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ABOUT THE CUMULATIVE IDLE TIME IN MULTIPHASE QUEUES

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Abstract. The paper is designated to the analysis of queueing systems, arising in the network theory and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). Also we note that multiphase queueing systems can be useful for modelling practical multi-stage service systems in a variety of disciplines, especially on manufacturing (assembly lines), computer networking (packet switch structures), and in telecommunications (*e.g.* cellular mobile networks), etc. This research presents heavy traffic limit theorems for the cumulative idle time in multiphase queues. In this work, functional limit theorems are proved for the values of important probability characteristics of the queueing system (a cumulative idle time of a customer).

Keywords. Queueing systems, multiphase queues, functional limit theorems, heavy traffic, a cumulative idle time of a customer.

1. Introduction

The modern queueing theory is one of the powerful tools for a quantitative and qualitative analysis of communication systems, computer networks, transportation systems, and many other technical systems. The paper is designated to the analysis of queueing systems, arising in the network theory and communications theory (called multiphase queueing systems, tandem queues or series of queueing systems). Also we note that multiphase queueing systems can be useful for modelling practical multi-stage service systems in a variety of disciplines, especially in

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manufacturing (assembly lines), computer networking (packet switch structures), and in telecommunications (*e.g.* cellular mobile networks), etc.

First, the references given below for applications of such queueing models to computer systems and communication networks can be cited. Particularly we note that a survey of applications of diffussion approximations to computer systems is presented in [8] (especially Chap. 4) and [9]. In [10], several diffussion approximations for the analysis of a packet-switching network and for the model of an interactive computer system are given. The behaviour of the paging drum or a computer hard disc has a significant impact on multiprogrammed and paging systems. A study on applying the diffussion approximations approach to the performance evaluation of such devices is presented in [1]. In [5], a survey of exact results on the networks of queues which are applicable to the evaluation of the performance of complex computer systems is given. In [6, 7], two important charakteristics of user of Quality of Service (QoS) in ATM (Asynchronous Transfer Mode) technology (call admission control procedure and cell loss estimate) are investigated. We can also mention several papers in this research trend (see, for example, [16–18]).

Finally, we try to present a survey of theoretical works for a cumulative idle time of a customer. In one of the first papers of this kind [22], numerical methods are used to study values of the mean of the cumulative idle time in single-server queues. In [27], limit theorems for the cumulative idle time in the systems $GI/G/1$ and $M/G/1$ were obtained. In [19], expressions for the cumulative idle time of a server are presented. In [26], the Laplace transform of the distribution of the cumulative idle time in a finite time interval for the $GI/G/1$ system is found. In [14], the author considers the Laplace transform of the expected cumulative idle time in an $M/G/1$ queue. In [24], the moderate-deviation behaviour of the cumulative idle time with single-server queues is investigated. These results complement the existing results on the heavy traffic behaviour of this process. In [28], the author established functional central limit theorems for a cumulative idle time process in a fluid queue. These limit processes have discontinuous sample paths (*e.g.*, a non-Brownian stable process, or a more general Levy process).

Thus, in this work, we investigate a cumulative idle time of a customer under heavy traffic in multiphase queueing systems. The functional limit theorems for a cumulative idle time of a customer under various heavy traffic conditions in multiphase queueing systems have been proved.

The natural setting for limit theorems in this paper is the weak convergence of probability measures on the function space $D[0,1](\equiv D)$. Since an excellent treatment of this subject has been recently published on [3], we shall only make a few remarks here about our terminology and notation. The stochastic processes characterizing the queueing system give rise to sequences of random functions in D, the space of all right-continuous functions on $[0,1]$ having left limits and endowed with a Skorohod metric, d. In [3], this metric is denoted by d_0. Together with d, D is a complete, separable metric space. Let $\mathcal{D}$ be the class of Borel sets of D.

