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ASYMPTOTIC ANALYSIS OF THE TRAJECTORIES OF THE LOGARITHMIC BARRIER ALGORITHM WITHOUT CONSTRAINT QUALIFICATIONS*

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Abstract. In this paper, we study the differentiability of the trajectories of the logarithmic barrier algorithm for a nonlinear program when the set Λ^* of the Karush-Kuhn-Tucker multiplier vectors is empty owing to the fact that the constraint qualifications are not satisfied.

Keywords. Logarithmic barrier, Penalty algorithms.

Mathematics Subject Classification. 90C30.

Introduction

In this paper, we analyze the differentiability of the trajectories of the logarithmic barrier algorithm for a nonlinear program when the constraint qualifications are not satisfied at the optimal solution. During our analysis, we establish sufficient conditions, which make it possible to conclude that the trajectory is differentiable.

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Consider the problem

$$\begin{cases} \min f(x) \\ \text{s.t.} \quad g_i(x) \leq 0, \ i \in I = \{1, 2, .., m\} \end{cases} \tag{1}$$

as well as the associated penalized subproblems

$$\phi(r, x) = f(x) - r \sum_{i=1}^{m} \log(-g_i(x)). \tag{2}$$

Fiacco and McCormick [5,6] showed, under some hypotheses, the existence of a differentiable function $x(r)$ in the neighborhood of $r = 0$ and such that $\lim\limits_{r \longrightarrow 0} x(r) = x^*$, where x^* is an optimal solution of the problem (1). Moreover, they showed that $x(r)$ is a strict local minimum of problem (2). Mifflin [9] showed that without the constraints qualification, any cluster point x^* satisfy the Fritz John [7] conditions.

The differentiability of the trajectories of the logarithmic barrier algorithm can be shown in two ways, either by using the Primal-Dual approach or Primal approach. These two approaches are respectively the subjects of the two following subsections.

0.1. Primal-Dual formulation

The first results of Fiacco and McCormick [5,6] concerning the differentiability properties of the trajectory $x(r)$ were obtained by using the implicit functions theorem to show that the following system:

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\ \lambda_i + \frac{r}{g_i(x)} = 0 \quad i \in J^* = \{i \in I \quad | \quad g_i(x^*) < 0\} \\ g_i(x) + \frac{r}{\lambda_i} = 0 \quad i \in I^* = \{i \in I \quad | \quad g_i(x^*) = 0\}, \end{cases}$$

implicitly defines $\mathcal{C}^1$ functions $(x(r), \lambda(r))$ in the neighborhood of $r = 0$.

Theorem 0.1 [5]**.** *If the sufficient optimality conditions for problem (1) hold at (x^*, λ^*); and moreover if:*

- *x^* is a regular point: $\{\nabla g_i(x^*) \quad | \quad i \in I^*\}$ is linearly independent;*
- *the strict complementarity is verified: $g_i(x^*) < 0 \iff \lambda_i^* = 0$ for $1 \leq i \leq m$,*

then the system

$$\begin{cases} \nabla_x \phi(r, x) = \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0 \\ \lambda_i + \dfrac{-r}{g_i(x)} = 0, \quad 1 \leq i \leq m \end{cases} \tag{3}$$

is equivalent to

$$\nabla_x \phi(r,x) = \nabla f(x) - r\sum_{i=1}^{m} \frac{\nabla g_i(x)}{g_i(x)} = 0 \tag{4}$$

and defines in a neighborhood V_0 *of* $r=0$ *isolated functions* $x(r)$ *and* $\lambda_i(r) = \dfrac{-r}{g_i(x(r))}$, $1 \le i \le m$ *of class* $\mathcal{C}^1$ *in* V_0 *such that when* $r \searrow 0$*:*

$$\begin{cases} x(r) \to x^*, \\ \lambda_i(r) \to \lambda_i^*. \end{cases}$$

Moreover $x(r)$ *is a strict local minimum of* $\phi(r,x)$*.*

0.2. Primal formulation

In this section, we suppose that the hypotheses of Theorem 0.1 are satisfied and we show that the preceding result can be obtained without the dual variables λ.

Since $x(r)$ is a solution of problem (2) then

$$\Phi(r,x(r)) = \nabla_x \phi(r,x(r)) = 0.$$

Thus, if the derivative $x'(r)$ exists then it must satisfy the following system:

$$\nabla_x \Phi(r,x(r))x'(r) + \nabla_r \Phi(r,x(r)) = 0,$$

where $\nabla_r \Phi(r,x(r))$ is the derivative of $\Phi(r,x(r))$ with respect to r, or that

$$\nabla_x \Phi(r,x(r))x'(r) = -\nabla_r \Phi(r,x(r)).$$

With an orthogonal transformation Q satisfying the relation

$$Q^T_{x(r)} \nabla g(x(r)) = \left(\begin{array}{c|c} U & R \\ \hline 0_{(n-m^*)\times m^*} & S \end{array} \right),$$

where U is a non singular upper triangular matrix and $m^* = |I^*|$, we can write

$$Q^T_{x(r)} \nabla_x \Phi(r,x(r)) Q_{x(r)} Q^T_{x(r)} x'(r) = -Q^T_{x(r)} \nabla_r \Phi(r,x(r)). \tag{5}$$

We introduce the notation

$$y(r) = Q^T_{x(r)} x'(r) \tag{6}$$

and

$$G = Q^T_{x(r)} (\nabla_x^2 f(x(r)) + \sum_{i=1}^{m} \lambda_i(r) \nabla_x^2 g_i(x(r))) Q_{x(r)}. \tag{7}$$

Then, by relations (6) and (7), relation (5) can be rewritten as

$$(G + \frac{1}{r} \begin{pmatrix} U & R \\ 0 & S \end{pmatrix} \begin{pmatrix} V_{I^*} & 0 \\ 0 & V_{J^*} \end{pmatrix} \begin{pmatrix} U & R \\ 0 & S \end{pmatrix}^T) y(r) \quad = \quad \frac{-1}{r} Q^T_{x(r)} \nabla g(x(r)) \lambda(r)$$

or

$$\begin{pmatrix} G_{11} + \frac{1}{r}(UV_{I^*}U^T + RV_{J^*}R^T) & G_{12} + \frac{1}{r}RV_{J^*}S^T \\ G_{21} + \frac{1}{r}SV_{J^*}R^T & G_{22} + \frac{1}{r}SV_{J^*}S^T \end{pmatrix} y(r) =$$

$$\begin{pmatrix} \frac{-1}{r}(U\lambda_{I^*}(r) + R\lambda_{J^*}(r)) \\ \frac{-1}{r}S\lambda_{J^*}(r) \end{pmatrix}, \quad (8)$$

where V_{I^*} and V_{J^*} are diagonal matrices with components λ_i^2 associated with respectively the active and nonactive constraints.

To show the differentiability of the trajectories generated by the logarithmic barrier algorithm, it is sufficient to show that system (8) possesses a solution at $r = 0$. However, the $\frac{1}{r}$ terms diverge when $r \longrightarrow 0$. Broyden and Attia [3] proposed a technique to solve the problem of the ill conditioning for the quadratic penalty. Dussault [4] generalized this technique so that system (8) is written in the form

$$My(r) = \bar{b},$$

where

$$\begin{cases} M = \begin{pmatrix} rU^{-1}G_{11} + V_{I^*}U^T + U^{-1}RV_{J^*}R^T & rU^{-1}G_{12} + U^{-1}RV_{J^*}S^T \\ G_{21} + \frac{1}{r}SV_{J^*}R^T & G_{22} + \frac{1}{r}SV_{J^*}S^T \end{pmatrix} \\ \bar{b} = \begin{pmatrix} \lambda_{I^*}(r) + U^{-1}R\lambda_{J^*}(r)) \\ \frac{-1}{r}S\lambda_{J^*}(r) \end{pmatrix}. \end{cases}$$

when $r \longrightarrow 0$ this system approaches

$$\begin{pmatrix} V_{I^*}U^T & 0 \\ {G_{21}}^* & {G_{22}}^* \end{pmatrix} y(0) = \begin{pmatrix} \lambda_{I^*}(0) \\ h \end{pmatrix}$$

where

- $V_{I^*}U^T$ is invertible because
  - the hypothesis of the linear independence of $\nabla g_{I^*}(x^*)$ ensures that U is an invertible matrix;
  - the strict complementarity hypothesis ensures that V_{I^*} is an invertible diagonal matrix;
- G_{22}^* is invertible since the second order sufficient condition is satisfied;
- $h = \lim\limits_{r \longrightarrow 0} \frac{-1}{r}S\lambda_{J^*}(r)$ is a bounded vector because $\forall j \in J^*$, $\lim\limits_{r \longrightarrow 0} \frac{-1}{r}\lambda_j(r) = \frac{1}{g_j(x^*)}$.

The developments above prove the following.

Theorem 0.2 [5]**.** *If the sufficient optimality conditions for problem (1) hold at* (x^*, λ^*) *, and moreover if:*

– x^* *is a regular point:* $\{\nabla g_i(x^*) \quad | \quad i \in I^*\}$ *is linearly independent,,*

– *strict complementarity is verified:* $g_i(x^*) < 0 \iff \lambda_i^* = 0$ *for* $1 \leq i \leq m$ *,*

then the system

$$\nabla_x \phi(r, x)) = 0 \tag{9}$$

defines in a neighborhood V_0 *of* $r = 0$ *an isolated function* $x(r)$ *of class* $\mathcal{C}^1$ *in* V_0 *such that when* $r \searrow 0$*:* $x(r) \to x^*$*. Moreover,* $x(r)$ *is a strict local minimum of* $\phi(r, x)$ *.*

We just presented two techniques to prove the differentiability of the logarithmic barrier trajectories under the traditional hypotheses used by Fiacco & McCormick. These hypotheses are

(*i*) the objective function is of class $\mathcal{C}^2$;
(*ii*) the gradients of the active constraints are linearly independent;
(*iii*) strict complementarity is satisfied;
(*iv*) the second order sufficient condition is satisfied;
(*v*) $\forall i \in I$, g_i is a $\mathcal{C}^2$ function.

Let us recall that, when hypotheses (*i*) and (*ii*) are satisfied, the set Λ^* of Karush-Kuhn-Tucker multiplier vectors [8] is nonempty. The study of the case $\Lambda^* = \emptyset$ where hypothesis (*ii*) is satisfied (which can occur only if the objective function is not differentiable at the optimum), is in [1,2] . In this paper, we study the case where $\Lambda^* = \emptyset$ and hypothesis (*i*) is satisfied. That can occur only if no constraint qualification is satisfied. The following example shows us that the trajectory may be differentiable even if no constraint qualifications is satisfied.

Example 0.1.

$$\begin{cases} Min \quad x_1 + x_3 + x_5^2 \\ \qquad x_4 - x_1^5 \leq 0 \\ \qquad x_2 - {x_1}^3 \leq 0 \\ \text{s.t.} \quad -x_3 \leq 0 \\ \qquad -x_4 \leq 0 \\ \qquad -x_2 \leq 0 \\ \qquad x_3 - 1 \leq 0. \end{cases}$$

Notice that the origin is the unique isolated solution, but that the active constraint gradients are linearly dependent there, and that $\Lambda^* = \emptyset$.

The penalized objective functions are written

$$\begin{aligned} \phi(r, x) \quad &= \quad x_1 + x_3 + x_5^2 - r\log(x_1^5 - x_4) - r\log({x_1}^3 - x_2) \\ &\quad -r\log(x_3) - r\log(x_4) - r\log(x_2) - r\log(1 - x_3), \end{aligned}$$

and their gradients are

$$\nabla\phi(r,x) = \begin{pmatrix} 1 - \frac{(3{x_1}^2 r)}{({x_1}^3 - x_2)} - \frac{(5{x_1}^4 r)}{({x_1}^5 - x_4)} \\ \frac{-r}{x_2} + \frac{r}{({x_1}^3 - x_2)} \\ 1 + \frac{-r}{x_3} - (\frac{r}{x_3 - 1}) \\ \frac{-r}{x_4} + \frac{r}{({x_1}^5 - x_4)} \\ 2x_5 \end{pmatrix},$$

where $x = (x_1, x_2, x_3, x_4, x_5)$. The values which cancel $\nabla_x\phi(r,x)$ and the associated $\lambda_i(r)$ quantities are

$$\begin{cases} x_1(r) = 16r, \qquad x_2(r) = 2048r^3 \\ x_3(r) = \frac{1}{2} + r - \frac{1}{2}\sqrt{1+4r^2}, \qquad x_4(r) = 524288r^5, \qquad x_5(r) = 0 \\ \lambda_1(r) = \frac{1}{524288r^4}, \qquad \lambda_2(r) = \frac{1}{2048r^2} \\ \lambda_3(r) = \frac{r}{\frac{1}{2}+r-\frac{1}{2}\sqrt{1+4r^2}} \\ \lambda_4(r) = \frac{1}{524288r^4}, \qquad \lambda_5(r) = \frac{1}{2048r^2} \\ \lambda_6(r) = \frac{r}{\frac{1}{2}-r+\frac{1}{2}\sqrt{1+4r^2}}. \end{cases}$$

Therefore the trajectories, $x(r) = (x_1(r), x_2(r), x_3(r), x_4(r), x_5(r))$, are differentiable at $r = 0$ but $\lambda_1(r)$, $\lambda_2(r)$, $\lambda_4(r)$ and $\lambda_5(r)$ diverge when $r \longrightarrow 0$. Notice that the optimal solution x^* satisfies the Fritz John [7] conditions. □

The paper is organized as follows. In Section 2, we will show some preliminary results. Then, in Section 3 we will give conditions to ensure that the trajectory of the barrier algorithm is differentiable even if $\Lambda^* = \emptyset$. An illustrative example and the proof of lemmas in Section 3.2.3 are presented respectivly in Appendix A2 et A1 and finally some concluding remarks close our paper.

