Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2008, Vol. 44, No. 2, 341–361
DOI: 10.1214/07-AIHP112

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Quenched non-equilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder

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Received 30 March 2006; revised 9 November 2006; accepted 15 January 2007

Abstract. For a sequence of i.i.d. random variables $\{\xi_x \colon x \in \mathbb{Z}\}$ bounded above and below by strictly positive finite constants, consider the nearest-neighbor one-dimensional simple exclusion process in which a particle at x (resp. $x+1$) jumps to $x+1$ (resp. x) at rate ξ_x. We examine a quenched non-equilibrium central limit theorem for the position of a tagged particle in the exclusion process with bond disorder $\{\xi_x \colon x \in \mathbb{Z}\}$. We prove that the position of the tagged particle converges under diffusive scaling to a Gaussian process if the other particles are initially distributed according to a Bernoulli product measure associated to a smooth profile $\rho_0 : \mathbb{R} \to [0, 1]$.

Résumé. Soit $\{\xi_x \colon x \in \mathbb{Z}\}$ une suite de variables aléatoires i.i.d. bornées supérieurement et inférieurement par des constantes finies et strictement positives. Nous étudions le théorème central limite "quenched" pour la position d'une particule marquée dans l'exclusion simple symmétrique unidimensionnelle où les variables d'occupation des sites x et $x+1$ sont échangés à taux ξ_x. Nous démontrons que la position de la particule marquée converge à l'échelle diffusive vers un processus Gaussien si les particules sont initiallement distribuées d'après une mesure de Bernoulli associée à un profil lisse $\rho_0 : \mathbb{R} \to [0, 1]$.

MSC: 60K35

Keywords: Hydrodynamic limit; Tagged particle; Non-equilibrium fluctuations; Random environment; Fractional Brownian motion

1. Introduction

A classical problem in statistical mechanics consists in proving that the dynamics of a single particle in a mechanical system are well approximated on a large scale by a Brownian motion [9,19]. In a seminal paper, Kipnis and Varadhan [10] proved an invariance principle for the position of a tracer particle in the symmetric simple exclusion process. The method relies on a central limit theorem for additive functionals of Markov processes and uses time reversibility and translation invariance. This approach has been extended to interacting particle systems whose generators satisfy a sector condition or, more generally, graded sector conditions ([12] and references therein).

In [8], we proved a non-equilibrium central limit theorem for the position of a tagged particle in the one-dimensional nearest-neighbor symmetric exclusion process. We assumed that the initial state is a product measure associated to a smooth profile. Observing that the position of the tagged particle can be recovered from the density field and the total current through a bond, we deduced a central limit theorem for the tagged particle from a joint non-equilibrium central limit theorem for the density field and the current.

[1]Partially supported by the John S. Guggenheim Memorial Foundation, FAPERJ and CNPq.

The evolution of random walks in a random environment has attracted some attention in these last years ([20] and references therein). Recently, a quenched central limit theorem has been proved for random conductance models [18]. Here, to each bond $\{x, y\}$ of $\mathbb{Z}^d$ is attached i.i.d. strictly positive random variables $\xi_{x,y}$. Under some conditions on the variables ξ, the authors proved, among other results, that for almost all environments ξ, a random walk on $\mathbb{Z}^d$ which jumps from x to y at rate $\xi_{x,y}$ converges, when diffusively rescaled, to a Brownian motion.

In this article we consider a one-dimensional nearest-neighbor exclusion process evolving on an environment ξ. Each particle behaves as the random walk described above, with the additional rule that a jump is suppressed whenever a particle decides to jump over a site already occupied. Under very mild assumptions on the environment, we prove that the density field converges to the solution of a heat equation, generalizing a previous result obtained by Nagy [16].