Then, if P_n and P are probability measures on $\mathcal{D}$ which satisfy

$$\lim_{n\to\infty}\int\limits_D f\mathrm{d}P_n = \int\limits_D f\mathrm{d}P$$

for every bounded, continuous, real-valued function f on D, we shall say that P_n *weakly converges* to P, as $n \to \infty$, and write $P_n \Rightarrow P$. A *random function* X is a measurable mapping from some probability space $(\Omega, \mathcal{B}, \mathcal{P})$ into D having *the distribution* $P = \mathcal{P}X^{-1}$ on $(D, \mathcal{D})$. We say a sequence of random functions $\{X_n\}$ weakly converges to the random function X, and write $X_n \Rightarrow X$ if the distribution P_n of X_n converges to the distribution P of X. A sequence of random functions $\{X_n\}$ converges to X in probability if X_n and X are defined on a common domain and, for all $\varepsilon > 0$, $\lim\limits_{n\to\infty} P\{d(X_n, X) \geq \varepsilon\} = 0$. When X is a constant function (not random), the convergence in probability is equivalent to the weak convergence. In such cases, we write $d(X_n, X) \Rightarrow 0$ or $X_n \Rightarrow X$. If X_n and Y_n have a common domain, we write $d(X_n, Y_n) \Rightarrow 0$, when for all $\varepsilon > 0$, $\lim\limits_{n\to\infty} P\{d(X_n, Y_n) \geq \varepsilon\} = 0$. We also use the uniform metric ρ which is defined by $\rho(x, y) = \sup\limits_{0\leq t\leq 1} |x(t) - y(t)|$ for $x, y \in D$. Also, note that $d(x, y) \leq \rho(x, y)$ for $x, y \in D$.

Next we state two theorems extremely useful for obtaining the weak convergence results in applications. The first one has come to be known as the "converging together theorem". For it we assume that X_n and Y_n are defined on a common domain and take values in a separable metric space (S, m). This result can be found in [3], Theorem 4.1.

Theorem 1.1.

$$\textit{If} \quad X_n \Rightarrow X \quad \textit{and} \quad d(X_n, Y_n) \Rightarrow 0, \quad \textit{then} \quad Y_n \Rightarrow X. \tag{1}$$

Now, suppose h is a measurable mapping of S into S', a second metric space with Borel sets $\mathcal{B}$. Each probability measure P on $(S, \mathcal{B})$ induces a unique probability measure $Ph^{-1}(A) = P(h^{-1}A)$ for $A \in \mathcal{B}'$ on $(S', \mathcal{B}')$. Let D_h be the set of discontinuities of h. The next result, known as the continuous mapping theorem, is an analog of the Mann-Wald theorem for Euclidean spaces (see [3], Th. 5.1). Define $h \circ X = h(X), X \in D$.

Theorem 1.2.

$$\textit{If} \quad X_n \Rightarrow X \quad \textit{and} \quad P\{X \in D_h\} = 0, \quad \textit{then} \quad h \circ X_n \Rightarrow h \circ X. \tag{2}$$

In practice we use this result as follows. First we show $X_n \Rightarrow X$, often by just quoting the known results. Then we find an appropriate mapping h which gives us the random elements we are really interested in, $h \circ X_n$, and finally apply (2).

In this paper, functional limit theorems are proved for values of the important probability characteristic of the multiphase queueing systems – a cumulative idle time of a customer. The main tool for the analysis of a multiphase queue in heavy traffic is a functional limit theorem for sums of independent identically distributed random variables (the proof can be found in [3]).

2. Statement of the problem

We investigate here a k-phase multiphase queue (*i.e.*, when a customer has been served in the j-th phase of the multiphase queue, he goes to the $j+1$-th phase of the multiphase queue, and after the customer has been served in the k-th phase of the multiphase queue, he leaves the multiphase queue). Let us denote t_n as the time of arrival of the n-th customer, $S_n^{(j)}$ as the service time of the n-th customer at the j-th phase of the multiphase queue, $z_n = t_{n+1} - t_n$; $\tau_{j,n}$ as the departure of the n-th customer after service in the j-th phase of the multiphase queue, $j = 1, 2, \ldots, k$.

