

## ON THE STEINHAUS AND BERGMAN PROPERTIES FOR INFINITE PRODUCTS OF FINITE GROUPS

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We study the relationship between the existence of nonprincipal ultrafilters over  $\omega$  and the failure of the automatic continuity, Steinhaus and Bergman properties for infinite products of finite groups.

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### 1. Introduction

In this paper, we will investigate the status of the Steinhaus and Bergman properties for infinite products of finite groups in various axiomatic frameworks. Our interest in these properties is partially motivated by the automatic continuity problem for Polish groups. More specifically, we will be interested in the question of which infinite products  $G = \prod G_n$  of nontrivial finite groups have the *automatic continuity property*, i.e. have the property that *every* homomorphism  $\varphi : G \rightarrow H$  from  $G$  into a Polish group  $H$  is necessarily continuous. In set theory with the Axiom of Choice, infinite products of finite groups typically fail to have this property; and, in fact, no examples of infinite products of finite groups with this property are

currently known. The basic example of a discontinuous homomorphism involves a nonprincipal ultrafilter  $\mathcal{U}$  over the set  $\omega$  of natural numbers.

**Example 1.1.** Suppose that there exists a fixed nontrivial finite group  $F$  such that  $G_n \cong F$  for all  $n \in \omega$ . Then the corresponding ultraproduct  $\prod_{\mathcal{U}} G_n$  is isomorphic to  $F$  and the associated homomorphism  $\varphi : \prod G_n \rightarrow F$  is not continuous, since  $\ker \varphi$  is a dense subgroup and hence is not closed.

The automatic continuity property for some more interesting infinite products of finite groups can be shown to fail for more complicated reasons.

**Example 1.2.** Let  $d \geq 2$  and suppose that  $G_n = \mathrm{SL}(d, p_n)$ , where  $(p_n \mid n \in \omega)$  is an increasing sequence of primes. If  $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$  is the corresponding ultraproduct of the fields  $\mathbb{F}_{p_n}$  of order  $p_n$ , then

$$\prod_{\mathcal{U}} \mathrm{SL}(d, p_n) \cong \mathrm{SL} \left( d, \prod_{\mathcal{U}} \mathbb{F}_{p_n} \right) = \mathrm{SL}(d, K)$$

and thus  $\mathrm{SL}(d, K)$  is a homomorphic image of  $\prod \mathrm{SL}(d, p_n)$ . Since  $K$  is a field of characteristic 0 and cardinality  $2^{\aleph_0}$ , it follows that  $K$  embeds into  $\mathbb{C}$  and hence  $\mathrm{SL}(d, K)$  embeds into  $\mathrm{SL}(d, \mathbb{C})$ . (The fact that an ultraproduct of finite structures, if infinite, actually has cardinality  $2^{\aleph_0}$  is due to Shelah [28].) Once again, it is clear that the associated homomorphism  $\varphi : \prod \mathrm{SL}(d, p_n) \rightarrow \mathrm{SL}(d, \mathbb{C})$  is not continuous.

**Remark 1.3.** In Sec. 3, we will present a more sophisticated construction involving an embedding of  $K$  into the field of Puiseux series over the field  $\overline{\mathbb{Q}}$  of algebraic numbers, which yields a discontinuous homomorphism of  $\prod \mathrm{SL}(d, p_n)$  into the infinite symmetric group  $\mathrm{Sym}(\omega)$ .

It is natural to ask whether the existence of a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$  is either necessary or sufficient in the above constructions of discontinuous homomorphisms. (The existence of a nonprincipal ultrafilter  $\mathcal{U}$  is clearly sufficient in Example 1.1. However, the construction in Example 1.2 also makes use of the existence of an embedding of the field  $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$  into  $\mathbb{C}$  and the usual proofs of this result rely on the existence of transcendence bases for both  $K$  and  $\mathbb{C}$ .) Of course, when considering this kind of question, we cannot work with the usual *ZFC* axioms of set theory since these already imply the existence of nonprincipal ultrafilters over arbitrary infinite sets. Instead we will work with the axiom system  $ZF + DC$ , where *DC* is the following weak form of the Axiom of Choice.

**Axiom of Dependent Choice.** (*DC*) Suppose that  $X$  is a nonempty set and that  $R$  is a binary relation on  $X$  such that for all  $x \in X$ , there exists  $y \in X$  with  $x R y$ . Then there exists a function  $f : \omega \rightarrow X$  such that  $f(n) R f(n + 1)$  for all  $n \in \omega$ .

The axiom system  $ZF + DC$  is sufficient to develop most of real analysis and descriptive set theory, but is insufficient to prove the existence of pathologies such

as nonmeasurable sets. (For example, see Moschovakis [23].) In particular, since nonprincipal ultrafilters over  $\omega$  are nonmeasurable when regarded as subsets of the Cantor space  $2^{\mathbb{N}}$ , it follows that  $ZF + DC$  does not prove the existence of such ultrafilters.

We will provide a structured answer to the above question. Firstly, the following result is well known. (For example, see Rosendal [26, Sec. 2].)

**Theorem 1.4.** *It is consistent with  $ZF + DC$  that if  $G, H$  are any Polish groups, then every homomorphism  $\varphi : G \rightarrow H$  is continuous.*

In fact, assuming the existence of suitable large cardinals, this is true in  $L(\mathbb{R})$ , the canonical minimal model of  $ZF$  which contains all of the ordinals and all of the real numbers. Of course, this implies the well known result that  $L(\mathbb{R})$  does not contain any nonprincipal ultrafilters over  $\omega$ . While it seems almost certain that the existence of a nonprincipal ultrafilter over  $\omega$  is necessary to prove the failure of the automatic continuity property for suitably chosen infinite products  $G = \prod G_n$  of finite groups, we have not completely settled this question. However, in Sec. 4, we will prove a number of partial results in this direction, including the following theorem.

**Theorem 1.5.** ( $ZF + DC$ ) *Suppose that  $d \geq 2$  and that  $(p_n \mid n \in \omega)$  is an increasing sequence of primes. If there exists a discontinuous homomorphism  $\varphi : \prod \text{SL}(d, p_n) \rightarrow \text{Sym}(\omega)$ , then there exists a nonprincipal ultrafilter over  $\omega$ .*

On the other hand, we will show that the existence of a nonprincipal ultrafilter over  $\omega$  is not sufficient to prove the failure of the automatic continuity property for suitably chosen infinite products of finite groups. In order to explain this result, it will be necessary in the remainder of this section to assume the existence of suitable large cardinals. We will not specify the precise large cardinal hypothesis that we need until it becomes necessary to do so in Sec. 7. (This paper has been written so that the first six sections can be read by mathematicians with no knowledge of advanced set theory, such as forcing, large cardinals, etc. It is only in the final section that some knowledge of advanced set theory is needed and this section can be omitted by mathematicians without the necessary background.) Following the usual convention [35], we will indicate the use of a large cardinal hypothesis by writing  $(LC)$  before the statement of the relevant theorem. The following result is a special case of a more general result that we will present in Sec. 5.

**Theorem 1.6.** ( $LC$ ) *It is consistent with  $ZF + DC$  that*

- (i) *there exists a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$ ; and*
- (ii) *for each  $d \geq 2$ , if  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then  $\prod \text{SL}(d, p_n)$  has the automatic continuity property.*

In fact, assuming the existence of suitable large cardinals, this is true in  $L(\mathbb{R})[\mathcal{U}]$ , the minimal model of  $ZF$  containing all of the ordinals and real numbers, together

with a Ramsey ultrafilter  $\mathcal{U}$  over  $\omega$ . Under a suitable large cardinal hypothesis,  $L(\mathbb{R})[\mathcal{U}]$  has canonicity features parallel to those of  $L(\mathbb{R})$ ; and, in particular, its theory does not depend on the choice of the Ramsey ultrafilter  $\mathcal{U}$ . Di Prisco–Todorćević [7] have shown that many of the regularity properties of  $L(\mathbb{R})$  continue to hold in  $L(\mathbb{R})[\mathcal{U}]$ . For example, in  $L(\mathbb{R})[\mathcal{U}]$ , every uncountable set of reals has a perfect subset. Thus it seems natural to regard  $L(\mathbb{R})[\mathcal{U}]$  as a canonical model of  $ZF + DC$  in which a minimal number of the pathological consequences of the Axiom of Choice holds, modulo the existence of a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$ . The results of this paper provide yet more evidence for this point of view.

Up until this point, we have considered two examples of infinite products of finite groups; namely, infinite products of a fixed finite group  $F$  and infinite products of the form  $\prod \mathrm{SL}(d, p_n)$  for various increasing sequences  $(p_n \mid n \in \omega)$  of primes. In the first example, we have seen that the existence of a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$  is sufficient to prove the failure of the automatic continuity property; while in the second example, this is not sufficient. Now we should also consider a third example; namely, the infinite product  $\prod \mathrm{Alt}(n)$  of the finite alternating groups. In this case, as we will explain in Sec. 6, it is natural to conjecture that the automatic continuity property does hold. So what is the essential difference between these three examples? Perhaps surprisingly, the key to our analysis of the infinite product  $\prod G_n$  of finite groups turns out to be the “asymptotic representation theory” of the sequence  $(G_n \mid n \in \omega)$ . In order to state this more precisely, it is necessary to introduce the following definitions.

**Definition 1.7.** Let  $H$  be a nontrivial finite group.

- (i) If  $K$  is a field, then  $d_K(H)$  denotes the minimal dimension of a nontrivial  $K$ -representation of  $H$ ; i.e. the least  $d$  such that there exists a nontrivial homomorphism  $\theta : H \rightarrow \mathrm{GL}(d, K)$ .
- (ii)  $d(H) = \min\{d_K(H) \mid K \text{ is a field}\}$ .

**Example 1.8.** Suppose that  $p \geq 5$  is a prime and that  $H = \mathrm{SL}(d, p)$ .

- (i) If  $d = 2$ , then  $d_{\mathbb{C}}(H) = (p - 1)/2$  and  $d(H) = 2$ .
- (ii) If  $d > 2$ , then  $d_{\mathbb{C}}(H) = (p^d - p)/(p - 1)$  and  $d(H) = d$ .

(For example, see Humphreys [12] and Tiep–Zalesskii [32].)

**Example 1.9.** If  $n > 8$  and  $H = \mathrm{Alt}(n)$ , then  $d_{\mathbb{C}}(H) = n - 1$  and  $d(H) = n - 2$ . (For example, see James [14] and Wagner [33, 34].)

Suppose that  $(G_n \mid n \in \omega)$  is an arbitrary sequence of nontrivial finite groups. Then we will prove that:

- (a) if  $\liminf d(G_n) < \infty$ , then  $\prod G_n$  does not have the automatic continuity property in the actual set-theoretic universe  $V$ .