Assuming that the environment is strictly elliptic, i.e., formed by i.i.d. random variables $\xi_{x,x+1}$ strictly bounded away from 0 and ∞, we prove a non-equilibrium central limit theorem for the density field, which holds for almost all realizations of the environment. Here the assumption of independence and identical distribution of the environment could be relaxed. In contrast with [6], where annealed central limit theorems are considered, we prove in this article a quenched statement.

From this result and from a non-equilibrium central limit theorem for the current, we prove the main result of the article which states a central limit theorem for the position of a tagged particle starting from a configuration in which particles are distributed according to a Bernoulli product measure associated to a smooth density profile. This central limit theorem holds for almost all environment ξ's.

The approach and the main technical difficulties can be summarized in a few words to the specialists. The model is in principle non-gradient due to the presence of the environment [4]. However, a functional transformation of the empirical measure (3.5) turns it into a gradient model. The proof that the transformed empirical measure is close to the original empirical measure imposes some conditions on the the environment.

The same strategy can be applied to derive a non-equilibrium central limit theorem for the density field. Here, however, to prove tightness and to show that the transformed density field is close to the original, some sharp estimates on the space time correlations are needed as well. The deduction of these estimates require a Nash type bound on the kernel of the random walk in the random conductance model, which has been proved only under a strict ellipticity condition of the environment. At this point, it remains to adapt the strategy introduced in [8] to prove the central limit theorem for the tagged particle.

While in Rome in April 2005, the second author showed to A. Faggionato the model and the method described in next section to derive the hydrodynamic behavior of this bond disorder model. At that time he thought that the approach required uniform ellipticity of the environment. A few months later, A. Faggionato [1], generalizing Nagy's method [16], and the authors proved independently the hydrodynamic behavior requiring only the assumptions stated in Theorem 2.1 below.

2. Main results

We state in this section the main results of the article. Denote by $\mathcal{X}$ the state space $\{0,1\}^{\mathbb{Z}}$ and by the Greek letter η the elements of $\mathcal{X}$ so that $\eta(x) = 1$ if there is a particle at site x for the configuration η and $\eta(x) = 0$ otherwise.

Consider a sequence $\{\xi_x \colon x \in \mathbb{Z}\}$ of strictly positive numbers. The symmetric nearest-neighbor simple exclusion process with bond disorder $\{\xi_x \colon x \in \mathbb{Z}\}$ is the Markov process $\{\eta_t \colon t \ge 0\}$ on $\mathcal{X}$ whose generator $L_{\xi,N}$ acts on cylinder functions f as

$$(L_{\xi,N} f)(\eta) = N^2 \sum_{x \in \mathbb{Z}} \xi_x (\nabla_x f)(\eta),$$

where $(\nabla_x f)(\eta) = f(\sigma^{x,x+1}\eta) - f(\eta)$ and

$$\sigma^{x,y}\eta(z) = \begin{cases} \eta(y), & z = x, \\ \eta(x), & z = y, \\ \eta(z), & z \neq x, y. \end{cases}$$

Notice that the process is sped up by N^2.

Existence and ergodic properties of this Markov process can be proved as in the space homogeneous case [14,16]. Moreover, the Bernoulli product measures ν_α in $\{0,1\}^{\mathbb{Z}}$, with marginals $\nu_\alpha\{\eta(x)=1\}=\alpha$ for $\alpha\in[0,1]$, are extremal, reversible measures.

For each profile $\rho_0:\mathbb{R}\to[0,1]$, denote by $\nu^N_{\rho_0(\cdot)}$ the product measure on $\mathcal{X}$ with marginals given by $\nu^N_{\rho_0(\cdot)}\{\eta(x)=1\}=\rho_0(x/N)$. For a measure μ on $\mathcal{X}$, let $\mathbb{P}^N_\mu$ stand for the probability measure on the path space $D(\mathbb{R}_+,\mathcal{X})$ induced by the Markov process η_t and the measure μ.