Let interarrival times (z_n) at the multiphase queue and service times $(S_n^{(j)})$ in every phase of the multiphase queue for $j = 1, 2, \ldots, k$ be mutually independent identically distributed random variables.

Next, denote by $BI_{j,n}$ the idle time of the n-th customer in the j-th phase of the multiphase queue; $I_{j,n} = \sum_{l=1}^{n} BI_{j,l}$ stands for a cummulative idle time in the n-th phase of the multiphase queue, $j = 1, 2, \ldots, k$, $n \geq k$ (see, for example, [23]).

Suppose that the idle time of a customer in each phase of the multiphase queue is unlimited, the service principle of customers is "first come, first served". All random variables are defined on the commmon probability space (Ω, F, P).

We form such a modified multiphase queue in which $BI_{j,n} = 0$, $j = 1, 2, \ldots, k$, $n < k$. Limit distributions for a modified multiphase queue and the usual multiphase queue which, working in heavy traffic conditions, are coincidental (see, for example, [12]). Thus, later we can investigate only the modified multiphase queue and admit that $n \geq k$.

When $j = 1, 2, \ldots, k$, let

$$\delta_{j,n} = \begin{cases} S_{n-(j-1)}^{(j)} - z_n, & \text{if} \quad n \geq k \\ 0, & \text{if} \quad n < k. \end{cases}$$

Denote $S_{j,n} = \sum_{l=1}^{n-1} \delta_{j,l}$, $S_{0,n} \equiv 0$, $\hat{S}_{j,n} = S_{j-1,n} - S_{j,n}$, $x_{j,n} = \tau_{j,n} - t_n$, $x_{0,n} \equiv 0$, $\hat{x}_{j,n+1} = x_{j,n} - \delta_{j,n+1}$, $\hat{x}_{0,n} \equiv 0$, $\alpha_j = M(z_n - S_n^{(j)})$, $y_{j,n} = \hat{x}_{j,n} - S_{j,n}$, $\hat{\delta}_n = \max_{1 \leq j \leq k} \max_{0 \leq l \leq 2n} |\delta_{j,l}|$, $\alpha_j = M\delta_{j,1}$, $\alpha_0 \equiv 0$, $Dz_n = \hat{\sigma}_0^2 > 0$, $DS_n^{(j)} = \hat{\sigma}_j^2 > 0$, $\sigma_j^2 = \hat{\sigma}_j^2 + \hat{\sigma}_0^2$, $j = 1, 2, \ldots, k$, $[x]$ as the integer part of the number.

Let $S_{j,0} = 0$, $j = 1, 2, \ldots, k$.

Let us consider, as in [20], a sequence of multiphase queues: $S_{m,n}^{(j)}$ are independent identically distributed random variables in the n-th multiphase queue, $j = 0, 1, 2, \ldots, k$, $S_{m,n}^{(0)} = z_{m,n}$, $m \geq 1$, $n \geq 1$.

Define $G_{j,n}(x) = \mathbf{P}(S_{1,n}^{(j)} < x),\ j = 0, 1, 2, \ldots, k$.
Let

$$\mathbf{D}S_{1,n}^{(j)} \to \hat{\sigma}_j^2 > 0,\ j = 0, 1, 2, \ldots, k. \tag{3}$$

Also let $S_{m,n}^{(j)}$ satisfy the Lindeberg condition: for each $\varepsilon > 0$

$$\lim_{n\to\infty} \int\limits_{|x| > \varepsilon\cdot\sqrt{n}} |x|^2 \mathrm{d}G_{j,n}(x) = 0,\ j = 0, 1, 2, \ldots, k. \tag{4}$$

For simplicity, we omit the index n in the sequel.

