \cap K(H)] = C(G) \cap K(H). \tag{8}$$

Let $v \in C_0(F) \cap [C(G) \cap K(H)]$ (this set is nonempty by assumption). Since $C_0(F)$ is open, there exists a neighborhood $B(v)$ of v, such that $B(v) \subset C_0(F)$, and from (8), $B(v) \cap [C_0(G) \cap K(H)] \neq \emptyset$. So, we can state that $C_0(F) \cap C_0(G) \cap K(H) \neq \emptyset$ and hence we have the global Cottle qualification.
b) Global Cottle $\Rightarrow$ PICMF. By hypothesis, $C_0(F) \cap C_0(G) \cap K(H) \neq \emptyset$, and therefore $C_0(F) \cap C(S) \neq \emptyset$ and $C_0(G) \cap K(H) \neq \emptyset$, that is, we have PICMF.
c) Global Cottle $\Leftrightarrow$ PIOMF. It is enough to note that f and g have a symmetric role in PIOMF and PICMF. So, if PICMF $\Leftrightarrow$ Global Cottle, then also PIOMF $\Leftrightarrow$ Global Cottle.
d) Global Cottle $\Leftrightarrow$ PCMF.
[$\Rightarrow$] It is immediate, because there exists $v \in C_0(F) \cap C_0(G) \cap K(H)$. Then, $Df_i(x_0, v) < 0$ $\forall i \in I$.
[$\Leftarrow$] By hypothesis, $\forall i \in I$ $\exists v_i \in C(F) \cap C_0(G) \cap K(H)$ such that $Df_i(x_0, v_i) < 0$. Let be $v = \sum_{i=1}^p \lambda_i v_i$ with $\lambda_i = 1/p$. We shall see that $v \in C_0(F) \cap C_0(G) \cap K(H)$, which implies global Cottle.
By the convexity of $Df_j(x_0, \cdot)$ we have $Df_j(x_0, v) \leq \sum_{i=1}^p \lambda_i Df_j(x_0, v_i) < 0$, since $Df_j(x_0, v_i) \leq 0$ $\forall j \neq i$ and $Df_i(x_0, v_i) < 0$. Analogously, $Dg_j(x_0, v) < 0$ $\forall j \in J_0$ and by the linearity of $\nabla h_k(x_0)$, $\nabla h_k(x_0)v = 0$.

5. a) Global Cottle $\Rightarrow$ QIOMF. As global Cottle is equivalent to PIOMF, it is enough to get that PIOMF implies QIOMF. But this is obvious, because if $C_0(F) \cap K(H) \neq \emptyset$, then for each $i = 1, \ldots, p$ $C_0(F^i) \cap K(H) \neq \emptyset$.

b) QIOMF $\Rightarrow$ Cottle. Assume that there exists $i \in I$ such that $C_0(S^i) = \emptyset$. This means that there is no solution $v \in \mathbb{R}^n$ of the system

$$\begin{cases} Df_j(x_0, v) < 0 \ \forall j \neq i \\ Dg_j(x_0, v) < 0 \ \forall j \in J_0 \\ \nabla h_k(x_0)v = 0 \ \forall k \in K. \end{cases}$$

Using Theorem 3.5 in [8] we obtain that there exists $(\lambda, \mu, \nu) \in \mathbb{R}^{p-1} \times \mathbb{R}^{J_0} \times \mathbb{R}^r$ such that $(\lambda, \mu) \geq 0, \ (\lambda, \mu) \neq 0$ and

$$\sum_{j \neq i} \lambda_j Df_j(x_0, v) + \sum_{j \in J_0} \mu_j Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geq 0 \ \forall v \in \mathbb{R}^n. \tag{9}$$