It seems reasonable to conjecture that the converse of (a) also holds, perhaps after imposing some suitable algebraic conditions (such as simplicity) on the groups  $G_n$ . In particular, it seems likely that  $\prod \text{Alt}(n)$  has the automatic continuity property in  $V$ . Now suppose that  $\liminf d(G_n) < \infty$ . Then we will prove that:

(b) if  $\liminf d_{\mathbb{C}}(G_n) < \infty$ , then the existence of a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$  is enough to prove that  $\prod G_n$  does not have the automatic continuity property.

On the other hand, if the groups  $G_n$  are members of a suitably chosen class  $\mathcal{C}$ , such as the class of groups of the form  $\text{SL}(d, p)$  for some fixed  $d \geq 2$ , then we will prove that:

(c) assuming  $(LC)$ , if  $(d_{\mathbb{C}}(G_n) \mid n \in \omega)$  grows sufficiently fast, then  $\prod G_n$  has the automatic continuity property in  $L(\mathbb{R})[\mathcal{U}]$ .

This paper is organized as follows. In Sec. 2, we will discuss the Steinhaus and Bergman properties for infinite products of finite groups. In Sec. 3, working with the usual  $ZFC$  axioms of set theory, we will prove that the Steinhaus and Bergman properties fail for various infinite products of finite groups. In Sec. 4, working with the axiom system  $ZF + DC$ , we will prove that the failure of the Bergman property for suitably chosen infinite products of finite groups implies the existence of a nonprincipal ultrafilter over  $\omega$ ; and we will show that the failure of a weak form of the Steinhaus property also implies the existence of such an ultrafilter. In Sec. 5, we will present a partition property  $PP$  for products of finite sets with measures; and we will show that  $ZF + DC + PP$  implies that various infinite products of finite groups have both the Bergman property and the Steinhaus property. In Sec. 6, we will briefly discuss the questions of which infinite products of nonabelian finite simple groups have either the Bergman property or the Steinhaus property in the actual set-theoretic universe  $V$ . Finally, in Sec. 7, assuming the existence of suitable large cardinals, we will prove that  $L(\mathbb{R})[\mathcal{U}]$  satisfies  $PP$ .

**Notation 1.10.** Let  $(H_n \mid n \in \omega)$  be a sequence of finite groups and let  $H = \prod H_n$ . Suppose that  $A \subseteq \omega$ .

- (i)  $\prod_{n \in A} H_n$  denotes the subgroup of  $H$  consisting of those elements  $(h_n) \in H$  such that  $h_n = 1$  for all  $n \in \omega \setminus A$ .
- (ii) If  $h = (h_n) \in H$ , then  $h \upharpoonright A$  denotes the element  $(g_n) \in \prod_{n \in A} H_n$  such that  $g_n = h_n$  for all  $n \in A$ .

Recall that  $H = \prod H_n$  is a Polish topological group with neighborhood basis of the identity given by  $\{\prod_{n \in A} H_n \mid A \text{ is a cofinal subset of } \omega\}$ .

Suppose that  $U$  is a subset of the group  $G$ . Then for each  $t \geq 1$ ,  $U^t$  denotes the set of elements  $g \in G$  which can be expressed as a product  $g = u_1 \cdots u_t$ , where each  $u_i \in U$ . The subset  $U$  is said to be *symmetric* if  $U = U^{-1}$  is closed under taking inverses.

## 2. The Steinhaus and Bergman Properties

In this section, we will discuss the Steinhaus and Bergman properties for infinite products of finite groups. The Steinhaus property was introduced by Rosendal–Solecki [25] in the context of the automatic continuity problem for homomorphisms between topological groups. In the following definition, a subset  $W$  of a group  $G$  is said to be *countably syndetic* if there exist elements  $g_n \in G$  for  $n \in \omega$  such that  $G = \bigcup_{n \in \omega} g_n W$ .

**Definition 2.1.** Let  $G$  be a topological group. Then  $G$  has the *Steinhaus property* if there exists a *fixed* integer  $k \geq 1$  such that for every symmetric countably syndetic subset  $W \subseteq G$ , the  $k$ -fold product  $W^k$  contains an open neighborhood of the identity element  $1_G$ .

**Proposition 2.2.** (Rosendal–Solecki [25]) *If  $G$  is a topological group with the Steinhaus property and  $\varphi : G \rightarrow H$  is a homomorphism into a separable group  $H$ , then  $\varphi$  is necessarily continuous.*

The class of groups with the Steinhaus property includes Polish groups with ample generics [16],  $\text{Aut}(\mathbb{Q}, <)$ ,  $\text{Homeo}(\mathbb{R})$  [25] and full groups of ergodic countable Borel equivalence relations [17]. However, no infinite product of finite groups is currently known to have the Steinhaus property. Of course, by Example 1.2 and Proposition 2.2, it follows that if  $(p_n \mid n \in \omega)$  is an increasing sequence of primes and  $d \geq 2$ , then  $\prod \text{SL}(d, p_n)$  does not have the Steinhaus property. We will prove the following more general result in Sec. 3.

**Theorem 2.3.** *Suppose that  $(G_n \mid n \in \omega)$  is a sequence of nontrivial finite groups. If  $\liminf d(G_n) < \infty$ , then  $\prod G_n$  does not have the automatic continuity property and hence does not have the Steinhaus property.*

As the reader has probably guessed, the proof of Theorem 2.3 involves the use of a suitable ultraproduct  $\prod_{\mathcal{U}} G_n$ . However, the following strengthening of Theorem 1.6, which we will prove in Sec. 5, shows that the existence of a nonprincipal ultrafilter over  $\omega$  is not always enough to prove that such a product  $\prod G_n$  does not have the Steinhaus property.

**Theorem 2.4.** (LC) *It is consistent with  $ZF + DC$  that*

- (i) *there exists a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$ ; and*
- (ii) *for each  $d \geq 2$ , if  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then  $\prod \text{SL}(d, p_n)$  has the Steinhaus property.*

Once again, assuming the existence of suitable large cardinals, this is true in  $L(\mathbb{R})[\mathcal{U}]$ , the minimal model of  $ZF$  containing all of the ordinals and real numbers, together with a Ramsey ultrafilter  $\mathcal{U}$  over  $\omega$ .

The Bergman property was introduced by Bergman [2] as a strengthening of the notion of uncountable cofinality which was introduced earlier by Macpherson–Neumann [21].

**Definition 2.5.** Suppose that  $G$  is a non-finitely generated group.

- (a)  $G$  has *countable cofinality* if  $G = \bigcup_{n \in \omega} G_n$  can be expressed as the union of a countable increasing chain of proper subgroups. Otherwise,  $G$  has *uncountable cofinality*.
- (b)  $G$  is *Cayley bounded* if for every symmetric generating set  $S$ , there exists an integer  $n \geq 1$  such that every element  $g \in G$  can be expressed as a product  $g = s_1 \cdots s_n$ , where each  $s_i \in S \cup \{1\}$ .
- (c)  $G$  has the *Bergman property* if  $G$  has uncountable cofinality and is Cayley bounded.

By de Cornulier [6], a group  $G$  has the Bergman property if and only if whenever  $G$  acts isometrically on a metric space, every  $G$ -orbit has a finite diameter. For this reason, groups with the Bergman property are often said to be “strongly bounded”. The class of groups with the Bergman property includes the symmetric groups over infinite sets [2], automorphism groups of various infinite structures [9, 13] and oligomorphic groups with ample generics [16]. The following easy observation is essentially contained in Bergman [2, Lemma 10].

**Lemma 2.6.** *If  $G$  is a non-finitely generated group, then the following conditions are equivalent.*

- (a)  $G$  has the Bergman property.
- (b) If  $G = \bigcup_{n \in \omega} U_n$  is the union of an increasing chain of symmetric subsets such that  $U_n U_n \subseteq U_{n+1}$  for all  $n \in \omega$ , then there exists an  $n \in \omega$  such that  $U_n = G$ .

In [6], improving an earlier result of Koppelberg–Tits [18], de Cornulier proved that if  $G$  is a product of infinitely many copies of a fixed finite perfect group, then  $G$  has the Bergman property; and Zalan Gyenis has recently checked that the arguments of Saxl–Shelah–Thomas [27] can be modified to prove that an infinite product  $\prod H_n$  of finite simple groups has the Bergman property if and only if  $\prod H_n$  has uncountable cofinality. This yields an explicit classification of the infinite products  $\prod H_n$  of finite simple groups satisfying the Bergman property, which we will discuss in Sec. 6. On the other hand, there are many infinite products of finite groups which are known not to have the Bergman property. In particular, the following result holds.

**Theorem 2.7.** *If  $d \geq 2$  and  $(p_n \mid n \in \omega)$  is an increasing sequence of primes, then:*

- (a)  $\prod \mathrm{SL}(d, p_n)$  has countable cofinality; and
- (b)  $\prod \mathrm{SL}(d, p_n)$  is not Cayley bounded.

Theorem 2.7(a) is essentially contained in Saxl–Shelah–Thomas [27]. However, for the sake of completeness, we will quickly sketch the very easy proof. (We will present the proof of Theorem 2.7(b) in Sec. 3.) Let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\omega$  and let  $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$  be the corresponding ultraproduct of the fields  $\mathbb{F}_{p_n}$  of order  $p_n$ . Then  $K$  is an uncountable field and

$$\prod_{\mathcal{U}} \mathrm{SL}(d, p_n) \cong \mathrm{SL}\left(d, \prod_{\mathcal{U}} \mathbb{F}_{p_n}\right) = \mathrm{SL}(d, K).$$

It follows that  $\mathrm{SL}(d, K)$  is a homomorphic image of  $\prod \mathrm{SL}(d, p_n)$  and hence Theorem 2.7(a) is an immediate consequence of the following observation.

**Proposition 2.8.** *If  $F$  is an uncountable field, then  $\mathrm{SL}(d, F)$  has countable cofinality.*

**Proof.** Let  $B$  be a transcendence basis of  $F$  over its prime subfield. Then  $B$  is uncountable and hence we can express  $B = \bigcup_{n \in \omega} B_n$  as the union of a countable strictly increasing chain of proper subsets. For each  $n \in \omega$ , let  $F_n$  be the algebraic closure of  $B_n$  in  $F$ . Then the strictly increasing chain of proper subgroups

$$\mathrm{SL}(d, F) = \bigcup_{n \in \omega} \mathrm{SL}(d, F_n)$$

witnesses that  $\mathrm{SL}(d, F)$  has countable cofinality. □

The following result, which will be proved in Sec. 4, shows that the existence of a nonprincipal ultrafilter over  $\omega$  is necessary in order to prove either Theorem 2.7(a) or Theorem 2.7(b).