Denote families of random functions in D as

$$I_j^n(t) = \frac{I_{j,[nt]}}{\sqrt{n}},\ y_j^n(t) = \frac{y_{j,[nt]}}{\sqrt{n}},\ j = 1, 2, \ldots, k,\ \ 0 \le t \le 1.$$

We see that if conditions (4) are fulfilled, then for each fixed $\varepsilon > 0$, (see [21])

$$\lim_{n\to\infty} P\left(\rho\left(I_j^n, y_j^n\right) > \varepsilon\right) = 0,\ \ j = 1, 2, \ldots, k. \tag{5}$$

3. Main results

Let us investigate a heavy traffic case, where

$$(\alpha_{j-1} - \alpha_j) \cdot \sqrt{n} \to -\infty,\ j = 1, 2, \ldots, k. \tag{6}$$

We prove such a functional limit theorem.

Theorem 3.1. *If conditions (3), (4), and (6) are fulfilled, then*

$$\left(\frac{I_{1,[nt]} - \alpha_1 \cdot [nt]}{\sqrt{n}}; \frac{I_{2,[nt]} - \alpha_2 \cdot [nt]}{\sqrt{n}}; \ldots; \frac{I_{k,[nt]} - \alpha_k \cdot [nt]}{\sqrt{n}}\right) \Rightarrow$$
$$\left(\sigma_1 \cdot z_1\left(t\right); \sigma_2 \cdot z_2\left(t\right); \ldots; \sigma_k \cdot z_k\left(t\right)\right),$$

where $z_j(t)$ are independent standard Wiener processes, $j = 1, 2, \ldots, k,\ \ 0 \le t \le 1$.

Proof. Note that it suffices to prove that (see (1) and (5))

$$\frac{y_{j,[nt]} - \alpha_j \cdot [nt]}{\sqrt{n}} \Rightarrow \sigma_j \cdot z_j(t), \tag{7}$$

where $z_j(t)$ are independent standard Wiener processes, $j = 1, 2, \ldots, k,\ \ 0 \le t \le 1$.

First we prove that

$$\frac{y_{j,n} - \left\{\sum\limits_{i=1}^{j} \hat{S}_{i,n}\right\}}{\sqrt{n}} = \frac{y_{j,n} - (-S_{j,n})}{\sqrt{n}} \Rightarrow 0,\ j = 1, 2, \ldots, k. \tag{8}$$

Let us investigate the case where $j = 1$. Thus,

$$y_{1,n} - \hat{S}_{1,n} = \max_{0 \le l \le n} (\hat{S}_{1,l}) - \hat{S}_{1,n}. \tag{9}$$

But, for every $\varepsilon > 0$,

$$P\left(\frac{\max_{0 \le l \le n} (\hat{S}_{1,l}) - \hat{S}_{1,n}}{\sqrt{n}} > \varepsilon \right) = P\left(\frac{\max_{0 \le l \le n} (-\hat{S}_{1,l})}{\sqrt{n}} > \varepsilon \right). \tag{10}$$

Similarly as in [11] and using conditions (6), we find that

$$\frac{\max_{0 \le l \le n} (-\hat{S}_{j,l})}{\sqrt{n}} \Rightarrow 0, \ j = 1, 2, \ldots, k. \tag{11}$$

From this, (9), and (10) we obtain that

$$\frac{y_{1,n} - \hat{S}_{1,n}}{\sqrt{n}} = \frac{\hat{y}_{1,n} - (-S_{1,n})}{\sqrt{n}} \Rightarrow 0. \tag{12}$$

Let us investigate the case where $j = 2$.