By hypothesis, there exists $u \in C(F) \cap C_0(G) \cap K(H)$. If for some $j \in J_0$, $\mu_j > 0$, then $\sum_{j \neq i} \lambda_j Df_j(x_0, u) + \sum_{j \in J_0} \mu_j Dg_j(x_0, u) < 0$, in contrast with the result obtained in (9) with $v = u$. Thus $\mu = 0$ and in (9) we have therefore that $\sum_{j \neq i} \lambda_j Df_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geq 0 \ \forall v \in \mathbb{R}^n$. By hypothesis, there exists $w \in C_0(F^i) \cap K(H)$ and an analogous argument shows that $\lambda = 0$, which is a contradiction.
6. Cottle and (H2) $\Rightarrow$ Generalized Abadie. It is enough to apply Proposition 2.6 to each set S^i.
7. Global Cottle and (H2) $\Rightarrow$ Kuhn-Tucker. To prove this result, it is sufficient to take the implication Global Cottle $\Rightarrow$ Cottle into account and to apply Proposition 2.6 to each S^i and to S^0.
8. a) Zangwill $\Rightarrow$ Kuhn-Tucker $\Rightarrow$ Abadie $\Rightarrow$ Generalized Abadie. Since $S^0 \subset S^i \ \forall i \in I$, from the isotonicity of the tangent cone and from (H3) it follows that

$$T(S^0, x_0) \subset \cap_{i=1}^{p} T(S^i, x_0) \subset \cap_{i=1}^{p} C(S^i) = C(S^0).$$

Now, from equation (2) applied to S^0, the three implications follow.
b) Generalized Abadie and (H4) $\Rightarrow$ Generalized Guignard. Obviously, $T(S^i, x_0) \subset \operatorname{cl}\operatorname{co} T(S^i, x_0) \subset C(S^i)$ (the last inclusion is due to the convexity of the derivatives and to (H3)). Therefore we get

$$\cap_{i=1}^{p} T(S^i, x_0) \subset \cap_{i=1}^{p} \operatorname{cl}\operatorname{co} T(S^i, x_0) \subset \cap_{i=1}^{p} C(S^i) = C(S^0)$$

and the implication is then evident. □

Remark 3.1.
(1) It is known that if f is Hadamard differentiable at x_0 and x_0 is a local Pareto minimum of f over S, then

$$C_0(F) \cap T(S, x_0) = \emptyset. \tag{10}$$

(2) If for some $i \in I$, f_i is Hadamard differentiable at x_0, (H0), (H2) and (H4) hold, and $x_0 \in \mathrm{LMin}(f, S)$, then $C_0(S^0) = \emptyset$ (and consequently, Global Cottle qualification is not satisfied at x_0).
Indeed one has that x_0 is a local solution to problem (P$_i$), *i.e.*, $x_0 \in \mathrm{LMin}(f_i, S^i)$, and by the previous remark,

$$C_0(f_i) \cap T(S^i, x_0) = \emptyset, \tag{11}$$

where $C_0(f_i) = \{v \in \mathbb{R}^n : \ df_i(x_0, v) < 0\}$.

Assume that $C_0(S^0) \neq \emptyset$, then $C_0(S^i) \neq \emptyset$. By Proposition 2.6, $T(S^i, x_0) = C(S^i)$. From (11), it follows that $C_0(f_i) \cap C_0(S^i) \subset C_0(f_i) \cap C(S^i) = \emptyset$. But $C_0(S^0) = C_0(f_i) \cap C_0(S^i) = \emptyset$ and we have a contradiction.

So, Global Cottle is not a true constraint qualification when some f_i is Hadamard differentiable and (H2) holds.
(3) If there is no equality constraints, then global Cottle is not verified at a local Pareto minimum (from Prop. 2.5 and Lem. 4.1 in the next section) and, consequently, neither is (AMFCQ) in [13].

4. Optimality conditions under generalized qualifications

In this section Kuhn-Tucker type necessary optimality conditions are given for a point to be local Pareto minimum. These conditions are obtained both in primal form and in dual form, with a feasible set defined by inequality and equality constraints, the objective functions and the constraints being, at least, Dini differentiable. In order to obtain the positivity of the multipliers associated with the vector-valued objective function, a generalized constraint qualification will be assumed. In this way we generalize Maeda's results [10], which are valid for differentiable functions and without equality constraints, and Preda and Chitescu's [13] who consider a problem with Dini differentiable functions and without equality constraints. We generalize also the results of Jiménez and Novo [7], valid for differentiable problems with equality constraints.

