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TWO NEW CLASSES OF TREES EMBEDDABLE INTO HYPERCUBES

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Abstract. The problem of embedding graphs into other graphs is much studied in the graph theory. In fact, much effort has been devoted to determining the conditions under which a graph G is a subgraph of a graph H, having a particular structure. An important class to study is the set of graphs which are embeddable into a hypercube. This importance results from the remarkable properties of the hypercube and its use in several domains, such as: the coding theory, transfer of information, multicriteria rule, interconnection networks ... In this paper we are interested in defining two new classes of embedding trees into the hypercube for which the dimension is given.

1. Introduction

For a graph G, V(G) and E(G) denote the sets of vertices and edges of G respectively. An embedding of the graph G into the graph H is an 1-1 mapping of V(G) into V(H) that preserves adjacency of vertices. Our particular interest concerns the case where G is a tree and H is a hypercube. A tree T is a connected graph without circuits. A root of a tree is a particular vertex. A binary tree is a tree with a maximum degree less than or equal to 3. A tree T is balanced if in the 2-coloring of V(T), both color sets have the same cardinality. A hypercube

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of dimension n (denoted by Q_n) is the graph whose vertices are all vectors of length n consisting of 0's and 1's such that two vertices are adjacent if and only if they differ in exactly one coordinate. We note that

$$|V(Q_n)| = 2^n \quad \text{and} \quad |E(Q_n)| = n2^{n-1}.$$

The dimension of a tree T, denoted by dim T, is the smallest n such that T is embeddable into Q_n. One of the results related to binary trees is given by Havel and Liebl [4]: for binary tree T having 2^n vertices with $n \geq 3$, if T is balanced and has 2 vertices of degree 3, then T is embeddable into Q_n. Havel [3] and Nebeski [8] analyze the embedding of trees $D_n, \#D_n, \hat{}D_n$ and $\check{D}_n$ defined as follows:

If $n = 1$ then $D_1 = K_{1,2}$ (the complete bigraph), if $n > 1$ D_n is the tree obtained from two disjoint copies T and T' of D_{n-1} and a new vertex v joined by one edge to the only vertex of degree 2 of T and by another edge to the analogous vertex of T'. Thus D_n has 2^n endvertices, one vertex of degree 2 called the root of D_n and $2^n - 2$ vertices of degree 3.

$\#D_n$ is obtained from two disjoint copies of D_n such that their roots are joined by a new edge, this edge will be referred to as the axial edge of $\#D_n$. $\#D_n$ has $2^{n+2} - 2$ vertices. $\hat{}D_n$ and $\check{D}_n$ are obtained from $\#D_n$ by inserting two new vertices of degree 2 into the axial edge or into end-edge respectively. The trees $\#D_n$, $\hat{}D_n$ and $\check{D}_n$ are embeddable into Q_{n+2} [8].

B_n is obtained from D_{n-1} by adding one edge to its root. B_n has $2^{n-1} + 1$ endvertices and $2^{n-1} - 1$ vertices of degree 3. In particular B_1 is the complete bigraph $K_{1,2}$. The cubical dimension of B_n is $n + 1$ [6].

A caterpillar is a tree with at least 3 vertices that becomes a path or single vertex when its endvertices are all removed. Havel and Liebl [4] have proved that any balanced caterpillar having 2^n vertices and maximum degree 3 is embeddable into Q_n. In the same way, Harrary, Lewinter and Widulski [2] provided identical result for the caterpillar of maximum degree 4.

A star is a tree with exactly one non endvertex u. A quasistar is a star such that its edges are subdivided. It is balanced if and only if it has just one odd ray. The degree of a quasistar is the number of edges incident to u. A quasistar of degree k having 2^n vertices is called k-quasistar. A double star is obtained from two stars such that the junctions u and v are joined. A double quasistar is a subdivision of a double star in which the edge joined u and v is not subdivided. A double quasistar is balanced if the number of paths of odd order incident to u is equal to the number of paths of odd order incident to v.