Then,

$$\begin{aligned}
y_{2,n} - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) &= \max_{0 \le l \le n} \left(y_{1,l} + \hat{S}_{2,l}\right) - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \\
&= \max_{0 \le l \le n} \left(y_{1,l} - \hat{S}_{1,l} + \left(\hat{S}_{1,l} + \hat{S}_{2,l}\right)\right) - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \\
&\ge \max_{0 \le l \le n} \left(\min_{0 \le m \le n} \left(y_{1,m} - \hat{S}_{1,m}\right) + \left(\hat{S}_{1,l} + \hat{S}_{2,l}\right)\right) - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \\
&= \min_{0 \le l \le n} \left(y_{1,l} - \hat{S}_{1,l}\right) + \left\{ \max_{0 \le l \le n} \left(\hat{S}_{1,l} + \hat{S}_{2,l}\right) - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \right\}.
\end{aligned} \tag{13}$$

On the another hand, similarly as in (13), we find that

$$\begin{aligned}
y_{2,n} - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \le \max_{0 \le l \le n} &\left(y_{1,l} - \hat{S}_{1,l}\right) \\
&+ \left\{ \max_{0 \le l \le n} \left(\hat{S}_{1,l} + \hat{S}_{2,l}\right)) - \left(\hat{S}_{1,n} + \hat{S}_{2,n}\right) \right\}.
\end{aligned} \tag{14}$$

However, if conditions (6) are fulfilled and applying (10) and (11), we get that

$$\begin{aligned} 0 &\le \frac{\max\limits_{0\le l\le n}\left(\hat{S}_{1,l}+\hat{S}_{2,l}\right)-\left(\hat{S}_{1,n}+\hat{S}_{2,n}\right)}{\sqrt{n}} \\ &\le \frac{\left\{\max\limits_{0\le l\le n}\hat{S}_{1,l}-\hat{S}_{1,n}\right\}+\left\{\max\limits_{0\le l\le n}\hat{S}_{2,l}-\hat{S}_{2,n}\right\}}{\sqrt{n}} \Rightarrow 0. \end{aligned} \tag{15}$$

Applying (2) to the minimum and maximum function in the case $j=1$ (see (12)), we have that

$$\frac{\max\limits_{0\le l\le n}\left(y_{1,l}-\hat{S}_{1,l}\right)}{\sqrt{n}} \Rightarrow 0$$

and

$$\frac{\min\limits_{0\le l\le n}\left(y_{1,l}-\hat{S}_{1,l}\right)}{\sqrt{n}} \Rightarrow 0.$$

From this and (13)–(15), we obtain that

$$\frac{y_{2,n}-\left(\hat{S}_{1,n}+\hat{S}_{2,n}\right)}{\sqrt{n}} \Rightarrow 0. \tag{16}$$

Let (8) be true for every $2<j\le k-1$. We prove that it is true for $j,\ j=3,\dots,k$. Analogously as in (13) and (14) we achieve that, for $j=3,\dots,k$,

$$\begin{aligned} y_{j,n}-\left(\sum_{i=1}^{j}\hat{S}_{i,n}\right) &\ge \min_{0\le l\le n}\left(y_{j-1,l}-\left\{\sum_{i=1}^{j-1}\hat{S}_{i,l}\right\}\right) \\ &\quad+\left(\max_{0\le l\le n}\left(\sum_{i=1}^{j}\hat{S}_{i,l}\right)-\left(\sum_{i=1}^{j}\hat{S}_{i,n}\right)\right) \end{aligned} \tag{17}$$

and

$$\begin{aligned} y_{j,n}-\left(\sum_{i=1}^{j}\hat{S}_{i,n}\right) &\le \max_{0\le l\le n}\left(y_{j-1,l}-\left\{\sum_{i=1}^{j-1}\hat{S}_{i,l}\right\}\right) \\ &\quad+\left(\max_{0\le l\le n}\left(\sum_{i=1}^{j}\hat{S}_{i,l}\right)-\left(\sum_{i=1}^{j}\hat{S}_{i,n}\right)\right). \end{aligned} \tag{18}$$