Theorem 4.1. *Let f be Hadamard differentiable at x_0, g_j, $j \in J_0$, Dini differentiable at x_0 and h Fréchet differentiable at x_0 and suppose that the generalized Abadie qualification is verified. If $x_0 \in \mathrm{LMin}(f, S)$, then there exists no solution $v \in \mathbb{R}^n$ of the system*

$$\begin{cases} Df(x_0, v) \leq 0, \ Df(x_0, v) \neq 0 \\ Dg_j(x_0, v) \leq 0 \ \forall j \in J_0 \\ \nabla h(x_0) v = 0, \end{cases} \tag{12}$$

i.e., x_0 is a proper local solution to problem (MOP) in the sense of Kuhn-Tucker.

Proof. Assume that the conclusion is not true. Then there exist $v \in \mathbb{R}^n$ and $i \in \{1, \dots, p\}$ such that

$$\begin{cases} Df_i(x_0, v) < 0 \\ Df_j(x_0, v) \le 0 \ \forall j \ne i \\ v \in C(S). \end{cases} \tag{13}$$

Thus $v \in C(S^0)$ and, by the generalized Abadie qualification, $v \in T(S^i, x_0)$. Since x_0 is a local Pareto minimum, it also is a local minimum of each scalar problem (P_j), in particular, $x_0 \in \mathrm{LMin}(f_i, S^i)$. As f_i is Hadamard differentiable, we have $Df_i(x_0, u) \ge 0 \ \forall u \in T(S^i, x_0)$. Taking $u = v$, then $Df_i(x_0, v) \ge 0$, in contradiction to (13). □

Theorem 4.2. *Let f and h be Fréchet differentiable at x_0 and g_j, $j \in J_0$, Dini differentiable at x_0 and suppose that the generalized Guignard qualification is verified. If $x_0 \in \mathrm{LMin}(f, S)$, then there is no solution $v \in \mathbb{R}^n$ of the system* (12).

Proof. Assume that (13) is true for some $i \in \{1, \dots, p\}$ and $v \in \mathbb{R}^n$. Thanks to the generalized Guignard qualification, $v \in \mathrm{cl}\,\mathrm{co}\, T(S^i, x_0)$. Since $x_0 \in \mathrm{LMin}(f_i, S^i)$ and f_i is Fréchet differentiable, we obtain $\nabla f_i(x_0)u \ge 0 \ \forall u \in T(S^i, x_0)$ $(Df_i(x_0, u) = \nabla f_i(x_0)u)$. By the linearity and the continuity of $\nabla f_i(x_0)(\cdot)$, it follows that $\nabla f_i(x_0)u \ge 0 \ \forall u \in \mathrm{cl}\,\mathrm{co}\, T(S^i, x_0)$. Taking $u = v$ we have a contradiction to (13). □

It is possible to obtain the dual form of these two last theorems by applying the generalized Tucker alternative theorem (Prop. 2.2).

Theorem 4.3. *Assume the hypothesis of Theorems 4.1 or 4.2 and let the derivatives $Df(x_0, \cdot)$ and $Dg_j(x_0, \cdot)$, $j \in J_0$, be convex. If the cones*

$$D_i = \mathrm{cone}\,\mathrm{co}(\cup_{j \ne i} \partial_D f_j(x_0)) + \mathrm{cone}\,\mathrm{co}(\cup_{j \in J_0} \partial_D g_j(x_0)) + \mathrm{lin}\{\nabla h_k(x_0) : \ k \in K\} \tag{14}$$

$i = 1, \dots, p$, are closed, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that

(*a*) $\lambda > 0, \ \mu \ge 0, \ \mu_j g_j(x_0) = 0, \ j = 1, \dots, m;$

(*b*) $\displaystyle\sum_{i=1}^{p} \lambda_i Df_i(x_0, v) + \sum_{j=1}^{m} \mu_j Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0)v \ge 0 \ \forall v \in \mathbb{R}^n.$ (15)

As usual, we take $\mu_j = 0$ if $g_j(x_0) < 0$.

Note that condition (b) is equivalent to

$$0 \in \sum_{i=1}^{p} \lambda_i \partial_D f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_D g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0). \tag{16}$$

If we denote by L the Lagrangian function: $L = \sum_{i=1}^{p} \lambda_i f_i + \sum_{j=1}^{m} \mu_j g_j + \sum_{k=1}^{r} \nu_k h_k$, then (16) is equivalent to

$$0 \in \partial_D L(x_0). \tag{17}$$

Theorem 4.3 generalizes Corollary 8 in [7] by Jiménez and Novo.