**Theorem 2.9.** *(ZF + DC) Let  $d \geq 2$  and let  $(p_n \mid n \in \omega)$  be an increasing sequence of primes. If  $\prod \mathrm{SL}(d, p_n)$  does not have the Bergman property, then there exists a nonprincipal ultrafilter over  $\omega$ .*

On the other hand, we will also show that the existence of a nonprincipal ultrafilter over  $\omega$  is not sufficient to prove either of the parts of Theorem 2.7.

**Theorem 2.10.** *(LC) If  $d \geq 2$  and  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then  $\prod \mathrm{SL}(d, p_n)$  has the Bergman property in  $L(\mathbb{R})[\mathcal{U}]$ .*

Examining the above proof of Theorem 2.7(a), we see that it relies upon the following three consequences of the Axiom of Choice:

- (i) the existence of a nonprincipal ultrafilter  $\mathcal{U}$  over  $\omega$ ;
- (ii) the existence of a transcendence basis  $B$  of the field  $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$ ; and
- (iii) the existence of an expression of  $B$  as the union of a countable strictly increasing chain of proper subsets.



Clearly  $L(\mathbb{R})[\mathcal{U}]$  satisfies (i); and since *DC* implies that every infinite set has a denumerably infinite subset, it follows easily that every infinite set can be expressed as the union of a countable strictly increasing chain of proper subsets in  $L(\mathbb{R})[\mathcal{U}]$ . Consequently, assuming *LC*, if  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then (ii) must fail in  $L(\mathbb{R})[\mathcal{U}]$ .

**Corollary 2.11.** (*LC*) *If  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then the field  $\prod_{\mathcal{U}} \mathbb{F}_{p_n}$  does not have a transcendence basis in  $L(\mathbb{R})[\mathcal{U}]$ .*

### 3. On the Failure of the Bergman and Steinhaus Properties

In this section, we will first show that if  $(p_n \mid n \in \omega)$  is an increasing sequence of primes and  $d \geq 2$ , then:

- There exists a discontinuous homomorphism of  $\prod \text{SL}(d, p_n)$  into  $\text{Sym}(\omega)$ .
- $\prod \text{SL}(d, p_n)$  is not Cayley bounded.

Then we will prove that if  $(G_n \mid n \in \omega)$  is a sequence of nontrivial finite groups such that  $\liminf d(G_n) < \infty$ , then  $\prod G_n$  does not have the automatic continuity property and hence does not have the Steinhaus property.

Once again, let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\omega$  and let  $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$  be the corresponding ultraproduct of the fields  $\mathbb{F}_{p_n}$  of order  $p_n$ . Our arguments depend upon the existence of a suitable valuation  $v : K \rightarrow \mathbb{Q} \cup \{\infty\}$ .

**Definition 3.1.** Let  $F$  be a field and let  $t$  be an indeterminate over  $F$ . Then  $F((t))$  denotes the corresponding field of formal power series; and

$$\mathbf{P}(F) = \bigcup_{n \geq 1} F((t^{1/n}))$$

denotes the corresponding field of Puiseux series. Let  $v_F : \mathbf{P}(F) \rightarrow \mathbb{Q} \cup \{\infty\}$  be the valuation such that if

$$0 \neq a = \sum_{k \geq M} a_k t^{k/n} \in \mathbf{P}(F),$$

where  $a_k \in F$ ,  $a_M \neq 0$ ,  $k, M \in \mathbb{Z}$  and  $n \geq 1$ , then  $v_F(a) = M/n$ . (As usual, we set  $v_F(0) = \infty$ .)

It is well known that if  $F$  is an algebraically closed field of characteristic 0, then  $\mathbf{P}(F)$  is algebraically closed. (For example, see Chevalley [4].) In particular, if  $\overline{\mathbb{Q}}$  is the field of algebraic numbers, then  $\mathbf{P}(\overline{\mathbb{Q}})$  is an algebraically closed field of cardinality  $2^{\aleph_0}$ . Hence, since  $K = \prod_{\mathcal{U}} \mathbb{F}_{p_n}$  is a field of characteristic 0 and cardinality  $2^{\aleph_0}$ , we can assume that  $K$  is a subfield of  $\mathbf{P}(\overline{\mathbb{Q}})$ . Furthermore, since  $K$  is uncountable and the automorphism group of  $\mathbf{P}(\overline{\mathbb{Q}})$  acts transitively on non-algebraic elements, we can assume that  $t \in K$ . From now on, we let  $v = v_{\overline{\mathbb{Q}}} \upharpoonright K$  denote the corresponding valuation of  $K$  and let  $R = \{a \in K \mid v(a) \geq 0\}$  be the

corresponding valuation ring. We will make use of the following result, which was proved in Thomas [31, Sec. 2].

**Theorem 3.2.**  $[\mathrm{SL}(d, K) : \mathrm{SL}(d, R)] = \omega$ .

**Corollary 3.3.** *There exists a discontinuous homomorphism of  $\prod \mathrm{SL}(d, p_n)$  into  $\mathrm{Sym}(\omega)$ .*

**Proof.** Let  $\pi : \prod \mathrm{SL}(d, p_n) \rightarrow \mathrm{SL}(d, K)$  be the canonical surjective homomorphism and let  $H = \pi^{-1}(\mathrm{SL}(d, R))$ . Then  $[\prod \mathrm{SL}(d, p_n) : H] = \omega$  and the action of  $\prod \mathrm{SL}(d, p_n)$  on the cosets of  $H$  induces a homomorphism

$$\varphi : \prod \mathrm{SL}(d, p_n) \rightarrow \mathrm{Sym}(\omega)$$

such that  $\varphi(H)$  is the stabilizer of 0 in  $\varphi(\prod \mathrm{SL}(d, p_n))$ . If  $S$  is the stabilizer of 0 in  $\mathrm{Sym}(\omega)$ , then  $S$  is an open subgroup of  $\mathrm{Sym}(\omega)$  and  $\varphi^{-1}(S) = H$ . Since  $H$  is clearly not an open subgroup of  $\prod \mathrm{SL}(d, p_n)$ , it follows that  $\varphi$  is not continuous.  $\square$

Next we will prove that  $\prod \mathrm{SL}(d, p_n)$  is not Cayley bounded. By the following easy observation, it is enough to show that  $\mathrm{SL}(d, K)$  is not Cayley bounded.

**Lemma 3.4.** *Suppose that  $G$  is a group and that  $N \trianglelefteq G$  is a normal subgroup. If  $G$  is Cayley bounded, then  $H = G/N$  is also Cayley bounded.*

**Proof.** Suppose that the symmetric generating set  $S \subseteq H$  witnesses that  $H$  is not Cayley bounded. Let  $\pi : G \rightarrow H$  be the canonical surjective homomorphism and let  $T = \pi^{-1}(S)$ . Then  $T$  witnesses that  $G$  is not Cayley bounded.  $\square$

From now on, in order to simplify notation, we will assume that  $d = 2$ . Recall that after identifying  $K$  with its image under a suitable embedding into the field  $\mathbf{P}(\overline{\mathbb{Q}})$  of Puiseux series in the indeterminate  $t$ , we have that  $t \in K$ . Also note that  $v(t) = 1$  and that  $v(t^{-1}) = -1$ . For each  $k \in K^* = K \setminus \{0\}$ , let

$$x(k) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad y(k) = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} \quad d(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}.$$

Then it is well known that  $T = \{x(k) \mid k \in K^*\} \cup \{y(k) \mid k \in K^*\}$  generates  $\mathrm{SL}(2, K)$ . (For example, see Lang [19, Lemma XIII.8.1].) Let

$$U = \{d(t), d(t^{-1})\} \cup \{x(k) \mid 0 \leq v(k) \leq 2\} \cup \{y(k) \mid 0 \leq v(k) \leq 2\}.$$

Since  $v(-k) = v(k)$  for all  $k \in K$ , it follows that  $U$  is a symmetric subset of  $\mathrm{SL}(2, K)$ . We claim that  $U$  generates  $\mathrm{SL}(2, K)$ . To see this, note that

$$d(t)x(k)d(t)^{-1} = x(t^2k), \quad d(t)^{-1}x(k)d(t) = x(t^{-2}k)$$

and that

$$v(t^2k) = v(t^2) + v(k) = v(k) + 2, \quad v(t^{-2}k) = v(t^{-2}) + v(k) = v(k) - 2.$$

Hence if  $k \in K^*$ , then there exists  $m \in \mathbb{Z}$  such that  $d(t)^m x(k) d(t)^{-m} \in U$ ; and similarly, there exists  $m \in \mathbb{Z}$  such that  $d(t)^m y(k) d(t)^{-m} \in U$ . It follows that  $T \subseteq \langle U \rangle$  and hence  $\langle U \rangle = \text{SL}(2, K)$ . Next for each matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in \text{SL}(2, K),$$

we define

$$\tau(A) = \min\{v(a_i) \mid 1 \leq i \leq 4\}.$$

Notice that since

$$\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 + a_2 b_3 & a_1 b_2 + a_2 b_4 \\ a_3 b_1 + a_4 b_3 & a_3 b_2 + a_4 b_4 \end{pmatrix}$$

and since, for example,

$$\begin{aligned} v(a_1 b_1 + a_2 b_3) &\geq \min\{v(a_1 b_1), v(a_2 b_3)\} \\ &= \min\{v(a_1) + v(b_1), v(a_2) + v(b_3)\}, \end{aligned}$$

it follows that  $\tau(AB) \geq \tau(A) + \tau(B)$  for all  $A, B \in \text{SL}(2, K)$ . Finally recall that for each  $m \in \mathbb{N}$ , we have that  $v(t^{-m}) = -m$  and so  $\tau(d(t^m)) = -m$ . It now follows easily that for each  $n \in \mathbb{N}$ , there exists  $m \in \mathbb{N}$  such that  $d(t^m)$  is not a product of  $n$  elements of  $U \cup \{1\}$ . Thus  $\text{SL}(2, K)$  is not Cayley bounded and it follows that  $\prod \text{SL}(2, p_n)$  is also not Cayley bounded.