Analogously as in (15), we see that

$$0 \le \max_{0\le l\le n}\left(\sum_{i=1}^{j}\hat{S}_{i,l}\right)-\left(\sum_{i=1}^{j}\hat{S}_{i,n}\right)\le\sum_{i=1}^{j}\left(\max_{0\le l\le n}\hat{S}_{i,l}-\hat{S}_{i,n}\right)$$
$$\le\sum_{i=1}^{k}\left(\max_{0\le l\le n}\hat{S}_{i,l}-\hat{S}_{i,n}\right). \tag{19}$$

Using (17)–(19), the proof is similar as in (16). So, we prove (8). But

$$\frac{-S_{j,[nt]}-\alpha_{j,[nt]}}{\sqrt{n}} \Rightarrow \sigma_j \cdot z_j(t), \tag{20}$$

where $z_j(t)$ are independent standard Wiener processes, $j=1,2,\dots,k,\ \ 0\le t\le 1$.
From this and applying (1), we prove (7). The proof is complete. □

Next let us investigate the heavy traffic case, where

$$(\alpha_{j-1}-\alpha_j)\cdot\sqrt{n}\to+\infty,\ \ j=1,2,\dots,k. \tag{21}$$

We prove such a functional limit theorem.

Theorem 3.2. *If conditions (3), (4), and (21) are fulfilled, then*

$$\left(\frac{I_{1,[nt]}}{\sqrt{n}};\frac{I_{2,[nt]}}{\sqrt{n}};\dots;\frac{I_{k,[nt]}}{\sqrt{n}}\right)\Rightarrow(0;0;\dots;0),\ 0\le t\le 1.$$

Proof. In this paper, we mostly use the equations which are presented in [20]:

$$\hat{x}_{j,n}=\max_{0\le l\le n}(\hat{x}_{j-1,l}-S_{j,l})+S_{j,n},\ \ \hat{x}_{0,n}\equiv 0,$$
$$x_{j,n}=\max(x_{j-1,n-1}+\delta_{j,n};\ x_{j,n-1}+\delta_{j,n}),\ x_{0,n}\equiv 0,$$
$$x_{j,n+1}=\max_{0\le l_1<l_2<\dots<l_j\le n}\left(\sum_{m=l_1}^{l_2-1}\delta_{1,m}+\sum_{m=l_2}^{l_3-1}\delta_{2,m}+\dots+\sum_{m=l_j}^{n}\delta_{j,m}\right),$$
$$x_{j,n}=\max_{0\le l\le n-1}(x_{j-1,l}-S_{j,l})+S_{j,n-1},\ j=1,2,\dots,k.$$

From this we obtain that

$$y_{j,n}=\max_{0\le l\le n}\left(y_{j-1,l}+\hat{S}_{j,l}\right)\le\max_{0\le l\le n}y_{j-1,l}+\max_{0\le l\le n}\hat{S}_{j,l}=y_{j-1,n}+\max_{0\le l\le n}\hat{S}_{j,l}\le\cdots$$
$$\le\sum_{i=1}^{j}\left\{\max_{0\le l\le n}\hat{S}_{i,l}\right\}\le\sum_{i=1}^{k}\left\{\max_{0\le l\le n}\hat{S}_{i,l}\right\},\ j=1,2,\dots,k. \tag{22}$$

On the other hand, we get that

$$y_{0,n} = \hat{x}_{0,n} - S_{0,n} \equiv 0 \quad \text{and} \quad y_{j,n} \geq \max_{0\leq l\leq n} \hat{S}_{j,l} \geq 0,\ j = 1,2,\ldots,k.$$

From this and (22), we achieve that

$$|y_{j,n}| \leq \sum_{i=1}^{k} \{\max_{0\leq l\leq n} \hat{S}_{i,l}\},\ j = 1,2,\ldots,k. \tag{23}$$