Now we investigate about the conditions on the functions of the problem, which assume that the cones (14) are closed. One of these criteria is given below.

Proposition 4.1. *If for each $i = 1, \ldots, p$, $C_0(S^i) \neq \emptyset$, then the cones D_i, $i = 1, \ldots, p$, given by* (14), *are closed.*

This follows from Proposition 3.6 in [8].

We remark that if the Cottle qualification holds, then it is unnecessary to use the generalized Tucker alternative theorem to obtain positive multipliers, since this result can directly be obtained.

Proposition 4.2. *Let f be Hadamard differentiable at x_0 with convex derivative and suppose that the Cottle qualification and (H2) are satisfied. If $x_0 \in \mathrm{LMin}(f, S)$, then conditions* (15) *hold.*

Proof. From Theorem 3.1, part 6, the generalized Abadie qualification is verified and, by Theorem 4.1, the system (12) does not admit solution.

For each $i = 1, \ldots, p$ we have $x_0 \in \mathrm{LMin}(f_i, S^i)$, hence $Df_i(x_0, v) \geq 0\ \forall v \in T(S^i, x_0)$. As it was seen in the proof of Theorem 3.1, part 6, for each $i = 1, \ldots, p$, $T(S^i, x_0) = C(S^i)$. Therefore, none of the p systems $(i = 1, \ldots, p)$:

$$\begin{cases} Df_i(x_0, v) < 0 \\ Df_j(x_0, v) \leq 0\ \forall j \neq i \\ Dg_j(x_0, v) \leq 0\ \forall j \in J_0 \\ \nabla h_k(x_0) v = 0\ \forall k \in K \end{cases} \tag{18}$$

has a solution $v \in \mathbb{R}^n$. Let us consider the convex problem

$$(\mathrm{CP}_i)\ \alpha_i = \mathrm{Min}\{Df_i(x_0, v) : Df_j(x_0, v) \leq 0\ \forall j \neq i,$$
$$Dg_j(x_0, v) \leq 0\ \forall j \in J_0,\ \nabla h_k(x_0) v = 0\ \forall k \in K\}.$$

Because of the incompatibility of the system (18) above, we have $\alpha_i \geq 0$. Since $v = 0$ is a feasible solution and $Df_i(x_0, 0) = 0$, it is $\alpha_i = 0$. From Theorem 28.2 in [14] (we can use it because $C_0(S^i) \neq \emptyset$), it follows that there exist $\lambda_{ij} \geq 0$, $j \neq i$; $\mu_{ij} \geq 0$, $j \in J_0$; $\nu_{ik} \in \mathbb{R}$, $k \in K$ such that

$$Df_i(x_0, v) + \sum_{j=1, j \neq i}^{p} \lambda_{ij} Df_j(x_0, v) + \sum_{j \in J_0} \mu_{ij} Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_{ik} \nabla h_k(x_0) v \geq 0$$

for all $v \in \mathbb{R}^n$, and for $i = 1, \ldots, p$. Adding over $i = 1, \ldots, p$, we have

$$\sum_{i=1}^{p} \lambda_i Df_i(x_0, v) + \sum_{j \in J_0} \mu_j Dg_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geq 0 \ \forall v \in \mathbb{R}^n,$$

where, in order to simplify, we have denoted $\lambda_i = 1 + \sum_{j=1, j\neq i}^{p} \lambda_{ji}$, $i = 1, \ldots, p$; $\mu_j = \sum_{i=1}^{p} \mu_{ij}$, $j \in J_0$; $\nu_k = \sum_{i=1}^{p} \nu_{ik}$, $k = 1, \ldots, r$, and obviously we have $\lambda > 0$, and $\mu \geq 0$. □

As a consequence of Theorem 4.3 we obtain the following corollary, which extends Maeda's Theorem 3.2 [10] for a problem with differentiable functions and also equality constraints.

Corollary 4.1 ([7], (Cor. 8))**.** *Let f, g and h be Fréchet differentiable at x_0 and suppose that the generalized Guignard qualification is verified. If $x_0 \in \mathrm{LMin}(f, S)$, then there exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that*
(a) $\lambda > 0$, $\mu \geq 0$, $\mu_j g(x_0) = 0$, $j = 1, \ldots, m$;
(b) $\sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) = 0$.

