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NONCOMMUTATIVE ZARISKI GEOMETRIES AND THEIR CLASSICAL LIMIT

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We undertake a case study of two series of nonclassical Zariski geometries and show that these geometries can be realised as representations of certain noncommutative C^*-algebras and introduce a natural limit construction which for each of the two series produces a classical U(1)-gauge field over a two-dimensional Riemann surface.

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1. Introduction

The notion of a Zariski geometry was introduced in [3] as a model-theoretic generalisation of objects of algebraic geometry and compact complex manifolds.

The main result of [3] was the classification of nonlinear (nonlocally modular) irreducible Zariski geometries of dimension one. The initial hope that every such geometry is definably isomorphic to an algebraic curve over an algebraically closed field F had to be corrected in the course of the study. The final classification theorem states that given a nonlinear irreducible one-dimensional Zariski geometry M there is an algebraically closed field F definable in M and an algebraic curve C over F such that M is a finite cover of $C(F)$, that is there is a Zariski continuous map $\mathbf{p} : M \to C(F)$ which is a surjection with finite fibres.

The paper [3] also provides a class of examples that demonstrates that in general we cannot hope to reduce $\mathbf{p}$ to a bijection. Given a smooth algebraic curve C with a big enough group G of regular automorphisms with a nonsplitting finite extension $\tilde{G}$, one can produce a "smooth" irreducible Zariski curve $\tilde{C}$ along with a finite cover $\mathbf{p} : \tilde{C} \to C$ and $\tilde{G}$ its group of Zariski-definable automorphisms.

Typically $\tilde{C}$ cannot be identified with any algebraic curve because $\tilde{G}$ is not embeddable into the group of regular automorphisms of an algebraic curve ([3], Sec. 10).

Taking into account known reductions of covers we can say that the above construction of $\tilde{C}$ is essentially the only way to produce a nonclassical Zariski curve. In other words, a general Zariski curve essentially looks like $\tilde{C}$ above.

A simple example of an unusual group $\tilde{G}$ for such a $\tilde{C}$, used in [3], is the class-2-nilpotent group of two generators $\mathbf{u}$ and $\mathbf{v}$ with the central commutator $[\mathbf{u}, \mathbf{v}]$ of finite order N. The correspondent G is then the free Abelian group on two generators. One can identify this $\tilde{G}$ as the quotient of the integer Heisenberg group $H_3(\mathbb{Z})$ by the subgroup of its centre of index N.

Also, since the group of regular isomorphisms of the smooth curve C must be infinite, we have very little freedom in choosing C; it has to be either the affine line over F, or the torus F^*, or an elliptic curve.

This paper undertakes the case study of the geometries of the corresponding $\tilde{C}$ for C an algebraic torus and an affine line.

The most comprehensive modern notion of a *geometry* is based on the consideration of a *coordinate algebra* of the geometric object. The classical meaning of a coordinate algebra comes from the algebra of *coordinate functions* on the object, that is, in our case, functions $\psi : \tilde{C}(F) \to F$ of a certain class. The most natural algebra of functions seems to be the algebra $F[\tilde{C}]$ of Zariski continuous (definable) functions. But by the virtue of the construction $F[\tilde{C}]$ is naturally isomorphic to $F[C]$, the algebra of regular functions on the algebraic curve C, that is the only geometry which we see by looking into $F[\tilde{C}]$ is the geometry of the algebraic curve C. To see the rest of the structure we had to extend $F[\tilde{C}]$ by introducing *semi-definable* functions, which satisfy certain *equations* but are not uniquely defined by these equations. The F-algebra of $\mathcal{H}(\tilde{C})$ of semi-definable functions contains the necessary information about $\tilde{C}$ but is not canonically defined. On the other hand, it is possible to define an F-algebra $\mathcal{A}(\tilde{C})$ of linear operators on $\mathcal{H}(\tilde{C})$ in a canonical way, depending on $\tilde{C}$ only. We proceed with this construction for both examples and write down explicit lists of generators and defining relations for algebras $\mathcal{A}(\tilde{C})$. One particular type of a semi-definable function which we call $*$-*functions*, of a clearly non-algebraic nature, plays a special role. The $*$-function induces an involution $*$ on $\mathcal{A}$. We show, for $F = \mathbb{C}$, that $\mathcal{A}$ thus gets the structure of a $*$-algebra, that is the involution $*$ associates with any $X \in \mathcal{A}$ its formal *adjoint* operator X^* satisfying usual formal requirements. Moreover, there is an $\mathcal{A}$-submodule of $\mathcal{H}(\tilde{C})$ with an inner product for which $*$ does indeed define adjoint operators.

Our first main theorem states that there is a reverse canonical construction which recovers $\tilde{C}$ from the algebra $\mathcal{A}$ uniquely. The points of $\tilde{C}$ correspond to one-dimensional eigenspaces (states) of certain self-adjoint operators, relations on C correspond to ideals of Cartesian powers of a commutative subalgebra of $\mathcal{A}$ and operations $\mathbf{u}$ and $\mathbf{v}$ correspond naturally to actions of certain operators of $\mathcal{A}$ on the states. This scheme is strikingly similar to the operator representations of quantum mechanics. Note that this construction is similar but not identical with the one we used in [7].

The final section of the paper concentrates on understanding the limit of the structures $\tilde{C} = \tilde{C}_N$, depending on N by the construction of $\tilde{G}$, as N tends to infinity. Among many possible ways to define the notion of the limit we found metric considerations most relevant. It turns out possible, when $F = \mathbb{C}$, to consider metric on each $\tilde{C}_N$ and to use correspondingly the notion of Hausdorff limit. Our main result in this section states that, for both types of examples, the Hausdorff limit $\tilde{C}_\infty$ of $\tilde{C}_N$, as N tends to infinity, is the structure identified as the principal U(1)-bundle over a Riemann surface with $\mathbf{u}$ and $\mathbf{v}$ defining a connection (covariant derivative) on the bundle. In physicists' terminology this is a gauge field with a connection of nonzero curvature (see e.g. [1] or [5]).

Combining with the results of the previous section one could speculate that $\tilde{C}_N$ are quantum deformations of the classical structure on $\tilde{C}_\infty$, and conversely, the latter is the classical limit of the quantum structures.

2. Non-Algebraic Zariski Geometries

2.0.1. The definitions on Zariski geometries in this section are all from [6].

Theorem 1. *There exists an irreducible pre-smooth Zariski structure* (*in particular of dimension* 1) *which is not interpretable in an algebraically closed field.*

The construction

Let $\mathbf{M} = (M, \mathcal{C})$ be an irreducible pre-smooth Zariski structure, $\mathrm{ZAut}\,\mathbf{M}$ the group of Zariski-continuous bijections of M, $G \leq \mathrm{ZAut}\,\mathbf{M}$ a subgroup acting freely on M and, for some $\tilde{G}$ with a finite subgroup $H \leq \tilde{G}$,

$$1 \to H \to \tilde{G} \overset{\mathrm{pr}}{\to} G \to 1,$$

a short exact sequence.

Consider a set $X \subseteq M$ of representatives of G-orbits: for each $a \in M$, $G \cdot a \cap X$ is a singleton.

Consider the formal set

$$\tilde{M}(\tilde{G}) = \tilde{M} = \tilde{G} \times X$$

and the projection map

$$\mathbf{p} : (g, x) \mapsto \mathrm{pr}(g) \cdot x.$$

Consider also, for each $f \in \tilde{G}$ the function

$$f : (g, x) \mapsto (fg, x).$$

We have thus obtained the structure

$$\tilde{\mathbf{M}} = (\tilde{M},\ \{f\}_{f \in \tilde{G}} \cup \mathbf{p}^{-1}(\mathcal{C}))$$

on the set $\tilde{M}$ with relations induced from $\mathbf{M}$ together with maps $\{f\}_{f \in \tilde{G}}$. We set the closed subsets of $\tilde{M}^n$ to be exactly those which are definable by positive quantifier-free formulas with parameters. Obviously, the structure $\mathbf{M}$ and the map $\mathbf{p} : \tilde{M} \to M$ are definable in $\tilde{\mathbf{M}}$. Since, for each $f \in \tilde{G}$,

$$\forall\, v,\ \mathbf{p}f(v) = f\mathbf{p}(v)$$

the image $\mathbf{p}(S)$ of a closed subset $S \subseteq \tilde{M}^n$ is closed in $\mathbf{M}$. We define $\dim S := \dim \mathbf{p}(S)$.

Lemma 1. *The isomorphism type of $\tilde{\mathbf{M}}$ is determined by M and $\tilde{G}$ only. The theory of $\tilde{\mathbf{M}}$ has quantifier elimination. $\tilde{\mathbf{M}}$ is an irreducible pre-smooth Zariski structure.*

Proof. One can use obvious automorphisms of the structure to prove quantifier elimination. The statement of the claim then follows by checking the definitions. The detailed proof is given in [3] Proposition 10.1. □

Lemma 2. *Suppose H does not split, that is for every proper $G_0 < \tilde{G}$*

$$G_0 \cdot H \neq \tilde{G}.$$

Then, every equidimensional Zariski expansion $\tilde{\mathbf{M}}'$ of $\tilde{\mathbf{M}}$ is irreducible.

Proof. Let $C = \tilde{M}'$ be an $|H|$-cover of the variety M, so $\dim C = \dim M$ and C has at most $|H|$ distinct irreducible components, say C_i, $1 \leq i \leq n$. For generic $y \in M$ the fibre $\mathbf{p}^{-1}(y)$ intersects every C_i (otherwise $\mathbf{p}^{-1}(M)$ is not equal to C).

Hence H acts transitively on the set of irreducible components. So, $\tilde{G}$ acts transitively on the set of irreducible components, so the setwise stabiliser G^0 of C_1 in $\tilde{G}$ is of index n in $\tilde{G}$ and also $H \cap G^0$ is of index n in H. Hence,

$$\tilde{G} = G^0 \cdot H,\ \text{with } H \nsubseteq G^0$$

contradicting our assumptions. □

Lemma 3. $\tilde{G} \leq \mathrm{ZAut}\,\tilde{\mathbf{M}}$, *that is $\tilde{G}$ is a subgroup of the group of Zariski-continuous bijecions of $\tilde{M}$.*

Proof. Immediate by construction. □

Lemma 4. *There is an integer μ such that given a rational or elliptic curve $\mathbf{M}$ (over an algebraically closed field F of characteristic zero) a subgroup G of the group of birational automorphisms of $\mathbf{M}$ and H and $\tilde{G}$ as above, with G nilpotent, without normal Abelian subgroups $G_0 \leq \tilde{G}$ such that*

$$|\tilde{G} : G_0| < \mu.$$

Then $\tilde{\mathbf{M}}$ is not interpretable in an algebraically closed field.