Let uv be an edge. We *design* by MD$(a_1, ..., a_k)$ the graph formed by the edge uv and k ($k \geq 1$) distinct paths of orders respectively $a_1, ..., a_k$ (a_i is a positive integer) where the extremities of each path are joined by an edge to u, the other by an edge to v with $a_1 + ... + a_k = 2^n - 2$. MD$(a_1, ..., a_k)$ is balanced if a_i is even for any i.

Recently Kobeissi [1] has shown that quasistar and double quasistar are embeddable into Q_n. The same Nebesky [9] has shown that any MD k balanced graph

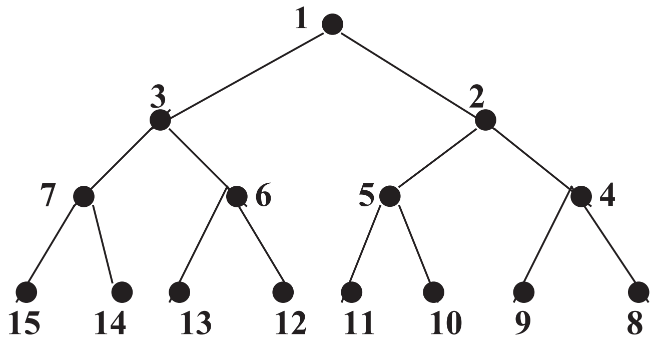

FIGURE 1. Complete binary tree D_3.

having 2^n vertices with $k \leq n-1$ is embeddable into Q_n. As a result: any k balanced quasistar (respectively double quasistar) with 2^n vertices such that $k \leq n$ (respectively $k \leq n-1$) is embeddable into Q_n.

A tree T is said to be C_n -valued, if the edges of T are labelled by integers from $\{1, ..., n\}$ in such a way that for any path P of T there is k from $\{1, ..., n\}$ such that an odd number of edges of P are assigned k [6].

Havel and Moravek [5] have used this notion in order to prove that a tree T is embeddable into Q_n if and only if there is a C_n -valuation of T.

2. EMBEDDING OF TWO NEW TREES

In this section, we introduce two new trees, denoted $D_n^{(k)}$ and $RD_n^{(2)}$ for which the cubical dimensions are given.

2.1. FIRST TYPE OF TREE

$D_n^{(k)}$ is obtained from the complete binary tree D_n by the removal of k endvertices where $k \leq 2^n$.

By removing only one vertex, the dimension of $D_n^{(1)}$ is equal to the dimension of D_n for $n > 2$ as illustrated by the example given in Figure 1.

The tree T_1 obtained by removing vertex 15 is not embeddable into Q_4.

Proof. Let $\{x,\ \mathrm{N}_0(x),\ \mathrm{N}_1(x),\ \mathrm{N}_2(x),...,\mathrm{N}_n(x)\}$ the decomposition of hypercube Q_n into levels from the vertex x then we have $|\mathrm{N}_i(x)| = \begin{pmatrix} n \\ i \end{pmatrix}$. Hence for Q_4

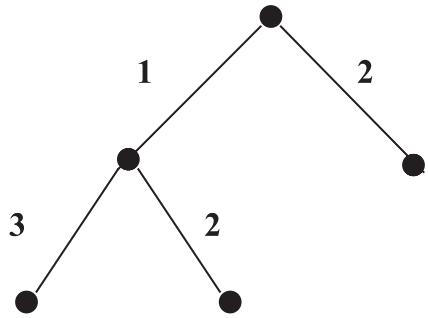

FIGURE 2. A C_3-valuation for $D_2^{(2)}$.

we have:

$$|\mathrm{N}_0(x)| = 1$$
$$|\mathrm{N}_1(x)| = 4$$
$$|\mathrm{N}_2(x)| = 6$$
$$|\mathrm{N}_3(x)| = 4$$
$$|\mathrm{N}_4(x)| = 1.$$