1. Preliminary results

It is possible to weaken hypothesis *ii*), but hypothesis *iii*) is a necessary condition as shown in the following theorem:

Theorem 1.1. *If there is an* $i \in I^*$ *such that* $\lambda_i(r) = \frac{-r}{g_i(x(r))}$ *approaches* $\lambda_i^* = 0$ *when* $r \longrightarrow 0$ *then the trajectory* $x(r)$ *is not differentiable at* $r = 0$.

Proof. Suppose that there is an $i \in I^*$ such that $\lambda_i(r) = \frac{-r}{g_i(x(r))}$ approaches $\lambda_i^* = 0$ when $r \longrightarrow 0$. If $x'(0)$ exists, then $\lim_{r \longrightarrow 0} \frac{x(r)-x(0)}{r} = x'(0)$ is bounded. By the mean value theorem we have that, since $i \in I*$,

$$\frac{g_i(x(r))}{r} = \frac{x(r) - x(0)}{r} \nabla g_i(\xi_r),$$

therefore

$$\lim_{r \longrightarrow 0} \frac{g_i(x(r))}{r} = x'(0) \nabla g_i(\xi^*) < \infty.$$

Thus

$$\lim_{r \longrightarrow 0} \lambda_i(r) = \lim_{r \longrightarrow 0} \frac{-r}{g_i(x(r))} = \frac{-1}{x'(0) \nabla g_i(\xi^*)} \neq 0,$$

which implies that $\lambda_i^* \neq 0$. □

The following example illustrates Theorem 1.1.

Example 1.1.

$$\begin{cases} \quad \min \quad x_1 \\ \text{s.t.} \quad \begin{array}{r} x_2^2 - x_1 \leq 0 \\ -x_2 \leq 0. \end{array} \end{cases}$$

The values which cancel $\nabla \phi(r, x)$ and the associated $\lambda_i(r)$ quantities are

$$\begin{cases} x_1(r) = \frac{3}{2} r \\ x_2(r) = \sqrt{\frac{r}{2}} \\ \lambda_1(r) = 1 \\ \lambda_2(r) = 2\sqrt{r}. \end{cases}$$

The optimal solution is $x^* = (0, 0)$. The second constraint is active and $\lambda_2(r)$ approaches 0 when $r \longrightarrow 0$. According to the preceding theorem, the trajectory is not differentiable. Indeed, we have $x_2(r) = \sqrt{\frac{r}{2}}$.

As previously stated, in this paper we are interested in the case $\Lambda^* = \emptyset$. In this context notice that necessarily some $\lambda_i(r) = \frac{-r}{g_i(x(r))}$ diverge as shown in the following theorem.

Theorem 1.2. *Let* $I^{*d} = \{i \in I^* : \lambda_i(r) \longrightarrow +\infty,$ *when* $\quad r \longrightarrow 0\}$.

(1) *If* $\Lambda^* = \emptyset$ *then* $I^{*d} \neq \emptyset$.
(2) *If* $I^{*d} \neq \emptyset$ *then there is a* $\lambda_0(r)$ *such that:*

$$\begin{cases} \lim_{r \longrightarrow 0} \lambda_0(r) = 0 \\ \lim_{r \longrightarrow 0} \lambda_0(r) \lambda_i(r) = \mu_i^* \\ \textit{where} \quad 0 \leq \mu_i^* < \infty \quad \textit{for} \qquad i = \{1, 2, .., m\}. \end{cases}$$

Proof. (1) Suppose that $I^{*d} = \emptyset$; then we have

$$\phi(r,x) = f(x) - r\sum_{i=1}^{m} \log(-g_i(x))$$

and its gradient

$$\nabla_x \phi(r,x) = \nabla f(x) - \sum_{i=1}^{m} \frac{r}{g_i(x)} \nabla g_i(x).$$

That is to say, if $x(r)$ is a local minimum of $\phi(r,x)$, then

$$\nabla_x \phi(r,x(r)) = \nabla f(x(r)) - \sum_{i=1}^{m} \frac{r}{g_i(x(r))} \nabla g_i(x(r)) = 0.$$

We use the notation $\lambda_i(r) = \frac{-r}{g_i(x(r))}$. Then $\forall i \in I\ \lambda_i(r) \geq 0$ and

$$\nabla f(x(r)) + \nabla g(x(r))^T \lambda(r) = 0.$$

- If $i \notin I^*$ then $\lim\limits_{r \longrightarrow 0} \lambda_i(r) = 0$.
- If $i \in I^*$ then, since $I^{*d} = \emptyset$, $\lim\limits_{r \longrightarrow 0} \lambda_i(r) = \begin{cases} 0 \\ or \\ \lambda_i^* > 0\,. \end{cases}$

Therefore,

$$\lim_{r \longrightarrow 0} \nabla f(x(r)) + \nabla g(x(r))^T \lambda(r) = \nabla f(x^*) + \nabla g(x^*)^T \lambda^* = 0$$

and $\lim\limits_{r \longrightarrow 0} \lambda(r) = \lambda^* \geq 0$, which implies that $\Lambda^* \neq \emptyset$.
(2) It is sufficient to take $\lambda_0(r) = \frac{1}{\max(\lambda_i(r), i \in I^{*d})}$ to have the result. □

Since we are interested in the case $\Lambda * = \emptyset$, the strict complementarity condition is adapted as follows:

Definition 1.1. The generalized strict complementarity condition is satisfied if $\lim\limits_{r \longrightarrow 0} \lambda_i(r) = 0$ implies $\lim\limits_{r \longrightarrow 0} g_i(x(r)) < 0$.

Remark 1.1. When $\Lambda^* \neq \emptyset$, the generalized strict complementarity condition is equivalent with the usual strict complementarity condition.

2. Differentiability of the primal trajectories $x(r)$

In this section, we propose to study the differentiability of the trajectory $x(r)$ even if the constraint qualifications are not satisfied, in particular the matrix

$\nabla g_{I^*}(x^*)$ is not of full rank. Let us recall that

$$\begin{array}{l} I^* = \{i : g_i(x^*) = 0\} \quad m^* = |I^*| \\ J^* = \{i : g_i(x^*) < 0\} \\ \lambda_i(r) = \frac{-r}{g_i(x(r))}. \end{array}$$

Suppose that $rank(\nabla g_{I^*}(x^*)) = k$ where $1 \leq k \leq m^*$ and that

$$\begin{array}{l} K = \{i_1, \cdots, i_k \in I^* \,|\, \nabla g_{i_1}(x^*), \cdots, \nabla g_{i_k}(x^*) \text{ are linearly independent}\}, \quad k = |K| \\ K^d = \{i \in K \,|\, \lim\limits_{r \longrightarrow 0} \lambda_i(r) = +\infty\}, \quad k^d = |K^d| \\ K^c = \{i \in K \,|\, \lim\limits_{r \longrightarrow 0} \lambda_i(r) = \lambda_i^* > 0\}, \quad k^c = |K^c| \\ I^* \backslash K = \{i \in I^* \,|\, \nabla g_i(x^*) \text{ and the columns of } \nabla g_K(x^*) \text{ are linearly dependent}\}, \\ m^* - k = |I^* \backslash K|. \end{array}$$

Remark 2.1. The set K is not unique, it can be chosen.

Remark 2.2. For all $i \in J^*$ we have:

$$\lim_{r \longrightarrow 0} \frac{-r}{g_i(x(r))} = \lim_{r \longrightarrow 0} \lambda_i(r) = 0.$$

Then $\lambda(r)$ can be written:

$$\lambda(r) = \begin{pmatrix} \lambda_{K^d}(r) \\ \lambda_{K^c}(r) \\ \lambda_{I^* \backslash K}(r) \\ \lambda_{J^*}(r) \end{pmatrix}. \tag{10}$$

Let

$$V(r) = \begin{pmatrix} V_{K^d}(r) & 0 & 0 & 0 \\ 0 & V_{K^c}(r) & 0 & 0 \\ 0 & 0 & V_{I^* \backslash K}(r) & 0 \\ 0 & 0 & 0 & V_{J^*}(r) \end{pmatrix}, \tag{11}$$

with

$$\begin{array}{rcl} V_{K^d}(r) & = & \mathrm{diag}(\lambda_i^2(r), i \in K^d), \\ V_{K^c}(r) & = & \mathrm{diag}(\lambda_i^2(r), i \in K^c), \\ V_{I^* \backslash K}(r) & = & \mathrm{diag}(\lambda_i^2(r), i \in I^* \backslash K), \\ V_{J^*}(r) & = & \mathrm{diag}(\lambda_i^2(r), i \in J^*). \end{array}$$

In the following we suppose, without loss of generality, that the components of $\lambda_{K^d}(r)$ are subscripted in descending order, and consider the following hypotheses:

H$_1$: x^* is an isolated stationary point **not necessarily regular** and the sequence $x(r_k)$ converges to x^*;

H$_2$: the generalized strict complementarity condition is satisfied (Def. 2.1);
H$_3$: f is a function of class $\mathcal{C}^2$;
H$_4$: $\forall i \in I$, g_i is a function of class $\mathcal{C}^2$;
H$_5$: $\lambda_{K^c}(r)$ approaches $\lambda^*_{K^c}$ when $r \longrightarrow 0$.

In what follows we suppose that these hypotheses are always satisfied. First, we recall a well-known lemma:

Lemma 2.1. *Let A be a $n \times m$ matrix, $(m < n)$. If $rank(A) = k < m$, then there exists a permutation B such that $AB = QZ$ where Q is an orthonormal matrix and*

$$Z = \left(\begin{array}{c|c} U & U_r \\ \hline 0_{(n-k)\times k} & 0_{(n-k)\times(m^*-k)} \end{array} \right),$$

with U an $k \times k$ non singular upper triangular matrix.

Let $x(r)$ be the solution of $\nabla_x \phi(r,x) = 0$; then $\nabla g(x(r))$ can be written

$$\nabla g(x(r)) = \left(\begin{array}{cccc} \nabla g_{K^d}(x(r)) & \nabla g_{K^c}(x(r)) & \nabla g_{I^*\setminus K}(x(r)) & \nabla g_{J^*}(x(r)) \end{array} \right)$$

and thus by the Gram-Schmidt orthogonalisation process, there exists an orthogonal matrix $Q_{x(r)}$ such that

$$Q^T_{x(r)} \nabla g(x(r)) = \begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & R_4(r) \end{pmatrix} \tag{12}$$

where

- **–:** $\begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) \\ 0 & U_{22}(r) & U_{23}(r) \\ 0 & 0 & U_{33}(r) \end{pmatrix}$ is a $m^* \times m^*$ upper triangular matrix.
- **–:** $U_{11}(r)$, $U_{22}(r)$ and $U_{33}(r)$ are respectively $k^d \times k^d$, $k^c \times k^c$ and $(m^* - k) \times (m^* - k)$ upper triangular matrices.
- **–:** $U_{12}(r)$, $U_{13}(r)$ and $U_{23}(r)$ are respectively $k^d \times k^c$, $k^d \times (m^* - k^d)$ and $k^c \times (m^* - k)$ matrices.

Then, according to the lemma 2.1,

- (i)**:** $\lim\limits_{r \longrightarrow 0} U_{33}(r) = 0_{(m^*-k)\times(m^*-k)}$,
- (ii)**:** $\begin{pmatrix} U_{11}(r) & U_{12}(r) \\ 0 & U_{22}(r) \end{pmatrix}$ is a $k \times k$ non singular upper triangular matrix,
- (iii)**:** $\lim\limits_{r \longrightarrow 0} \begin{pmatrix} U_{11}(r) & U_{12}(r) \\ 0 & U_{22}(r) \end{pmatrix} = \begin{pmatrix} U^*_{11} & U^*_{12} \\ 0 & U^*_{22} \end{pmatrix}$ is a $k \times k$ non singular upper triangular matrix.

According to the same decomposition, we write:

$$Q_{x(r)} = \begin{pmatrix} Q_{K^d} & Q_{K^c} & Q_{I^*\setminus K} & Q_{J^*} \end{pmatrix} \tag{13}$$

where Q_{K^d}, Q_{K^c} and $Q_{I^*\setminus K}$ are matrices associated respectively with the gradients of the active constraints, ∇g_{K^d}, ∇g_{K^c}, $\nabla g_{I^*\setminus K}$, and Q_{J^*} being a matrix associated with the gradients of the inactive constraints ∇g_{J^*}.

In what follows we first show the existence of $x'(r)$ for $r \in]0, \bar{r}]$ and $\bar{r} > 0$ and then that $x'(0)$ exists.

2.1. Trajectory analysis of $x(r)$ at $r = \bar{r} > 0$

If the derivative $x'(r)$ exists, then it must satisfy the following system:

$$\nabla_x \Phi(r, x(r))x'(r) + \nabla_r \Phi(r, x(r)) = 0,$$

where $\nabla_r \Phi(r, x(r))$ is the derivative of $\Phi(r, x(r))$ with respect to r. We have

$$\nabla_x \Phi(r, x(r))x'(r) = -\nabla_r \Phi(r, x(r)). \tag{14}$$

To show the differentiability of the trajectory $x(r)$ at $r = \bar{r} > 0$, it is sufficient to show that $\nabla_x \Phi(r, x(r))$ is an invertible matrix for $r > 0$. With this intention, let

$$D_K^* = \{d \quad | \quad \nabla g_K(x^*)^T d = 0\}.$$

Lemma 2.2. *If* $d \in D_K^*$ *then* $\nabla g_{I^*\setminus K}(x^*)^T d = 0$.