The remainder of this section is devoted to the proof of Theorem 2.3. Suppose that  $(G_n \mid n \in \omega)$  is a sequence of nontrivial finite groups with  $\liminf d(G_n) < \infty$ . Then there exists an infinite subset  $I \subseteq \omega$  and a fixed  $d \geq 1$  such that for each  $n \in I$  there exists a nontrivial homomorphism

$$\varphi_n : G_n \rightarrow \text{GL}(d, F_n)$$

for some field  $F_n$ . In order to simplify notation, we will assume that  $I = \omega$ . Let  $\mathcal{U}$  be a nonprincipal ultrafilter over  $\omega$  and let

$$\varphi : \prod_{\mathcal{U}} G_n \rightarrow \prod_{\mathcal{U}} \text{GL}(d, F_n)$$

be the homomorphism defined by  $(g_n)_{\mathcal{U}} \mapsto (\varphi_n(g_n))_{\mathcal{U}}$ . Let  $F = \prod_{\mathcal{U}} F_n$  and  $H = \varphi(\prod_{\mathcal{U}} G_n)$ . By Thomas [31, Theorem 2.1], since  $F$  is a field of cardinality at most  $2^{\aleph_0}$  and

$$1 \neq H \leq \prod_{\mathcal{U}} \text{GL}(d, F_n) \cong \text{GL}(d, F),$$

it follows that there exists a proper subgroup  $H_0 < H$  such that  $1 < [H : H_0] \leq \omega$ . (As with our earlier arguments, the proof of Thomas [31, Theorem 2.1] involves defining a suitable valuation on  $F$ .) Let  $L = \varphi^{-1}(H_0)$ . Then  $L$  is a proper subgroup of  $\prod_{\mathcal{U}} G_n$  of countable (possibly finite) index. Let  $\pi : \prod G_n \rightarrow \prod_{\mathcal{U}} G_n$  be the

canonical surjective homomorphism and let  $M = \pi^{-1}(L)$ . Then  $M$  is a proper subgroup of  $\prod_{\mathcal{U}} G_n$  of countable (possibly finite) index. If  $M$  has infinite index, then arguing as in the proof of Corollary 3.3, it follows that there exists a discontinuous homomorphism of  $\prod G_n$  into  $\text{Sym}(\omega)$ . Hence we can assume that  $[\prod G_n : M] = \ell > 1$  is finite. Since  $\mathcal{U}$  is nonprincipal, it follows that if  $P$  is *any* open subgroup of  $\prod G_n$ , then  $\pi(P) = \prod_{\mathcal{U}} G_n$ . In particular,  $M$  is not an open subgroup and hence there exists a discontinuous homomorphism from  $\prod G_n$  into the finite group  $\text{Sym}(\ell)$ . This completes the proof of Theorem 2.3.

#### 4. On the Existence of Nonprincipal Ultrafilters

In this section, working with the axiom system  $ZF + DC$ , we will prove that the failure of the Bergman property for suitably chosen infinite products  $\prod H_n$  of finite groups implies the existence of a nonprincipal ultrafilter over  $\omega$ . It is currently not known whether failures of the Steinhaus property also imply the existence of a nonprincipal ultrafilter over  $\omega$ . However, we will show that failures of a weak form of the Steinhaus property do indeed imply the existence of such an ultrafilter.

**Theorem 4.1.** (*ZF + DC*) *Let  $(H_n \mid n \in \omega)$  be a sequence of nontrivial finite groups which satisfies the following condition:*

(†) *There is a fixed integer  $t \geq 1$  such that for all  $n \in \omega$ , there is a conjugacy class  $C_n \subseteq H_n$  such that  $C_n^t = H_n$ .*

*If  $\prod H_n$  does not have the Bergman property, then there exists a nonprincipal ultrafilter over  $\omega$ .*

**Proof.** Suppose that  $G = \prod H_n$  does not have the Bergman property. Then we can express  $G = \bigcup_{k \in \omega} U_k$  as the union of a strictly increasing chain of symmetric proper subsets such that  $U_k U_k \subseteq U_{k+1}$  for all  $k \in \omega$ . Consider

$$\mathcal{I} = \left\{ A \subseteq \omega \mid \prod_{n \in A} H_n \subseteq U_k \text{ for some } k \in \omega \right\}.$$

Then clearly  $\mathcal{I}$  is an ideal which contains all the finite subsets of  $\omega$ . Hence it is enough to prove that there exists a set  $B \notin \mathcal{I}$  such that  $\mathcal{I} \cap \mathcal{P}(B)$  is a prime ideal over  $B$ .

Suppose that no such set  $B$  exists. Then for each  $A \notin \mathcal{I}$ , there exists  $A' \subseteq A$  such that  $A' \notin \mathcal{I}$  and  $A \setminus A' \notin \mathcal{I}$ ; and hence we can inductively find pairwise disjoint subsets  $\{A_k \mid k \in \omega\}$  of  $\omega$  such that  $A_k \notin \mathcal{I}$  and  $\omega \setminus \bigcup_{\ell \leq k} A_\ell \notin \mathcal{I}$  for all  $k \in \omega$ .

**Claim 4.2.** *There exists  $k \in \omega$  such that for every  $h \in \prod_{n \in A_k} H_n$ , there exists  $g \in U_k$  such that  $g \upharpoonright A_k = h$ .*

**Proof of Claim 4.2.** If not, then there exists  $h \in G$  such that for all  $k \in \omega$  and  $g \in U_k$ , we have that  $g \upharpoonright A_k \neq h \upharpoonright A_k$ . But this means that  $h \notin \bigcup_{k \in \omega} U_k$ , which is a contradiction.  $\square$

Fix some such  $k \in \omega$ . For each  $n \in A_k$ , let  $C_n$  be the conjugacy class of  $H_n$  given by condition (†) and let  $h = (h_n) \in \prod_{n \in A_k} H_n$  be such that  $h_n \in C_n$  for all  $n \in A_k$ . Let  $h \in U_\ell$  and let  $m = \max\{k, \ell\}$ . Then it follows that the conjugacy class  $C$  of  $h$  in  $\prod_{n \in A_k} H_n$  is contained in  $U_m^3$ ; and hence  $\prod_{n \in A_k} H_n$  is contained in  $U_m^{3t}$ . But this means that  $\prod_{n \in A_k} H_n \subseteq U_s$  for some  $s \geq m$ , which contradicts the fact that  $A_k \notin \mathcal{I}$ . This completes the proof of Theorem 4.1.  $\square$

Clearly Theorem 2.9 is an immediate consequence of Theorem 4.1, together with the following result.

**Proposition 4.3.** (Ellers–Gordeev–Herzog [10]) *Suppose that  $K$  is any field such that  $|K| > 5$  and that  $C$  is any noncentral conjugacy class of  $\text{SL}(d, K)$ .*

- (i) *If  $d = 2$ , then  $C^8 = \text{SL}(2, K)$ .*
- (ii) *If  $d > 2$ , then  $C^{2d} = \text{SL}(d, K)$ .*

In the remainder of this section, we will consider the following weak form of the Steinhaus property.

**Definition 4.4.** The Polish group  $G$  is said to have the *weak Steinhaus property* if for every symmetric countably syndetic subset  $W \subseteq G$ , there exists an integer  $k \geq 1$  such that  $W^k$  contains an open neighborhood of the identity element  $1_G$ .

For example, if the Polish group  $G$  has a non-open subgroup of countable index, then clearly  $G$  does not have the weak Steinhaus property. In particular, if we work with  $ZFC$ , then the results of Sec. 3 show that  $\prod \text{SL}(d, p_n)$  does not have the weak Steinhaus property. The rest of this section is devoted to the proof of the following result.

**Theorem 4.5.** ( $ZF + DC$ ) *Suppose that  $d \geq 2$  and that  $(p_n \mid n \in \omega)$  is an increasing sequence of primes. If  $\prod \text{SL}(d, p_n)$  does not have the weak Steinhaus property, then there exists a nonprincipal ultrafilter over  $\omega$ .*

Notice that Theorem 1.5 is an easy consequence of Theorem 4.5. For assume that  $\varphi : \prod \text{SL}(d, p_n) \rightarrow \text{Sym}(\omega)$  is a discontinuous homomorphism. Then there exists an open subgroup  $U \leq \text{Sym}(\omega)$  such that  $\varphi^{-1}(U)$  is not open in  $\prod \text{SL}(d, p_n)$ . Since  $U$  has countable index in  $\text{Sym}(\omega)$ , it follows that  $\varphi^{-1}(U)$  is a non-open subgroup of countable index in  $\prod \text{SL}(d, p_n)$  and hence  $\prod \text{SL}(d, p_n)$  does not have the weak Steinhaus property.

Most of our effort will go into proving the following special case of Theorem 4.5.

**Theorem 4.6.** ( $ZF + DC$ ) *Suppose that  $d \geq 2$  and that  $(p_n \mid n \in \omega)$  is an increasing sequence of primes. If there exists a subgroup  $H < \prod \text{SL}(d, p_n)$  such that  $[\prod \text{SL}(d, p_n) : H] = \omega$ , then there exists a nonprincipal ultrafilter over  $\omega$ .*

The proof of Theorem 4.6 makes use of some of the basic properties of primitive permutation groups. Recall that if  $\Omega$  is any nonempty set and  $G \leq \text{Sym}(\Omega)$ , then

$G$  is said to act *primitively* on  $\Omega$  if:

- (i)  $G$  acts transitively on  $\Omega$ ; and
- (ii) there does not exist a nontrivial  $G$ -invariant equivalence relation on  $\Omega$ .

It is well known that if  $G \leq \text{Sym}(\Omega)$  is a transitive subgroup, then  $G$  acts primitively on  $\Omega$  if and only if the stabilizer  $G_\alpha = \{g \in G \mid g(\alpha) = \alpha\}$  is a maximal subgroup of  $G$  for some (equivalently every)  $\alpha \in \Omega$ . Also if  $G$  acts primitively on  $\Omega$  and  $1 \neq N \trianglelefteq G$  is a nontrivial normal subgroup, then it follows that  $N$  must act transitively on  $\Omega$ . (For example, see Cameron [3, Theorem 1.7].)

The proof of Theorem 4.6 also makes use of the following easy consequence of Proposition 4.3.

**Lemma 4.7.** (*ZF + DC*) *Suppose that  $d \geq 2$  and that  $(p_n \mid n \in \omega)$  is an increasing sequence of primes. Then every normal subgroup  $N$  of countable index in  $\prod \text{SL}(d, p_n)$  is open.*

**Proof.** Let  $G = \prod \text{SL}(d, p_n)$  and let  $\mathcal{F} = \{g_\tau = (g_\tau(n)) \mid \tau \in 2^{\mathbb{N}}\} \subseteq G$  be a family such that for each  $\tau \neq \sigma \in 2^{\mathbb{N}}$ , there exists an integer  $n_{\tau, \sigma} \geq 0$  such that

- $g_\tau(n) = g_\sigma(n)$  for all  $n < n_{\tau, \sigma}$ ; and
- $g_\tau(n)^{-1}g_\sigma(n)$  is a noncentral element of  $\text{SL}(d, p_n)$  for all  $n \geq n_{\tau, \sigma}$ .