The empirical measure associated to the process η_t is defined by

$$\pi^N_t(\mathrm{d}u)=\frac{1}{N}\sum_{x\in\mathbb{Z}}\eta_t(x)\delta_{x/N}(\mathrm{d}u).$$

Fix $0<\gamma<\infty$. Let $C^2_0(\mathbb{R})$ be the set of twice continuously differentiable functions $G:\mathbb{R}\to\mathbb{R}$ with compact support. Fix a profile $\rho_0:\mathbb{R}_+\to[0,1]$. A bounded function $\rho:\mathbb{R}_+\times\mathbb{R}\to[0,1]$ is said to be a weak solution of the heat equation

$$\begin{cases}\partial_t\rho=\gamma^{-1}\Delta\rho,\\ \rho(0,\cdot)=\rho_0(\cdot)\end{cases}\tag{2.1}$$

if

$$\langle\rho_t,G\rangle=\langle\rho_0,G\rangle+\int_0^t\mathrm{d}s\big\langle\rho_s,\gamma^{-1}\Delta G\big\rangle\tag{2.2}$$

for all $t\ge 0$ and all G in $C^2_0(\mathbb{R})$. In these equations, Δ stands for the Laplacian and $\langle\rho,H\rangle$ for the integral of H with respect to the measure $\rho(u)\,\mathrm{d}u$. It is well known that for any bounded profile $\rho_0:\mathbb{R}\to[0,1]$, there exists a unique weak solution of (2.1). The first main result of the article states a quenched law of large numbers for the empirical measure under weak assumptions on the environment $\{\xi_x:\ x\in\mathbb{Z}\}$.

Theorem 2.1. *Assume that* $\sup_{x\in\mathbb{Z}}\xi_x<\infty$ *and*

$$\lim_{K\to\infty}\frac{1}{K}\sum_{x=1}^{K}\xi_x^{-1}=\gamma,\qquad \lim_{K\to\infty}\frac{1}{K}\sum_{x=-K}^{-1}\xi_x^{-1}=\gamma\tag{2.3}$$

for some $0<\gamma<\infty$. *Fix a profile* $\rho_0:\mathbb{R}\to[0,1]$. *Under* $\mathbb{P}^N_{\nu^N_{\rho_0(\cdot)}}$, π^N_t *converges in probability to the weak solution of* (2.1): *For every continuous function with compact support* G, *every* $t\ge 0$ *and every* $\delta>0$,

$$\lim_{N\to\infty}\mathbb{P}^N_{\nu^N_{\rho_0(\cdot)}}\big[\big|\langle\pi^N_t,G\rangle-\langle\rho_t,G\rangle\big|>\delta\big]=0,$$

where $\rho(t,u)$ *is the weak solution of* (2.1).

To prove a quenched non-equilibrium central limit theorem for the empirical measure, assume that $\{\xi_x:\ x\in\mathbb{Z}\}$ is a sequence of i.i.d. random variables defined on a probability space $(\Omega,P,\mathcal{F})$ such that

$$P\big[\varepsilon\le\xi_0\le\varepsilon^{-1}\big]=1\tag{2.4}$$

for some $\varepsilon>0$. This strong ellipticity condition is needed in Section 6 to prove sharp estimates of the decay of the space–time correlation functions. All other arguments require the weaker integrability condition: $E[\xi_0^{-6}]<\infty$.

Fix a profile $\rho_0:\mathbb{R}\to[0,1]$ and an environment ξ. Let $\rho^{N,\xi}_t(x)=\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\eta_t(x)]$. A trivial computation shows that $\rho^{N,\xi}_t:\mathbb{Z}\to[0,1]$ is the solution of the discrete linear equation

$$\begin{cases}\partial_t\rho_t(x)=N\big\{\xi_x(\nabla_N\rho_t)(x)-\xi_{x-1}(\nabla_N\rho_t)(x-1)\big\},\\ \rho_0(x)=\rho_0(x/N),\end{cases}\tag{2.5}$$

where $(\nabla_N h)(x) = N\{h(x+1) - h(x)\}$. We denote frequently $\rho_t^{N,\xi}$ by ρ_t^N.

Denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R})$ its dual, the space of distributions. Let $\{Y_t^N, t \geq 0\}$ be the density fluctuation field, a $\mathcal{S}'(\mathbb{R})$-valued process given by

$$Y_t^N(G) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} G\left(\frac{x}{N}\right)\{\eta_t(x) - \rho_t^{N,\xi}(x)\}$$

for G in $\mathcal{S}(\mathbb{R})$. Next theorem states the almost sure convergence of the finite dimensional distributions of Y_t^N to the marginals of a centered Gaussian field.

Theorem 2.2. *Let $\{\xi_x : x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables satisfying assumption* (2.4). *Let $\rho_0 : \mathbb{R} \to [0,1]$ be a profile with first derivative in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. There exists a set of environments Ω_0 with total measure such that for every ξ in Ω_0, every $k \geq 1$ and every $0 \leq t_1 < \cdots < t_k$, $(Y_{t_1}^N, \dots, Y_{t_k}^N)$ converges to a centered Gaussian vector $(Y_{t_1}, \dots, Y_{t_k})$ with covariance given by*

$$\begin{aligned} E\big[Y_s(G)Y_t(H)\big] = &\int_{\mathbb{R}} \chi\big(\rho_0(u)\big) T_s G(u) T_t H(u) \\ &+ 2\gamma^{-1} \int_0^s \mathrm{d}r \int_{\mathbb{R}} \chi\big(\rho(r,u)\big) \nabla T_{s-r} G(u) \nabla T_{t-r} H(u) \end{aligned} \tag{2.6}$$

for all $0 \leq s \leq t$, G, H in $\mathcal{S}(\mathbb{R})$. Here ρ stands for the solution of the heat equation (2.1), *$\{T_r : r \geq 0\}$ for the semigroup associated to $\gamma^{-1}\Delta$ and $\chi(\alpha) = \alpha(1-\alpha)$ for the compressibility in the exclusion process.*

Denote by $\nu_{\rho_0(\cdot)}^{N,*}$ the measure $\nu_{\rho_0(\cdot)}^N$ conditioned to have a particle at the origin and by X_t^N the position at time t of the particle initially at the origin. Define $u_t^N = u_t^{N,\xi}$ by the relation

$$\sum_{x=0}^{u_t^N} \rho_t^{N,*}(x) \leq \xi_{-1} \int_0^t N^2\{\rho_s^{N,*}(-1) - \rho_s^{N,*}(0)\}\,\mathrm{d}s < \sum_{x=0}^{u_t^N+1} \rho_t^{N,*}(x), \tag{2.7}$$

where $\rho_t^{N,*}$ is the solution of (2.5) with initial condition $\rho_0^{N,*}(0) = 1$, $\rho_0^{N,*}(x) = \rho_0(x/N)$, $x \neq 0$. Let $W_t^N = (X_t^N - u_t^N)/\sqrt{N}$.

Theorem 2.3. *Let $\{\xi_x : x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables satisfying assumption* (2.4). *Let ρ_0 be an initial profile with first derivative in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and second derivative in $L^\infty(\mathbb{R})$. There exists a set of environments Ω_0 with total measure and the following property. For every ξ in Ω_0, every $k \geq 1$ and every $0 \leq t_1 < \cdots < t_k$, under $\mathbb{P}_{\nu_{\rho_0(\cdot)}^{N,*}}$, $(W_{t_1}^N, \dots, W_{t_k}^N)$ converges in law to a Gaussian vector $(W_{t_1}, \dots, W_{t_k})$ with covariances given by*