Proof. Since for each $i \in I*\setminus K$, $\nabla g_i(x*)$ and some columns of the matrix $\nabla g_K(x^*)$ are linearly dependent, then there exists a matrix Z such that $\nabla g_{I^*\setminus K}(x^*) = \nabla g_K(x^*)Z$ and thus we have

$$\begin{aligned}\nabla g_{I^*\setminus K}^T(x^*)d &= Z^T \nabla g_K^T(x^*)d \\ &= 0.\end{aligned}$$ □

As we are presently interested in the case $\Lambda(x^*) = \emptyset$ and since according to Theorem 1.2 some $\lambda_i(r)$ diverge, then $\nabla^2 L(x(r), \lambda(r))$ may diverge and thus we cannot satisfy the second order sufficient condition quoted by Fiacco and McCormick. This motivates us to consider the following hypothesis:

- $\mathbf{H}_6$**:** there exists an $\hat{r}$ such that $\forall d \in D_K^*$, $d \neq 0$ and $\forall r \in]0, \hat{r}]$ we have $d^T \nabla^2 L(x(r), \lambda(r))d > 0$.

Theorem 2.1. *If hypothesis* $\mathbf{H}_6$ *is satisfied then the trajectory* $x(r)$ *is differentiable on* $]0, \hat{r}]$.

Proof. To show that the trajectory $x(r)$ is differentiable on $]0,\hat{r}]$ it is sufficient to check that $\forall d \in \mathbb{R}^n\backslash\{0\}$, $d^T\nabla_x\Phi(r,x(r))d > 0$ $\forall r \in]0,\hat{r}]$. We have

$$\begin{aligned}
\nabla_x\Phi(r,x(r)) &= \nabla^2 f(x(r)) - \sum_{i=1}^{m} \frac{r}{g_i(x(r))}\nabla^2 g_i(x(r)) \\
&\quad + \sum_{i=1}^{m} \frac{r}{g_i(x(r))^2}\nabla g_i(x(r))\nabla g_i(x(r))^T \\
&= \nabla^2 f(x(r)) + \sum_{i=1}^{m}\lambda_i(r)\nabla^2 g_i(x(r)) + \sum_{i=1}^{m}\frac{\lambda_i(r)^2}{r}\nabla g_i(x(r))\nabla g_i(x(r))^T \\
&= \nabla^2 L(x(r),\lambda(r)) + \sum_{i=1}^{m}\frac{\lambda_i(r)^2}{r}\nabla g_i(x(r))\nabla g_i(x(r))^T.
\end{aligned}$$

We notice that $\sum_{i=1}^{m}\frac{\lambda_i(r)^2}{r}d^T\nabla g_i(x(r))\nabla g_i(x(r))^T d \geq 0$ for $d \in \mathbb{R}^n\backslash\{0\}$ and thus we have two cases:

- **(1):** if $d \in D_K^*$ then $d^T\nabla^2 L(x(r),\lambda(r))d > 0$ for $r \in]0,\hat{r}]$;
- **(2):** if $d \notin D_K^*$ then $\forall r \in]0,\hat{r}]$ we have

$$\begin{aligned}
d^T\nabla_x\Phi(r,x(r))d &= d^T\nabla^2 L(x(r),\lambda(r))d + \sum_{i=1}^{m}\frac{\lambda_i(r)^2}{r}d^T\nabla g_i(x(r))\nabla g_i(x(r))^T d \\
&= \sum_{i=1}^{m}\lambda_i(r)(d^T\nabla^2 g_i(x(r))d + \frac{\lambda_i(r)^2}{r}d^T\nabla g_i(x(r))\nabla g_i(x(r))^T d) \\
&\quad + d^T\nabla^2 f(x(r))d.
\end{aligned}$$

However the term $\frac{\lambda_i(r)^2}{r}\nabla g_i(x(r))\nabla g_i(x(r))^T > 0$ diverges and dominates the converging term $d^T\nabla^2 g_i(x(r))d$, as well as the term $\sum_{i=1}^{m}\lambda_i(r)$ $(d^T\nabla^2 g_i(x(r))\ d + \frac{\lambda_i(r)^2}{r}\nabla g_i(x(r))\nabla g_i(x(r))^T) > 0$ dominates the term $d^T\nabla^2 f(x(r))d$, which implies that $d^T\nabla_x\Phi(r,x(r))d > 0$, $\forall r \in]0,\hat{r}]$.

Therefore we have $\forall d \neq 0$, $d^T\nabla_x\Phi(r,x(r))d > 0$, $\forall r \in]0,\hat{r}]$. □

2.2. Trajectory analysis of $x(r)$ at $r = 0$

With the orthogonal transformation introduced above, we have

$$Q_{x(r)}^T\nabla_x\Phi(r,x(r))Q_{x(r)}Q_{x(r)}^T x'(r) = -Q_{x(r)}^T\nabla_r\Phi(r,x(r)).$$

Let us put

$$\begin{cases} y(r) = \begin{pmatrix} y_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{pmatrix} = Q^T_{x(r)} x'(r) \\ \Omega(r) = Q^T_{x(r)} \nabla_x \Phi(r, x(r)) Q_{x(r)} \\ b(r) = -Q^T_{x(r)} \nabla_r \Phi(r, x(r)); \end{cases}$$

then we proceed by analyzing the following system:

$$\Omega(r) y(r) = b(r). \tag{15}$$

2.2.1. *Analysis of $y_1(r)$ and $y_2(r)$*

Consider the following lemma the object of which is to show that $\lim_{r \longrightarrow 0} y_1(r)$ and $\lim_{r \longrightarrow 0} y_2(r)$ exists.

Lemma 2.3. *If hypotheses* $\mathbf{H}_6$ *and* $\mathbf{H}_5$ *are satisfied then the vector $y_1(r)$ approaches 0 and vector $y_2(r)$ approaches a bounded vector when $r \longrightarrow 0$.*

Proof. For each $i \in K$, consider the function $h_i(r) = g_i \circ x(r)$. According to Theorem 2.1, there exists $\hat{r}$ such that $h_i(r)$ is differentiable on $]0, \hat{r}]$. By the mean value theorem, there exists $\bar{r} \in]0, r[$ such that

$$\begin{aligned} h_i(r) - h_i(0) &= g_i(x(r)) - g_i(x(0)) \\ &= (r - 0) h'_i(\bar{r}) \\ &= r \nabla g_i(x(\bar{r}))^T x'(\bar{r}), \end{aligned}$$

which implies

$$\begin{aligned} \frac{-1}{\lambda_i(r)} = \frac{g_i(x(r))}{r} &= \nabla g_i(x(\bar{r}))^T x'(\bar{r}) \\ &= \nabla g_i(x(\bar{r}))^T Q_{x(r)} Q^T_{x(r)} x'(\bar{r}) \\ &= (Q^T_{x(r)} \nabla g_i(x(\bar{r})))^T y(\bar{r}). \end{aligned}$$

Thus, we have

$$\begin{array}{rcl}
\begin{pmatrix} -V_{K^d}^{-1/2}(r) & 0 \\ 0 & -V_{K^c}^{-1/2}(r) \end{pmatrix} \begin{pmatrix} e_{K^d} \\ e_{K^c} \end{pmatrix} & = & (Q_{x(r)}^T \nabla g_K(x(\bar{r})))^T y(\bar{r}) \\
& = & \begin{pmatrix} U_{11}(r) & U_{12}(r) \\ 0 & U_{22}(r) \\ 0 & 0 \\ 0 & 0 \end{pmatrix}^T \begin{pmatrix} y_1(\bar{r}) \\ y_2(\bar{r}) \\ y_3(\bar{r}) \\ y_4(\bar{r}) \end{pmatrix} \\
& = & \begin{pmatrix} U_{11}^T(r) & 0 \\ U_{12}^T(r) & U_{22}^T(r) \end{pmatrix} \begin{pmatrix} y_1(\bar{r}) \\ y_2(\bar{r}) \end{pmatrix},
\end{array}$$

where e_{K^d} and e_{K^c} are unit vectors with respectively k^d and k^c components. Since $U_{11}(r)$ and $U_{22}(r)$ are nonsingular matrices, then $y_1(\bar{r})$ and $y_2(\bar{r})$ can be expressed as follows:

$$\begin{cases} y_1(\bar{r}) = -(U_{11}^{-1}(r))^T V_{K^d}^{-1/2}(r) e_{K^d} \\ y_2(\bar{r}) = (U_{22}^{-1}(r))^T (-V_{K^c}^{-1/2}(r) e_{K^c} - U_{12}^T(r) y_1(\bar{r})). \end{cases} \tag{16}$$

When $r \longrightarrow 0$, $\bar{r} \longrightarrow 0$ and $V_{K^d}^{-1/2}(r) e_{K^d}$ approaches a null vector, therefore

$$\begin{cases} y_1(0) = 0 \\ y_2(0) = (U_{22}^{*-1})^T V_{K^c}^{-1/2}(0) e_{K^c}. \end{cases} \qquad \square$$

Remark 2.3. Equation (16) expresses $y_1(\bar{r})$ according to $V_{K^d}^{-1/2}(r)$. However, $\bar{r}$ approaches 0 as $r \longrightarrow 0$, but we do not know yet how $\bar{r}$ behaves close to $r = 0$; also, we have that

$$\lambda_i(r) \nabla g_i(x(\bar{r}))^T x'(\bar{r}) = -1.$$

2.2.2. *Analysis of the term $b(r)$*

We have

$$\begin{array}{rcl}
b(r) & = & -Q_{x(r)}^T \nabla_r (\nabla_x \phi(r, x(r))) \\
& = & \dfrac{1}{r} Q_{x(r)}^T \nabla g(x(r)) \lambda(r).
\end{array}$$

By using relations (10) and (12), we obtain:

$$b(r) = \frac{1}{r}\begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & R_4(r) \end{pmatrix}\lambda(r)$$

$$= \begin{pmatrix} \frac{1}{r}(U_{11}(r)\lambda_{K^d}(r) + U_{12}(r)\lambda_{K^c}(r) + U_{13}(r)\lambda_{I^*\setminus K}(r) + R_1(r)\lambda^*_{J^*}(r)) \\ \frac{1}{r}(U_{22}(r)\lambda_{K^c}(r) + U_{23}(r)\lambda_{I^*\setminus K}(r) + R_2(r)\lambda^*_{J^*}(r)) \\ \frac{1}{r}(U_{33}(r)\lambda_{I^*\setminus K}(r) + R_3(r)\lambda^*_{J^*}(r)) \\ \frac{1}{r}R_4(r)\lambda^*_{J^*}(r) \end{pmatrix}$$

$$= \begin{pmatrix} b_1(r) \\ b_2(r) \\ b_3(r) \\ b_4(r) \end{pmatrix}.$$

Consider the following lemma which enables us to show that $rb_1(r)$, $rb_2(r)$, $rb_3(r)$, and $b_4(r)$ are bounded in the neighborhood of $r = 0$.

Lemma 2.4. *When $r \longrightarrow 0$, the vectors $U_{11}(r)\lambda_{K^d}(r) + U_{13}(r)\lambda_{I^*\setminus K}(r)$, $U_{23}(r)$ $\lambda_{I^*\setminus K}(r)$ and $U_{33}(r)\lambda_{I^*\setminus K}(r)$ approach bounded vectors.*

Proof. We have

$$\begin{aligned} Q^T_{x(r)}\Phi(r, x(r)) &= Q^T_{x(r)}(\nabla f(x(r)) + \nabla g(x(r))\lambda(r)) \\ &= Q^T_{x(r)}\nabla f(x(r)) + Q^T_{x(r)}\nabla g(x(r))\lambda(r) \\ &= Q^T_{x(r)}\nabla f(x(r)) + \begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & R_4(r) \end{pmatrix} \\ &\quad \begin{pmatrix} \lambda_{K^d}(r) \\ \lambda_{K^c}(r) \\ \lambda_{I^*\setminus K}(r) \\ \lambda_{J^*}(r) \end{pmatrix} \\ &= 0. \end{aligned}$$

Let us define

$$F(r)=\begin{pmatrix}F_1(r)\\F_2(r)\\F_3(r)\\F_4(r)\end{pmatrix}=Q^T_{x(r)}\nabla f(x(r))$$

to express the system in the following form:

$$\begin{cases}U_{11}(r)\lambda_{K^d}(r)+U_{13}(r)\lambda_{I^*\setminus K}(r)=-F_1(r)-U_{12}(r)\lambda_{K^c}(r)-R_1(r)\lambda_{J^*}(r)\\ U_{23}(r)\lambda_{I^*\setminus K}(r)=-F_2(r)-U_{22}(r)\lambda_{K^c}(r)-R_2(r)\lambda_{J^*}(r)\\ U_{33}(r)\lambda_{I^*\setminus K}(r)=-F_3(r)-R_3(r)\lambda_{J^*}(r)\\ F_4(r)+R_4(r)\lambda_{J^*}(r)=0.\end{cases}$$

Since $F(r)$ is a continuous function at the points $x(r)$ and by the hypothesis $x(r)\longrightarrow x^*$, $F(r)$ is bounded when r approaches 0. Moreover $Q^T_{x(r)}\Phi(r,x(r))=0$ and $\lambda_{J^*}(r)$ approaches 0 so that

$$\begin{cases}\lim\limits_{r\longrightarrow 0}(U_{11}(r)\lambda_{K^d}(r)+U_{13}(r)\lambda_{I^*\setminus K}(r))=-(F_1(0)+U^*_{12}\lambda^*_{K^c})\\ \lim\limits_{r\longrightarrow 0}(U_{23}(r)\lambda_{I^*\setminus K}(r))=-(F_2(0)+U^*_{22}\lambda^*_{K^c})\\ \lim\limits_{r\longrightarrow 0}U_{33}(r)\lambda_{I^*\setminus K}(r)=-F_3(0).\end{cases}$$ □

On the basis of Lemma 2.4 we can prove the following corollary:

Corollary 2.1. *If hypotheses* $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, *and* $\mathbf{H}_4$ *are satisfied then* $\begin{pmatrix}F_1(r)\\F_2(r)\\F_3(r)\\\frac{1}{r}F_4(r)\end{pmatrix}=\begin{pmatrix}rb_1(r)\\rb_2(r)\\rb_3(r)\\b_4(r)\end{pmatrix}$ *is a bounded vector in the neighborhood of* $r=0$.