Proof. Assume that $\tilde{\mathbf{M}}$ is definable in an algebraically closed field F'. Then F is definable in F'. The latter is known (B. Poizat) to imply that F' is definably isomorphic to F, so we may assume that $F' = F$.

Also, since $\dim \tilde{\mathbf{M}} = \dim \mathbf{M} = 1$, it follows that $\tilde{\mathbf{M}}$, up to finitely many points, is in a bijective definable correspondence with a smooth projective algebraic curve, say $C = C(F)$, and every bijection $f \in \tilde{G}$ on $\tilde{\mathbf{M}}$ induces a birational transformation on C. Every birational transformation of a smooth projective algebraic curve $C(F)$ has a unique extension to a regular automorphism of $C(F)$, so $\tilde{G}$ is embedded into the group of regular automorphisms of $C(F)$.

The automorphism group of a curve is finite if genus of the curve is 2 or higher, so we can have only rational or elliptic curve for C.

Consider first the case when C is rational, that is C is a projective line. The group of regular automorphisms of $C(F)$ then is $\mathrm{PGL}(2, F)$. Since $\tilde{G}$ is nilpotent, by the well-known fact of the theory of linear groups $[M]$, for some positive integer μ_0, which does not depend on F, the group $\tilde{G}$ must have a normal unipotent (hence Abelian) subgroup of index less than μ_0. If we choose $\mu \geq \mu_0$, this contradicts the assumptions of the lemma.

In case C is an elliptic curve, the group of automorphisms is a semidirect product of a finitely generated Abelian group (the group of "complex multiplications") acting freely on the Abelian group of the elliptic curve. This group has no nilpotent non-Abelian subgroups. This finishes the proof of Lemma 4. $\square$

In general it is harder to analyse the situation when $\dim M > 1$ since the group of birational automorphisms is not so immediately reducible to the group of biregular automorphisms of a smooth variety in higher dimensions. But nevertheless the same method can prove the useful fact that the construction produces examples essentially of non-algebro-geometric nature.

Proposition 1. *Suppose* $\mathbf{M}$ *is an Abelian variety without complex multiplication, H does not split and $\tilde{G}$ is nilpotent non-Abelian. Then $\tilde{M}$ cannot be an algebraic variety with $p : \tilde{M} \to M$ a regular map.*

Proof. If $\mathbf{M}$ is an Abelian variety and $\tilde{\mathbf{M}}$ were algebraic, the map $p : \tilde{M} \to M$ has to be unramified since all its fibres are of the same order (equal to $|H|$). Hence $\tilde{\mathbf{M}}$ being a finite unramified cover must have the same universal cover as $\mathbf{M}$. So, $\tilde{\mathbf{M}}$ must be an Abelian variety as well. The group of automorphisms of an Abelian variety without complex multiplication is the canonical Abelian group of the variety. The contradiction. $\square$

Proposition 2. *Suppose* $\mathbf{M}$ *is an F-variety and, in the construction of* $\tilde{\mathbf{M}}$, *the group $\tilde{G}$ is finite. Then* $\tilde{\mathbf{M}}$ *is definable in any expansion of the field F by a total linear order.*

In particular, if $\mathbf{M}$ *is a complex variety,* $\tilde{\mathbf{M}}$ *is definable in the reals.*

Proof. Extend the ordering of F to a linear order of M and define

$$X := \{x \in M : \ x = \min G \cdot x\}.$$

The rest of the construction of $\tilde{\mathbf{M}}$ is definable. ◻

Problem. (i) *Classify Zariski structures definable in the reals.*
(ii) *Classify Zariski structures definable in the reals as a smooth real manifold.*
(iii) *Find new Zariski structures definable in* $\mathbb{R}_{an}$ *as a smooth real manifold.*

3. Examples

Let N be a positive integer and F an algebraically closed field of characteristic prime to N. Consider the groups given by generators and defining relations,

$$G = \langle u, v : \ uv = vu \rangle,$$
$$\tilde{G} = \tilde{G}_N = \langle \mathbf{u}, \mathbf{v} : \ [\mathbf{u}, [\mathbf{u}, \mathbf{v}]] = [\mathbf{v}, [\mathbf{u}, \mathbf{v}]] = 1 = [\mathbf{u}, \mathbf{v}]^N \rangle,$$

where $[\mathbf{u}, \mathbf{v}]$ stands for the commutator $\mathbf{v}\mathbf{u}\mathbf{v}^{-1}\mathbf{u}^{-1}$.

We will consider two examples of the construction of a one-dimensional $\tilde{\mathbf{M}}$ from an algebraic curve $\mathbf{M}$ using the groups G and $\tilde{G}$. By Sec. 2, G is going to be a subgroup of the group of rational automorphisms of $\mathbf{M}$, so $\mathbf{M}$ has to be of genus 0 or 1. In our examples $\mathbf{M}$ is the algebraic torus F^* and the affine line F.

3.1. *The N-cover of the affine line*

3.1.1. *We assume here that the characteristic of F is* 0.

Let $a, b \in F$ be additively independent.

G acts on F:

$$ux = a + x, \ \ vx = b + x.$$

Taking $\mathbf{M}$ to be F this determines, by Sec. 2, a presmooth non-algebraic Zariski curve $\tilde{M}$ which from now on we denote P_N, and P_N will stand for the universe of this structure.

The correspondent definition for the covering map $\mathbf{p} : \tilde{M} \to M = F$ then gives us

$$\mathbf{p}(\mathbf{u}t) = a + \mathbf{p}(t), \quad \mathbf{p}(\mathbf{v}t) = b + \mathbf{p}(t). \tag{1}$$

3.1.2. *Semi-definable functions on* P_N

Lemma. *Let ϵ be a primitive root of* 1 *of order N. There are functions* $\mathbf{y}$ *and* $\mathbf{z}$

$$P_N \to F$$

satisfying the following functional equations, *for any* $t \in P_N$,

$$\mathbf{y}^N(t) = 1, \ \mathbf{y}(\mathbf{u}t) = \epsilon\mathbf{y}(t), \ \mathbf{y}(\mathbf{v}t) = \mathbf{y}(t), \tag{2}$$
$$\mathbf{z}^N(t) = 1, \ \mathbf{z}(\mathbf{u}t) = \mathbf{z}(t), \ \mathbf{z}(\mathbf{v}t) = \mathbf{y}(t) \cdot \mathbf{z}(t). \tag{3}$$

Proof. Choose a subset $S \subseteq M = F$ of representatives of G-orbits, that is $F = G + S$. By the construction in Sec. 2 we can identify $P_N = \tilde{M}$ with $\tilde{G} \times S$ in such a way that $\mathbf{p}(g, s) = \mathrm{pr}(g) + s$. This means that, for any $s \in S$, a t in $\tilde{G} \cdot s$ is of the form $t = \mathbf{u}^m \mathbf{v}^n [\mathbf{u}, \mathbf{v}]^\ell \cdot s$ and

$$\mathbf{p}(\mathbf{u}^m \mathbf{v}^n [\mathbf{u}, \mathbf{v}]^\ell \cdot s) := ma + nb + s.$$

Set

$$\mathbf{y}(\mathbf{u}^m \mathbf{v}^n [\mathbf{u}, \mathbf{v}]^\ell \cdot s) := \epsilon^m,$$
$$\mathbf{z}(\mathbf{u}^m \mathbf{v}^n [\mathbf{u}, \mathbf{v}]^\ell \cdot s) := \epsilon^l.$$

This satisfies (2) and (3). □

Remark. Notice that it follows from (1)–(3):

(a) $\mathbf{p}$ is surjective and N-to-1, with fibres of the form

$$\mathbf{p}^{-1}(\lambda) = Ht, \quad H = \{[\mathbf{u}, \mathbf{v}]^\ell : 0 \leq l < N\}.$$

(b) $\mathbf{y}([\mathbf{u}, \mathbf{v}]t) = \mathbf{y}(t)$,
(c) $\mathbf{z}([\mathbf{u}, \mathbf{v}]t) = \epsilon \mathbf{z}(t)$.

3.1.3. Denote $F[N] = \{\xi \in F : \xi^N = 1\}$ and define the *band function* on F as a function $\mathrm{bd} : F \to F[N]$.

Set for $\lambda \in F$

$$\mathrm{bd}(\lambda) = \mathbf{y}(t), \text{ if } \mathbf{p}(t) = \lambda,$$

This is well defined by the remark in Sec. 3.1.2.

Acting by $\mathbf{u}$ on t and using (1) and (2) we have

$$\mathrm{bd}(a + \lambda) = \epsilon \mathrm{bd}\, \lambda. \tag{4}$$

Acting by $\mathbf{v}$ we obtain

$$\mathrm{bd}(b + \lambda) = \mathrm{bd}\, \lambda. \tag{5}$$

Remark. In a more general context we are going to call the band function and the angular function of the next section $*$-*functions*, explaining the reasons for this in Sec. 3.1.6.

Proposition. *The structure P_N is definable in*

$$(F, +, \cdot, \mathrm{bd}).$$

Proof. Set

$$P_N = F \times F[N] = \{\langle x, \epsilon^\ell \rangle : x \in F,\ \ell = 0, \ldots, N - 1\}$$

and define the maps

$$\mathbf{p}(\langle x, \epsilon^\ell \rangle) := x,\ \mathbf{u}(\langle x, \epsilon^\ell \rangle) := \langle a + x, \epsilon^\ell \rangle),\ \mathbf{v}(\langle x, \epsilon^\ell \rangle) := \langle b + x, \epsilon^\ell \mathrm{bd}(x) \rangle.$$

One checks easily that the action of $\tilde{G}$ is well-defined and (1) holds. □

Remark. One can easily define in $(F, +, \cdot, \mathrm{bd})$ functions $\mathbf{p}, \mathbf{y}$ and $\mathbf{z}$ satisfying (2) and (3)

Assuming that $F = \mathbb{C}$ and for simplicity that $a \in i\mathbb{R}$ and $b \in \mathbb{R}$, both nonzero, we may define, for $z \in \mathbb{C}$,

$$\mathrm{bd}\, z := \exp\left(\frac{2\pi i}{N}\left[\mathrm{Re}\left(\frac{z}{a}\right)\right]\right)$$

($[\mathrm{Re}(\frac{z}{a})]$ stands for the entire part of $\mathrm{Re}(\frac{z}{a})$).

This satisfies (4) and (5) and so P_N over $\mathbb{C}$ is definable in $\mathbb{C}$ equipped with the measurable but not continuous function $\mathrm{bd}\, z$ above.

Question. Does there exist a supersimple structure of the form $(F, +, \cdot, \mathrm{bd})$ satisfying (4) and (5)?

3.1.4. *The space of semi-definable functions*

Let $\mathcal{H}$ be an F-algebra containing all the functions $P_N \to F$ which are definable in the expansion of P_N by $\mathbf{y}$ and $\mathbf{z}$.