Since  $[G : N] \leq \omega$ , there exist  $\tau \neq \sigma \in 2^{\mathbb{N}}$  such that  $g_\tau N = g_\sigma N$  and hence  $g = g_\tau^{-1}g_\sigma \in N$ . Since  $N$  is a normal subgroup, the conjugacy class  $C = g^G$  is contained in  $N$ . Applying Proposition 4.3, it follows easily that  $N$  contains the open subgroup  $\prod_{n \geq n_{\tau, \sigma}} \text{SL}(d, p_n)$  and hence  $N$  is open.  $\square$

**Proof of Theorem 4.6.** Let  $G = \prod \text{SL}(d, p_n)$  and let  $\{P_j \mid j \in J\}$  be the set of open subgroups of  $G$  such that  $H \leq P_j$ . Since  $H \leq \bigcap_{j \in J} P_j$  and the intersection of infinitely many open subgroups of  $G$  has index  $2^\omega$ , it follows that  $J$  is finite. Let

$$G' = \prod_{n \geq n_0} \text{SL}(d, p_n) \leq \bigcap_{j \in J} P_j.$$

Then after replacing  $G$  by  $G'$  and  $H$  by its projection  $H'$  into  $G'$  if necessary, we can assume that  $H$  is not contained in any proper open subgroups of  $G$ .

Let  $G = \bigsqcup_{n \in \omega} g_n H$  be the coset decomposition of  $H$  in  $G$ . Then we can construct a strictly increasing chain  $H_n$  of proper subgroups of  $G$  as follows.

- $H_0 = H$ .
- Suppose inductively that  $H_n$  has been defined and that  $H \leq H_n < G$ . If  $H_n$  is a maximal proper subgroup of  $G$ , then the construction terminates with  $H_n$ . Otherwise, let  $k_n$  be the least integer  $k$  such that  $H_n < \langle H_n, g_k \rangle < G$  and let  $H_{n+1} = \langle H_n, g_{k_n} \rangle$ .

First suppose that there exists an integer  $n$  such that  $H_n$  is a maximal proper subgroup of  $G$ . Then we claim that  $[G : H_n] = \omega$ . Otherwise,  $[G : H_n] < \omega$  and

hence  $N = \bigcap_{g \in G} gH_n g^{-1}$  is a normal subgroup of  $G$  such that  $N \leq H_n$  and  $[G : N] < \omega$ . Applying Lemma 4.7, it follows that  $N$  is an open subgroup of  $G$  and hence  $H_n$  is also an open subgroup of  $G$ . But this contradicts the fact that  $H$  is not contained in any proper open subgroups of  $G$ . Next assume that the construction does not terminate after finitely many steps and let  $H_\omega = \bigcup_{n \in \omega} H_n$ . Then either  $H_\omega = G$  or else  $H_\omega$  is a maximal proper subgroup of  $G$ . In the former case,  $G$  has countable cofinality and hence, by Theorem 2.9, there exists a nonprincipal ultrafilter over  $\omega$ . Thus we can assume that  $H_\omega$  is a maximal proper subgroup of  $G$  and our earlier argument shows that  $[G : H_\omega] = \omega$ .

In order to simplify notation, we will assume that  $H$  is a maximal subgroup of  $G$ . Hence, by considering the left translation action of  $G$  on the set  $\{g_n H \mid n \in \mathbb{N}\}$ , we obtain a homomorphism

$$\psi : G \rightarrow \text{Sym}(\omega)$$

such that  $\psi(G)$  acts primitively on  $\omega$ . It follows that if  $N \trianglelefteq G$  is any normal subgroup, then either  $\psi(N) = 1$  or  $\psi(N)$  acts transitively on  $\mathbb{N}$ . Let

$$\mathcal{I} = \left\{ A \subseteq \omega \mid \psi \left( \prod_{n \in A} \text{SL}(d, p_n) \right) = 1 \right\}.$$

Then  $\mathcal{I}$  is clearly an ideal on  $\omega$ . Furthermore, if  $F \subseteq \omega$  is a finite subset, then  $\psi(\prod_{n \in F} \text{SL}(d, p_n))$  cannot act transitively on  $\mathbb{N}$  and so  $F \in \mathcal{I}$ . We will show that  $\mathcal{I}$  is a prime ideal.

So assume that there exists a subset  $A \subseteq \omega$  such that both  $A \notin \mathcal{I}$  and  $\omega \setminus A \notin \mathcal{I}$ . Let  $P = \prod_{n \in A} \text{SL}(d, p_n)$  and let  $Q = \prod_{n \in \omega \setminus A} \text{SL}(d, p_n)$ . Then both  $\psi(P)$  and  $\psi(Q)$  act transitively on  $\mathbb{N}$ . Suppose that  $g \in P$  is such that  $\psi(g)$  fixes some integer  $n \in \mathbb{N}$ . If  $k \in \mathbb{N}$  is arbitrary, then there exists  $h \in Q$  such that  $\psi(h)(n) = k$ ; and since  $g$  and  $h$  commute, it follows that

$$\psi(g)(k) = (\psi(g) \circ \psi(h))(n) = (\psi(h) \circ \psi(g))(n) = \psi(h)(n) = k.$$

Thus  $g \in \ker \psi$ . It follows that  $N = \ker \psi \cap P$  is a normal subgroup of  $P$  such that  $[P : N] = \omega$ , which contradicts Lemma 4.7.  $\square$

**Proof of Theorem 4.5.** Let  $G = \prod \text{SL}(d, p_n)$  and assume that the symmetric countably syndetic subset  $W \subseteq G$  witnesses the failure of the weak Steinhaus property. Let  $H = \langle W \rangle$  be the subgroup generated by  $W$ . Then clearly  $[G : H] \leq \omega$ . If  $[G : H] = \omega$ , then the result follows from Theorem 4.6 and so we can assume that  $[G : H] < \omega$ . Applying Lemma 4.7, it follows easily that  $H$  is an open subgroup of  $G$ . Let

$$G' = \prod_{n \geq n_0} \text{SL}(d, p_n) \leq H$$

and let  $\pi : G \rightarrow G'$  be the canonical projection. Consider the set  $W' = \pi(W)$  of generators of  $G'$ . If  $W'$  witnesses that  $G'$  is not Cayley bounded, then the result

follows from Theorem 2.9. Hence we can suppose that there exists an integer  $k \geq 1$  such that  $(W')^k = G'$ . Let  $g = (g_n) \in G'$  be such that  $g_n$  is a noncentral element of  $\text{SL}(d, p_n)$  for all  $n \geq n_0$ . Then  $g \in W^\ell$  for some  $\ell \geq 1$ ; and Proposition 4.3 implies that

$$G' \subseteq \underbrace{W^k W^\ell W^k \dots W^k W^\ell W^k}_{m \text{ times}} = W^{2km + \ell m},$$

where  $m = 8$  if  $d = 2$  and  $m = 2d$  if  $d > 2$ . But this contradicts the assumption that  $W$  witnesses the failure of the weak Steinhaus property.  $\square$

### 5. The Bergman and Steinhaus Properties in $L(\mathbb{R})[\mathcal{U}]$

In this section, we will present a partition property  $PP$  for products of finite sets with measures; and we will show that  $ZF + DC + PP$  implies that if  $(H_n \mid n \in \omega)$  is a sequence of nontrivial finite groups such that  $(d_{\mathbb{C}}(H_n) \mid n \in \omega)$  grows sufficiently fast, then  $\prod H_n$  has both the Bergman property and the Steinhaus property.

**The Partition Property.** ( $PP$ ) If  $(\langle a_n, \mu_n \mid n \in \omega \rangle)$  is a sufficiently fast growing sequence of finite sets  $a_n$  with measures  $\mu_n$ , then for every partition

$$\prod a_n = \bigsqcup_{m \in \omega} X_m,$$

there exists an integer  $m \in \omega$  such that  $\prod b_n \subseteq X_m$  for some sequence of subsets  $b_n \subseteq a_n$  such that  $\lim_{n \rightarrow \infty} \mu_n(b_n) = \infty$ .

Here the words “sufficiently fast growing” should be interpreted in the sense that there is a *fixed* function  $f$  that assigns a natural number to every finite sequence of finite sets with measures  $(\langle a_m, \mu_m \mid m < n \rangle)$  and that an infinite sequence  $(\langle a_n, \mu_n \mid n \in \omega \rangle)$  is sufficiently fast growing if

$$\mu_n(a_n) > f(\langle \langle a_m, \mu_m \mid m < n \rangle \rangle)$$

for all  $n \in \omega$ . The exact formula for the function  $f$  is immaterial for the purposes of this paper. We will only mention that it is primitive recursive with a growth rate approximately that of a tower of exponentials of linear height. (The primitive recursive bound can be extracted from the argument in Shelah–Zapletal [30].)

The partition property  $PP$  fails in  $ZFC$ , since the Axiom of Choice can be used to construct highly irregular partitions. However, it does hold in  $ZFC$  if we restrict our attention to partitions into Borel sets; and it also holds for arbitrary partitions in many models of set theory in which the Axiom of Choice fails. In particular, in Sec. 7, we will prove the following result, which extends the work of Di Prisco–Todorćević [8, Sec. 7].

**Theorem 5.1.** ( $LC$ )  $L(\mathbb{R})[\mathcal{U}]$  satisfies  $PP$ .

We will also make use of the following recent result of Babai–Nikolov–Pyber [1] in the newly flourishing area of “arithmetic combinatorics”. Recall that if  $H$  is a



nontrivial finite group and  $K$  is a field, then  $d_K(H)$  denotes the minimal dimension of a nontrivial  $K$ -representation of  $H$ ; i.e. the least  $d$  such that there exists a nontrivial homomorphism  $\theta : H \rightarrow \text{GL}(d, K)$ .

**Theorem 5.2.** (Babai–Nikolov–Pyber [1]) *Let  $H$  be a nontrivial finite group and let  $k$  be an integer such that  $1 \leq k^3 \leq d_{\mathbb{C}}(H)$ . If  $A \subseteq H$  is a subset such that  $|A| \geq |H|/k$ , then  $A^3 = H$ .*

**Proof.** By Babai–Nikolov–Pyber [1, Corollary 2.6], if  $1 \leq k^3 \leq d_{\mathbb{R}}(H)$  and  $A \subseteq H$  with  $|A| \geq |H|/k$ , then  $A^3 = H$ . Since  $d_{\mathbb{C}}(H) \leq d_{\mathbb{R}}(H)$ , the result follows.  $\square$

**Remark 5.3.** If  $H$  is a nontrivial finite group, then either  $d_{\mathbb{R}}(H) = d_{\mathbb{C}}(H)$  or  $d_{\mathbb{R}}(H) = 2d_{\mathbb{C}}(H)$ . For the purposes of this paper, it does not matter whether we work with  $d_{\mathbb{C}}(H)$  or  $d_{\mathbb{R}}(H)$ . Since most of the literature on the representation theory of finite groups deals with complex representations, we have chosen to state our results in terms of  $d_{\mathbb{C}}(H)$ .