Proof. Since

$$\begin{pmatrix}rb_1(r)\\rb_2(r)\\rb_3(r)\\b_4(r)\end{pmatrix}=$$

$$\begin{pmatrix}U_{11}(r)\lambda_{K^d}(r)+U_{12}(r)\lambda_{K^c}(r)+U_{13}(r)\lambda_{I^*\setminus K}(r)+R_1(r)\lambda^*_{J^*}(r)\\ U_{22}(r)\lambda_{K^c}(r)+U_{23}(r)\lambda_{I^*\setminus K}(r)+R_2(r)\lambda^*_{J^*}(r)\\ U_{33}(r)\lambda_{I^*\setminus K}(r)+R_3(r)\lambda^*_{J^*}(r)\\ \frac{1}{r}R_4(r)\lambda^*_{J^*}(r)\end{pmatrix}$$

and

$$\begin{pmatrix} F_1(r) \\ F_2(r) \\ F_3(r) \\ F_4(r) \end{pmatrix} = Q_{x(r)}^T \nabla f(x(r))$$

$$= -Q_{x(r)}^T \nabla g(x(r))\lambda(r)$$

$$= \begin{pmatrix} U_{11}(r)\lambda_{K^d}(r) + U_{12}(r)\lambda_{K^c}(r) + U_{13}(r)\lambda_{I^*\setminus K}(r) + R_1(r)\lambda_{J^*}^*(r) \\ U_{22}(r)\lambda_{K^c}(r) + U_{23}(r)\lambda_{I^*\setminus K}(r) + R_2(r)\lambda_{J^*}^*(r) \\ U_{33}(r)\lambda_{I^*\setminus K}(r) + R_3(r)\lambda_{J^*}^*(r) \\ R_4(r)\lambda_{J^*}^*(r) \end{pmatrix},$$

we have

$$\begin{pmatrix} F_1(r) \\ F_2(r) \\ F_3(r) \\ \frac{1}{r}F_4(r) \end{pmatrix} = \begin{pmatrix} rb_1(r) \\ rb_2(r) \\ rb_3(r) \\ b_4(r) \end{pmatrix}.$$

Since $\lim_{r\longrightarrow 0} \frac{1}{r} R_4(r)\lambda_{J^*}^*(r)$ is a bounded vector, $\frac{1}{r}F_4(r)$ is a bounded vector in the neighborhood of $r = 0$ and thus according to Lemma 2.4 the result follows. □

2.2.3. *Analysis of the term* $\Omega(r)$

We present here the high level analysis of $\Omega(r)$. Details of the proofs of several lemmas will be found in Appendix A1. Now, we study the matrix $\Omega(r)$ from system (15). Let us define

$$H = H(x(r), r) = \nabla_x^2 f(x(r)) + \sum_{i=1}^{m} \lambda_i(r)\nabla_x^2 g_i(x(r)), \tag{17}$$

$$G(r) = Q_{x(r)}^T H Q_{x(r)} = \begin{pmatrix} G_{11}(r) & G_{12}(r) & G_{13}(r) & G_{14}(r) \\ G_{21}(r) & G_{22}(r) & G_{23}(r) & G_{24}(r) \\ G_{31}(r) & G_{32}(r) & G_{33}(r) & G_{34}(r) \\ G_{41}(r) & G_{42}(r) & G_{43}(r) & G_{44}(r) \end{pmatrix}, \tag{18}$$

and, recalling our splitting of variables into four part,

$$S(r) = \begin{pmatrix} S_{11}(r) & S_{12}(r) & S_{13}(r) & S_{14}(r) \\ S_{21}(r) & S_{22}(r) & S_{23}(r) & S_{24}(r) \\ S_{31}(r) & S_{32}(r) & S_{33}(r) & S_{34}(r) \\ S_{41}(r) & S_{42}(r) & S_{43}(r) & S_{44}(r) \end{pmatrix} \tag{19}$$

with $S_{ij}(r) = R_i(r)V_{J^*}(r)R_j(r)$, and

$$W(r) = \left(\begin{array}{c|c} \begin{pmatrix} W_{11}(r) & W_{12}(r) & W_{13}(r) \\ W_{21}(r) & W_{22}(r) & W_{23}(r) \\ W_{31}(r) & W_{32}(r) & W_{33}(r) \end{pmatrix} & 0_{m^*\times(n-m^*)} \\ \hline 0_{(n-m^*)\times m^*} & 0_{(n-m^*)\times(n-m^*)} \end{array} \right) \tag{20}$$

where

$$\begin{array}{rcl} W_{11}(r) & = & U_{11}(r)V_{K^d}(r)U_{11}^T(r)+U_{12}(r)V_{K^c}(r)U_{12}^T(r)+U_{13}(r)V_{I^*\setminus K}(r)U_{13}^T(r) \\ W_{12}(r) & = & U_{12}(r)V_{K^c}(r)U_{22}^T(r)+U_{13}(r)V_{I^*\setminus K}(r)U_{23}^T(r), \\ W_{13}(r) & = & U_{13}(r)V_{I^*\setminus K}(r)U_{33}^T(r), \\ W_{21}(r) & = & U_{22}(r)V_{K^c}(r)U_{12}^T(r)+U_{23}(r)V_{I^*\setminus K}(r)U_{13}^T(r), \\ W_{22}(r) & = & U_{22}(r)V_{K^c}(r)U_{22}^T(r)+U_{23}(r)V_{I^*\setminus K}(r)U_{23}^T(r), \\ W_{23}(r) & = & U_{23}(r)V_{I^*\setminus K}(r)U_{33}^T(r), \\ W_{31}(r) & = & U_{33}(r)V_{I^*\setminus K}(r)U_{13}^T(r), \\ W_{32}(r) & = & U_{33}(r)V_{I^*\setminus K}(r)U_{23}^T(r), \\ \text{and}\, W_{33}(r) & = & U_{33}(r)V_{I^*\setminus K}(r)U_{33}^T(r). \end{array}$$

Recall that $\Omega(r) = Q_{x(r)}^T \nabla_x^2 \phi_r(r, x(r)) Q_{x(r)}$, which allows to write

$$\begin{array}{rcl} \Omega(r) & = & Q_{x(r)}^T(\nabla_x^2 f(x(r)) + \sum_{i=1}^{m} \lambda_i(r)\nabla_x^2 g_i(x(r)))Q_{x(r)} \\ & & +\dfrac{1}{r}Q_{x(r)}^T \nabla g(x(r))V(r)(Q_{x(r)}^T\nabla g(x(r)))^T \\ & = & G(r) + \dfrac{1}{r}\begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & R_4(r) \end{pmatrix} \\ & & \times V(r)\begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & R_4(r) \end{pmatrix}^T. \end{array}$$

Next by using relations (11), (18), (19), (20), we obtain the following form of $\Omega(r)$

$$\begin{aligned}\Omega(r) &= G(r) + \frac{1}{r}S(r) + \frac{1}{r}W(r)\\ &= \begin{pmatrix} \Omega_{11}(r) & \Omega_{12}(r) & \Omega_{13}(r) & \Omega_{14}(r)\\ \Omega_{21}(r) & \Omega_{22}(r) & \Omega_{23}(r) & \Omega_{24}(r)\\ \Omega_{31}(r) & \Omega_{32}(r) & \Omega_{33}(r) & \Omega_{34}(r)\\ \Omega_{41}(r) & \Omega_{42}(r) & \Omega_{43}(r) & \Omega_{44}(r)\end{pmatrix}.\end{aligned}$$

So that the system (15) is writen as:

$$\begin{pmatrix} \frac{1}{r}F_1(r)\\ \frac{1}{r}F_2(r)\\ \frac{1}{r}F_3(r)\\ \frac{1}{r}F_4(r)\end{pmatrix} = \begin{pmatrix} \Omega_{11}(r) & \Omega_{12}(r) & \Omega_{13}(r) & \Omega_{14}(r)\\ \Omega_{21}(r) & \Omega_{22}(r) & \Omega_{23}(r) & \Omega_{24}(r)\\ \Omega_{31}(r) & \Omega_{32}(r) & \Omega_{33}(r) & \Omega_{34}(r)\\ \Omega_{41}(r) & \Omega_{42}(r) & \Omega_{43}(r) & \Omega_{44}(r)\end{pmatrix}\begin{pmatrix} y_1(r)\\ y_2(r)\\ y_3(r)\\ y_4(r)\end{pmatrix}.$$

Let us put

$$y_1(r) = {U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r)z_1(r);$$

then we have the following equivalent modified system:

$$\begin{aligned}\begin{pmatrix} \frac{1}{r}F_1(r)\\ \frac{1}{r}F_2(r)\\ \frac{1}{r}F_3(r)\\ \frac{1}{r}F_4(r)\end{pmatrix} &= \begin{pmatrix} \Omega_{11}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r) & \Omega_{12}(r) & \Omega_{13}(r) & \Omega_{14}(r)\\ \Omega_{21}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r) & \Omega_{22}(r) & \Omega_{23}(r) & \Omega_{24}(r)\\ \Omega_{31}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r) & \Omega_{32}(r) & \Omega_{33}(r) & \Omega_{34}(r)\\ \Omega_{41}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r) & \Omega_{42}(r) & \Omega_{43}(r) & \Omega_{44}(r)\end{pmatrix}\\ &\times\begin{pmatrix} z_1(r)\\ y_2(r)\\ y_3(r)\\ y_4(r)\end{pmatrix}.\end{aligned}$$

Multiply the first block of line and the three blocks of lines of the preceding system respectively by $V_{K^d}^{-1/2}(r)U_{11}^{-1}(r)$ and r to obtain the following equivalent modified system:

$$\begin{pmatrix} \bar{F}_1(r) \\ F_2(r) \\ F_3(r) \\ \frac{1}{r}F_4(r) \end{pmatrix} = \begin{pmatrix} \bar{\Omega}_{11}(r) & \bar{\Omega}_{12}(r) & \bar{\Omega}_{13}(r) & \bar{\Omega}_{14}(r) \\ \bar{\Omega}_{21}(r) & r\Omega_{22}(r) & r\Omega_{23}(r) & r\Omega_{24}(r) \\ \bar{\Omega}_{31}(r) & r\Omega_{32}(r) & r\Omega_{33}(r) & r\Omega_{34}(r) \\ \bar{\Omega}_{41}(r) & \Omega_{42}(r) & \Omega_{43}(r) & \Omega_{44}(r) \end{pmatrix} \begin{pmatrix} z_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{pmatrix}$$

with

$$\begin{aligned} \bar{F}_1(r) &= V_{K^d}^{-1/2}(r)U_{11}^{-1}(r)F_1(r), \\ \bar{\Omega}_{11}(r) &= rV_{K^d}^{-1/2}(r)U_{11}^{-1}(r)\Omega_{11}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r), \\ \bar{\Omega}_{1i}(r) &= rV_{K^d}^{-1/2}(r)U_{11}^{-1}(r)\Omega_{1i}(r) \quad i = 2,3,4, \\ \bar{\Omega}_{i1}(r) &= r\Omega_{i1}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r) \quad i = 2,3, \\ \bar{\Omega}_{41}(r) &= \Omega_{41}(r){U_{11}^{-1}}^T(r){V_{K^d}}^{-1/2}(r). \end{aligned}$$

We now consider regrouping the blocks 1 and 2 as well as 3 and 4. We wish to analyze separately the diagonal block subsystems. In order to analyze the systems separately, we introduce:

$$N_1(r) = \begin{pmatrix} \bar{F}_1(r) \\ F_2(r) \end{pmatrix} - \begin{pmatrix} \bar{\Omega}_{13}(r) & r\Omega_{23}(r) \\ \bar{\Omega}_{14}(r) & r\Omega_{24}(r) \end{pmatrix} \begin{pmatrix} y_3(r) \\ y_4(r) \end{pmatrix},$$

$$N_2(r) = \begin{pmatrix} F_3(r) \\ \frac{1}{r}F_4(r) \end{pmatrix} - \begin{pmatrix} \bar{\Omega}_{31}(r) & r\Omega_{32}(r) \\ \bar{\Omega}_{41}(r) & \Omega_{42}(r) \end{pmatrix} \begin{pmatrix} z_1(r) \\ y_2(r) \end{pmatrix}.$$

We can now analyze the two systems separately:

$$(P) \qquad \begin{cases} \bar{\Omega}_1(r) \begin{pmatrix} z_1(r) \\ \\ y_2(r) \end{pmatrix} = N_1(r) \\ \\ \\ \bar{\Omega}_2(r) \begin{pmatrix} y_3(r) \\ \\ y_4(r) \end{pmatrix} = N_2(r). \end{cases}$$

where $\bar{\Omega}_1(r)$ and $\bar{\Omega}_2(r)$ are the following matrices:

$$\begin{aligned} \bar{\Omega}_1(r) &= \begin{pmatrix} \bar{\Omega}_{11}(r) & \bar{\Omega}_{12}(r) \\ \\ \bar{\Omega}_{21}(r) & r\Omega_{22}(r) \end{pmatrix} \\ \text{and } \bar{\Omega}_2(r) &= \begin{pmatrix} r\Omega_{33}(r) & r\Omega_{34}(r) \\ \\ \Omega_{43}(r) & \Omega_{44}(r) \end{pmatrix}. \end{aligned}$$

Consider the following definition which weakens the regularity assumption used by Fiacco and McCormick (linear independence of the gradients of the active constraints).