The term *semi-definable* corresponds to the fact that $\mathbf{y}$ and $\mathbf{z}$ are chosen to satisfy certain functional equations which do not determine these uniquely. On the other hand, these functional equations is all we need to know about these functions.

We define linear operators $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{U}$ and $\mathbf{V}$ on $\mathcal{H}$:

$$\begin{array}{rcl} \mathbf{X}: \ \psi(t) & \mapsto & \mathbf{p}(t) \cdot \psi(t), \\ \mathbf{Y}: \ \psi(t) & \mapsto & \mathbf{y}(t) \cdot \psi(t), \\ \mathbf{Z}: \ \psi(t) & \mapsto & \mathbf{z}(t) \cdot \psi(t), \\ \mathbf{U}: \ \psi(t) & \mapsto & \psi(\mathbf{u}t), \\ \mathbf{V}: \ \psi(t) & \mapsto & \psi(\mathbf{v}t). \end{array} \tag{6}$$

Denote $\tilde{G}^*$ the group generated by the operators $\mathbf{U}$, $\mathbf{V}$, $\mathbf{U}^{-1}$, $\mathbf{V}^{-1}$, denote $\mathfrak{X}_\epsilon$ (or simply $\mathfrak{X}$) the F-algebra $F[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]$ and $\mathcal{A}_\epsilon$ (or simply $\mathcal{A}$) the extension of the F algebra $\mathfrak{X}_\epsilon$ by $\tilde{G}^*$. We let $\mathbf{E}$ to stand for $\mathbf{VUV}^{-1}\mathbf{U}^{-1}$.

Lemma. *The generators of $\mathcal{A}_\epsilon$ satisfy the following relations.*

$$\begin{array}{ll} \mathbf{XY} & = \mathbf{YX}; \mathbf{XZ} = \mathbf{ZX}; \mathbf{YZ} = \mathbf{ZY}; \\ \mathbf{Y}^N & = 1; \mathbf{Z}^N = 1; \\ \mathbf{UX} & -\mathbf{XU} = a\mathbf{U}; \mathbf{VX} - \mathbf{XV} = b\mathbf{V}; \\ \mathbf{UY} & = \epsilon\mathbf{YU}; \mathbf{YV} = \mathbf{VY}; \\ \mathbf{ZU} & = \mathbf{UZ}; \\ \mathbf{VZ} & = \mathbf{YZV}; \\ \mathbf{UE} & = \mathbf{EU}; \mathbf{VE} = \mathbf{EV}; \mathbf{E}^N = 1. \end{array} \tag{7}$$

Proof. Immediate from (6), (1), (2) and (3). □

While elements of $\mathcal{H}$ and $\mathcal{H}$ as a whole are not uniquely defined, we prove in Sec. 3.1.6 that $\mathcal{A}$ is exactly the algebra of operators on $\mathcal{H}$ generated by $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}$ and $\mathbf{V}$ satisfying the defining relations (7).

We prove below, in the theorem of Sec. 3.1.5, that the algebra determined by the relations (7) is exactly $\mathcal{A}$ and so the definition of $\mathcal{A}$ does not depend on the arbitrariness in the construction of $\mathcal{H}$.

3.1.5. Let $\mathrm{Max}(\mathfrak{X})$ be the set of isomorphism classes of one-dimensional irreducible $\mathfrak{X}$-modules.

Lemma 5. $\mathrm{Max}(\mathfrak{X})$ *can be represented by one-dimensional modules* $\langle e_{\mu,\xi,\zeta}\rangle$ ($e_{\mu,\xi,\zeta}$ *generating the module*) *for* $\mu \in F, \xi, \zeta \in F[N]$, *defined by the action on the generating vector as follows*:

$$\mathbf{X}e_{\mu,\xi,\zeta} = \mu e_{\mu,\xi,\zeta}, \ \mathbf{Y}e_{\mu,\xi,\zeta} = \xi e_{\mu,\xi,\zeta}, \ \mathbf{Z}e_{\mu,\xi,\zeta} = \zeta e_{\mu,\xi,\zeta}.$$

Proof. This is a standard fact of commutative algebra. □

Remark. We can find some of the $e_{\mu,\xi,\zeta}$ in $\mathcal{H}$, which by definition contains the following *delta-functions*, for any $t \in P_N$,

$$\delta_t(s) = \begin{cases} 1, & \text{if } t = s; \\ 0, & \text{otherwise.} \end{cases}$$

One checks that

$$\mathbf{X}\delta_p = \mathbf{p}(t) \cdot \delta_t, \quad \mathbf{Y}\delta_t = \mathbf{y}(t) \cdot \delta_t, \quad \mathbf{Z}\delta_t = \mathbf{z}(t) \cdot \delta_t.$$

That is, up to a scalar, we get $\delta_t = e_{\mathbf{p}(t),\mathbf{y}(t),\mathbf{z}(t)}$ in this way.

Assuming F is endowed with a fixed function $\mathrm{bd} : F \to F[N]$, we call $\langle \mu, \xi, \zeta\rangle$ as above *real oriented* if

$$\mathrm{bd}\mu = \xi.$$

Correspondingly, we call the module $\langle e_{\mu,\xi,\zeta}\rangle$ real oriented if $\langle \mu, \xi, \zeta\rangle$ is.

$\mathrm{Max}^+(\mathfrak{X})$ will denote the subspace of $\mathrm{Max}(\mathfrak{X})$ consisting of real oriented modules $\langle e_{\mu,\xi,\zeta}\rangle$.

Lemma 6. $\langle \mu, \xi, \zeta\rangle$ *is real oriented if and only if*

$$\langle \mu, \xi, \zeta\rangle = \langle \mathbf{p}(t), \mathbf{y}(t), \mathbf{z}(t)\rangle,$$

for some $t \in P_N$. *Such a* t *is unique.*

Proof. It follows from the definition of bd that $\langle \mathbf{p}(t), \mathbf{y}(t), \mathbf{z}(t)\rangle$ is real oriented.

Assume now that $\langle \mu, \xi, \zeta\rangle$ is real oriented. Since $\mathbf{p}$ is a surjection, there is $t' \in P_N$ such that $\mathbf{p}(t') = \mu$. By the definition of bd, $\mathbf{y}(t') = \mathrm{bd}\,\mu$. By the remark in Sec. 3.1.2

both values stay the same if we replace t' by $t = [\mathbf{u}, \mathbf{v}]^k\, t'$. By the same remark, for some k, $\mathbf{z}(t) = \zeta$. $\square$

Now we introduce an infinite-dimensional $\mathcal{A}$-module $\mathcal{H}_R$. As a vector space $\mathcal{H}_R$ is spanned by

$$\{e_{\mu,\xi,\zeta} : \mu = \mathbf{p}(t), \xi = \mathbf{y}(t), \zeta = \mathbf{z}(t), \text{ for } t \in P_N\}.$$

The action of the generators of $\mathcal{A}$ on $\mathcal{H}_R$ is defined on $e_{\mu,\xi,\zeta}$. The action of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ is already defined, so we need only to define the action of $\mathbf{U}$ and $\mathbf{V}$.

Since we may identify $e_{\mathbf{p}(t),\mathbf{y}(t),\mathbf{z}(t)} = \delta_t$, we have by definitions

$$\mathbf{U}e_{\mathbf{p}(t),\mathbf{y}(t),\mathbf{z}(t)} = \mathbf{U}\delta_t = \delta_t(\mathbf{u}s) = \delta_{\mathbf{u}^{-1}t} = e_{\mathbf{p}(\mathbf{u}^{-1}t),\mathbf{y}(\mathbf{u}^{-1}t),\mathbf{z}(\mathbf{u}^{-1}t)},$$
$$\mathbf{V}e_{\mathbf{p}(t),\mathbf{y}(t),\mathbf{z}(t)} = \mathbf{V}\delta_t = \delta_t(\mathbf{v}s) = \delta_{\mathbf{v}^{-1}t} = e_{\mathbf{p}(\mathbf{v}^{-1}t),\mathbf{y}(\mathbf{v}^{-1}t),\mathbf{z}(\mathbf{v}^{-1}t)}.$$

Equivalently,

$$\mathbf{U}e_{\mu,\xi,\zeta} := e_{\mu-a,\epsilon^{-1}\xi,\zeta}$$

and

$$\mathbf{V}e_{\mu,\xi,\zeta} := e_{\mu-b,\xi,\xi^{-1}\zeta}.$$

From now on we identify $\mathrm{Max}^+(\mathfrak{X})$ with the family of one-dimensional $\mathfrak{X}$-eigenspaces of $\mathcal{H}_R$.

Theorem 2. (i) *There is a bijective correspondence* $\Xi : \mathrm{Max}^+(\mathfrak{X}) \to P_N$ *between the set of* $\mathfrak{X}$*-eigensubspaces of* $\mathcal{H}_R$ *and* P_N.
(ii) *The action of* $\tilde{G}^*$ *on* $\mathcal{H}_R$ *induces an action on* $\mathrm{Max}^+(\mathfrak{X}) \to \mathrm{Max}^+(\mathfrak{X})$. *The correspondence* Ξ *transfers anti-isomorphically the natural action of* $\tilde{G}^*$ *on* $\mathrm{Max}^+(\mathfrak{X})$ *to a natural action of* $\tilde{G}$ *on* P_N.
(iii) *The map*

$$\mathbf{p}_{\mathfrak{X}} : \langle e_{\mu,\xi,\zeta} \rangle \mapsto \mu$$

is an N*-to-*1*-surjection* $\mathrm{Max}^+(\mathfrak{X}) \to F$ *such that*

$$\left(\mathrm{Max}^+(\mathfrak{X}), \mathbf{U}, \mathbf{V}, \mathbf{p}_{\mathfrak{X}}, F\right) \cong_\xi (P_N, \mathbf{u}, \mathbf{v}, \mathbf{p}, F)\,.$$

(iv) *The action of the algebra* $\mathcal{A}$ *on* $\mathcal{H}_R$ *are faithful, that is an operator* T *of the algebra annihilates* $\mathcal{H}_R$ *if and only if* $T = 0$.
(v) $\mathcal{A}$ *is represented by defining relations* (7).