Proof. Under these assumptions, the condition "for each $i = 1, \ldots, p$, D_i is closed" is verified, since

$$D_i = \operatorname{cone}\operatorname{co}(\{\nabla f_j(x_0) : \ j \neq i\} \cup \{\nabla g_j(x_0) : \ j \in J_0\}) + \operatorname{lin}\{\nabla h_k(x_0) : \ k \in K\}$$

is a polyhedral convex cone and, therefore, it is closed. □

The following example shows that Cottle qualification may not be verified; however, we can apply Theorem 4.3.

Example 4.1. In $\mathbb{R}^3$, let $x_0 = (0, 0, 0)$, $f_1 = 2x - 2z$, $f_2 = -2y$ and let g be the support function of the set $B = \{(x, y, z) : \ x^2 + (y-2)^2 + z^2 \leq 2, \ z \geq 0\}$. We obtain the following expression of g:

$$g(x, y, z) = \begin{cases} 2y + \sqrt{2x^2 + 2y^2 + 2z^2} & \text{if} \quad z \geq 0 \\ 2y + \sqrt{2x^2 + 2y^2} & \text{if} \quad z < 0. \end{cases}$$

Obviously $Dg(x_0, v) = g(v)$ and $\partial_D g(x_0) = B$. The feasible set is $G = \{(x, y, z) : g(x, y, z) \leq 0\}$ and the point x_0 is a Pareto minimum of $f = (f_1, f_2)$ over G. Cottle qualification is not verified because $C_0(S^1) = \{v : \ \nabla f_2(x_0)v < 0, \ Dg(x_0, v) < 0\} = \emptyset$, but we can apply Theorem 4.3, since the cones $D_1 = \operatorname{cone}\operatorname{co}(\{(0, -2, 0)\} \cup B) = \{(x, y, z) : \ z \geq 0\}$ and $D_2 = \operatorname{cone}\operatorname{co}(\{(2, 0, -2)\} \cup B)$ are closed (D_2 is closed because $C_0(S^2) \neq \emptyset$, since $(-1, -2, 0) \in C_0(S^2)$). So (16) holds, with $(\lambda_1, \lambda_2, \mu) = (1, 2, 2)$ and $b = (-1, 2, 1) \in \partial_D g(x_0)$.

Finally, we establish necessary optimality conditions without equality constraints; moreover, we do not require the objective functions to be Hadamard differentiable as in Theorems 4.1 and 4.2.

In Lemma 4.1 and Theorem 4.4 the following definition is used: $\varphi : \mathbb{R}^n \to \mathbb{R}$ is quasiconvex at x_0 on a neighborhood of x_0 if there exists $\delta > 0$ such that

$$x \in B(x_0, \delta),\ \varphi(x) \le \varphi(x_0)\ \Rightarrow\ \varphi(\lambda x_0 + (1-\lambda)x) \le \varphi(x_0)\ \forall \lambda \in (0,1).$$

Lemma 4.1. *If f has continuous Dini derivative at x_0, g_j, $j \in J_0$, are quasiconvex at x_0 on a neighborhood of x_0 and $x_0 \in \mathrm{LWMin}(f, G)$, then*

$$T(G, x_0) \cap C_0(F) = \emptyset.$$

Proof. Assume that $T(G,x_0) \cap C_0(F) \neq \emptyset$ and choose $v \in T(G,x_0) \cap C_0(F)$. Thus, $Df(x_0, v) < 0$. Using Lemma 4.10 in [8], $\mathrm{cl}\, Z(G, x_0) = T(G, x_0)$. Hence there exist $v_n \in Z(G, x_0)$ such that $v_n \to v$. By the continuity of $Df(x_0, \cdot)$, we can suppose that $Df(x_0, v_n) < 0\ \forall n \in \mathbb{N}$. By the definition of Dini derivative, $Df(x_0, v_n) = \lim_{t \to 0^+} (f(x_0 + tv_n) - f(x_0))/t < 0$. Therefore, there exists $\delta_n > 0$ such that $\forall t \in (0, \delta_n]$, $f(x_0 + tv_n) - f(x_0) < 0$. As $v_n \in Z(G, x_0)$, there exists $\eta_n > 0$ such that $\forall t \in (0, \eta_n]$, $x_0 + tv_n \in G$. Let us choose ε_n such that $0 < \varepsilon_n \le \mathrm{Min}\{\delta_n, \eta_n\}$ and $\varepsilon_n \to 0^+$. The sequence $x_n = x_0 + \varepsilon_n v_n \to x_0$, $x_n \in G$ and $f(x_n) < f(x_0)$, in contradiction with the minimality of x_0. □