Definition 2.1. Let

$$E = \{i \in I^* \backslash K \quad | \quad \lim_{r \longrightarrow 0} \lambda_i(r) \nabla g_i(x(r)) \quad \text{is not bounded}\}.$$

x^* is called a weak regular point when one of the two following conditions is satisfied:

(1) $E = \emptyset$,
(2) if $E \neq \emptyset$ then there is a partition of K^d, $\mathbb{P} = \{K_1^d, \cdots, K_l^d\}$ and a bijective function $\alpha : E \longrightarrow \mathbb{P}$ defined by:

$$\alpha(i) = K_i^d \Longleftrightarrow \quad \begin{cases} \forall \bar{r} > 0, \ \lambda_i(r) \nabla g_i(x(r)) + \sum_{j \in K_i^d} \lambda_j(r) \nabla g_j(x(r)) \\ \qquad \text{is a bounded vector on } [0, \bar{r}]. \end{cases}$$

Remark 2.4. We notice that:

(1) If x^* is a weak regular point, then $\lim_{r \longrightarrow 0} B(r)$ is a bounded matrix where

$$B(r) = \nabla g_{I^* \backslash K}(x(r)) {V_{I^* \backslash K}}^{1/2} + \nabla g_{K^d}(x(r)) A(r)$$

with $A(r) = (A_1(r) \quad | \quad 0_{k^d \times (m^* - k^d - |E|)})$ such that

$$A_1(r) = \begin{pmatrix} \lambda_{J_1}(r) & 0 & \cdots & 0 \\ 0 & \lambda_{J_2}(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{J_l}(r) \end{pmatrix}$$

and $J_1, J_2 \cdots, J_l$ form a partition of K^d;

(2) $\nabla g_E(x^*) = -\nabla g_{K^d}(x^*) \lim_{r \longrightarrow 0} A_1(r) V_E^{-1/2}(r)$ where $\lim_{r \longrightarrow 0} A_1(r) V_E^{-1/2}(r)$ is a bounded matrix;

(3) $\forall i \in I^* \backslash K$ such that $i \notin E$, $\nabla g_i(x^*)$ is a null vector,

(4) if x^* is regular then it is weak regular.

We first observe that if there exists diverging multipliers, then either $E \neq \emptyset$ or an active constraint gradient vanishes.

Lemma 2.5. *If* $\forall i \in I^*$, $\nabla g_i(x^*) \neq 0$ *and* $E = \emptyset$ *then* $K^d = \emptyset$.

Next,

Lemma 2.6. *If* x^* *is a weak regular point and* $E \neq \emptyset$ *then* $\nabla g_E(x^*)$ *is a full rank matrix.*

Lemma 2.7. *If* x^* *is a weak regular point then* $\forall i \in E$, $\nabla g_i(x^*)$ *and the columns of the matrix* $\nabla g_{K^c}(x^*)$ *are linearly independent.*

Consider now the two following lemmas. The first will be useful in the proof of the second, for which the object is to show that some submatrices of the matrix $\Omega(r)$ are bounded when r is close to 0.

Lemma 2.8. *Let* M *be a* $n \times n$ *matrix. If*

$$M = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \qquad \textit{Then} \quad I + M \textit{ is a nonsingular matrix.}$$

Proof. The lemma is shown easily by using the Gauss-Jordan process. □

Lemma 2.9. *If* x^* *is a weak regular point then*

(*i*) $U_{13}(r) {V_{I^* \backslash K}}^{1/2}(r) = -U_{11}(r) A(r) + \bar{B}_1(r)$ *where* $\lim_{r \longrightarrow 0} \bar{B}_1(r)$ *is a bounded matrix,*

(*ii*) $\lim_{r \longrightarrow 0} U_{23}(r) {V_{I^* \backslash K}}^{1/2}(r) = U_{23}^*$ *where* U_{23}^* *is a bounded matrix,*

(iii) $\lim_{r\longrightarrow 0} U_{33}(r){V_{I^*\setminus K}}^{1/2}(r) = U_{33}^*$ *where* U_{33}^* *is a bounded matrix,*

(iv) $\lim_{r\longrightarrow 0} {V_{K^d}}^{-1/2}(r)A(r) = A^*$ *where* A^* *is a bounded matrix,*

(v) $I + A^*{A^*}^T$ *is a nonsingular matrix,*

(vi) $\lim_{r\longrightarrow 0} {V_{K^d}}^{-1/2}(r)U_{11}(r){V_{K^d}}^{1/2}(r) = \bar{U}_{11}$ *is a nonsingular matrix,*

(vii) $\lim_{r\longrightarrow 0} {V_{K^d}}^{-1/2}(r)U_{11}(r)A(r) = \bar{A}$ *where* $\bar{A}$ *is a bounded matrix.*

Let us recall that

$$y_1(r) = {U_{11}^{-1}}^T(r)V_{K^d}^{-1/2}(r)z_1(r),$$

and consider the tow following lemmas, for which the object is to show, under some hypotheses, that $z_1(r)$ et $y_1(r)$ are bounded in the neighborhood of $r = 0$.

Lemma 2.10. *If* $\lim_{r\longrightarrow 0} V_{K^d}(r)^{1/2}y_1(r)$ *is a bounded vector then* $\lim_{r\longrightarrow 0} z_1(r)$ *is a bounded vector.*

Now,

Lemma 2.11. *If* $\forall i \in K^d$, $\lim_{r\longrightarrow 0} \frac{r\lambda_i'(r)}{\lambda_i(r)}$ *is bounded then* $\lim_{r\longrightarrow 0} V_{K^d}(r)^{1/2}y_1(r)$ *is a bounded vector.*

Define

$$\begin{aligned} D_K^* &= \{d \quad | \quad \nabla g_K(x^*)^T d = 0\}, \\ \bar{D} &= \lim_{r\longrightarrow 0}\{d \quad | \quad \nabla g_{I^*}(x(r))^T d = 0\}, \\ \bar{D}^\perp &= \text{orthogonal complement of } \bar{D} \end{aligned}$$

and let us consider the following definition which weakens the usual second order sufficient condition for our case where $\Lambda^* = \emptyset$.

Definition 2.2. We shall say that the weak second order sufficient condition (W.S.O.S.C. for short) holds at x^* when

(i) $\mathbf{H}_6$ is satisfied;

(ii) $\exists\, \hat{r}$ such that $\forall d \in \bar{D}^\perp \cap D_K^*$, $d \neq 0$ and $\forall r \in]0, \hat{r}]$,

$$rd^T(\nabla^2 L(x(r), \lambda(r)) + \sum_{i=1}^{m} \lambda_i^2(r)\nabla g_i(x(r))\nabla g_i(x(r))^T)d > 0.$$

Consider the following technical hypothesis which will be useful in the following section:

$\mathbf{H}_7$: $\lim_{r\longrightarrow 0} rd^T\nabla^2 L(x, \lambda(r))d = \begin{cases} \text{is bounded if} \quad d \in \bar{D}^\perp \cap D_K^* \\ 0 \quad \text{elsewhere.} \end{cases}$

Lemma 2.12. *If hypotheses* $\mathbf{H}_3$*,* $\mathbf{H}_4$ *and* $\mathbf{H}_7$ *are satisfied at* x^* *then we have*

$$\lim_{r\longrightarrow 0} \begin{pmatrix} rG_{11}(r) & rG_{12}(r) & rG_{13}(r) & rG_{14}(r) \\ rG_{21}(r) & rG_{22}(r) & rG_{23}(r) & rG_{24}(r) \\ rG_{31}(r) & rG_{32}(r) & rG_{33}(r) & rG_{34}(r) \\ G_{41}(r) & G_{42}(r) & G_{43}(r) & G_{44}(r) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G_{33}^* & 0 \\ G_{41}^* & G_{42}^* & G_{43}^* & G_{44}^* \end{pmatrix}$$

where G_{33}^**,* G_{41}^**,* G_{42}^**,* G_{43}^* *et* G_{44}^* *are bounded matrices.*

Lemma 2.13. *If x^* is a weak regular point, satisfying the W.S.O.S.C. and hypotheses $\mathbf{H}_3$, $\mathbf{H}_4$ and $\mathbf{H}_7$, then* $\begin{pmatrix} G^*_{33} + U^*_{33} {U^*_{33}}^T & 0 \\ G^*_{43} & G^*_{44} \end{pmatrix}$ *is a nonsingular matrix.*

Lemma 2.14. *If x^* is a weak regular point satisfying the weak second order sufficient condition, hypotheses $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_7$ and furthermore $\lim_{r\longrightarrow 0} V^{1/2}_{K^d}(r) y_1(r)$ is a bounded vector, then*

(*i*) $\lim_{r\longrightarrow 0} \bar{\Omega}_2(r) = \bar{\Omega}_2(0)$ *where $\bar{\Omega}_2(0)$ is a nonsingular matrix,*
(*ii*) $\lim_{r\longrightarrow 0} N_2(r) = N_2(0)$ *where $N_2(0)$ is a bounded vector.*

Lemma 2.15. *If x^* is a weak regular point, satisfying the weak second order sufficient condition, hypotheses $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_7$ and furthermore that $\lim_{r\longrightarrow 0} V^{1/2}_{K^d}(r) y_1(r)$ is a bounded vector, then the sequences of vectors $y_3(r)$ and $y_4(r)$ approach respectively $y_3(0)$ and $y_4(0)$, which are bounded vectors.*

2.3. Differentiability of $x(r)$ in the neighborhood of $r = 0$

In this section, we show the principal theorem concerning the differentiability of the trajectories when $\Lambda^* = \emptyset$. Consider the following definition:

Definition 2.3. We say that x^* is a linear weak regular point if the two following conditions are satisfied:

(*i*) x^* is a weak regular point,
(*ii*) for all $i \in K^d$, $\lim_{r\longrightarrow 0} \frac{r\lambda_i'(r)}{\lambda_i(r)}$ is bounded.

Remark 2.5. If x^* is a linear weak regular point then according to Lemma 2.11, $\lim_{r\longrightarrow 0} V^{1/2}_{K^d} y_1(r)$ is a bounded vector.

Theorem 2.2. *If x^* is a linear weak regular point, satisfying the weak second order sufficient condition and hypotheses $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, $\mathbf{H}_4$, $\mathbf{H}_6$, $\mathbf{H}_5$, $\mathbf{H}_7$, then the trajectory $x(r)$ is differentiable at the point $r = 0$.*

Proof. According to Theorem 2.1, the trajectory $x(r)$ is differentiable at points $r > 0$. Close to 0 let us denote

$$\begin{aligned} y(r) &= Q^T_{x(r)} x'(r) \\ &= \begin{pmatrix} y_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{pmatrix}. \end{aligned}$$

According to Lemmas 2.3 and 2.15, the vector $\lim_{r\longrightarrow 0}\begin{pmatrix} y_1(r) \\ y_2(r) \\ y_3(r) \\ y_4(r) \end{pmatrix}$ exists and since $Q_{x(r)}$ is an orthogonal matrix then the vector $\lim_{r\longrightarrow 0} x'(r)$ exists also, and we have the result. □

Let us now show that the theorem is a genuine generalization of the usual results by checking that the theorems of Fiacco and McCormick constitute in fact a corollary of Theorem 2.2.

Corollary 2.2. *If x^* satisfies the usual hypotheses used by Fiacco and McCormick then the trajectory $x(r)$ is differentiable.*

Proof. Again let us point out the usual hypotheses used by Fiacco and McCormick:

- $\forall i \in I$, g_i is a $\mathcal{C}^2$ function;
- the objective function is of class $\mathcal{C}^2$;
- *linear independence*: $\nabla g_i(x^*), i \in I^*$, are linearly independent;
- x^* is a cluster point of the sequence $x(r)$;
- *strict complementarity*: $g_i(x^*) = 0 \Longleftrightarrow \lambda_i^* > 0$;
- *second order sufficient condition*: $Z^t \nabla^2 L(x^*, \lambda^*) Z > 0, \forall Z \in Ker$ $(\nabla g_{I^*}(x^*))$.

The first two hypotheses imply that hypotheses $\mathbf{H}_3$, $\mathbf{H}_4$ are satisfied. Since $\nabla g_i(x^*), i \in I^*$, are linearly independent then x^* is a weak regular point, hypotheses $\mathbf{H}_1$ and $\mathbf{H}_5$ are satisfied and $K^d = \emptyset$. Thus x^* is a linear weak regular point and hypothesis $\mathbf{H}_7$ is satisfied. Strict complementarity implies that hypothesis $\mathbf{H}_2$ is satisfied and the second order sufficient condition implies that the weak second order sufficient condition is satisfied, and we have the result. □

3. Conclusion

This work falls within the field of Interior Point Methods intensively studied in the context of linear and quadratic programming in the last fifteen years. Our work relates to the application of these methods in a non-linear programming context and generalizes results originating in 1968. Motivated by the success of penalty algorithm methods on non-linear programming problems, in particular to the log barrier algorithm, Fiacco and McCormick [5,6] showed, under some hypotheses, the existence of a differentiable trajectory in the optimal solution neighborhood. When the usual hypotheses are used by Fiacco & McCormick, the set Λ^* of the multiplier vectors of Karush-Kuhn-Tucker [8] is nonempty.

In this paper we analysed the behavior of the log barrier trajectory in the neighborhood of a degenerate solution (*i.e.* the contraint qualification are not satisfied), which implies that the set Λ^* of Karush-Kuhn-Tucker multipliers [8] is empty. To show the existence of a differentiable trajectory even if the optimal solution is not regular, we defined new weak hypotheses. The weak second order sufficient condition (W.S.O.S.C.) is a natural generalization of the usual second order sufficient condition . The weak linear regularity definition is a natural generalization of the usual regularity and ensures that some submatrices of the matrix $\sum_{i=1}^{m} \lambda_i^2(r) Q_{x(r)}^T \nabla g_i(x(r)) Q_{x(r)} \nabla g_i(x(r))^T$ are bounded and $\lim_{r \longrightarrow 0} V_{K^d}^{1/2}(r) y_1(r)$ is a bounded vector. However, $\mathbf{H}_7$ is a technical hypothesis, which may not be necessary to the result but nonetheless useful in our proofs. This paper shows us, under the weakened hypothesis, that a differentiable trajectory exists in the optimal solution neighborhood. This coroborates the robustness of the log barrier algorithm method and enlarges its field of application.

Our work opens new perspectives toward the development of robust algorithms to solve degenerate problems. An interesting extension of our work would weaken the strict complementarity condition. Miflin[9] has shown that in that case (under constraint qualifications), the trajectories behave as $\mathcal{O}(\sqrt{r})$, so that a possible extension would involve a parametrization in $t = r^2$.