Proof. (i) Immediate by Lemma 6.
(ii) Indeed, by the definition above the action of $\mathbf{U}$ and $\mathbf{V}$ corresponds to the action on real oriented N-tuples:

$$\mathbf{U} : \langle \mathbf{p}(t), \mathbf{y}(t), \mathbf{z}(t)\rangle \mapsto \langle \mathbf{p}(t) - a, \epsilon^{-1}\mathbf{y}(t), \mathbf{z}(t)\rangle = \langle \mathbf{p}(\mathbf{u}^{-1}t), \mathbf{y}(\mathbf{u}^{-1}t), \mathbf{z}(\mathbf{u}^{-1}t)\rangle,$$
$$\mathbf{V} : \langle \mathbf{p}(t) - b, \mathbf{y}(t), \mathbf{y}(t)^{-1}\mathbf{z}(t)\rangle \mapsto \langle \mathbf{p}(\mathbf{v}^{-1}t), \mathbf{y}(\mathbf{v}^{-1}t), \mathbf{z}(\mathbf{v}^{-1}t)\rangle.$$

(iii) Immediate from (i) and (ii).
(iv) Using the relations (7) we can represent

$$T = \sum_{i \in I} c_i \mathbf{X}^{i_1} \mathbf{Y}^{i_2} \mathbf{Z}^{i_3} \mathbf{U}^{i_4} \mathbf{V}^{i_5} \mathbf{E}^{i_6} \tag{8}$$

for some finite $I \subset \mathbb{Z}^6$, $i = \langle i_1 \dots i_6 \rangle$ and $c_i \in \mathbb{C}$.

Given an element $e_{\mu,\xi,\zeta}$ of the basis, the action of T on it produces

$$Te_{\mu,\xi,\zeta} = \sum_{i \in I} c_i \mathbf{X}^{i_1} \mathbf{Y}^{i_2} \mathbf{Z}^{i_3} e_{\mu(i),\xi(i),\zeta(i)},$$

where

$$e_{\mu(i),\xi(i),\zeta(i)} = \mathbf{U}^{i_4} \mathbf{V}^{i_5} \mathbf{E}^{i_6} e_{\mu,\xi,\zeta}$$

is a basis element by definition of the action of $\mathbf{U}$ and $\mathbf{V}$, moreover one can check that $e_{\mu(i),\xi(i),\zeta(i)}$ are distinct for distinct $\mathbf{U}^{i_4} \mathbf{V}^{i_5} \mathbf{E}^{i_6}$.

Since the basis elements are eigenvectors of $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$

$$Te_{\mu,\xi,\zeta} = \sum_{i \in I} c_i \cdot d_i(\mu,\xi,\zeta) e_{\mu(i),\xi(i),\zeta(i)}$$

for some nonzero $d_i(\mu,\xi,\zeta) \in \mathbb{C}$.

Now assume that T annihilates $\mathcal{H}_R$. Then the right-hand side of the above must be zero and by linear independence all $c_i \cdot d_i(\mu,\xi,\zeta) = 0$, which can only happen if all $c_i = 0$ and $T = 0$.
(v) Let $\mathcal{B}$ be the algebra given by the abstract generators $\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{U}, \mathbf{V}, \mathbf{U}^{-1}, \mathbf{V}^{-1}$ satisfying the relations (7). Every element T of this algebra can be represented in the form (8). Let $\pi : \mathcal{B} \to \mathcal{A}$ be the obvious homomorphism onto $\mathcal{A}$. Suppose there is an element T of the algebra $\mathcal{B}$ such that $\pi(T) = 0$. T can be represented in the form (8). By (iv) all the coefficients c_i must be zero, so $T = 0$. This proves that π is an isomorphism. □

3.1.6. *$*$-representation*

Our aim here is to introduce a natural $*$-algebra structure on $\mathcal{A}$. Recall that a $\mathbb{C}$-algebra $\mathcal{A}$ is called a $*$-algebra if there is a map $* : \mathcal{A} \to \mathcal{A}$ (taking adjoints) satisfying the following properties: for all $X, Y \in \mathcal{A}$:

(1) $(X^*)^* = X$,
(2) $(XY)^* = Y^* X^*$,
(3) $(X + Y)^* = X^* + Y^*$,
(4) for every $\lambda \in \mathbb{C}$ and every $X \in \mathcal{A}$:

$$(\lambda X)^* = \overline{\lambda} X^*.$$

Of course, this definition is inherently linked with the real–complex structure of $\mathbb{C}$, which one would not expect to interact well with a ω-stable structure. In our case the $*$-structure results from the $*$-data we identified while representing P_N in Sec. 3.1.3–3.1.5, namely the band function bd and the basis $\{e_{\mu,\xi,\zeta}\}$ of $\mathcal{H}_R$. We tend to think of an operator X as self-adjoint, that is $X^* = X$, if for any eigenvalue x of X, $\operatorname{bd} x = 1$ (compare this with the remark about bd at the end of Sec. 3.1.3). We think of the basis $\{e_{\mu,\xi,\zeta}\}$ as an orthonormal one. Multiplications by roots of unity preserves orthonormality. Correspondingly, a unitary operator is the one which transforms this orthonormal basis to an orthonormal one. In particular, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Y}$ and $\mathbf{Z}$ should be assumed unitary. These are the principles that induce our definitions below.

We will assume $F = \mathbb{C}$ and find a $*$-algebra structure on an extension $\mathcal{A}^\#$ of $\mathcal{A}$, so $\mathcal{A}$ is a subalgebra of a $*$-algebra, but not necessarily a $*$-algebra itself.

We will also assume $a = 2\pi i/N = \epsilon$, $b \in \mathbb{R}$ and start by extending the space $\mathcal{H}$ of semi-definable functions with a function $\mathbf{w} : P_N \to \mathbb{C}$ such that

$$\exp \mathbf{w} = \mathbf{y},\ \ \mathbf{w}(\mathbf{u}t) = \frac{2\pi i}{N} + \mathbf{w}(t),\ \ \mathbf{w}(\mathbf{v}t) = \mathbf{w}(t).$$

We can easily do this by setting as in Sec. 3.1.2

$$\mathbf{w}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u},\mathbf{v}]^\ell \cdot s) := \frac{2\pi i m}{N}.$$

Now we extend $\mathcal{A}$ to $\mathcal{A}^\#$ by adding the new operator

$$\mathbf{W} : \psi \mapsto \mathbf{w}\psi$$

which obviously satisfies

$$\mathbf{W}\mathbf{X} = \mathbf{X}\mathbf{W},\ \ \mathbf{W}\mathbf{Y} = \mathbf{Y}\mathbf{W},\ \ \mathbf{W}\mathbf{Z} = \mathbf{Z}\mathbf{W}.$$
$$\mathbf{U}\mathbf{W} = \frac{2\pi i}{N} + \mathbf{W}\mathbf{U},\ \ \mathbf{V}\mathbf{W} = \mathbf{W}\mathbf{V}.$$

We set

$$\mathbf{U}^* := \mathbf{U}^{-1},\ \ \mathbf{V}^* := \mathbf{V}^{-1},\ \ \mathbf{Y}^* = \mathbf{Y}^{-1},$$
$$\mathbf{Z}^* := \mathbf{Z}^{-1},\ \ \mathbf{W}^* := -\mathbf{W},\ \ \mathbf{X}^* := \mathbf{X} - 2\mathbf{W},$$

implying that $\mathbf{U}, \mathbf{V}, \mathbf{Y}$ and $\mathbf{Z}$ are unitary and $i\mathbf{W}$ and $\mathbf{X} - \mathbf{W}$ are formally self-adjoint.

Lemma. *There is a representation of $\mathcal{A}^\#$ in an inner product space such that $\mathbf{U}, \mathbf{V}$ and $\mathbf{Y}$ act as unitary and $i\mathbf{W}$ and $\mathbf{X} - \mathbf{W}$ as self-adjoint operators.*

Proof. Let $\mathcal{H}_R^0$ be the inner product space spanned (finite sums) by vectors $e_{\mu,\xi,\zeta}$, assumed to be an orthonormal system, such that

$$\mu = x + \frac{2\pi i k}{N},\quad \xi = e^{\frac{2\pi i k}{N}},\quad \zeta = e^{\frac{2\pi i m}{N}},\qquad \text{for } x \in \mathbb{R},\ \ k, m \in \mathbb{Z}. \tag{9}$$

One checks that $\mathcal{H}_R^0$ is closed under the action of $\mathcal{A}$ defined in Sec. 3.1.5, that is $\mathcal{H}_R^0$ is an $\mathcal{A}$-module. We also define the action by $\mathbf{W}$

$$\mathbf{W} : e_{\mu,\xi,\zeta} \mapsto \frac{2\pi i k}{N} e_{\mu,\xi,\zeta}$$

for $\mu = x + 2\pi ik/N$. This obviously agrees with the defining relations of $\mathcal{A}^{\#}$. So $\mathcal{H}_R^0$ is an $\mathcal{A}^{\#}$-module.

Now $\mathbf{U}, \mathbf{V}, \mathbf{Y}$ and $\mathbf{Z}$ are unitary operators on $\mathcal{H}_R^0$ since they transform the orthonormal basis into an orthonormal one. $i\mathbf{W}$ and $\mathbf{X} - \mathbf{W}$ are self-adjoint since their eigenvalues on the orthonormal basis are the reals $-\frac{2\pi k}{N}$ and x, correspondingly. $\square$

Corollary. *The $*$-operation on the generators of $\mathcal{A}^{\#}$ defined above extends uniquely to $*$-operation on the whole $\mathcal{A}^{\#}$ and $(\mathcal{A}^{\#}, *)$ satisfies all the identities of a $\mathbb{C}$-algebra with adjoints. Moreover, since $\mathcal{A}^{\#}$ has a faithful representation on an inner product space we can introduce the usual operator norm on $\mathcal{A}^{\#}$ with $\mathbf{Y}, \mathbf{Z}, \mathbf{W}, \mathbf{U}$ and $\mathbf{V}$ bounded operators and $\mathbf{X}$ unbounded.*

Remark 1. Our choice of the $*$-structure on $\mathcal{A}^{\#}$ has been motivated by

(i) the need to encode the fact that the relevant $e_{\mu,\xi,\zeta}$ must be "real oriented", that is $\mathrm{bd}\,\mu = \xi$;
(ii) the natural interpretation of the band function and the related function $\mathbf{w}$ (for $a \in i\mathbb{R}$ and $b \in \mathbb{R}$ and $N \to \infty$) as functions indicating when μ is "almost real". More precisely, as remarked in Sec. 3.1.3 bd can be interpreted, for $a = 2\pi i/N$, $b \in \mathbb{R}$, as

$$\mathrm{bd}(x + 2\pi iy) = \exp 2\pi i \frac{[yN]}{N},$$

where $x, y \in \mathbb{R}$, and $[yN]$ is the entire part of yN. Since $[yN]/N$ converges to y the condition $\mathrm{bd}\,\mu = 1$ says that μ is "almost real".

Remark 2. The natural interpretation of the band function is used in Sec. 4 to obtain "the classical limit" P_∞ of the P_N.

Comments. (1) We have seen that in the representation $\mathcal{H}_R$ the $e_{\mu,\xi,\zeta}$ are eigenvectors of the self-adjoint operator $\mathbf{X} - \mathbf{W}$. So in physics jargon $\langle e_{\mu,\xi,\zeta}\rangle$ would be called *states*.
(2) The discrete nature of the imaginary part of μ in (9) is necessitated by two conditions: the interpretation of $*$ as taking adjoints and the noncontinuous form of the band function. The first condition is crucial for any physical interpretation and the second one follows from the description of the Zariski structure P_N. Comparing this with the real differentiable structure P_∞ constructed in Sec. 4 as the limit of the P_N, we suggest to interpret the latter along with its representation via $\mathcal{A}$ in this section as the *quantisation* of the former.