$$\begin{aligned} \rho(s,u_s)\rho(t,u_t) E[W_s W_t] = &\int_{-\infty}^0 \mathrm{d}v\, P[Z_s \leq v] P[Z_t \leq v] \chi\big(\rho_0(v)\big) \\ &+ \int_0^\infty \mathrm{d}v\, P[Z_s \geq v] P[Z_t \geq v] \chi\big(\rho_0(v)\big) \\ &+ \frac{2}{\gamma} \int_0^s \mathrm{d}r \int_{-\infty}^\infty \mathrm{d}v\, p_{t-r}(u_t, v) p_{s-r}(u_s, v) \chi\big(\rho(r,v)\big) \end{aligned}$$

provided $s \leq t$. In this formula, $Z_t = u_t + B_{t/\gamma}^0$, where B_t^0 is a standard Brownian motion starting from the origin, and $p_t(v,w)$ stands for the kernel of $B_{t/\gamma}^0$.

3. Hydrodynamic limit

We prove in this section Theorem 2.1. Fix an environment satisfying (2.3) and denote by $\mathcal{M}_+(\mathbb{R})$ the set of positive Radon measures in $\mathbb{R}$. Fix $T \geq 0$ and a bounded profile $\rho_0 : \mathbb{R} \to [0, 1]$. Let $\{Q_N : N \geq 1\} = \{Q_{N,\xi} : N \geq 1\}$ be the sequence of measures on $D([0, T], \mathcal{M}_+(\mathbb{R}))$ induced by the Markov process π_t^N and the initial state $\nu^N_{\rho_0(\cdot)}$.

The proof of Theorem 2.1 is divided in two steps. We first prove tightness of the sequence $\{Q_N\}_N$, and then that all limit points of $\{Q_N\}_N$ are supported on weak solutions of the hydrodynamic equation. It follows from these two results and the uniqueness of weak solutions of the heat equation (2.1) that π_t^N converges in probability to the absolutely continuous measure $\rho(t, u)\,\mathrm{d}u$ whose density is the solution of (2.1) (cf. [9]).

It turns out that this program cannot be accomplished for the empirical measure π_t^N, but for a "corrected by the environment" process X_t^N, which is close enough to the empirical measure π_t^N.

3.1. Corrected empirical measure

Denote by $C_0^2(\mathbb{R})$ the space of twice continuously differentiable functions with compact support. For a function G in $C_0^2(\mathbb{R})$ and an environment ξ, let $T_\xi G : \mathbb{Z} \to \mathbb{R}$ be the sequence defined by

$$(T_\xi G)(x) = \sum_{j<x} \xi_j^{-1} \left\{ G\left(\frac{j+1}{N}\right) - G\left(\frac{j}{N}\right) \right\}. \tag{3.1}$$

For each $N \geq 1$ and each function G in $C_0^2(\mathbb{R})$, the series $\sum_x \xi_x^{-1}[G((x+1)/N) - G(x/N)]$ is absolutely summable because G has compact support. Moreover, it follows from (2.3) that

$$T_{\xi,G} = T_{\xi,G}^N := \sum_{x\in\mathbb{Z}} \xi_x^{-1} \left\{ G\left(\frac{(x+1)}{N}\right) - G\left(\frac{x}{N}\right) \right\} \tag{3.2}$$

converges to 0 as $N \uparrow \infty$.

We introduce $T_\xi G$ for two reasons. On the one hand, we expect $(T_\xi G)(x)$ to be close to $\gamma G(x/N)$, which is the content of Lemma 3.1. On the other hand,

$$N\big\{(T_\xi G)(x+1) - (T_\xi G)(x)\big\}\xi_x = (\nabla_N G)\left(\frac{x}{N}\right),$$

where ∇_N stands for the discrete derivative: $(\nabla_N G)(x/N) = N\{G(x + 1/N) - G(x/N)\}$. Hence, formally,

$$L_{\xi,N} \frac{1}{N} \sum_{x\in\mathbb{Z}} (T_\xi G)(x)\eta(x) = \frac{1}{N} \sum_{x\in\mathbb{Z}} (\Delta_N G)\left(\frac{x}{N}\right)\eta(x), \tag{3.3}$$

where Δ_N stands for the discrete Laplacian.