**Theorem 5.4.** (ZF + DC + PP) *Let  $(H_n \mid n \in \omega)$  be a sequence of nontrivial finite groups; and for each  $n \in \omega$ , let  $k_n = \lfloor d_{\mathbb{C}}(H_n)^{1/3} \rfloor$  and let  $\mu_n$  be the measure on  $H_n$  defined by  $\mu_n(A) = k_n(|A|/|H_n|)$ . If  $(\langle H_n, \mu_n \rangle \mid n \in \omega)$  grows sufficiently fast, then  $\prod H_n$  has both the Bergman property and the Steinhaus property.*

**Proof.** To see that  $G = \prod H_n$  has the Steinhaus property, suppose that  $W \subseteq G$  is a symmetric countably syndetic subset and let  $G = \bigcup_{m \in \omega} g_m W$ . Then *PP* implies that there exists  $m \in \omega$  such that  $\prod A_n \subseteq g_m W$  for some sequence of subsets  $A_n \subseteq H_n$  such that  $\lim_{n \rightarrow \infty} \mu_n(A_n) = \infty$ ; and after replacing  $\prod A_n$  by  $g_m^{-1} \prod A_n$ , we can suppose that  $\prod A_n \subseteq W$ . Let  $n_0 \in \omega$  be such that  $\mu_n(A_n) \geq 1$  and hence  $|A_n| \geq |H_n|/k_n$  for all  $n \geq n_0$ . Clearly we can suppose that  $A_n = \{a_n\}$  is a singleton for each  $n < n_0$ . Applying Theorem 5.2, it follows that  $W^3 \supseteq gG'$ , where

- $g = (a_0^3, \dots, a_{n_0-1}^3, 1, 1, \dots)$  and
- $G'$  is the open subgroup  $\prod_{n \geq n_0} H_n$ .

Since  $W$  is symmetric, it follows that  $W^6 \supseteq (gG')^{-1}gG' = G'$ . This completes the proof that  $\prod H_n$  has the Steinhaus property.

To see that  $G = \prod H_n$  has the Bergman property, suppose that  $G = \bigcup_{m \in \omega} U_m$  is the union of an increasing chain of symmetric subsets such that  $U_m U_m \subseteq U_{m+1}$  for all  $m \in \omega$ . Arguing as above, it follows that there exists  $m \in \omega$  such that  $U_m^6$  contains an open subgroup  $G'$  and hence  $G' \subseteq U_{m+3}$ . Since  $[G : G'] < \omega$ , this implies that there exists  $k \in \omega$  such that  $G = U_k$ , as required.  $\square$

Unfortunately the statement of Theorem 5.4 is a little involved. However, we can obtain a much cleaner statement if we impose the following natural condition on the collection  $\{G_n \mid n \in \omega\}$  of groups.

**Definition 5.5.** A class  $\mathcal{C}$  of nontrivial finite groups is said to be a *Jordan class* if for each integer  $d \geq 1$ , there exist only finitely many groups  $H \in \mathcal{C}$  up to isomorphism such that  $d_{\mathbb{C}}(H) = d$ .

The following result is an immediate consequence of Theorem 5.4. Here the phrase “grows sufficiently fast” means that there is a function  $f_{\mathcal{C}} : \mathbb{N} \rightarrow \mathbb{N}$  such that  $(d_{\mathbb{C}}(H_n) \mid n \in \omega)$  grows sufficiently fast if  $d_{\mathbb{C}}(H_{n+1}) > f_{\mathcal{C}}(d_{\mathbb{C}}(H_n))$  for all  $n \in \omega$ . And the phrase has a similarly natural interpretation in the statements of Corollaries 5.7 and 5.8.

**Corollary 5.6.** (*ZF + DC + PP*) Let  $\mathcal{C}$  be a Jordan class of nontrivial finite groups. If  $H_n \in \mathcal{C}$  for all  $n \in \omega$  and  $(d_{\mathbb{C}}(H_n) \mid n \in \omega)$  grows sufficiently fast, then  $\prod H_n$  has both the Bergman property and the Steinhaus property.

**Corollary 5.7.** (*LC*) If  $d \geq 2$  and  $(p_n \mid n \in \omega)$  is a sufficiently fast growing sequence of primes, then  $\prod \mathrm{SL}(d, p_n)$  has both the Bergman property and the Steinhaus property in  $L(\mathbb{R})[\mathcal{U}]$ .

**Proof.** Recall that if  $p \geq 5$  is a prime, then  $d_{\mathbb{C}}(\mathrm{SL}(2, p)) = (p - 1)/2$ ; and that if  $d > 2$ , then  $d_{\mathbb{C}}(\mathrm{SL}(d, p)) = (p^d - p)/(p - 1)$ . Thus the result is an immediate consequence of Theorem 5.1 and Corollary 5.6.  $\square$

It is perhaps worth pointing out that the proof of the following corollary does *not* make use of the classification of the finite simple groups.

**Corollary 5.8.** (*LC*) Suppose that  $(H_n \mid n \in \omega)$  is a sequence of nonabelian finite simple groups. If  $(|H_n| \mid n \in \omega)$  is sufficiently fast growing, then  $\prod H_n$  has both the Bergman property and the Steinhaus property in  $L(\mathbb{R})[\mathcal{U}]$ .

**Proof.** By Jordan’s theorem, there exists a function  $\varphi : \mathbb{N} \rightarrow \mathbb{N}$  such that if  $H$  is a finite subgroup of  $\mathrm{GL}(n, \mathbb{C})$ , then  $H$  contains an abelian normal subgroup  $N$  with  $[H : N] \leq \varphi(n)$ . (For example, see Curtis–Reiner [5, Theorem 36.13].) Thus the class  $\mathcal{C}$  of nonabelian finite simple groups is a Jordan class; and if  $|H_n|$  grows sufficiently fast, then  $d_{\mathbb{C}}(H_n)$  also grows sufficiently fast.  $\square$

We will conclude this section with a result which shows that it is necessary to impose some condition on the growth rate of the sequence  $(d_{\mathbb{C}}(H_n) \mid n \in \omega)$  if we wish to obtain the conclusion of Theorem 5.4.

**Theorem 5.9.** (*ZF + DC*) Suppose that there exists a nonprincipal ultrafilter over  $\omega$ . Then whenever  $(H_n \mid n \in \omega)$  is a sequence of nontrivial finite groups such that  $\liminf d_{\mathbb{C}}(H_n) < \infty$ , then  $\prod H_n$  does not have the automatic continuity property and hence does not have the Steinhaus property.

**Proof.** Recall that every complex representation of a finite group is similar to a unitary representation. (For example, see Curtis–Reiner [5, Exercise 10.6].) Hence

there exist an infinite subset  $I \subseteq \omega$  and a fixed integer  $d \geq 1$  such that for each  $n \in I$ , there exists a nontrivial homomorphism  $\varphi_n : H_n \rightarrow U(d, \mathbb{C})$ , where  $U(d, \mathbb{C})$  denotes the compact group of  $d \times d$  unitary matrices. In order to simplify notation, we will assume that  $I = \omega$ .

For each  $g_n \in H_n$  and  $1 \leq i, j \leq d$ , let  $\varphi_n(g_n)_{ij}$  denote the  $ij$  entry of the matrix  $\varphi_n(g_n) \in U(d, \mathbb{C})$ . Then if  $\mathcal{U}$  is a nonprincipal ultrafilter over  $\omega$ , we can define a homomorphism

$$\begin{aligned} \psi : \prod H_n &\rightarrow U(d, \mathbb{C}) \\ (g_n) &\mapsto (z_{ij}), \end{aligned}$$

where  $z_{ij} = \lim_{\mathcal{U}} \varphi_n(g_n)_{ij}$ . We claim that  $\psi$  is not continuous. To see this, suppose that  $\psi$  is continuous and let  $W \subseteq U(d, \mathbb{C})$  be an open neighborhood of the identity element which contains no nontrivial subgroups. (For the existence of such a neighborhood, see Helgason [11, II.B.5].) Then there exists an open subgroup  $H \subseteq \prod H_n$  such that  $\psi(H) \subseteq W$  and hence  $H \leq \ker \psi$ . In particular, there exists a cofinite subset  $A \subseteq \omega$  such that  $\prod_{k \in A} H_k \leq \ker \psi$ . For each  $k \in A$ , choose  $g_k \in H_k$  such that  $\varphi_k(g_k) \notin W$ . Then, letting  $g = (g_k) \in \prod_{k \in A} H_k$ , we have that  $\psi(g) \notin W$ , which is a contradiction.  $\square$

**Remark 5.10.** Recall that de Cornulier [6] has shown that if  $G$  is a product of infinitely many copies of a *fixed* finite perfect group  $H$ , then  $G$  has the Bergman property. Thus the analog of Theorem 5.9 is false for the Bergman property.

## 6. The Bergman and Steinhaus Properties in $V$

Suppose that  $(H_n \mid n \in \omega)$  is a sequence of nonabelian finite simple groups. Then, in the previous section, assuming the existence of suitable large cardinals, we proved that if  $(|H_n| \mid n \in \omega)$  is sufficiently fast growing, then  $\prod H_n$  has both the Bergman property and the Steinhaus property in  $L(\mathbb{R})[\mathcal{U}]$ . In this section, we will briefly discuss the question of when  $\prod H_n$  has either the Bergman property or the Steinhaus property in the actual set-theoretic universe  $V$ . In particular, throughout this section, we will work with the usual *ZFC* axioms of set theory.

Recall that the classification of the finite simple groups says that if  $S$  is a nonabelian finite simple group, then one of the following cases must hold.

- (i)  $S$  is one of the 26 sporadic finite simple groups.
- (ii)  $S$  is an alternating group  $\text{Alt}(n)$  for some  $n \geq 5$ .
- (iii)  $S$  is a group  $L(q)$  of (possibly twisted) Lie type  $L$  over a finite field  $\mathbb{F}_q$  for some prime power  $q$ .

The following condition is the key to understanding when the product  $\prod H_n$  has countable cofinality.

**Definition 6.1.** A sequence  $(H_n \mid n \in \omega)$  of nonabelian finite simple groups satisfies the *Malcev condition* if there exists an infinite subset  $I$  of  $\omega$  such that the following properties hold.

- (a) There exists a fixed (possibly twisted) Lie type  $L$  such that for all  $n \in I$ ,  $H_n = L(q_n)$  for some prime power  $q_n$ .
- (b) If  $n, m \in I$  and  $n < m$ , then  $q_n < q_m$ .