3.2. *The algebraic torus case*

3.2.1. Let F be an algebraically closed field of any characteristic prime to N and $a, b \in F^*$ be multiplicatively independent.

G acts on F^* :

$$ux = ax, \; vx = bx.$$

Taking M to be F^* this determines, by Sec. 2, a presmooth non-algebraic Zariski curve $\tilde{M}$ which from now on we denote T_N.

The correspondent definition for the covering map $\mathbf{p} : \tilde{M} = T_N \to M = F^*$ then gives us

$$\mathbf{p}(\mathbf{u}t) = a\mathbf{p}(t), \quad \mathbf{p}(\mathbf{v}t) = b\mathbf{p}(t). \tag{10}$$

We also note that there exists the well-defined function $\mathbf{p}' : T_N \to F$ given by

$$\mathbf{p}'(t)\mathbf{p}(t) = 1. \tag{11}$$

For the rest of the section fix ϵ to be a primitive root of unity of order N, $\alpha = a^{1/N}$ and $\beta = b^{1/N}$, roots of a and b of order N.

3.2.2. *Semi-definable functions in T_N*

Lemma. *There exist functions*

$$\mathbf{x}, \mathbf{x}', \mathbf{y} : T_N \to \mathrm{F}$$

satisfying the following functional equations, *for any $t \in T_N$,*

$$\mathbf{x}^N(t) = \mathbf{p}(t), \quad \mathbf{x}(\mathbf{u}t) = \alpha\mathbf{x}(t), \quad \mathbf{x}(\mathbf{v}t) = \beta\mathbf{y}(t)\mathbf{x}(t), \tag{12}$$

$$\mathbf{x}(t)\mathbf{x}'(t) = 1, \tag{13}$$

$$\mathbf{y}^N(t) = 1, \quad \mathbf{y}(\mathbf{u}t) = \epsilon\mathbf{y}(t), \quad \mathbf{y}(\mathbf{v}t) = \mathbf{y}(t). \tag{14}$$

Proof. Choose a subset $S \subseteq \mathrm{F}^*$ of representatives of G-orbits, that is $\mathrm{F} = G \cdot S$. By the construction in Sec. 2.0.1 we can identify $T_N = \tilde{M}$ with $\tilde{G} \times S$ in such a way that $\mathbf{p}(\gamma s) = \mathrm{pr}(\gamma) \cdot s$. This means that, for any $s \in S$ and $t \in \tilde{G} \cdot s$ of the form $t = \mathbf{u}^m\mathbf{v}^n[\mathbf{u}, \mathbf{v}]^\ell \cdot s$,

$$\mathbf{p}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u}, \mathbf{v}]^\ell \cdot s) := a^m \cdot b^n \cdot s.$$

Fix (randomly) a root $s^{1/N}$ of order N for each $s \in S$. Set

$$\mathbf{x}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u}, \mathbf{v}]^\ell \cdot s) := \alpha^m \cdot \beta^n \cdot \epsilon^{-\ell} s^{\frac{1}{N}}.$$

Set also

$$\mathbf{y}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u}, \mathbf{v}]^\ell \cdot s) := \epsilon^m.$$

This satisfies (12)–(14). $\square$

Remark. Notice that it follows from (12) and (14) that
$\mathbf{x}([\mathbf{u}, \mathbf{v}]t) = \epsilon^{-1}\mathbf{x}(t)$,
$\mathbf{y}([\mathbf{u}, \mathbf{v}]t) = \mathbf{y}(t)$.

3.2.3. Define the *angular function* on F as a function $\mathrm{ang} : F \to F[N]$.

Set for $\lambda \in F$

$$\mathrm{ang}(\lambda) = \mathbf{y}(t), \quad \text{if } \mathbf{p}(t) = \lambda.$$

This is well defined by the remark in Sec. 3.2.2.

Acting by $\mathbf{u}$ on t and using (10) and (14) we have

$$\mathrm{ang}(a\lambda) = \epsilon \, \mathrm{ang}\, \lambda. \tag{15}$$

Acting by $\mathbf{v}$ we obtain

$$\mathrm{ang}(b\lambda) = \mathrm{ang}\, \lambda. \tag{16}$$

Proposition. The structure T_N is definable in

$$(F, +, \cdot, \mathrm{ang}).$$

Indeed, set $T_N = F^*$ and define the maps

$$\mathbf{p}(t) := t^N$$

and

$$\mathbf{u}(t) := \alpha t, \ \mathbf{v}(t) := \mathrm{ang}(t^N)\beta t.$$

One checks easily that

$$\mathbf{vu}(t) = \epsilon \cdot \mathbf{uv}(t)$$

and so the action of $\tilde{G}$ is well-defined and (10) holds.

Remark 3. Assuming that $F = \mathbb{C}$ and $\epsilon = \exp(2\pi i/N)$, let for an $r \in \mathbb{R}$,

$$a = \exp\left(\frac{2\pi i}{N} + r\right), \quad \text{and } b \in \mathbb{R}_+, \ b \neq 1.$$

Then we may define, for $z \in \mathbb{C}$,

$$\mathrm{ang}\, z := \exp\left(\frac{2\pi i}{N}\left[\frac{N}{2\pi}\arg z\right]\right).$$

This is a well-defined function satisfying also (15) and (16), and so T_N over $\mathbb{C}$ is definable in $\mathbb{C}$ equipped with the measurable but not continuous function above.

It is also interesting to remark that, for this angular function,

$$\left|\arg z - \frac{2\pi}{N}\left[\frac{N}{2\pi}\arg z\right]\right| \leq \frac{2\pi}{N}$$

and so $\mathrm{ang}\, z$ converges uniformly on z to $\exp(i \arg z)$ as N tends to ∞.

Remark 4. In the context of noncommutative geometry it is interesting to see whether there exists an abstract, model-theoretic interpretation of ang which allows a *measure theory* for the semi-definable functions introduced above. David Evans proved the following theorem.

Theorem. ([2]) *The class of fields* $(\mathrm{F}, +, \cdot, \mathrm{ang})$ *of a fixed characteristic endowed with a function* ang *satisfying* (15) *and* (16) *has a model companion, which is a supersimple theory. The models of the theory allow a nontrivial finite measure such that all definable sets are measurable.*

3.2.4. *The space of semi-definable functions and the operator algebra*

Let $\mathcal{H}$ be an F-algebra containing all the functions $T_N \to F$ which are definable in T_N expanded by $\mathbf{x}$ and $\mathbf{y}$.

We define linear operators $\mathbf{X}, \mathbf{X}^{-1}, \mathbf{Y}$, $\mathbf{U}$ and $\mathbf{V}$ on $\mathcal{H}$:

$$\begin{array}{ll} \mathbf{X} & : \psi(t) \mapsto \mathbf{x}(t) \cdot \psi(t), \\ \mathbf{X}^{-1} & : \psi(t) \mapsto \mathbf{x}'(t) \cdot \psi(t), \\ \mathbf{Y} & : \psi(t) \mapsto \mathbf{y}(t) \cdot \psi(t), \\ \mathbf{U} & : \psi(t) \mapsto \psi(\mathbf{u}t), \\ \mathbf{V} & : \psi(t) \mapsto \psi(\mathbf{v}t). \end{array} \tag{17}$$

Denote $\tilde{G}^*$ the group generated by the operators $\mathbf{U}$, $\mathbf{V}$, $\mathbf{U}^{-1}$, $\mathbf{V}^{-1}$, denote $\mathfrak{X}_\epsilon$ the F-algebra $F[\mathbf{X}, \mathbf{X}^{-1}, \mathbf{Y}]$ and $\mathcal{A}_\epsilon$ (or simply $\mathcal{A}$) the extension of the F algebra $\mathfrak{X}_\epsilon$ by $\tilde{G}^*$.

The generators of the algebra $\mathcal{A}_\epsilon$ obviously satisfy the following relations, for $\mathbf{E}$ standing for $\mathbf{VUV}^{-1}\mathbf{U}^{-1}$.

$$\begin{array}{l} \mathbf{XY} = \mathbf{YX}; \\ \mathbf{Y}^N = 1;\ \mathbf{XX}^{-1} = 1; \\ \mathbf{XU} = \alpha^{-1}\mathbf{UX}; \\ \mathbf{XV} = \beta^{-1}\mathbf{Y}^{-1}\mathbf{VX}; \\ \mathbf{YU} = \epsilon^{-1}\mathbf{UY}; \\ \mathbf{YV} = \mathbf{VY}; \\ \mathbf{UE} = \mathbf{EU}; \mathbf{VE} = \mathbf{EV}; \mathbf{E}^N = 1. \end{array} \tag{18}$$

By the argument in Theorem 2(iv) and (v), the algebra determined by the relations (18) is exactly $\mathcal{A}$ and so the definition of $\mathcal{A}$ does not depend on the arbitrariness in the construction of $\mathcal{H}$.

3.2.5. Let $\mathrm{Max}(\mathfrak{X})$ be the set of isomorphism classes of one-dimensional irreducible $\mathfrak{X}$-modules.

Lemma 7. $\mathrm{Max}(\mathfrak{X})$ *can be represented by one-dimensional modules* $\langle e_{\mu,\xi} \rangle$ $(= \mathrm{F}e_{\mu,\xi})$ *for* $\mu \in F, \xi \in F[N]$, *defined by the action on the corresponding generating vector:*

$$\mathbf{X}e_{\mu,\xi} = \mu e_{\mu,\xi}, \quad \mathbf{Y}e_{\mu,\xi} = \xi e_{\mu,\xi}.$$

Proof. This is a standard fact of commutative algebra. $\square$

Assuming F is endowed with an angular function ang, we call $\langle \mu, \xi\rangle$ as above *positively oriented* if

$$\operatorname{ang} \mu^N = \xi.$$

Correspondingly, we call the $\mathfrak{X}$-module (state) $\langle e_{\mu,\xi}\rangle$ positively oriented if $\langle \mu, \xi\rangle$ is.

$\mathcal{H}_0^+$ will denote the linear space spanned by the positively oriented states $\langle e_{\mu,\xi}\rangle$.

Lemma 8. *$\langle \mu, \xi\rangle$ is positively oriented if and only if*

$$\langle \mu, \xi\rangle = \langle \mathbf{x}(t), \mathbf{y}(t)\rangle,$$

for some $t \in T$. Such a t is unique.