Hopes to address non-regular solutions with equality constraints seem more difficult since in that case, stationary points for the constraints may not be feasible. Extension to other penalties, such as the exponential penalty function appears to fail: the K-K-T multiplier estimates for the exponential penalty function ($e^{\frac{g_i(x(r))}{r}}$) will diverge, so $\frac{g_i(x(r))}{r} \to \infty$ destroying the hope for a differentiable primal trajectory.

Appendix A1: Proof of lemmas in section 2.2.3

Proof of Lemma 2.5. Suppose that $E = \emptyset$ and $K^d \neq \emptyset$. We have

$$\begin{aligned} \sum_{i=1}^{m} \lambda_i(r) \nabla g_i(x(r)) &= \sum_{i \in K^d} \lambda_i(r) \nabla g_i(x(r)) + \sum_{i \in K^c} \lambda_i(r) \nabla g_i(x(r)) \\ &+ \sum_{i \in I^* \setminus K} \lambda_i(r) \nabla g_i(x(r)) + \sum_{i \in J^*} \lambda_i(r) \nabla g_i(x(r)). \end{aligned}$$

However $\lim_{r \longrightarrow 0} \sum_{i \in K^d} \lambda_i(r) \nabla g_i(x(r))$ is a bounded vector, since

(i) $\lim_{r \longrightarrow 0} \sum_{i \in K^c} \lambda_i(r) \nabla g_i(x(r))$ is a bounded vector,

(ii) $\lim_{r \longrightarrow 0} \sum_{i \in J^*} \lambda_i(r) \nabla g_i(x(r))$ is a bounded vector,

(iii) $\lim_{r \longrightarrow 0} \sum_{i \in I^* \setminus (K \cup E)} \lambda_i(r) \nabla g_i(x(r))$ is a bounded vector because $E = \emptyset$,

(iv) $\lim_{r \longrightarrow 0} \sum_{i=1}^{m} \lambda_i(r) \nabla g_i(x(r))$ is a bounded vector by Corollary 2.1.

If $K^d \neq \emptyset$ there are two possible cases: $k^d = 1$ or $k^d > 1$. If $k^d = 1$ then $K^d = \{i_0\}$, $\lim_{r \longrightarrow 0} \lambda_{i_0}(r) \nabla g_{i_0}(x(r))$ is a bounded vector and thus $\nabla g_{i_0}(x^*)$ is a null vector. If $k^d > 1$, suppose there exists $i_0 \in K^d$ such that $\lambda_{i_0}(r) = \max(\lambda_i(r), i \in K^d)$, thus we have

$$\forall i \in K^d \backslash \{i_0\}, \lim_{r \longrightarrow 0} \frac{\lambda_i(r)}{\lambda_{i_0}(r)} = \mu_i^* \geq 0$$

which implies that

$$\begin{aligned}\nabla g_{i_0}(x^*) &= -\lim_{r\longrightarrow 0}\sum_{i\in K^d}\frac{\lambda_i(r)}{\lambda_{i_0}(r)}\nabla g_i(x(r)) \\ &= -\sum_{i\in K^d\setminus\{i_0\}}\mu_i^*\nabla g_i(x^*),\end{aligned}$$

and we have: either
$\forall i \in K^d\setminus\{i_0\}$ $\mu_i^* = 0$ then $\nabla g_{i_0}(x^*)$ is a null vector, or
$\exists i \in K^d\setminus\{i_0\}$ such that $\mu_i^* > 0$ then $\nabla g_{i_0}(x^*)$ is written as a linear combination of the columns of $\nabla g_K(x^*)$. Both contradict the definition of K, and we have the result. □

Proof of Lemma 2.6. Pick some $\alpha_i \in \mathbb{R}$ such that $\sum_{i\in E}\alpha_i\nabla g_i(x^*) = 0$. Since x^* is a weak regular point then according to Remark 2.4, there are scalars $a_j \in \mathbb{R}$ such that

$$\begin{aligned}\sum_{i\in E}\alpha_i\nabla g_i(x^*) &= \sum_{i\in E}\alpha_i\sum_{j\in K_i^d}a_j\nabla g_j(x^*) \\ &= \sum_{i\in E}\sum_{j\in K_i^d}\alpha_i a_j\nabla g_j(x^*) \\ &= \sum_{j\in K_1^d}\alpha_1 a_j\nabla g_j(x^*)+\sum_{j\in K_2^d}\alpha_2 a_j\nabla g_j(x^*)+\cdots+\sum_{j\in K_l^d}\alpha_l a_j\nabla g_j(x^*) \\ &= 0,\end{aligned}$$

where $\forall j \in K^d$, a_j is nonzero. However $\{K_1^d,\dots,K_l^d\}$ forms a partition of K^d and $\nabla g_{K^d}(x^*)$ is a full rank matrix, which implies that

$$\begin{aligned}\forall j \in K_1^d, \quad \alpha_1 a_j &= 0 \\ \forall j \in K_2^d, \quad \alpha_2 a_j &= 0 \\ \vdots &= \vdots \\ \forall j \in K_l^d, \quad \alpha_l a_j &= 0\end{aligned}$$

so that we have $\alpha_1 = \alpha_2 = \cdots = \alpha_l = 0$. Thus the columns of the matrix $\nabla g_E(x^*)$ are linearly independent, and we have the result. □

Proof of Lemma 2.7. Suppose that there is an $i \in E$ such that $\nabla g_i(x^*)$ and the columns of the matrix $\nabla g_{K^c}(x^*)$ are linearly dependent, which implies that there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i\nabla g_i(x^*) + \sum_{j\in K^c}\alpha_j\nabla g_i(x^*) = 0$. Since x^* is a weak regular point then

$$\begin{aligned}\alpha_i \nabla g_i(x^*) + \sum_{j\in K^c} \alpha_j \nabla g_i(x^*) &= \alpha_i \sum_{j\in K_i^d} a_j \nabla g_j(x^*) + \sum_{j\in K^c} \alpha_j \nabla g_i(x^*) \\ &= \sum_{j\in K_i^d} \alpha_i a_j \nabla g_j(x^*) + \sum_{j\in K^c} \alpha_j \nabla g_i(x^*) \\ &= 0,\end{aligned}$$

where $\forall j \in K_i^d$, a_j is a nonzero scalar. However $K_i^d \subseteq K$ and $K^c \subseteq K$ where

$$K = \{i_1, \cdots, i_k \in I^* \,|\, \nabla g_{i_1}, \cdots, \nabla g_{i_k}(x^*) \text{ are linearly independent}\},$$

which implies that

$$\begin{aligned}\forall j \in K_i^d, \quad \alpha_i a_j &= 0 \\ \forall j \in K^c, \quad \alpha_j &= 0,\end{aligned}$$

and so $\alpha_i = 0$, and we have the result. □

Proof of Lemma 2.9. Recall that

$$B(r) = \nabla g_{I^*\setminus K}(x(r)) {V_{I^*\setminus K}}^{1/2} + \nabla g_{K^d}(x(r))A$$

where $A = (A_1 \quad | \quad 0_{k^d\times(m^*-k^d-|E|)})$ such that

$$A_1(r) = \begin{pmatrix} \lambda_{J_1}(r) & 0 & \cdots & 0 \\ 0 & \lambda_{J_2}(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{J_l}(r) \end{pmatrix}$$

and $J_1, J_2 \cdots, J_l$ form a partition of K^d. Since x^* is a weak regular point then $\lim_{r\longrightarrow 0} B(r)$ is a bounded matrix.

Multiply $B(r)$ by $Q_{x(r)}^T$ to obtain

$$\begin{aligned}Q_{x(r)}^T B(r) &= Q_{x(r)}^T \nabla g_{I^*\setminus K}(x(r)) {V_{I^*\setminus K}}^{1/2}(r) + Q_{x(r)}^T \nabla g_{K^d}(x^*)A(r) \\ &= \begin{pmatrix} U_{13}(r){V_{I^*\setminus K}}^{1/2}(r) \\ U_{23}(r){V_{I^*\setminus K}}^{1/2}(r) \\ U_{33}(r){V_{I^*\setminus K}}^{1/2}(r) \end{pmatrix} + \begin{pmatrix} U_{11}(r)A(r) \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \bar{B}_1(r) \\ \bar{B}_2(r) \\ \bar{B}_3(r) \end{pmatrix},\end{aligned}$$

thus

$$\begin{cases} \lim_{r \longrightarrow 0} \begin{pmatrix} \bar{B}_2(r) \\ \bar{B}_3(r) \end{pmatrix} = \begin{pmatrix} U_{23}^* \\ U_{33}^* \end{pmatrix} \\ \\ \lim_{r \longrightarrow 0} \bar{B}_1(r) \quad \text{is a bounded matrix} \end{cases}$$

which shows points *i*), *ii*) and *iii*).
iv) Since $\nabla g_{K^d}(x^*)$ is of full rank then $\forall J_i \subseteq K^d$, $\nabla g_{J_i}(x^*)$ is of full rank. This enables us to say that the components of each column of the matrix $A_1(r)$ diverge at the same rate when $r \longrightarrow 0$, *i.e.*

$$\begin{aligned}
& \lim_{r \longrightarrow 0} V_{K^d}{}^{-1/2} A_1(r) \\
= \quad & \lim_{r \longrightarrow 0} \begin{pmatrix} V_{J_1}^{-1/2}(r)\lambda_{J_1}(r) & 0 & \cdots & 0 \\ 0 & V_{J_2}^{-1/2}(r)\lambda_{J_2}(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_{J_l}^{-1/2}(r)\lambda_{J_l}(r) \end{pmatrix} \\
= \quad & \lim_{r \longrightarrow 0} \begin{pmatrix} e_{J_1} & 0 & \cdots & 0 \\ 0 & e_{J_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{J_l} \end{pmatrix} \\
= \quad & \begin{pmatrix} e_{J_1} & 0 & \cdots & 0 \\ 0 & e_{J_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_{J_l} \end{pmatrix} \\
= \quad & A_1^*
\end{aligned}$$

where $e_{J_1}, e_{J_2}, \cdots, e_{J_l}$ are unit vectors. Then we have

$$\begin{aligned}
\lim_{r \longrightarrow 0} V_{K^d}{}^{-1/2}(r) A(r) &= \lim_{r \longrightarrow 0} V_{K^d}{}^{-1/2}(r)(A_1(r) \quad | \quad 0_{k^d \times (m^* - k^d - |E|)}) \\
&= (\lim_{r \longrightarrow 0} V_{K^d}{}^{-1/2}(r) A_1(r) \quad | \quad 0_{k^d \times (m^* - k^d - |E|)}) \\
&= (A_1^* \quad | \quad 0_{k^d \times (m^* - k^d - |E|)}) \\
&= A^*.
\end{aligned}$$

(v)

$$\begin{aligned} I + A^* A^{*T} &= I + {A_1}^* {A_1}^{*T} \\ &= \begin{pmatrix} I_{J_1} + e_{J_1} e_{J_1}^T & 0 & \cdots & 0 \\ 0 & I_{J_2} + e_{J_2} e_{J_2}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{J_l} + e_{J_l} e_{J_l}^T \end{pmatrix} \end{aligned}$$

therefore $I + A^* A^{*T}$ is a block diagonal matrix such that $e_{J_1} e_{J_1}^T$, $e_{J_2} e_{J_2}^T$, $\cdots$, $e_{J_l} e_{J_l}^T$ are matrices with all their components equal to 1, and thus by lemma 2.8, $I + A^* A^{*T}$ is a nonsingular matrix, and we have the proof of v).
(vi) Let us define

$$U_{11}(r) = \begin{pmatrix} U_{J_1}^1(r) & U_{J_2}^1(r) & \cdots & U_{J_l}^1(r) \\ 0 & U_{J_2}^2(r) & \cdots & U_{J_l}^2(r) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{J_l}^l(r) \end{pmatrix},$$

which enables us to write

$$\begin{gathered} {V_{K^d}}^{-1/2}(r) U_{11}(r) {V_{K^d}}^{1/2}(r) = \\ \begin{pmatrix} V_{J_1}^{-1/2}(r) U_{J_1}^1(r) {V_{J_1}}^{1/2}(r) & V_{J_1}^{-1/2}(r) U_{J_2}^1(r) {V_{J_2}}^{1/2}(r) & \cdots & V_{J_1}^{-1/2}(r) U_{J_l}^1(r) {V_{J_l}}^{1/2}(r) \\ 0 & V_{J_2}^{-1/2}(r) U_{J_2}^2(r) {V_{J_2}}^{1/2}(r) & \cdots & V_{J_2}^{-1/2}(r) U_{J_l}^2(r) {V_{J_l}}^{1/2}(r) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_{J_l}^{-1/2}(r) U_{J_l}^l(r) {V_{J_l}}^{1/2}(r) \end{pmatrix}. \end{gathered}$$