Following to this propriety and the definition of embedding, a tree T is embeddable into Q_4 if for any x of T the decomposition of T into levels from x satisfied the conditions cited below:

$$|\mathrm{N}_0(x)| \leq 1$$
$$|\mathrm{N}_1(x)| \leq 4$$
$$|\mathrm{N}_2(x)| \leq 6$$
$$|\mathrm{N}_3(x)| \leq 4$$
$$|\mathrm{N}_4(x)| \leq 1.$$

From these conditions, the tree T_1 is not embeddable into Q_4 (see Tab. 1 given in Appendix).

Note that dim $D_2^{(2)} = 3$ and dim $D_2^{(3)} =$ dim $D_2^{(4)} = 2$. □

Theorem 1. *dim* $D_n^{(k)} = n + 1$ *for* $n > 1$, $2 \leq k \leq 2^n$.

Proof. Firstly we prove that dim $D_n^{(2)} = n + 1$.

According to the proposition of HAVEL it is sufficient to determine a C_{n+1}-valuation of $D_n^{(2)}$. The proof is by induction on n. For $n = 2$, a C_3-valuation is given in Figure 2. □

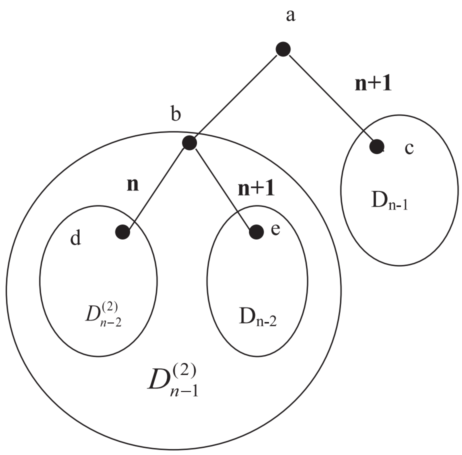

FIGURE 3. A C_{n+1}valuation for $D_n^{(2)}$.

As the complete binary trees are defined inductively, it is valuable for $D_n^{(2)}$ and we can obtain $D_n^{(2)}$ from D_{n-1} and $D_{n-1}^{(2)}$ as shown in Figure 3.

If P belongs to D_{n-1} then $\exists\, k \in \{1, ..., n+1\}$ such that an odd number of edges of P are assigned k because D_{n-1} is C_{n+1} -valued.

If P belongs to $D_{n-1} \cup$ ac, this tree is a B_n and then $\exists\, k \in \{1, ..., n+1\}$ such that an odd number of edges of P are assigned k.

If P $=$ $P_1 \cup$ ab $\cup$ ac $\cup$ bd $\cup$ P_2 where $P_1 \subseteq D_{n-2}^{(2)}$ and $P_2 \subseteq D_{n-1}$ then we have the following (see Fig. 4):

- for the subpath P_1 : $\exists\, k_1 \in \{1, ..., n-1\}$ such that an odd number of edges of P_1 are assigned k_1 ...(1);
- for the subpath P_2 :$\exists\, k_2 \in \{1, ..., n+1\}$ such that an odd number of edges of P_2 are assigned k_2...(2);
- $k_2 \neq n+1$ otherwise the condition will not be satisfied for the tree B_n which is C_{n+1} valued...(3);
- hence from (1), (2) and (3) k may have the value $n+1$.

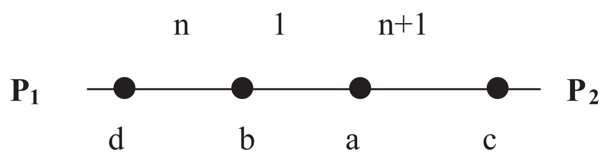

FIGURE 4. A subgraph of $D_n^{(2)}$.