Proof. Indeed, since p is a surjection, there is $t' \in T$ such that $\mathbf{p}(t') = \mu^N$. Hence, by the definition of $\mathbf{x}(t')$ and $\operatorname{ang}(t')$ we have $\mathbf{x}(t') = \epsilon^k \mu$, $\mathbf{y}(t') = \xi$, for some k. By the remark in Sec. 3.2.2 we have $\mathbf{x}([\mathbf{u}, \mathbf{v}]^k t') = \epsilon^{-k}\mathbf{x}(t') = \mu$ and $\mathbf{y}([\mathbf{u}, \mathbf{v}]^k t') = \mathbf{y}(t') = \xi$. So $t = [\mathbf{u}, \mathbf{v}]^k t'$ is as required. $\square$

Remark. It is immediate from the lemma and Remark 3 that all the positively oriented $e_{\mu,\xi}$ are represented by the delta-functions δ_t, $t \in T_N$.

Using the representation of the $e_{\mu,\xi}$ by the delta-functions and the action of $\mathbf{U}$ and $\mathbf{V}$ on the space of functions defined in (17) we get

$$\mathbf{U}e_{\mu,\xi} := e_{\nu,\zeta}, \text{ with } \nu = \alpha\mu, \quad \zeta = \epsilon^{-1}\xi$$

and

$$\mathbf{V}e_{\mu,\xi} := e_{\nu,\zeta}, \text{ with } \nu = \beta\xi^{-1}\mu, \quad \zeta = \xi.$$

We denote $\mathcal{H}_0^+$ the linear space spanned by all the positively oriented $e_{\mu,\xi}$, and denote $\operatorname{Max}^+(\mathfrak{X})$ the family of one-dimensional positively oriented $\mathfrak{X}$-eigenspaces of $\mathcal{H}_0^+$ or *states* as such things are referred to in physics literature.

Theorem 3. (i) *There is a bijective correspondence $\Xi:\ \operatorname{Max}^+(\mathfrak{X}) \to T_N$ between the set of positively oriented states and T_N.*
(ii) *The action of $\tilde{G}^*$ on $\mathcal{H}$ leaves $\mathcal{H}_0^+$ and $\operatorname{Max}^+(\mathfrak{X})$ setwise invariant. The correspondence Ξ transfers anti-isomorphically the natural action of $\tilde{G}^*$ on $\operatorname{Max}^+(\mathfrak{X})$ to the natural action of $\tilde{G}$ on T_N.*
(iii) *The map*

$$\mathbf{p}_{\mathfrak{X}} : \langle e_{\mu,\xi}\rangle \mapsto \mu^N$$

is an N-to-1-surjection $\operatorname{Max}^+(\mathfrak{X}) \to F$ such that

$$\left(\operatorname{Max}^+(\mathfrak{X}), \mathbf{U}, \mathbf{V}, \mathbf{p}_{\mathfrak{X}}, F\right) \cong_{\Xi} (T_N, \mathbf{u}, \mathbf{v}, \mathbf{p}, F).$$

(iv) *The action of the algebra $\mathcal{A}$ on $\mathcal{H}_0^+$ are faithful, that is an operator T of the algebra annihilates $\mathcal{H}_0^+$ if and only if $T = 0$.*
(v) *$\mathcal{A}$ is represented by defining relations (18).*

Proof. (i) Immediate by Lemma 8.
(ii) Indeed, by the definition above the action of $\mathbf{U}$ and $\mathbf{V}$ corresponds to the action on positively oriented pairs:

$$\mathbf{u} : e_{\mathbf{x}(t),\mathbf{y}(t)} \mapsto \mathbf{U}^{-1} e_{\mathbf{x}(t),\mathbf{y}(t)} = e_{\alpha\mathbf{x}(t),\epsilon\mathbf{y}(t)} = e_{\mathbf{x}(\mathbf{u}t),\mathbf{y}(\mathbf{u}t)},$$
$$\mathbf{v} : e_{\mathbf{x}(t),\mathbf{y}(t)} \mapsto \mathbf{V}^{-1} e_{\mathbf{x}(t),\mathbf{y}(t)} = e_{\beta\mathbf{y}(t)\mathbf{x}(t),\ \mathbf{y}(t)} = e_{\mathbf{x}(\mathbf{v}t),\mathbf{y}(\mathbf{v}t)}.$$

(iii), (iv) and (v). Same as for Theorem 2 in Sec. 3.1.5. □

3.2.6. $*$-*structure*

We add to Sec. 3.2.2 the new semi-definable function $\mathbf{w}$ satisfying, for some δ, such that $\delta^N = \epsilon$,

$$\mathbf{y} = \mathbf{w}^N, \quad \mathbf{w}(\mathbf{u}t) = \delta\mathbf{w}(t), \quad \mathbf{w}(\mathbf{v}t) = \mathbf{y}(t)^{-1}\mathbf{w}(t).$$

In accordance with Sec. 3.2.2 we can define

$$\mathbf{w}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u},\mathbf{v}]^l) = \delta^m\epsilon^l.$$

Now we introduce

$$\hat{\mathrm{ang}}\, x := \mathbf{w}(t), \quad \text{for } x = \mathbf{x}(t).$$

Since $\mathbf{x}$ is a bijection, this is well defined on F. Moreover, using the unique representation

$$x = \mathbf{x}(\mathbf{u}^m\mathbf{v}^n[\mathbf{u},\mathbf{v}]^l) = \alpha^m\beta^n\epsilon^{-l}s^{1/N}$$

of Sec. 3.2.2 we have

$$\hat{\mathrm{ang}}(\alpha^m\beta^n\epsilon^{-l}s^{\frac{1}{N}}) = \delta^m\epsilon^l.$$

Taking $a = \epsilon\rho$, $\rho \in \mathbb{R}_+$ (positive reals), $\rho \neq 1$, as suggested in Sec. 3.2.3, and $\alpha^{-1}\delta \in \mathbb{R}_+$, we have

$$\hat{\mathrm{ang}}(\alpha x) = \delta\hat{\mathrm{ang}}\, x, \quad \hat{\mathrm{ang}}(\beta x) = \hat{\mathrm{ang}}\, x.$$

Extend the list of operators on $\mathcal{H}$ to include

$$\mathbf{W} : \psi \mapsto \mathbf{w} \cdot \psi.$$

Obviously $\mathbf{W}$ commutes with $\mathbf{X}$. As in Sec. 3.2.5 denote $e_{\mu,w}$ an eigenvector of $\mathbf{X}$ and $\mathbf{W}$ with eigenvalues μ and w correspondingly. The action of $\mathbf{U}$ and $\mathbf{V}$ is defined on $e_{\mu,w}$ similarly to Sec. 3.2.5:

$$\begin{aligned} \mathbf{U} &: e_{\mu,w} \mapsto e_{\alpha^{-1}\mu,\,\delta^{-1}w}, \\ \mathbf{V} &: e_{\mu,w} \mapsto e_{\beta^{-1}w^{-N}\mu,w^{1-N}}. \end{aligned} \tag{19}$$

Consider the algebra $\mathcal{A}$ as a $*$-algebra with the condition that $\mathbf{X}\mathbf{W}^{-1}$ is *self-adjoint* and $\mathbf{W}$, $\mathbf{U}$ and $\mathbf{V}$ are *unitary*.

Set

$$\mathbf{W}^* := \mathbf{W}^{-1},\ \mathbf{U}^* := \mathbf{U}^{-1},\ \mathbf{V}^* := \mathbf{V}^{-1}$$

that is define these operators as *unitary*. Set

$$\mathbf{X}^* := \mathbf{W}^{-1}\mathbf{X}\mathbf{W}^{-1} = \mathbf{X}\mathbf{W}^{-2},$$

so

$$(\mathbf{X}\mathbf{W}^{-1})^* = \mathbf{W}^{*-1}\mathbf{X}^* = \mathbf{W}\mathbf{X}^* = \mathbf{X}\mathbf{W}^{-1}$$

that is $\mathbf{X}\mathbf{W}^{-1}$ is *self-adjoint*.

Lemma. *There is an inner product space $\mathcal{H}_+$ with the faithful action of $\mathcal{A}$ on it such that $*$ corresponds to taking adjoint operators.*

Proof. Consider $\mathcal{H}_+ \subseteq \mathcal{H}$ generated by all $e_{\mu,w}$ satisfying the condition

$$\mu \cdot w^{-1} \in \mathbb{R}_+,\ w = \exp\frac{2\pi i k}{N^2}, \quad \text{for } k \in \mathbb{Z}. \tag{20}$$

We introduce the inner product in $\mathcal{H}_+$ assuming the $e_{\mu,w}$ to form an orthonormal basis.

Now, by definition $\mathbf{X}\mathbf{W}^{-1}$ acts as a positive self-adjoint operator

$$\mathbf{X}\mathbf{W}^{-1} : e_{\mu,w} \mapsto \mu w^{-1} e_{\mu,w}.$$

$\mathbf{W}$ acts as unitary since w is a root of unity.

$\mathcal{H}_+$ is closed under $\mathbf{U}$ and $\mathbf{V}$ since $\alpha^{-1}\mu\delta w^{-1}$ and $\beta^{-1}\mu\delta w^{-1}$ are in $\mathbb{R}_+$.

The fact that the action is faithful is essentially proved in Theorem 3. ◻

Comment. Using the representation on $\mathcal{H}_+$ one can clearly interpret the angular function $\hat{\text{ang}}\,\mu$ as $\exp\arg\mu$, for μ satisfying (20). For general μ we can use the interpretation as in Sec. 3.2.3:

$$\hat{\text{ang}}\,\mu = \exp\frac{2\pi i}{N^2}\left[\frac{N^2}{2\pi}\arg\mu\right],$$

where $[r]$ stands for the integer part of a real number r. Of course, we stress again that $\hat{\text{ang}}\,\mu$ is very well approximated by $\exp\arg\mu$:

$$\left|\frac{2\pi i}{N^2}\left[\frac{N^2}{2\pi}\arg\mu\right] - \arg\mu\right| \le \frac{2\pi}{N^2}.$$

In other words, the condition on the states being positively oriented in Theorem 3 is similar to conditions usually stated in terms of C^*-algebras. This must justify the name $*$-functions for ang, $\hat{\text{ang}}$ and bd.

4. The Metric Limit

Our aim in this section is to find an interpretation of the limit, as N tends to ∞, of structures T_N and P_N in "classical" terms. "Classical" here is supposed to mean "using function and relations given in terms of real manifolds and analytic functions". Of course, we have to define the meaning of the "limit" first. We found a satisfactory solution to this problem in case of P_N which is presented below.