Since $\forall i \in \{1, 2, .., l\}$, the components of the vector $\lambda_{j_i}(r)$ diverge at the same rate then
$\lim\limits_{r \longrightarrow 0} V_{J_1}^{-1/2}(r) U_{J_1}^1(r) {V_{J_1}}^{1/2}(r)$, $\lim\limits_{r \longrightarrow 0} V_{J_2}^{-1/2}(r) U_{J_2}^2(r) {V_{J_2}}^{1/2}(r)$, $\cdots$, $\lim\limits_{r \longrightarrow 0} V_{J_l}^{-1/2}(r)$ $U_{J_l}^l(r)$ ${V_{J_l}}^{1/2}(r)$ are bounded upper triangular and nonsingular matrices. Since the components of the diagonal matrix ${V_{K^d}}^{1/2}(r)$ are subscripted in descending order then

$$\forall i, j \in \{1, 2, .., l\} \quad \text{such that} \quad i < j, \lim_{r \longrightarrow 0} {V_{J_i}}^{-1/2}(r) {U^i}_{J_j}(r) {V_{J_j}}^{1/2}(r) = 0,$$

which implies that $\lim\limits_{r \longrightarrow 0} {V_{K^d}}^{-1/2}(r) U_{11}(r) {V_{K^d}}^{1/2}(r) = \bar{U}_{11}$ are bounded upper triangular and nonsingular matrices.
(vii) Let

$$U_{11}(r) = \begin{pmatrix} U_{J_1}^1(r) & U_{J_2}^1(r) & \cdots & U_{J_l}^1(r) \\ 0 & U_{J_2}^2(r) & \cdots & U_{J_l}^2(r) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{J_l}^l(r) \end{pmatrix},$$

which enables us to write

$$V_{K^d}{}^{-1/2}(r)U_{11}(r)A_1(r) =$$
$$\begin{pmatrix} V_{J_1}^{-1/2}(r)U_{J_1}^1(r)\lambda_{J_1}(r) & V_{J_1}^{-1/2}(r)U_{J_2}^1(r)\lambda_{J_2}(r) & \cdots & V_{J_1}^{-1/2}(r)U_{J_l}^1(r)\lambda_{J_l}(r) \\ 0 & V_{J_2}^{-1/2}(r)U_{J_2}^2(r)\lambda_{J_2}(r) & \cdots & V_{J_2}^{-1/2}(r)U_{J_l}^2(r)\lambda_{J_l}(r) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_{J_l}^{-1/2}(r)U_{J_l}^l(r)\lambda_{J_l}(r) \end{pmatrix}.$$

Since the components of the diagonal matrix $V_{K^d}{}^{1/2}(r)$ are subscripted in descending order then the columns of the matrix A_1 are arranged in descending order with respect to the rate of divergence, which implies that

$$\lim_{r\longrightarrow 0} V_{K^d}{}^{-1/2}(r)U_{11}(r)A_1(r) =$$
$$\begin{pmatrix} \lim\limits_{r\longrightarrow 0} V_{J_1}^{-1/2}(r)U_{J_1}^1(r)\lambda_{J_1}(r) & 0 & \cdots & 0 \\ 0 & \lim\limits_{r\longrightarrow 0} V_{J_2}^{-1/2}(r)U_{J_2}^2(r)\lambda_{J_2}(r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lim\limits_{r\longrightarrow 0} V_{J_l}^{-1/2}(r)U_{J_l}^l(r)\lambda_{J_l}(r) \end{pmatrix},$$

from which follows $\lim\limits_{r\longrightarrow 0} V_{K^d}{}^{-1/2}(r)U_{11}(r)A_1(r) = \bar{A}_1$ is a bounded matrix. Consequently

$$\lim_{r\longrightarrow 0} V_{K^d}{}^{-1/2}(r)U_{11}(r)A(r) = \bar{A}$$

where $\bar{A} = (\bar{A}_1 \quad | \quad 0_{k^d\times(m^*-k^d-|E|)})$ is a bounded matrix, which concludes the proof of (vii). □

Proof of Lemma 2.10. Since $y_1(r) = (U_{11}^{-1})^T V_{K^d}^{-1/2} z_1(r)$ then

$$\begin{aligned} \lim_{r\longrightarrow 0} z_1(r) &= \lim_{r\longrightarrow 0} V_{K^d}^{1/2}(r){U_{11}}^T(r)y_1(r) \\ &= \lim_{r\longrightarrow 0} V_{K^d}^{1/2}(r){U_{11}}^T(r)V_{K^d}^{-1/2}(r)V_{K^d}^{1/2}(r)y_1(r). \end{aligned}$$

However, according to (vi) of lemma 2.9, $\lim\limits_{r\longrightarrow 0} V_{K^d}^{1/2}(r){U_{11}}^T(r)V_{K^d}^{-1/2}(r)$ is a bounded upper triangular and nonsingular matrix, and we have the result. □

Proof of Lemma 2.11. Let $i \in K^d$. Since $\forall r > 0$, $\lambda_i(r)$ is a differentiable function then

$$\begin{aligned} \lambda_i'(r) &= \frac{-g_i(x(r)) + r\nabla g_i(x(r))^T x'(r)}{g_i(x(r))^2} \\ &= \frac{\lambda_i(r)}{r} + \frac{\lambda_i(r)^2}{r}\nabla g_i(x(r))^T x'(r) \\ &= \frac{\lambda_i(r)}{r}(1 + \lambda_i(r)\nabla g_i(x(r))^T x'(r)) \end{aligned}$$

which yields

$$\frac{r\lambda_i'(r)}{\lambda_i(r)} = 1 + \lambda_i(r)\nabla g_i(x(r))^T x'(r). \tag{21}$$

Therefore $\lim_{r\longrightarrow 0} \lambda_i(r)\nabla g_i(x(r))^T x'(r)$ is bounded. According to Equation (16) we have

$$\begin{array}{rcl} V_{K^d}(r)^{1/2}y_1(r) & = & -V_{K^d}(r)^{1/2}(U_{11}^{-1})^T(r)V^{-1/2}{}_{K^d}(\hat{r})e_{K^d} \quad \text{where} \quad \hat{r} > r \\ & = & -V_{K^d}(r)^{1/2}(U_{11}^{-1})^T V_{K^d}(r)^{-1/2}V_{K^d}(r)^{1/2}V^{-1/2}{}_{K^d}(\hat{r})e_{K^d}. \end{array}$$

To show that $\lim_{r\longrightarrow 0} V_{K^d}(r)^{1/2}y_1(r)$ is a bounded vector it suffices to show that $\lim_{r\longrightarrow 0} V_{K^d}(r)^{1/2}V^{-1/2}{}_{K^d}(\hat{r})$ is a bounded matrix and

$$V_{K^d}(r)^{1/2}V^{-1/2}{}_{K^d}(\hat{r}) = \begin{pmatrix} \frac{\lambda_{i_1}(r)}{\lambda_{i_1}(\hat{r})} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_{i_2}(r)}{\lambda_{i_2}(\hat{r})} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_{i_l}(r)}{\lambda_{i_l}(\hat{r})} \end{pmatrix}.$$

According to Remark 2.3 and Equation (21), for $i \in K^d$ we have

$$\begin{array}{rcl} \dfrac{\lambda_i(r)}{\lambda_i(\hat{r})} & = & -\lambda_i(r)\nabla g_i(x(r))^T x'(r) \\ & = & 1 - \dfrac{r\lambda_i'(r)}{\lambda_i(r)} \end{array}$$

which implies that $\lim_{r\longrightarrow 0} \frac{\lambda_i(r)}{\lambda_i(\hat{r})}$ is bounded, and we have the result. □

Proof of Lemma 2.12. If hypotheses $\mathbf{H}_3$ and $\mathbf{H}_4$ are satisfied at x^* then according to relations (17) and (13), respectively,

$$\begin{array}{rcl} H(r) & = & \nabla^2 f(x(r)) + \sum_{i=1}^{m} \lambda_i(r)\nabla^2 g_i(x(r)) = \nabla^2 L(x(r), \lambda(r)) \\ \text{and } Q_{x(r)} & = & \begin{pmatrix} Q_{K^d}(r) & Q_{K^c}(r) & Q_{I^*\setminus K}(r) & Q_{J^*}(r) \end{pmatrix}, \end{array}$$

we have

$$Q_{x(r)}^T H Q_{x(r)} = \begin{pmatrix} {Q_{K^d}}^T(r) \\ {Q_{K^c}}^T(r) \\ {Q_{I^*\setminus K}}^T(r) \\ {Q_{J^*}}^T(r) \end{pmatrix} H\left(Q_{K^d}(r)\ Q_{K^c}(r)\ Q_{I^*\setminus K}(r)\ Q_{J^*}(r)\right)$$

$$= \begin{pmatrix} {Q_{K^d}}^T(r)H(r)Q_{K^d}(r) & {Q_{K^d}}^T(r)H(r)Q_{K^c}(r) & {Q_{K^d}}^T(r)H(r)Q_{I^*\setminus K}(r) & {Q_{K^d}}^T(r)H(r)Q_{J^*}(r) \\ {Q_{K^c}}^T(r)H(r)Q_{K^d}(r) & {Q_{K^c}}^T(r)H(r)Q_{K^c}(r) & {Q_{K^c}}^T(r)H(r)Q_{I^*\setminus K}(r) & {Q_{K^c}}^T(r)H(r)Q_{J^*}(r) \\ {Q_{I^*\setminus K}}^T(r)H(r)Q_{K^d}(r) & {Q_{I^*\setminus K}}^T(r)H(r)Q_{K^c}(r) & {Q_{I^*\setminus K}}^T(r)H(r)Q_{I^*\setminus K}(r) & {Q_{I^*\setminus K}}^T(r)H(r)Q_{J^*}(r) \\ {Q_{J^*}}^T(r)H(r)Q_{K^d}(r) & {Q_{J^*}}^T(r)H(r)Q_{K^c}(r) & {Q_{J^*}}^T(r)H(r)Q_{I^*\setminus K}(r) & {Q_{J^*}}^T(r)H(r)Q_{J^*}(r) \end{pmatrix}$$

$$= \begin{pmatrix} G_{11}(r) & G_{12}(r) & G_{13}(r) & G_{14}(r) \\ G_{21}(r) & G_{22}(r) & G_{23}(r) & G_{24}(r) \\ G_{31}(r) & G_{32}(r) & G_{33}(r) & G_{34}(r) \\ G_{41}(r) & G_{42}(r) & G_{43}(r) & G_{44}(r) \end{pmatrix}.$$

Since hypothesis $\mathbf{H}_7$ is satisfied and the columns of the matrix $Q_{I^*\setminus K}$ are the vectors of $\bar{D}^\perp \cap D_K^*$ then we have:

$$\lim_{r\longrightarrow 0} \begin{pmatrix} rG_{11}(r) & rG_{12}(r) & rG_{13}(r) & rG_{14}(r) \\ rG_{21}(r) & rG_{22}(r) & rG_{23}(r) & rG_{24}(r) \\ rG_{31}(r) & rG_{32}(r) & rG_{33}(r) & rG_{34}(r) \\ G_{41}(r) & G_{42}(r) & G_{43}(r) & G_{44}(r) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & G_{33}^* & 0 \\ G_{41}^* & G_{42}^* & G_{43}^* & G_{44}^* \end{pmatrix}.$$

□

Proof of Lemma 2.13. Lemmas 2.9 and 2.12 imply that $\begin{pmatrix} G_{33}^* + U_{33}^*{U_{33}^*}^T & 0 \\ G_{43}^* & G_{44}^* \end{pmatrix}$ is a bounded matrix. Condition $i)$ of W.S.O.S.C. implies that G_{44}^* is a nonsingular matrix. Condition $ii)$ of W.S.O.S.C. implies that $G_{33}^* + U_{33}^*{U_{33}^*}^T$ is a nonsingular matrix , which implies that $\begin{pmatrix} G_{33}^* + U_{33}^*{U_{33}^*}^T & 0 \\ G_{43}^* & G_{44}^* \end{pmatrix}$ is a nonsingular matrix. □

Proof of Lemma 2.14. If x^* is a weak regular point then, according to Lemma 2.9, we have

$$\lim_{r\longrightarrow 0} U_{23}(r){V_{I^*\setminus K}}^{1/2}(r) = U_{23}^*,$$
$$\lim_{r\longrightarrow 0} U_{33}(r){V_{I^*\setminus K}}^{1/2}(r) = U_{33}^*,$$
$$\lim_{r\longrightarrow 0} V_{K^d}^{-1/2}(r)A(r) = A^*,$$

and
(i)

$$\lim_{r\longrightarrow 0}\bar{\Omega}_2(r) = \lim_{r\longrightarrow 0}\begin{pmatrix} rG_{33}(r)+U_{33}(r)V_{I^*\setminus K}(r)U_{33}^T(r)+S_{33}(r) & rG_{34}(r)+S_{34}(r) \\ G_{43}(r)+\frac{1}{r}S_{43}(r) & G_{44}(r)+\frac{1}{r}S_{44}(r)\end{pmatrix}$$
$$= \begin{pmatrix} G_{33}^*+U_{33}^*{U_{33}^*}^T & 0 \\ G_{43}^* & G_{44}^*\end{pmatrix}.$$

According to Lemma 2.13, $\bar{\Omega}_2(0)$ is a nonsingular matrix.
(ii) Since we have

(1) $\lim_{r\longrightarrow 0}(rG_{31}(r)V_{K^d}^{-1/2}(r) + S_{31}(r)V_{K^d}^{-1/2}(r) + U_{33}(r)V_{I^*\setminus K}(r){U_{13}}^T(r) V_{K^d}^{-1/2}(r))$ is a bounded vector, indeed

$$\begin{aligned}
\lim_{r\longrightarrow 0}(rG_{31}(r)V_{K^d}^{-1/2}(r)) &= 0 \quad \text{by hypothesis } \mathbf{H}_7,\\
\lim_{r\longrightarrow 0}(S_{31}(r)V_{K^d}^{-1/2}(r)) &= 0,\\
\lim_{r\longrightarrow 0}(U_{33}(r)V_{I^*\setminus K}(r){U_{13}}^T(r)V_{K^d}^{-1/2}(r)) &= \lim_{r\longrightarrow 0}(U_{33}(r){V_{I^*\setminus K}}^{1/2}(r){V_{I^*\setminus K}}^{1/2}(r)\\
&\quad\times {U_{13}}^T(r)V_{K^d}^{-1/2}(r))\\
&= U_{33}^*\lim_{r\longrightarrow 0}(-A^T(r)U_{11}^T(r)+\bar{B}_1(r)^T)V_{K^d}^{-1/2}(r),\\
&\quad i) \text{ and } iii) \text{ from Lemma 2.9}\\
&= -U_{33}^*\lim_{r\longrightarrow 0}A^T(r)U_{11}^T(r)V_{K^d}^{-1/2}(r)\\
&\quad + U_{33}^*\lim_{r\longrightarrow 0}\bar{B}_1(r)^TV_{K^d}^{-1/2}(r)\\
&= -U_{33}^*\lim_{r\longrightarrow 0}A^T(r)U_{11}^T(r)V_{K^d}^{-1/2}(r)\\
&= -U_{33}^*\bar{A}^T,
\end{aligned}$$