If P $=$ $P_1 \cup$ eb $\cup$ ba $\cup$ ac $\cup$ P_2 where $P_1 \subseteq D_{n-2}$ and $P_2 \subseteq D_{n-1}$ then we obtain the following (see Fig. 5):

- for the subpath P1 : $\exists\, k_1 \in \{1, ..., n\}$ such that an odd number of edges of P_1 are assigned k_1...(1);

- for the subpath P2 : $\exists\ k_2 \in \{1, ..., n+1\}$ such that an odd number of edges of P_2 are assigned k_2...(2);
- $k_2 \neq n+1$ otherwise the condition will not be satisfied for the tree B_n which is C_{n+1} valued...(3);
- hence from (1), (2) and (3) k may have the value 1.

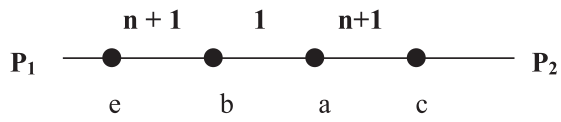

FIGURE 5. A subgraph of $D_n^{(2)}$.

Also, there exists a C_{n+1}- valuation for $D_n^{(2)}$ and consequently this tree is embeddable into Q_{n+1} for $n > 1$.

Secondly for $2 \leq k \leq 2^n$ we have

- $D_{n-1} = D_n^{(k)}$ for $k = 2^n$ so $D_{n-1} \subseteq D_n^{(k)}$ for $2 \leq k \leq 2^n$...(1);
- in other way $D_n^{(k)} \subseteq D_n^{(2)}$ for $2 \leq k \leq 2^n$ ...(2);
- from (1) and (2) we have $D_{n-1} \subseteq D_n^{(k)} \subseteq D_n^{(2)}$ then dim $D_{n-1} \leq$ dim $D_n^{(k)} \leq$ dim $D_n^{(2)}$;
- as dim $D_n^{(2)} =$ dim $D_{n-1} = n+1$;
- so dim $D_n^{(k)} = n+1$, $n > 2$, k $(2 \leq k \leq 2^n)$.

2.2. The second new type of tree

$RD_n^{(2)}$ is obtained from the complete binary tree D_n by adding two edges to its root.

Theorem 2. *$RD_n^{(2)}$ is embeddable into Q_{n+2} and dim $RD_n^{(2)} = n+2$, $n \geq 2$.*

Note that dim $RD_2^{(2)} = 4$.

Proof. By induction on n.

For $n = 2$.

A C_4-valuation is given in Figure 6. □

A C_{n+2}- valuation of $RD_n^{(2)}$ can be constructed as follows, see Figure 7.

Remark 3. We keep the same valuation of $RD_{n-1}^{(2)}$ for the subgraphs D_{n-1} and show that for any path of $RD_n^{(2)}$, there exists an integer k for which an odd number of edges are assigned k.

- If P belongs to the subgraph in Figure 8:
  There exists an integer $k \in \{1, ..., n+2\}$ such that the propriety is verified because this graph is a B_{n+1}.
- If P belongs to the subgraph of Figure 9:
  So $P = P_1 \cup \mathrm{ac} \cup \mathrm{ce} \cup \mathrm{ef}$ or $P_2 \cup \mathrm{bc} \cup \mathrm{ce} \cup \mathrm{ef}$ with P_1, $P_2 \in D_{n-2}$.

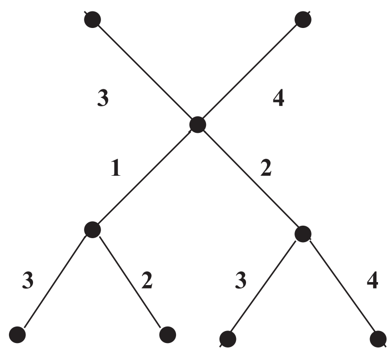

FIGURE 6. A C_4-valuation of $RD_2^{(2)}$.