4.0.1. First we want to establish a connection of the group $\tilde{G}_N$ with the *integer Heisenberg group* $H(\mathbb{Z})$ which is the group of matrices of the form

$$\begin{pmatrix} 1 & k & m \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

with $k, l, m \in \mathbb{Z}$. More precisely, $\tilde{G}_N$ is isomorphic to the group

$$H(\mathbb{Z})_N = H(\mathbb{Z})/N.Z,$$

where $N.Z$ is the central subgroup

$$N.Z = \left\{ \begin{pmatrix} 1 & 0 & Nm \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \ m \in \mathbb{Z} \right\}.$$

Similarly the *real Heisenberg group* $H(\mathbb{R})$ is defined as the group of matrices of the form (21) with $k, l, m \in \mathbb{R}$. The analogue (or the limit case) of $H(\mathbb{Z})_N$ is the factor-group

$$H(\mathbb{R})_\infty := \ H(\mathbb{R}) \Big/ \begin{pmatrix} 1 & 0 & Z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact there is the natural group embedding

$$i_N : \begin{pmatrix} 1 & k & m \\ 0 & 1 & \ell \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \frac{k}{\sqrt{N}} & \frac{m}{N} \\ 0 & 1 & \frac{\ell}{\sqrt{N}} \\ 0 & 0 & 1 \end{pmatrix}$$

inducing the embedding $H(\mathbb{Z})_N \subset H(\mathbb{R})_\infty$.

Notice the following

Lemma 9. *Given the embedding i_N for every $\langle u, v, w \rangle \in H(\mathbb{R})_\infty$ there is $\langle \frac{k}{\sqrt{N}}, \frac{\ell}{\sqrt{N}}, \frac{m}{N} \rangle \in i_N(H(\mathbb{Z})_N)$ such that*

$$\left| u - \frac{k}{\sqrt{N}} \right| + \left| v - \frac{\ell}{\sqrt{N}} \right| + \left| w - \frac{m}{N} \right| < \frac{3}{\sqrt{N}}.$$

In other words, the distance (given by the sum of absolute values) between any point of $H(\mathbb{R})_\infty$ and the set $i_N(H(\mathbb{Z})_N)$ is at most $3/\sqrt{N}$. Obviously, also the distance between any point of $i_N(H(\mathbb{Z})_N)$ and the set $H(\mathbb{R})_\infty$ is 0, because of the embedding. In other words, this defines that *the Hausdorff distance between the two sets is at most* $3/\sqrt{N}$.

In situations when the pointwise distance between sets M_1 and M_2 is defined, we also say that the Hausdorff distance between two L-structures on M_1 and M_2 is at most α if the Hausdorff distance between the universes M_1 and M_2 as well as between $R(M_1)$ and $R(M_2)$, for any L-predicate or graph of an L-operation R, is at most α.

Finally, we say that an L-structure M is the *Hausdorff limit* of L-structures M_N, $N \in \mathbb{N}$, if for each positive α there is N_0 such that for all $N > N_0$ the distance between M_N and M is at most α.

Remark. The notion of limit that we use differs from the very similar and now standard notion of the Gromov–Hausdorff limit in that we do not require that our metric spaces be compact. This can be amended by choosing an appropriate compactification of the spaces involved.

Lemma 10. *The group structure $H(\mathbb{R})_\infty$ is the Hausdorff limit of its substructures $H(\mathbb{Z})_N$, where the distance is defined by the embeddings i_N.*

Proof. Lemma 1 proves that the universe of $H(\mathbb{R})_\infty$ is the limit of the corresponding sequence. Since the group operation is uniformly continuous in the topology determined by the distance, the graphs of the group operations converge as well. □

4.0.2. Given nonzero real numbers a, b, c the integer Heisenberg group $H(\mathbb{Z})$ acts on $\mathbb{R}^3$ as follows:

$$\langle k, l, m\rangle\langle x, y, s\rangle = \langle x + ak, y + bl, s + acky + abcm\rangle, \tag{22}$$

where $\langle k, l, m\rangle$ is the matrix (21).

We can also define the action of $H(\mathbb{Z})$ on $\mathbb{C} \times S^1$, equivalently on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}/\mathbb{Z}$, as follows:

$$\langle k, l, m\rangle\langle x, y, \exp 2\pi is\rangle = \langle x + ak, y + bl, \exp 2\pi i(s + acky + abcm)\rangle, \tag{23}$$

where $x, y, s \in \mathbb{R}$.

In the discrete version intended to model Sec. 3.1.1 we consider q/N, $q \in \mathbb{Z}$, in place of $s \in \mathbb{R}$ and take $a = b = 1/\sqrt{N}$, $c = 1$. We replace (23) by

$$\langle k, l, m\rangle\langle x, y, e^{\frac{2\pi iq}{N}}\rangle = \left\langle x + \frac{k}{\sqrt{N}},\ y + \frac{\ell}{\sqrt{N}},\ \exp 2\pi i\frac{q + k[y\sqrt{N}] + m}{N}\right\rangle. \tag{24}$$

Check that this is still an action:

$$\langle k',\ell',m'\rangle \left(\langle k,\ell,m\rangle \left\langle x,y,\exp\frac{2\pi iq}{N}\right\rangle\right)$$

$$= \langle k',\ell',m'\rangle \left\langle x+\frac{k}{\sqrt{N}}, y+\frac{\ell}{\sqrt{N}}, \exp 2\pi i\frac{q+k[y\sqrt{N}]+m}{N}\right\rangle$$

$$= \left\langle x+\frac{k}{\sqrt{N}}+\frac{k'}{\sqrt{N}}, y+\frac{\ell}{\sqrt{N}}+\frac{\ell'}{\sqrt{N}},\right.$$

$$\left.\exp 2\pi i\frac{q+k[y\sqrt{N}]+m+k'[(y+\frac{\ell}{\sqrt{N}})\sqrt{N}]+m'}{N}\right\rangle$$

$$= \left\langle x+\frac{k+k'}{\sqrt{N}}, y+\frac{\ell+\ell'}{\sqrt{N}}, \exp 2\pi i\frac{q+(k+k')[y\sqrt{N}]+k'l+m+m'}{N}\right\rangle$$

$$= (\langle k',l',m'\rangle\langle k,l,m\rangle)\left\langle x,y,\exp\frac{2\pi iq}{N}\right\rangle.$$

Moreover, we may take m modulo N in (24), that is $\langle k,l,m\rangle \in H(\mathbb{Z})_N$, and simple calculations similar to the above show the following:

Lemma 11. *The formula* (24) *defines the free action of $H(\mathbb{Z})_N$ on $\mathbb{R}\times\mathbb{R}\times\exp\frac{2\pi i}{N}\mathbb{Z}$ (equivalently on $\mathbb{C}\times\exp\frac{2\pi i}{N}\mathbb{Z}$).*

We think of $\langle x,y,\exp\frac{2\pi iq}{N}\rangle$ as an element t of P_N (see Sec. 3.1.1), $x+iy$ as $p(t)\in\mathbb{C}$. The actions $x+iy\mapsto a+x+iy$ and $x+iy\mapsto x+i(y+b)$ are obvious rational automorphisms of the affine line $\mathbb{C}$.

We interpret the action of $\langle 1,0,0\rangle$ and $\langle 0,1,0\rangle$ by (24) on $\mathbb{C}\times\exp\frac{2\pi i}{N}\mathbb{Z}$ as $\mathbf{u}$ and $\mathbf{v}$ correspondingly. Then the commutator $[\mathbf{u},\mathbf{v}]$ corresponds to $\langle 0,0,-1\rangle$, which is the generating element of the centre of $H(\mathbb{Z})_N$. In other words, the subgroup $\mathrm{gp}(\mathbf{u},\mathbf{v})$ of $H(\mathbb{Z})_N$ generated by the two elements is isomorphic to $\tilde{G}$. We thus get, using Lemma 1 of Sec. 2.0.1.

Lemma 12. *Under the above assumption and notation the structure on $\mathbb{C}\times\exp\frac{2\pi i}{N}\mathbb{Z}$ in the language of Sec.* 3.1.1 *described by* (24) *is isomorphic to the example P_N of Sec.* 3.1.1 *with $F=\mathbb{C}$.*

Below we identify P_N with the structure above based on $\mathbb{C}\times\{\exp\frac{2\pi i}{N}\mathbb{Z}\}$.

Note that every group word in $\mathbf{u}$ and $\mathbf{v}$ gives rise to a definable map in P_N. We want to introduce a uniform notation for such definable functions.

Let α be a monotone nondecreasing converging sequence of the form

$$\alpha = \left\{\frac{k_N}{\sqrt{N}} : k_N, N\in\mathbb{Z},\ N>0\right\}.$$

We call such a sequence *admissible* if there is an $r \in \mathbb{R}$ such that

$$\left| r - \frac{k_N}{\sqrt{N}} \right| \leq \frac{1}{\sqrt{N}}. \tag{25}$$

Given $r \in \mathbb{R}$ and $N \in \mathbb{N}$ one can easily find k_N satisfying (25) and so construct an α converging to r, which we denote $\hat{\alpha}$,

$$\hat{\alpha} := \lim \alpha = \lim_N \frac{k_N}{\sqrt{N}}.$$

We denote I by the set of all admissible sequences converging to a real on $[0,1]$, so

$$\{\hat{\alpha} : \alpha \in I\} = \mathbb{R} \cap [0,1].$$

For each $\alpha \in I$ we introduce two operation symbols $\mathbf{u}_\alpha$ and $\mathbf{v}_\alpha$. We denote $P_N^{\#}$ the definable expansion of P_N by all such symbols with the interpretation

$$\mathbf{u}_\alpha = \mathbf{u}^{k_N}, \quad \mathbf{v}_\alpha = \mathbf{v}^{k_N} \quad (k_N\text{-multiple of the operation}),$$

if $\frac{k_N}{\sqrt{N}}$ stands in the Nth position in the sequence α.

Note that the sequence

$$dt := \left\{ \frac{1}{\sqrt{N}} : N \in \mathbb{N} \right\}$$

is in I and $\mathbf{u}_{dt} = \mathbf{u}$, $\mathbf{v}_{dt} = \mathbf{v}$ in all $P_N^{\#}$.

4.0.3. We now define the structure P_∞ to be the structure on sorts $\mathbb{C} \times S^1$ (denoted P_∞) and sort $\mathbb{C}$, with the field structure on $\mathbb{C}$ and the projection map $\mathbf{p} : \langle x, y, e^{2\pi i s} \rangle \mapsto \langle x, y \rangle \in \mathbb{C}$, and definable maps $\mathbf{u}_\alpha$ and $\mathbf{v}_\beta$, $\alpha, \beta \in I$, acting on $\mathbb{C} \times S^1$ (in accordance with the action by $H(\mathbb{R})_\infty$) as follows:

$$\begin{aligned} \mathbf{u}_\alpha(\langle x, y, e^{2\pi i s} \rangle) &= \langle \hat{\alpha}, 0, 0 \rangle \langle x, y, e^{2\pi i s} \rangle = \langle x + \hat{\alpha}, y, e^{2\pi i (s + \hat{\alpha} y)} \rangle, \\ \mathbf{v}_\beta(\langle x, y, e^{2\pi i s} \rangle) &= \langle 0, \hat{\beta}, 0 \rangle \langle x, y, e^{2\pi i s} \rangle = \langle x, y + \hat{\beta}, e^{2\pi i s} \rangle. \end{aligned} \tag{26}$$

Theorem 4. *P_∞ is the Hausdorff limit of structures* $\mathrm{P}_N^{\#}$.