(2) $\lim_{r\longrightarrow 0}(G_{41}(r)V_{K^d}^{-1/2}(r)+\frac{1}{r}S_{41}(r)V_{K^d}^{-1/2}(r))$ is a bounded vector, indeed

$$\begin{aligned}
&\lim_{r\longrightarrow 0}(G_{41}(r)V_{K^d}^{-1/2}(r)) = 0 \quad \text{by hypothesis } \mathbf{H}_7,\\
&\lim_{r\longrightarrow 0}(\tfrac{1}{r}S_{41}(r)V_{K^d}^{-1/2}(r)) = 0.
\end{aligned}$$

Then

$$\begin{aligned}\lim_{r\longrightarrow 0} N_2(r) &= \begin{pmatrix} F_3(0) \\ b_4(0) \end{pmatrix} - \lim_{r\longrightarrow 0} \begin{pmatrix} \bar{\Omega}_{13}(r) & r\Omega_{23}(r) \\ \bar{\Omega}_{14}(r) & r\Omega_{24}(r) \end{pmatrix} \begin{pmatrix} y_3(r) \\ y_4(r) \end{pmatrix} \\ &= \begin{pmatrix} F_3(0) \\ b_4(0) \end{pmatrix} - \begin{pmatrix} -U_{33}^*\bar{A}^T & {G^*}_{32} + U_{33}^*{U_{23}^*}^T \\ 0 & {G^*}_{42} \end{pmatrix} \begin{pmatrix} z_1(0) \\ y_2(0) \end{pmatrix} \\ &= N_2(0).\end{aligned}$$

According to Lemma 2.10, $z_1(0)$ is a bounded vector. Thus $N_2(0)$ is a bounded vector. □

Proof of Lemma 2.15. Vectors $y_3(r)$ and $y_4(r)$ are solutions of the following system:

$$\bar{\Omega}_2(r) \begin{pmatrix} y_3(r) \\ y_4(r) \end{pmatrix} = N_2(r).$$

When $r \longrightarrow 0$, this system approaches

$$\bar{\Omega}_2(0) \begin{pmatrix} y_3(0) \\ y_4(0) \end{pmatrix} = N_2(0).$$

According to lemma 2.14, $\bar{\Omega}_2(0)$ is a nonsingular matrix and $N_2(0)$ is a bounded vector, then the vector $\begin{pmatrix} y_3(0) \\ y_4(0) \end{pmatrix}$ exists and is bounded. □

Appendix A2: an illustrative example

In this section, we consider the following example

$$\begin{cases} \min & x_1 + x_3 + x_5 + {x_6}^2 \\ \text{s.t} & \begin{array}{c} -x_4 \le 0 \\ -x_2 \le 0 \\ -x_3 \le 0 \\ x_4 - {x_1}^5 \le 0 \\ x_2 - {x_5}^3 \le 0 \\ x_3 - 1 \le 0. \end{array} \end{cases}$$

For this problem, $x^* = (0,0,0,0,0,0)$ is the unique isolated solution. The gradients of the active constraints at x^* are linearly dependent, whereas $\nabla g_{I^*}(x(r))$ is a full rank matrix for all $r > 0$.

The penalized problems are written:

$$\begin{aligned}\phi(r,x) &= x_1+x_3+x_5+{x_6}^2-r\log(x_4)-r\log(x_2)-r\log(x_3)\\ &\quad -r\log(-x_4+{x_1}^5)-r\log(-x_2+{x_5}^3)-\log(-x_3+1)\end{aligned}$$

and their gradients are

$$\begin{aligned}\nabla\phi(r,x) &= \begin{pmatrix}1\\0\\1\\0\\1\\2x_6\end{pmatrix}-\left(\frac{r}{x_4}\right)\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}\\ &\quad -\left(\frac{r}{x_2}\right)\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}-\left(\frac{r}{x_3}\right)\begin{pmatrix}0\\0\\-1\\0\\0\\0\end{pmatrix}\\ &\quad -\left(\frac{r}{{x_1}^5-x_4}\right)\begin{pmatrix}5{x_1}^4\\0\\0\\-1\\0\\0\end{pmatrix}-\left(\frac{r}{{x_5}^3-x_2}\right)\begin{pmatrix}0\\-1\\0\\0\\3{x_5}^2\\0\end{pmatrix}\\ &\quad -\left(\frac{r}{x_3-1}\right)\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}\\ &= \begin{pmatrix}1-\frac{5{x_1}^4r}{({x_1}^5-x_4)}\\ \frac{-r}{x_2}+\frac{r}{({x_5}^3-x_2)}\\ 1+\frac{-r}{x_3}-\frac{r}{x_3-1}\\ \frac{-r}{x_4}+\frac{r}{({x_1}^5-x_4)}\\ 1+\frac{3{x_5}^2r}{{x_5}^3-x_2}\\ 2x_6\end{pmatrix}.\end{aligned}$$

The values which cancel $\nabla\phi(r,x)$ and the associated $\lambda_i(r)$ quantities are

$$\begin{cases} x_1(r) = 10r, \quad x_2(r) = 108r^3, \quad x_3(r) = \dfrac{1}{2} + r - \dfrac{1}{2}\sqrt{1+4r^2} \\ x_4(r) = 50000r^5, \quad x_5(r) = 6r, \quad x_6(r) = 0 \\ \lambda_1 = \dfrac{1}{50000}\dfrac{1}{r^4}, \quad \lambda_2 = \dfrac{1}{108}\dfrac{1}{r^2}, \quad \lambda_3 = \dfrac{r}{\dfrac{1}{2} + r - \dfrac{1}{2}\sqrt{1+4r^2}} \\ \lambda_4 = \dfrac{1}{50000}\dfrac{1}{r^4}, \quad \lambda_5 = \dfrac{1}{108}\dfrac{1}{r^2}, \quad \lambda_6 = -\dfrac{r}{\dfrac{1}{2} - r + \dfrac{1}{2}\sqrt{1+4r^2}}. \end{cases}$$

The multipliers $\lambda_4(r)$ and $\lambda_5(r)$ diverge when $r \longrightarrow 0$ whereas the trajectories, $x(r) = (x_1(r), x_2(r), x_3(r), x_4(r), x_5(r))$, are differentiable at the point $r = 0$.
Let us recall that

$$\begin{array}{rcl} \nabla g_I(x(r)) & = & \left(\nabla g_{K^d}(x(r)) \quad \nabla g_{K^c}(x(r)) \quad \nabla g_{I^*\setminus K}(x(r)) \quad \nabla g_{J^*}(x(r)) \right) \\ & = & \begin{pmatrix} 0 & 0 & 0 & -50000r^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -108r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

where

$$\begin{array}{l} K = \{i_1, \cdots, i_k \in I^* \,|\, \nabla g_{i_1}(x^*), \cdots, \nabla g_{i_k}(x^*) \text{ are linearly independent}\}, \\ K^d = \{i \in K \,|\, \lim\limits_{r \longrightarrow 0} \lambda_i(r) = +\infty\}, \\ K^c = \{i \in K \,|\, \lim\limits_{r \longrightarrow 0} \lambda_i(r) = \lambda_i^* > 0\}, \\ I^*\setminus K = \{i \in I^* \,|\, \nabla g_i(x^*) \text{ and the columns of } \nabla g_K(x^*) \text{ are linearly dependent}\}. \end{array}$$

In this example we have

$$\begin{array}{l} K = \{1, 2, 3\}, \\ K^d = \{1, 2\}, \\ K^c = \{3\}, \\ I^*\setminus K = \{4, 5\}, \end{array}$$

which implies that the associated canonical subspaces are

$$
\begin{array}{rcl}
D_K^* & = & \{d \quad | \quad \nabla g_K(x^*)^T d = 0\} \\
 & = & \{(d_1, 0, 0, 0, d_5, d_6) \quad | \quad d_1, d_5, d_6 \in \mathbb{R}\}, \\
\bar{D} & = & \lim\limits_{r \longrightarrow 0} \{d \quad | \quad \nabla g_{I^*}(x(r))^T d = 0\} \\
 & = & \{(0, 0, 0, 0, 0, d_6) \quad | \quad d_6 \in \mathbb{R}\}, \\
\bar{D}^{\perp} & = & \lim\limits_{r \longrightarrow 0} \{d \quad | \quad \nabla g_{I^*}(x(r))^T d = 0\}^{\perp} \\
 & = & \{(d_1, d_2, d_3, d_4, d_5, 0) \quad | \quad d_1, d_2, d_3, d_4, d_5 \in \mathbb{R}\}.
\end{array}
$$

Therefore

$$
\lambda_1(r) \nabla g_1(x(r)) + \lambda_4 \nabla g_4(x(r)) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{is a bounded vector,}
$$

and

$$
\lambda_2(r) \nabla g_2(x(r)) + \lambda_5 \nabla g_5(x(r)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{is a bounded vector,}
$$

which implies that $(0, 0, 0, 0, 0)$ is a weak regular point. Since

$$
\lambda(r) = \begin{pmatrix} \lambda_{K^d}(r) \\ \lambda_{K^c}(r) \\ \lambda_{I^* \setminus K}(r) \\ \lambda_{J^*}(r) \end{pmatrix},
$$

then we have

$$V(r) = \begin{pmatrix} V_{K^d}(r) & 0 & 0 & 0 \\ 0 & V_{K^c}(r) & 0 & 0 \\ 0 & 0 & V_{I^*\setminus K}(r) & 0 \\ 0 & 0 & 0 & V_{J^*}(r) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2500000000}\frac{1}{r^8} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{11664}\frac{1}{r^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2}{(-\frac{1}{2}-r+\frac{1}{2}\sqrt{1+4r^2})^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2500000000}\frac{1}{r^8} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{11664}\frac{1}{r^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{r^2}{(-\frac{1}{2}+r-\frac{1}{2}\sqrt{1+4r^2})^2} \end{pmatrix}$$

which gives us

- $V_{K^d}(r) = \begin{pmatrix} \frac{1}{2500000000}\frac{1}{r^8} & 0 \\ 0 & \frac{1}{11664}\frac{1}{r^4} \end{pmatrix}$,
- $V_{I^*\setminus K}(r) = \begin{pmatrix} \frac{1}{2500000000}\frac{1}{r^8} & 0 \\ 0 & \frac{1}{11664}\frac{1}{r^4} \end{pmatrix}$,
- $V_{K^c}(r) = \dfrac{r^2}{(-\frac{1}{2}-r+\frac{1}{2}\sqrt{1+4r^2})^2}$,
- $V_{J^*}(r) = \dfrac{r^2}{(-\frac{1}{2}+r-\frac{1}{2}\sqrt{1+4r^2})^2}$.

Moreover,

$$Q_{x(r)}{}^T\nabla g(x(r)) = \begin{pmatrix} U_{11}(r) & U_{12}(r) & U_{13}(r) & R_1(r) \\ 0 & U_{22}(r) & U_{23}(r) & R_2(r) \\ 0 & 0 & U_{33}(r) & R_3(r) \\ 0 & 0 & 0 & S(r) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 50000\, r^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 108\, r^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$Q_{x(r)} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which gives us

- $U_{11}(r) = \begin{pmatrix} 1 & 0 \\ & \\ 0 & 1 \end{pmatrix}$,
- $U_{12}(r) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,
- $U_{13}(r) = \begin{pmatrix} -1 & 0 \\ & \\ 0 & -1 \end{pmatrix}$,
- $U_{33}(r) = \begin{pmatrix} 50000\, r^4 & 0 \\ 0 & 108\, r^2 \end{pmatrix}$,
- $U_{22}(r) = 1$ and $U_{23}(r) = (0, 0)$.

We notice that

- $\lim\limits_{r \longrightarrow 0} U_{33}(r) V_{I^* \setminus K}^{1/2}(r) = \lim\limits_{r \longrightarrow 0} \begin{pmatrix} 50000\, r^4 & 0 \\ 0 & 108\, r^2 \end{pmatrix} \begin{pmatrix} \frac{1}{50000 r^4} & 0 \\ 0 & \frac{1}{108\, r^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
- $\lim\limits_{r \longrightarrow 0} U_{23}(r) V_{I^* \setminus K}^{1/2}(r) = \lim\limits_{r \longrightarrow 0} (0, 0) \begin{pmatrix} \frac{1}{50000 r^4} & 0 \\ 0 & \frac{1}{108\, r^2} \end{pmatrix} = (0, 0)$,

- $U_{13}(r)V_{I^*\setminus K}^{1/2}(r) = -U_{11}(r)A(r) + B_1(r)$ où $B_1(r) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

  et $A(r) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,

which illustrates Lemma 2.9. Also

$$\begin{aligned} G(x,r) &= Q_{x(r)}^T(\nabla^2 f(x(r)) + \sum_{i=1}^{i=6} \lambda_i(r)\nabla^2 g_i(x(r)))Q_{x(r)} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2}{5r} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{3r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

and we have

$$(i)\ \lim_{r\longrightarrow 0} \begin{pmatrix} rG_{11}(r) & rG_{12}(r) & rG_{13}(r) & rG_{14}(r) \\ rG_{21}(r) & rG_{22}(r) & rG_{23}(r) & rG_{24}(r) \\ rG_{31}(r) & rG_{32}(r) & rG_{33}(r) & rG_{34}(r) \\ G_{41}(r) & G_{42}(r) & G_{43}(r) & G_{44}(r) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-2}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \text{ is bounded,}$$

(ii) $G^*{}_{33} + U^*{}_{33} U^*{}^T_{33} = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$ is nonsingular,

(iii) $\begin{pmatrix} G^*{}_{33} + U^*{}_{33} U^*{}^T_{33} & 0 \\ G^*{}_{43} & G^*{}_{44} \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is nonsingular.

Then the weak second order sufficient condition and hypothesis $\mathbf{H}_7$ are satisfied.

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