Theorem 4.4. *Let* S = G *and assume that:*
(a) f_i, $i \in I$, g_j, $j \in J_0$, are quasiconvex at x_0 on a neighborhood of x_0.
(b) f_i, $i \in I$, g_j, $j \in J_0$, are Dini differentiable at x_0, with $Df_i(x_0, \cdot)$, $i \in I$, linear and $Dg_j(x_0, \cdot)$, $j \in J_0$, convex.
(c) The generalized Guignard qualification holds.
If $x_0 \in \mathrm{LMin}(f, G)$, then the system

$$\begin{cases} Df(x_0, v) \le 0,\ Df(x_0, v) \neq 0 \\ Dg_j(x_0, v) \le 0\ \forall j \in J_0 \end{cases}$$

has no solution $v \in \mathbb{R}^n$.

Proof. Assume the thesis does not hold. Then there exist $v \in \mathbb{R}^n$ and $i \in \{1, \ldots, p\}$ such that

$$\begin{cases} Df_i(x_0, v) < 0 \\ Df_j(x_0, v) \le 0\ \forall j \neq i \\ Dg_j(x_0, v) \le 0\ \forall j \in J_0. \end{cases} \tag{19}$$

Therefore $v \in C(S^0)$ and, using condition (c), $v \in \cap_{j=1}^{p} \mathrm{cl\,co}\, T(S^j, x_0)$. Consequently, $v \in \mathrm{cl\,co}\, T(S^i, x_0)$. Since $x_0 \in \mathrm{LMin}(f, S)$, it follows that $x_0 \in \mathrm{LMin}(f_i, S^i)$. Since $Df_i(x_0, \cdot)$ is continuous (it is linear) and as hypothesis (a) holds, we can apply Lemma 4.1, obtaining $Df_i(x_0, u) \ge 0\ \forall u \in T(S^i, x_0)$. Moreover, by the linearity, we deduce that $Df_i(x_0, u) \ge 0\ \forall u \in \mathrm{cl\,co}\, T(S^i, x_0)$. In particular, taking $u = v$, we obtain $Df_i(x_0, v) \ge 0$, contradicting (19). □

This theorem improves Theorem 3.1 in Preda and Chitescu [13] because quasiconcavity of f_i is not required.

The following theorem can be proved in a similar. From this result, we obtain subsequently (Th. 4.6) necessary optimality conditions in dual form.

Theorem 4.5. *Assume the hypotheses of Theorem 4.4, with* $Df_i(x_0,\cdot)$, $i \in I$, *convex (instead of linear) and*
(c′) The generalized Abadie qualification holds,
(instead of (c)). Then the same conclusion of Theorem 4.4 holds.

Theorem 4.6. *Suppose that the hypotheses of Theorems 4.4 or 4.5 hold true. If for each* $i = 1,\dots,p$ *the cone*

$$D_i = \operatorname{cone}\operatorname{co}(\cup_{j\neq i}\partial_D f_j(x_0)) + \operatorname{cone}\operatorname{co}(\cup_{j\in J_0}\partial_D g_j(x_0))$$

is closed, then there exists $(\lambda,\mu) \in \mathbb{R}^p \times \mathbb{R}^m$ *such that*
(a) $\lambda > 0$, $\mu \geq 0$, $\mu_j g_j(x_0) = 0$, $j = 1,\dots,m$;
(b) $\sum_{i=1}^p \lambda_i Df_i(x_0,v) + \sum_{j=1}^m \mu_j Dg_j(x_0,v) \geq 0 \quad \forall v \in \mathbb{R}^n$.