Proof. The sort $\mathbb{C}$ is the same in all structures.

The sort P_∞ is the limit of its substructures P_N since S^1 $(= \exp i\mathbb{R})$ is the limit of $\exp \frac{2\pi i}{N}\mathbb{Z}$ in the standard metric of $\mathbb{C}$. Also, the graph of the projection map $\mathbf{p} : P_\infty \to \mathbb{C}$ is the limit of $\mathbf{p} : P_N \to \mathbb{C}$ for the same reason.

Finally it remains to check that the graphs of $\mathbf{u}$ and $\mathbf{v}$ in P_∞ are the limits of those in P_N. It is enough to see that for any $\langle x, y, \exp \frac{2\pi i q}{N} \rangle \in P_N$ the result of the action by $\mathbf{u}_\alpha$ and $\mathbf{v}_\beta$ calculated in $P_N^{\#}$ is at most at the distance $2/\sqrt{N}$ from the ones calculated in P_∞, for any $\langle x, y, \exp \frac{2\pi i q}{N}\rangle \in P_\infty$. And indeed, the action in $P_N^\#$ by definition is

$$\begin{aligned}
&\mathbf{u}_\alpha : \left\langle x, y, \exp \tfrac{2\pi i q}{N}\right\rangle \mapsto \left\langle x + \tfrac{k_N}{\sqrt{N}}, y, \exp \tfrac{2\pi i}{N}(q + k_N[y\sqrt{N}])\right\rangle,\\
&\mathbf{v}_\beta : \left\langle x, y, \exp \tfrac{2\pi i q}{N}\right\rangle \mapsto \left\langle x, y + \tfrac{l_N}{\sqrt{N}}, \exp 2\pi i \tfrac{q}{N}\right\rangle.
\end{aligned} \tag{27}$$

Obviously,

$$\left|\frac{k_N y}{\sqrt{N}} - \frac{k_N[y\sqrt{N}]}{N}\right| = \frac{k_N}{\sqrt{N}}\left|\frac{y\sqrt{N} - [y\sqrt{N}]}{\sqrt{N}}\right| < \frac{k_N}{\sqrt{N}}\frac{1}{\sqrt{N}} \le \frac{1}{\sqrt{N}},$$

which together with (25) proves that the right-hand side of (27) is at the distance at most $2/\sqrt{N}$ from the right-hand side of (26) uniformly on the point $\langle x, y, \exp \frac{2\pi i q}{N}\rangle$. □

4.0.4. The structure P_∞ can be seen as the principal bundle over $\mathbb{R} \times \mathbb{R}$ with the structure group U(1) (the rotations of S^1) and the projection map $\mathbf{p}$. The action by the Heisenberg group allows to define a *connection* on the bundle. A connection determines "a smooth transition from a point in a fibre to a point in a nearby fibre". As noted above $\mathbf{u}$ and $\mathbf{v}$ in the limit process correspond to infinitesimal actions (in a nonstandard model of P_∞) which can be written in the form

$$\begin{aligned}
\mathbf{u}(\langle x, y, e^{2\pi i s}\rangle) &= \langle x + dx, y, e^{2\pi i(s + ydx)}\rangle,\\
\mathbf{v}(\langle x, y, e^{2\pi i s}\rangle) &= \langle x, y + dy, e^{2\pi i s}\rangle,
\end{aligned}$$

where dx and dy are infinitesimals equal to the dt of Sec. 4.0.2.

These formulas allow one to calculate the derivative of a section

$$\psi : \langle x, y\rangle \mapsto \langle x, y, e^{2\pi i s(x,y)}\rangle$$

of the bundle in any direction on $\mathbb{R} \times \mathbb{R}$. In general moving infinitesimally from the point $\langle x, y\rangle$ along x we get $\langle x+dx, y, \exp 2\pi i(s+ds)\rangle$. We need to compare this to *the parallel transport along* x given by the formulas above, $\langle x+dx, y, \exp 2\pi i(s+ydx)\rangle$. So the difference is

$$\langle 0, 0, \exp 2\pi i(s + ds) - \exp 2\pi i(s + ydx)\rangle.$$

Using the usual laws of differentiation one gets for the third term

$$\begin{aligned}
&\exp 2\pi i(s + ds) - \exp 2\pi i(s + ydx)\\
&= (\exp 2\pi i(s + ds) - \exp 2\pi i s) - (\exp 2\pi i(s + ydx) - \exp 2\pi i s)\\
&= d \exp 2\pi i s - 2\pi i y \exp 2\pi i s\, dx\\
&= \left(\frac{d \exp 2\pi i s}{dx} - 2\pi i y \exp 2\pi i s\right) dx
\end{aligned}$$

which gives for a section $\psi = \exp 2\pi i s$ the following *covariant derivative* along x,

$$\nabla_x \psi = \frac{d}{dx}\psi - 2\pi i y \psi = \left(\frac{d}{dx} + A_x\right)\psi.$$

Similarly, the covariant derivative along y

$$\nabla_y \psi = \frac{d}{dy}\psi = \left(\frac{d}{dy} + A_y\right)\psi$$

with the second term $A_y = 0$.

The *curvature* of the connection is by definition the commutator

$$[\nabla_x, \nabla_y] = \frac{dA_y}{dx} - \frac{dA_x}{dy} = 2\pi i,$$

that is in physicists' terms this pictures an U(1)-gauge field theory over $\mathbb{R}^2$ with a connection of constant nonzero curvature.

4.1. *Algebraic torus*

4.1.1. We
think of elements of $\mathbb{C}^* \times S^1$ as pairs $\langle z, \exp is\rangle$, where $z = \exp(ix + y) \in \mathbb{C}^*$ $x, y, s \in \mathbb{R}$.

The action of $H(\mathbb{Z})$ on $\mathbb{C}^* \times S^1$, can be given, following (22) by

$$\begin{aligned} \mathbf{u}\langle \exp(ix + y),\ \exp is\rangle &= \langle \exp(ix + ia + y),\ \exp i(s + ay)\rangle, \\ \mathbf{v}\langle \exp ix + y,\ \exp is\rangle &= \langle \exp(ix + y + b),\ \exp is\rangle. \end{aligned} \tag{28}$$

The action by $\mathbf{v}$ is well defined since it simply takes the pair $\langle z, t\rangle$ to $\langle e^b z, t\rangle$.

To calculate $\mathbf{u}\,\langle z, t\rangle$ one first takes

$$\ln z = ix + y + 2\pi i n = i(x + 2\pi n) + y, \quad n \in \mathbb{Z}.$$

This recovers y uniquely and so $\mathbf{u}$ is well-defined.

The corresponding discrete version will be

$$\begin{aligned} &\langle k, l, m\rangle \left\langle \exp(2\pi i x + y),\ \exp 2\pi i \frac{q}{N}\right\rangle \\ &\quad = \left\langle \exp\left(2\pi i\left(x + \frac{k}{N}\right) + y + \frac{\ell}{N}\right),\ \exp 2\pi i \frac{q + k[Ny] + m}{N}\right\rangle. \end{aligned} \tag{29}$$

This is a group action, by the same calculation as in Sec. 4.0.2.

In this discrete version $t = \langle \exp(2\pi i x + y),\ \exp 2\pi i \frac{q}{N}$ is an element of T_N and correspondingly $\mathbf{p}(t) = \exp(2\pi i x + y)$. The a and b of Sec. 3.2 will be $e^{2\pi i/N}$ and $e^{1/N}$ correspondingly.

Theorem 5. *The structure on* $\mathbb{C}^* \times \{\exp \frac{2\pi i \mathbb{Z}}{N}\}$ *in the language of Sec.* 3.2 *described by* (29) *is isomorphic to the example* T_N *of Sec.* 3.2 *with* $F = \mathbb{C}$.

We want to calculate the covariant derivative following the method of 4.0.4. We use similar notation for the infinitesimal action

$$dx = \left\{ \frac{1}{N} : N \in \mathbb{N} \right\} = dy,$$

the infinitesimal corresponding to the sequence. But the actual coordinates on $\mathbb{C}^*$ are

$$z^1 = e^{2\pi i x} \text{ and } z^2 = e^y,$$

so

$$dz^1 = 2\pi i z^1 dx, \quad dz^2 = z^2 dy.$$

Now for

$$\psi : z \mapsto \langle x, y, e^{2\pi i s(z)} \rangle$$

the difference between the shift dz^1 and the parallel transport along the same shift will be, by the same formulas as in Sec. 4.0.4,

$$\exp 2\pi i(s + ds) - \exp 2\pi i(s + y dx).$$

This is equal to

$$\left(\frac{d \exp 2\pi i s}{dx} - 2\pi i y \exp 2\pi i s \right) dx = \left(\frac{d \exp 2\pi i s}{dz^1} - \frac{\ln z^2}{z^1} \exp 2\pi i s \right) dz^1$$

which gives the covariant derivative along z^1

$$\nabla_{z^1} \psi = \frac{d}{dz^1} \psi - \frac{\ln z^2}{z^1} \psi.$$

Similarly, ∇_{z^2} the covariant derivative along z^2 is just $\frac{d}{dz^2}\psi$, the second term being zero.

The curvature of the connection is

$$[\nabla_{z^1}, \nabla_{z^2}] = \frac{1}{z^1 z^2},$$

which is a nonconstant curvature (note also that $z^1 z^2 = \exp(2\pi i x + y)$ does not vanish on $\mathbb{C}^*$).

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References

1. B. A. Dubrovin, A. T. Fomenko and S. P. Novikov, *Modern Geometry: Methods and Applications* (Springer-Verlag, 1990).
2. D. Evans, Expansions of fields by angular functions, *J. Inst. Math. Jussieu* **7** (2008) 735–751.
3. E. Hrushovski and B. Zilber, Zariski geometries, *J. Amer. Math. Soc.* **9** (1996) 1–56.
4. A. I. Mal'tsev, On some classes of infinite soluble groups, *Mat. Sb.* **28** (1951) 567–588 (in Russian).
5. G. Svetlichny, Preparation for gauge theory, arXiv:math-ph/9902027v1.
6. B. Zilber, *Zariski Geometries*, LMS Lecture Notes Series, Vol. 360 (Cambridge Univ. Press, 2010).
7. B. Zilber, A class of quantum Zariski geometries, in *Model Theory with Applications to Algebra and Analysis I*, LMS Lect Notes, Vol. 350, eds. H. D. Macpherson, A. Pillay and A. J. Wilkie (Cambridge Univ. Press, 2008), pp. 293–326.