Arguing as in the proof of Theorem 2.7(a), it follows easily that if  $(H_n \mid n \in \omega)$  satisfies the Malcev condition, then  $\prod H_n$  has countable cofinality. Conversely, by Saxl–Shelah–Thomas [27, Theorem 1.9], if  $(H_n \mid n \in \omega)$  does not satisfy the Malcev condition, then  $\prod H_n$  has uncountable cofinality. Furthermore, as we mentioned earlier, Zalan Gyenis has recently checked that the arguments of Saxl–Shelah–Thomas [27] can be modified to prove that an infinite product  $\prod H_n$  of finite simple groups has the Bergman property if and only if  $\prod H_n$  has uncountable cofinality. Consequently, we have the following classification of the infinite products  $\prod H_n$  of finite simple groups satisfying the Bergman property.

**Theorem 6.2.** *If  $(H_n \mid n \in \omega)$  is a sequence of nonabelian finite simple groups, then the following are equivalent:*

- (a)  $(H_n \mid n \in \omega)$  does not satisfy the Malcev condition.
- (b)  $\prod H_n$  has the Bergman property.

The proof of Theorem 2.3 shows that if  $(H_n \mid n \in \omega)$  satisfies the Malcev condition, then there exists a subgroup  $K \leq \prod H_n$  with  $[\prod H_n : K] = \omega$  and hence  $\prod H_n$  does not have the Steinhaus property. Also, it is clear that if  $(H_n \mid n \in \omega)$  satisfies the following condition, then  $\prod H_n$  has a non-open subgroup of finite index and so once again the Steinhaus property fails. (See Example 1.1.)

**Definition 6.3.** A sequence  $(H_n \mid n \in \omega)$  of nonabelian finite simple groups satisfies the *Saxl–Wilson condition* if there exists an infinite subset  $I$  of  $\omega$  and a fixed group  $S$  such that  $H_n = S$  for all  $n \in I$ .

In Thomas [31], it was shown that  $\prod H_n$  has a non-open subgroup  $K$  such that  $[\prod H_n : K] < 2^{\aleph_0}$  if and only if  $(H_n \mid n \in \omega)$  satisfies either the Malcev condition or the Saxl–Wilson condition. Consequently, it seems natural to make the following conjecture.

**Conjecture 6.4.** *If  $(H_n \mid n \in \omega)$  is a sequence of nonabelian finite simple groups, then the following are equivalent:*

- (a)  $(H_n \mid n \in \omega)$  satisfies neither the Malcev condition nor the Saxl–Wilson condition.
- (b)  $\prod H_n$  has the Steinhaus property.

**Remark 6.5.** Using the classification of the finite simple groups, it is easily seen that condition (a) is equivalent to:

$$(a)' \liminf d(H_n) = \infty.$$

## 7. The Partition Property (PP)

Suppose that the Ramsey ultrafilter  $\mathcal{U}$  is  $L(\mathbb{R})$ -generic for the notion of forcing  $\mathcal{P}(\omega)/\text{Fin}$ . In this section, assuming the existence of suitable large cardinals, we will prove that the Partition Property (PP) holds in  $L(\mathbb{R})[\mathcal{U}]$ . More specifically, we will make use of the following large cardinal assumption.

(LC) *There exist infinitely many Woodin cardinals below a measurable cardinal.*

**Remark 7.1.** It is perhaps worth pointing out that LC is certainly not the minimal large cardinal assumption necessary to prove that PP holds in  $L(\mathbb{R})[\mathcal{U}]$ . For example, it is enough to assume the statement  $A_\kappa$  of Neeman–Zapletal [24] for a sufficiently large cardinal  $\kappa$ , and this is weaker in consistency strength than LC.

If we merely want to prove the consistency of  $ZF + DC + PP$ , then it is only necessary to assume the existence of an inaccessible cardinal. In more detail, suppose that  $\kappa \in V$  is an inaccessible cardinal and that  $\text{Coll}(\omega, < \kappa)$  is the usual Lévy collapse. (For the basic properties of the Lévy collapse, see Jech [15, Chap. 26].) Let  $G \subseteq \text{Coll}(\omega, < \kappa)$  be a  $V$ -generic filter and let  $\bar{\mathbb{R}}$  be the set of reals in the generic extension  $V[G]$ . Then the corresponding Solovay model  $V(\bar{\mathbb{R}})$  consists of the sets  $z \in V[G]$  which are hereditarily definable in  $V[G]$  from parameters in  $\mathbb{R} \cup V$ . Let the Ramsey ultrafilter  $\bar{\mathcal{U}}$  be  $V[G]$ -generic for  $\mathcal{P}(\omega)/\text{Fin}$ . Then clearly  $\bar{\mathcal{U}}$  is also  $V(\bar{\mathbb{R}})$ -generic for  $\mathcal{P}(\omega)/\text{Fin}$ . Most of our effort in this section will be devoted to proving the following result.

**Theorem 7.2.** *With the above hypotheses,  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  satisfies PP.*

To transfer this result from  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  to  $L(\mathbb{R})[\mathcal{U}]$ , we will make use of the fact that, assuming LC, if  $\kappa \in V$  is the least inaccessible cardinal, then the theory of  $L(\mathbb{R})[\mathcal{U}]$  is not altered by forcing with  $\text{Coll}(\omega, < \kappa)$ . In more detail, assuming LC, the theory of  $L(\mathbb{R})$  is not altered by forcing with  $\text{Coll}(\omega, < \kappa)$ ; i.e.  $L(\bar{\mathbb{R}})$  is elementarily equivalent to  $L(\mathbb{R})$ . (For example, see Larson [20, Corollary 3.1.16].) Since  $\mathcal{P}(\omega)/\text{Fin}$  is a homogeneous notion of forcing, it follows that if  $\varphi$  is any sentence in the language of set theory, then

$$\begin{aligned} L(\mathbb{R})[\mathcal{U}] \models \varphi &\iff L(\mathbb{R}) \models \mathcal{P}(\omega)/\text{Fin} \Vdash \varphi \\ &\iff L(\bar{\mathbb{R}}) \models \mathcal{P}(\omega)/\text{Fin} \Vdash \varphi \\ &\iff L(\bar{\mathbb{R}})[\bar{\mathcal{U}}] \models \varphi. \end{aligned}$$

**Proof of Theorem 5.1.** (LC) Let  $\kappa$  be the least inaccessible cardinal and let  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  be as in Theorem 7.2. Then the Ramsey ultrafilter  $\bar{\mathcal{U}}$  is also  $L(\bar{\mathbb{R}})$ -generic

for  $\mathcal{P}(\omega)/\text{Fin}$ . Working inside  $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ , suppose that  $(\langle a_n, \mu_n \mid n \in \omega \rangle)$  is a sufficiently fast growing sequence of finite sets  $a_n$  with measures  $\mu_n$  and that

$$\prod a_n = \bigsqcup_{m \in \omega} X_m$$

is any partition. Since  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  satisfies *PP*, there exists an integer  $m \in \omega$  and a sequence of subsets  $(b_n \subseteq a_n \mid n \in \omega) \in V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  such that  $\prod b_n \subseteq X_m$  and  $\lim_{n \rightarrow \infty} \mu_n(b_n) = \infty$ . Since  $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  and  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  have the same reals, it follows that  $(b_n \subseteq a_n \mid n \in \omega) \in L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ . Thus  $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  satisfies *PP*. Finally since the theory of  $L(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  is not altered by forcing with  $\text{Coll}(\omega, < \kappa)$ , it follows that  $L(\mathbb{R})[\bar{\mathcal{U}}]$  also satisfies *PP*.  $\square$

The remainder of this section will be devoted to the proof of Theorem 7.2. As usual, we will identify the notion of forcing  $\mathcal{P}(\omega)/\text{Fin}$  with the quasi-order  $([\omega]^\omega, \subseteq^*)$  of infinite subsets of  $\omega$ , quasi-ordered by  $c \subseteq^* d$  if and only if  $|c \setminus d| < \omega$ . The key element of the proof is the work of Shelah–Zapletal [30] showing that for every sufficiently fast growing sequence  $(\langle a_n, \mu_n \mid n \in \omega \rangle)$  of finite sets  $a_n$  with measures  $\mu_n$ , there is a notion of forcing  $\mathbb{P}$  with the following properties:

- (1)  $\mathbb{P}$  adds a new element  $\dot{x} \in \prod a_n$ .
- (2)  $\mathbb{P}$  is proper,  ${}^\omega\omega$ -bounding and adds no independent reals.
- (3)  $\mathbb{P}$  is defined in a way which depends only on the reals; i.e. if  $M \subseteq N$  are transitive models of set theory with the same reals, then  $\mathbb{P}^M = \mathbb{P}^N$ .
- (4) Suppose that  $M$  is a transitive model of set theory such that  $\mathcal{P}(\mathcal{P}(\mathbb{R}))^M$  is countable. Then for every  $p \in \mathbb{P}^M$ , there exist a sequence of sets  $(b_n \mid n \in \omega)$  with  $b_n \subseteq a_n$  and  $\mu_n(b_n) \rightarrow \infty$  such that the product  $\prod b_n$  consists only of  $M$ -generic points for the poset  $\mathbb{P}_p^M = \{q \in \mathbb{P}^M \mid q \leq p\}$ .

Here an *independent real* is an infinite subset  $a \subseteq \omega$  in the generic extension such that neither  $a$  nor  $\omega \setminus a$  contains an infinite ground model subset.

Let  $\kappa \in V$  be an inaccessible cardinal and let  $G \subseteq \text{Coll}(\omega, < \kappa)$  be a  $V$ -generic filter. Suppose that  $V(\bar{\mathbb{R}})$  is the corresponding Solovay model and that the Ramsey ultrafilter  $\bar{\mathcal{U}}$  is  $V[G]$ -generic (and hence also  $V(\bar{\mathbb{R}})$ -generic) for  $\mathcal{P}(\omega)/\text{Fin}$ . Let  $(\langle a_n, \mu_n \mid n \in \omega \rangle) \in V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$  be a sufficiently fast growing sequence of finite sets  $a_n$  with measures  $\mu_n$  and let  $\prod a_n = \bigsqcup_{m \in \omega} X_m$  be a partition of the product into countably many pieces within the model  $V(\bar{\mathbb{R}})[\bar{\mathcal{U}}]$ . Working inside  $V[G]$ , let  $c_0 \in [\omega]^\omega$  be any infinite subset of  $\omega$ . Then it is enough to find a subset  $c \in [\omega]^\omega$  with  $c \subseteq^* c_0$  and a sequence of subsets  $(b_n \subseteq a_n \mid n \in \omega) \in V[G]$  with  $\lim_{n \rightarrow \infty} \mu_n(b_n) = \infty$  such that for some  $m \in \omega$ ,

$$c \Vdash \prod b_n \subseteq \dot{X}_m.$$

Let  $f : \prod a_n \rightarrow \omega$  be the function defined by

$$f(x) = m \iff x \in X_m$$

and let  $\dot{f} \in V(\bar{\mathbb{R}})$  be a  $\mathcal{P}(\omega)/\text{Fin}$ -name for  $f$ . By the standard homogeneity arguments with respect to the Lévy collapse  $\text{Coll}(\omega, < \kappa)$ , we can assume that  $c_0, (a_n \mid n \in \omega) \in V$  and that the  $\mathcal{P}(\omega)/\text{Fin}$ -name  $\dot{f}$  is definable from the elements of the ground model  $V$ . In particular, it follows that there exists a formula  $\varphi(v_0, v_1, v_2)$  with parameters in  $V$  such that for every  $V$ -generic filter  $H \subseteq \text{Coll}(\omega, < \kappa)$  and every  $c \in [\omega]^\omega$ ,  $x \in \prod a_n$  and  $m \in \omega$ ,

$$V[H] \models c \Vdash_{\mathcal{P}(\omega)/\text{Fin}} \dot{x} \in \dot{X}_m \iff V[H] \models \varphi(c, x, m). \quad (7.3)$$