Of course, $T_\xi G$ may not belong to $\ell_1(\mathbb{Z})$, the space of summable series, and the left-hand side of the previous formula may not be defined. To overcome this difficulty, we modify $T_\xi G$ in order to integrate it with respect to the empirical measure. Fix an arbitrary integer $l > 0$ which will remain fixed in this section. Let $g = g_l : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(u) = \begin{cases} 0, & u < 0, \\ u/l, & 0 \leq u < l, \\ 1, & u \geq l. \end{cases}$$

For each function G in $C_0^2(\mathbb{R})$, let

$$(T_{\xi,l} G)(x) := (T_\xi G)(x) - \frac{T_{\xi,G}}{T_{\xi,g}} (T_\xi g)(x). \tag{3.4}$$

Notice that $T_{\xi,g}$ converges to γ almost surely, as $N \uparrow \infty$. In particular, by (3.2) the ratio $T_{\xi,G}/T_{\xi,g}$ vanishes almost surely as $N \uparrow \infty$. At the end of this section we prove the following statement.

Lemma 3.1. *For each function G in $C_0^2(\mathbb{R})$, and each environment ξ satisfying* (2.3), *$T_{\xi,l}G$ belongs to $\ell_1(\mathbb{Z})$ and*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{x\in\mathbb{Z}}\left|T_{\xi,l}G(x)-\gamma G\left(\frac{x}{N}\right)\right|=0.$$

Denote by X_t^N the corrected empirical measure defined by

$$X_t^N(G)=X_t^{N,l,\xi}(G)=\frac{1}{N}\sum_{x\in\mathbb{Z}}T_{\xi,l}G(x)\eta_t^N(x) \tag{3.5}$$

for each function G in $C_0^2(\mathbb{R})$.

As mentioned before, the sequence $T_{\xi,l}G(x)$ has two properties. On the one hand, in view of Lemma 3.1, it is close to $\gamma G(x/N)$ in $\ell_1(\mathbb{Z})$. In particular, the integral of G with respect to the empirical measure is close to $\gamma^{-1}X_t^N(G)$ uniformly in time. On the other hand, by (3.3), the martingale associated to $\gamma^{-1}X_t^N(G)$ has an integral term which can be expressed as a function of the empirical measure. Indeed, for a function G in $C_0^2(\mathbb{R})$, let $M_t^N(G)=M_t^{N,l,\xi}(G)$ be the martingale defined by

$$\begin{aligned}M_t^N(G)&=X_t^N(G)-X_0^N(G)-\int_0^t \mathrm{d}s\, N^2 L X_s^N(G)\\&=X_t^N(G)-X_0^N(G)-\int_0^t \mathrm{d}s\left\{\langle\pi_s^N,\Delta_N G\rangle-\frac{T_{\xi,G}}{T_{\xi,g}}\langle\pi_s^N,\Delta_N g\rangle\right\}.\end{aligned} \tag{3.6}$$

3.2. Tightness of π_t^N

It is well known that a sequence of probability measures $\{Q_N\}_N$ on $D([0,T],\mathcal{M}_+(\mathbb{R}))$ is tight if and only if the sequence $\{Q_N(G)\}_N$ is tight for all $G\in C_0^2(\mathbb{R})$, where $Q_N(G)$ is the probability measure in $D([0,T],\mathbb{R})$ corresponding to the process $\langle\pi_t^N,G\rangle$.