Thus, we have (see (23)) that, for every fixed $\varepsilon > 0$,

$$P\left(\frac{\sup\limits_{0\leq t\leq 1} |y_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right) \leq P\left(\frac{\max\limits_{0\leq l\leq n} |y_{j,l}|}{\sqrt{n}} > \varepsilon\right) \leq$$

$$P\left(\frac{\max\limits_{0\leq l\leq n}\left(\sum\limits_{i=1}^{k}\left\{\max\limits_{0\leq m\leq l} \hat{S}_{i,m}\right\}\right)}{\sqrt{n}} > \varepsilon\right) \leq P\left(\frac{\sum\limits_{i=1}^{k}\left\{\max\limits_{0\leq l\leq n}\max\limits_{0\leq m\leq l} \hat{S}_{i,m}\right\}}{\sqrt{n}} > \varepsilon\right) \leq$$

$$P\left(\frac{\sum\limits_{i=1}^{k}\left\{\max\limits_{0\leq l\leq n} \hat{S}_{i,l}\right\}}{\sqrt{n}} > \varepsilon\right) \leq \sum_{i=1}^{k} P\left(\frac{\max\limits_{0\leq l\leq n} \hat{S}_{i,l}}{\sqrt{n}} > \frac{\varepsilon}{k}\right),\ j = 1,2,\ldots,k. \tag{24}$$

But, if conditions (21) are fulfilled, then (see [11])

$$\frac{\max\limits_{0\leq l\leq n} \hat{S}_{j,l}}{\sqrt{n}} \Rightarrow 0,\ j = 1,2,\ldots,k. \tag{25}$$

From this and (24), we prove that, for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\frac{\sup\limits_{0\leq t\leq 1} |y_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right) = 0,\ j = 1,2,\ldots,k. \tag{26}$$

Thus, for every fixed $\varepsilon > 0$,

$$P\left(\frac{\sup\limits_{0\leq t\leq 1} |I_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right) \leq P\left(\frac{\sup\limits_{0\leq t\leq 1} |I_{j,[nt]} - y_{j,[nt]}|}{\sqrt{n}} > \frac{\varepsilon}{2}\right)$$

$$+ P\left(\frac{\sup\limits_{0\leq t\leq 1} |y_{j,[nt]}|}{\sqrt{n}} > \frac{\varepsilon}{2}\right),\ j = 1,2,\ldots,k. \tag{27}$$

From (27), (5), and (26) we find that, for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} P\left(\frac{\sup\limits_{0\le t\le 1} |I_{j,[nt]}|}{\sqrt{n}} > \varepsilon\right) = 0,\ j = 1, 2, \ldots, k. \tag{28}$$

The proof is complete (see (28) and (1)).

□

4. Concluding remarks

1. We note that $\hat{x}_{j,n}, j = 1, 2, \ldots, k$ is an analog of the classical Lindley equation in the multiphase queue case and represents the waiting time in phases of the multiphase queue (see [11]). But $y_{j,n}, j = 1, 2, \ldots, k$ is a dual Lindley equation with respect to $\hat{x}_{j,n}, j = 1, 2, \ldots, k$ and represents a cumulative idle time in the multiphase queue (see [2]).

2. Diffusion approximations are continuous approximations to discontinuous arrival and service processes in the queueing model. Previously two different approaches to diffusion approximations for queueing models have been proposed. In both cases, the queueing length or the waiting time distribution is approximated by solving a partial differential eguation. However the two methods differ according to the choice of boundary conditions. The simpler one uses reflecting boundaries so that no probability mass accumulates at the boundaries (see, for example, this paper, [11], [13], etc.). A more sophisticated approach is based on the instantaneous return process (see [4, 10]), which combines the partial differential equation formulation for the process strictly inside the boundaries, with a discrete state-space model at the boundaries themselves. In some cases of queueing systems this approach is more accurate than the classical method (see [15, 16, 25]). Of course, the instantaneous return process is an important idea. Extension of the idle process investigated in this paper to the instantaneous return process is an important future research problem.

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