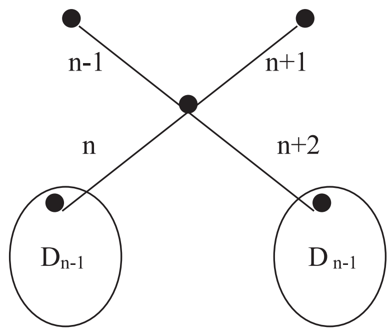

FIGURE 7. A C_{n+2}-valuation of $RD_n^{(2)}$.

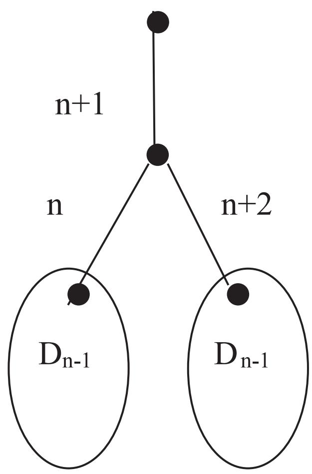

FIGURE 8. A subgraph of $RD_n^{(2)}$.

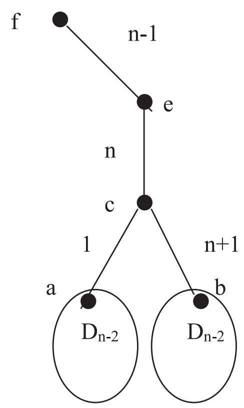

FIGURE 9. A subgraph of $RD_n^{(2)}$.

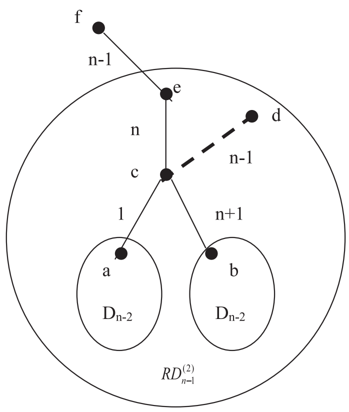

FIGURE 10. A C_{n+1} valuation of $RD_{n-1}^{(2)}$.

Note that the C_{n+1}-valuation of $RD_{n-1}^{(2)}$ is defined as follows (see Fig. 10) then only one case is posed:

If for the path $P_1 \cup$ ac $\cup$ cd or $P_2 \cup$ bc $\cup$ cd $k = n - 1$ so for the path $P_1 \cup$ ac $\cup$ ce $\cup$ ef or $P_2 \cup$ bc $\cup$ ce $\cup$ ef the k may have the value $n - 1$.

Then for the path P of $RD_n^{(2)}$ there always exists an odd number of edges assigned k, so dim $RD_n^{(2)} = n + 2n \geq 2$.

2.3. Conclusion

In this paper, we presented two types of tree that are embeddable in a hypercube. The special characteristics are given and a analytical results provided.

The question we pose is how many edges can de added to the root of D_n such that the graph obtained will be embeddable into Q_{n+2}?

Table 1. The cardinality of levels of T_1.

vertex x/level	$\|N_0(x)\|$	$\|N_1(x)\|$	$\|N_2(x)\|$	$\|N_3(x)\|$	$\|N_4(x)\|$	$\|N_5(x)\|$	$\|N_6(x)\|$
1	1	2	4	7	0	0	0
2	1	3	5	2	3	0	0
3	1	3	4	2	4	0	0
4	1	3	2	3	2	3	0
5	1	3	2	3	2	3	0
6	1	3	2	2	2	4	0
7	1	2	3	2	2	4	0
8	1	1	1	2	3	2	3
9	1	1	1	2	3	2	3
10	1	1	1	2	3	2	3
11	1	1	1	2	3	2	1
12	1	1	2	2	2	2	4
13	1	1	2	2	2	2	4
14	1	1	1	2	3	2	4

Appendix

T_1 is the tree obtained from D_3 by removing one endvertex.

The cardinality of levels obtained from the decomposition of T_1 from the vertex x such $x \in \{1, 2, 3, ..., 14\}$ is given in Table 1.

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