Expressions similar to equations (16) and (17) can also be obtained for (b).

This theorem corrects Theorem 3.2 in Preda and Chitescu [13], which is not true, as the following counterexample shows.

Example 4.2. Let us consider the problem

$$\operatorname{Min} f(x) \text{ subject to } g(x) \leq 0,$$

where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $f(x,y,z) = (x,-y)$ and $g(x,y,z) = y + z + \sqrt{x^2+y^2+z^2}$. We are going to study the minimality conditions at the point $x_0 = (0,0,0)$.
The feasible set $S = \{(x,y,z) : \ g \leq 0\}$ is the regular cone (in the sense of elementary geometry) with axis the halfline $(x,y,z) = \lambda(0,-1,-1)$, $\lambda \geq 0$ and whose generatrices form an angle of $\pi/4$ with the axis of the cone. We have

$$\begin{aligned} S^1 &= \{(x,y,z) : \ f_2 \leq 0,\ g \leq 0\} = \{(0,0,z) : \ z \leq 0\},\\ S^2 &= \{(x,y,z) : \ f_1 \leq 0,\ g \leq 0\} = S \cap \{x \leq 0\},\\ S^0 &= \{(x,y,z) : \ f_1 \leq 0,\ f_2 \leq 0,\ g \leq 0\} = S^1,\\ \nabla f_1(x_0) &= (1,0,0),\ \nabla f_2(x_0) = (0,-1,0),\ Dg(x_0,v) = g(v). \end{aligned}$$

We are under the hypotheses of Theorem 3.2 in Preda and Chitescu [13]:
1. The point x_0 is a Pareto minimum, since $f(x_0) = (0,0)$ and if $(x,y,z) \in S$, then $f(x,y,z) - f(x_0) \in -\mathbb{R}^2_+ \setminus \{0\}$ is not true (note that S is inside the dihedral $\{(x,y,z) : \ y \leq 0,\ z \leq 0\}$ and it cuts the plane $y = 0$ only in the halfline OZ^-).
2. The generalized Guignard qualification is true, because $\operatorname{cl}\operatorname{co} T(S^1,x_0) \cap \operatorname{cl}\operatorname{co} T(S^2,x_0) = S^1$ and $C(S^0) = S^0 = S^1$.
3. f_1, f_2 are linear, and consequently they are quasiconvex and quasiconcave; g is convex, and therefore, it also is quasiconvex.
4. f_1, f_2 are Fréchet differentiable, thus their Dini derivatives are linear, and hence, concave and convex. g is Hadamard differentiable and its derivative at x_0 is g itself, which is a convex function.

However, there exist no $\lambda_1 > 0,\ \lambda_2 > 0,\ \mu \geq 0$ such that

$$\lambda_1 \nabla f_1(x_0)v + \lambda_2 \nabla f_2(x_0)v + \mu Dg(x_0, v) \geq 0 \quad \forall v \in \mathbb{R}^3. \tag{20}$$

In fact, the unique solution with $\mu = 0$ is $\lambda_1 = \lambda_2 = 0$. Then, let $\mu > 0$; we can assume $\mu = 1$ and put $v = (x, y, z)$. Then (20) is equivalent to

$$\langle(-\lambda_1, \lambda_2 - 1, -1), (x, y, z)\rangle \leq \|(x, y, z)\| \quad \forall (x, y, z) \in \mathbb{R}^3.$$

This expression means that $(-\lambda_1, \lambda_2 - 1, -1)$ is a subgradient at 0 of the convex function $x \mapsto \|x\|$. But $\partial\|\cdot\|(0) = \operatorname{cl} B(0, 1)$ and therefore $\lambda_1^2 + (\lambda_2 - 1)^2 + 1 \leq 1$, and the only solution is $(\lambda_1, \lambda_2) = (0, 1)$.

Note that Theorem 4.6 cannot be applied, because the cone

$$D_1 = \operatorname{cone}\{\nabla f_2(x_0)\} + \operatorname{cone} \partial_D g(x_0) = \{(x, y, z) :\ z > 0 \text{ or } (z = 0,\ x = 0)\}$$

is not closed, being $\partial_D g(x_0) = \operatorname{cl} B(b_0, 1)$ with $b_0 = (0, 1, 1)$.

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