Working inside the ground model  $V$ , consider the product of the forcing  $\mathbb{P}$  with  $\mathbb{Q} = \mathcal{P}(\omega)/\text{Fin}$ . Then the poset  $\mathbb{Q}$  adds a Ramsey ultrafilter  $\mathbf{u}$  and  $\mathbb{P}$  adds a point  $x \in \prod a_n$ . Since the definition of the forcing  $\mathbb{P}$  only depends on the real numbers, it follows that  $\mathbb{P}^V = \mathbb{P}^{V[\mathbf{u}]}$ . Hence if  $\mathbf{u}, x$  are mutually generic, then  $x$  will be  $\mathbb{P}^{V[\mathbf{u}]}$ -generic over the model  $V[\mathbf{u}]$ .

**Lemma 7.4.** *In  $V[\mathbf{u}][x]$ ,  $\mathbf{u}$  still generates a Ramsey ultrafilter.*

**Proof.** By Shelah [29, VI.5.1], since  $\mathbb{P}^V = \mathbb{P}^{V[\mathbf{u}]}$  is proper and  ${}^\omega\omega$ -bounding in  $V[\mathbf{u}]$ , it is enough to show that  $\mathbf{u}$  still generates an ultrafilter in  $V[\mathbf{u}][x]$ . First note that since  $\mathbb{Q}$  is  $\sigma$ -closed and  $\mathbb{P}$  is proper, it follows that  $\mathcal{P}(\omega) \cap V[\mathbf{u}][x] = \mathcal{P}(\omega) \cap V[x]$ . (Since  $\mathbb{P}$  is proper, each real  $r \in V[\mathbf{u}][x]$  is obtained from a countable collection  $\mathcal{C} = \{C_n \mid n \in \omega\} \in V[\mathbf{u}]$  of countable subsets  $C_n \subseteq \mathbb{P}$  such that each  $C_n$  is predense below some condition  $p \in \mathbb{P}$ ; and since  $\mathbb{Q}$  is  $\sigma$ -closed, it follows that  $\mathcal{C} \in V$  and hence  $r \in V[x]$ .) Now suppose that  $p \in \mathbb{P}, q \in \mathbb{Q}$  are conditions and that  $p \Vdash \tau \subseteq \omega$ . Since  $\mathbb{P}$  does not add any independent reals, there exists a condition  $p' \leq p$  and an infinite subset  $q' \subseteq q$  such that either  $p' \Vdash q' \subseteq \tau$  or  $p' \Vdash \tau \cap q' = \emptyset$ . Hence either  $\langle q', p' \rangle \Vdash \tau \in \dot{\mathbf{u}}$  or  $\langle q', p' \rangle \Vdash \omega \setminus \tau \in \dot{\mathbf{u}}$ . It follows that  $\mathbf{u}$  still generates an ultrafilter in  $V[\mathbf{u}][x]$ .  $\square$

From now on, fix some  $\mathbf{u} \in V[G]$  such that  $\mathbf{u}$  is  $V$ -generic for  $\mathbb{Q}$  and  $c_0 \in \mathbf{u}$ . Let  $\mathbb{D} \in V[\mathbf{u}]$  be the poset consisting of the conditions  $(s, S)$ , where  $s \in [\omega]^{<\omega}$  and  $S \in \mathbf{u}$ , partially ordered by

$$(s, S) \leq (t, T) \iff s \supseteq t \text{ and } s \setminus t \subseteq T.$$

Then  $\mathbb{D}$  adds an infinite subset  $\dot{c} \subseteq \omega$  which diagonalizes the Ramsey ultrafilter  $\mathbf{u}$ ; i.e. a subset  $\dot{c}$  such that  $|\dot{c} \setminus S| < \omega$  for all  $S \in \mathbf{u}$ . In fact, by Mathias [22], every set diagonalizing  $\mathbf{u}$  is  $V[\mathbf{u}]$ -generic for the poset  $\mathbb{D}$ . By Lemma 7.4, if  $\bar{\mathbf{u}}$  is the upwards closure of  $\mathbf{u}$  in the model  $V[\mathbf{u}][x]$ , then  $\bar{\mathbf{u}}$  is a Ramsey ultrafilter in  $V[\mathbf{u}][x]$ . Hence if  $\bar{\mathbb{D}} \in V[x, \mathbf{u}]$  is the corresponding poset diagonalizing  $\bar{\mathbf{u}}$ , then  $\mathbb{D}$  is dense in  $\bar{\mathbb{D}}$  and every set diagonalizing  $\mathbf{u}$  is  $V[\mathbf{u}][x]$ -generic for both  $\bar{\mathbb{D}}$  and  $\mathbb{D}$ . Let  $\varphi(v_0, v_1, v_2)$  be the formula with parameters in  $V$  given by (7.3).

**Lemma 7.5.** *In  $V[\mathbf{u}][x]$ , there exists a natural number  $m \in \omega$  such that*

$$\langle 1, 1 \rangle \Vdash_{\mathbb{D} \times \text{Coll}(\omega, < \kappa)} \varphi(\dot{c}, \dot{x}, \check{m});$$

in other words, in  $V[\mathbf{u}][x]$ , we have that

$$\langle 1, 1 \rangle \Vdash_{\mathbb{D} \times \text{Coll}(\omega, < \kappa)} \dot{c} \Vdash_{\mathcal{P}(\omega)/\text{Fin}} \check{x} \in \dot{X}_m.$$

**Proof.** Suppose not. Then there exist distinct numbers  $m_0, m_1 \in \omega$  and conditions for the associated 3-step iteration

$$\langle r_0, s_0, \dot{d}_0 \rangle, \langle r_1, s_1, \dot{d}_1 \rangle \leq \langle 1, 1, \dot{c} \rangle$$

such that  $\langle r_0, s_0, \dot{d}_0 \rangle \Vdash \check{x} \in \dot{X}_{m_0}$  and  $\langle r_1, s_1, \dot{d}_1 \rangle \Vdash \check{x} \in \dot{X}_{m_1}$ . Choose mutually  $V[\mathbf{u}][x]$ -generic filters  $H_0 \subseteq \mathbb{D}$ ,  $K_0 \subseteq \text{Coll}(\omega, < \kappa)$  such that  $\langle r_0, s_0 \rangle \in H_0 \times K_0$ ; and note that, since  $d = \dot{d}_0 / (H_0 \times K_0) \subseteq^* \dot{c} / H_0$ , it follows that  $d$  is  $V[\mathbf{u}][x]$ -generic for the poset  $\mathbb{D}$ . Hence, after making a finite adjustment to the set  $d$  if necessary, we can find a  $V[\mathbf{u}][x]$ -generic filter  $H_1 \subseteq \mathbb{D}$  such that  $r_1 \in H_1$  and  $d = \dot{c} / H_1$ . A standard homogeneity argument with respect to the Lévy collapse  $\text{Coll}(\omega, < \kappa)$  now shows that there exists a  $V[\mathbf{u}][x][H_1]$ -generic filter  $K_1 \subseteq \text{Coll}(\omega, < \kappa)$  such that  $s_1 \in K_1$  and  $V[\mathbf{u}][x][H_0 \times K_0] = V[\mathbf{u}][x][H_1 \times K_1]$ . Let  $d' = \dot{d}_1 / (H_1 \times K_1)$ . Consider the notion of forcing  $\mathcal{P}(\omega)/\text{Fin}$  inside the model  $V[\mathbf{u}][x][H_0 \times K_0] = V[\mathbf{u}][x][H_1 \times K_1]$ . Working with the  $V[\mathbf{u}][x]$ -generic filter  $H_0 \times K_0$ , it follows that  $d \Vdash \check{x} \in \dot{X}_{m_0}$  and so  $V[\mathbf{u}][x][H_0 \times K_0] \models \varphi(d, x, m_0)$ . Furthermore, since

$$d' = \dot{d}_1 / (H_1 \times K_1) \subseteq^* \dot{c} / H_1 = d,$$

it follows that  $V[\mathbf{u}][x][H_0 \times K_0] \models \varphi(d', x, m_0)$  and so  $V[\mathbf{u}][x][H_0 \times K_0] \not\models \varphi(d', x, m_1)$ . On the other hand, working with the  $V[\mathbf{u}][x]$ -generic filter  $H_1 \times K_1$ , it follows that  $d' \Vdash \check{x} \in \dot{X}_{m_1}$  and so  $V[\mathbf{u}][x][H_1 \times K_1] \models \varphi(d', x, m_1)$ , which is a contradiction.  $\square$

Next, working in the model  $M = V[\mathbf{u}]$ , let  $p \in \mathbb{P}^M$  be a condition that identifies the natural number  $m$  in the statement of Lemma 7.5. Then, since  $\mathcal{P}(\mathcal{P}(\mathbb{R}))^M$  is countable in  $V[G]$ , there exists a sequence of sets  $(b_n \mid n \in \omega) \in V[G]$  with  $b_n \subseteq a_n$  and  $\mu_n(b_n) \rightarrow \infty$  such that the product  $\prod b_n$  consists only of  $M$ -generic points for the poset  $\mathbb{P}_p^M = \{q \in \mathbb{P}^M \mid q \leq p\}$ . Let  $c \in V[G]$  be an infinite subset of  $\omega$  which diagonalizes the ultrafilter  $\mathbf{u} \in M$ . Then clearly  $c \subseteq^* c_0$  and we claim that  $c \Vdash \prod b_n \subseteq \dot{X}_m$ . To see this, suppose that  $x \in \prod b_n$ . Then  $\mathbf{u}, x$  are mutually  $V$ -generic and  $c$  is  $V[\mathbf{u}][x]$ -generic for  $\mathbb{D}$ . Hence, working in  $V[G]$ , Lemma 7.5 implies that the condition  $c \in \mathcal{P}(\omega)/\text{Fin}$  forces  $\check{x} \in \dot{X}_m$ . This completes the proof of Theorem 7.2.

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