We claim that the process $X_t^N(G)$ is tight. Recall Aldous criteria for tightness in $D([0,T],\mathbb{R})$:

Lemma 3.2. *A sequence of probability measures $\{P_N\}_N$ in $\mathcal{D}([0,T],\mathbb{R})$ is tight if*

(i) *For all $0\le t\le T$ and for all $\varepsilon>0$ there exists a finite constant A such that* $\sup_N P_N(|x_t|>A)<\varepsilon$,
(ii) *For all* $\delta>0$,

$$\lim_{\beta\to 0}\limsup_{N\to\infty}\sup_{\substack{\tau\in\mathcal{T}\\ \theta\le\beta}}P_N\big(|x_{\tau+\theta}-x_\tau|>\delta\big)=0,$$

where $\mathcal{T}$ is the set of stopping times with respect to the canonical filtration bounded by T.

To prove tightness of $X_t^N(G)$ note that (i) is automatically satisfied because the number of particles per site is bounded and $T_{\xi,l}G$ converges to γG in $\ell_1(\mathbb{Z})$.

To check condition (ii), fix a stopping time τ bounded by T and $\theta\le\beta$. Recall from formula (3.6) that we may express $X_{\tau+\theta}^N(G)-X_\tau^N(G)$ as the sum of a martingale difference and an integral. On the one hand, computing the quadratic variation of the martingale $M_t^N(G)$, we obtain that

$$\begin{aligned}&\mathbb{E}_N\big[\big(M_{\tau+\theta}^N(G)-M_\tau^N(G)\big)^2\big]\\&\quad=\mathbb{E}_N\left[\int_\tau^{\tau+\theta}\mathrm{d}s\,\frac{1}{N^2}\sum_{x\in\mathbb{Z}}\left\{\nabla_N G\left(\frac{x}{N}\right)-\frac{T_{\xi,G}}{T_{\xi,g}}\nabla_N g\left(\frac{x}{N}\right)\right\}^2\big(\eta_s^N(x+1)-\eta_s^N(x)\big)^2\right].\end{aligned}$$

The previous expression is bounded above by $\theta N^{-1}\{C(G)+l^{-1}(T_{\xi,G}/T_{\xi,g})^2\}$, which vanishes as $N\uparrow\infty$ in view of (2.3) and (3.2).

On the other hand, since there is at most one particle per site and since G belongs to $C_0^2(\mathbb{R})$,

$$\left|\int_\tau^{\tau+\theta} \mathrm{d}s\,\frac{1}{N}\sum_{x\in\mathbb{Z}}\left\{\Delta_N G\left(\frac{x}{N}\right) - \frac{T_{\xi,G}}{T_{\xi,g}}\Delta_N g\left(\frac{x}{N}\right)\right\}\eta_s^N(x)\right| \le C_0\beta + \frac{2\beta}{l}\left|\frac{T_{\xi,G}}{T_{\xi,g}}\right|$$

for some finite constant C_0 depending only on G. As $N\uparrow\infty$, the second term vanishes in view of (2.3), (3.2). This proves condition (ii) of Lemma 3.2 and tightness of the process $X_t^N(G)$.

In view of Lemma 3.1,

$$\sup_{0\le t\le T}\left|\gamma\langle\pi_t^N, G\rangle - X_t^N(G)\right| \le \frac{1}{N}\sum_{x\in\mathbb{Z}}\left|\gamma G\left(\frac{x}{N}\right) - T_{\xi,l}G(x)\right| \tag{3.7}$$

converges to 0 as $N\uparrow 0$. In particular, $\langle\pi_t^N, G\rangle$ is also tight, with the same limit points of $X_t^N(G)$. Since this statement holds for all G in $C_0^2(\mathbb{R})$, the sequence Q_N is tight.

3.3. Uniqueness of limit points

Let Q be a limit point of the sequence $\{Q_N\}_N$. Since there is at most one particle per site, Q is concentrated on absolutely continuous paths $\pi(t,\mathrm{d}u) = \rho(t,u)\,\mathrm{d}u$, with positive density bounded by 1: $0\le\rho(t,u)\le 1$.

We have seen in the last subsection that Q is also a limit point of X_t^N. Fix a function G in $C_0^2(\mathbb{R})$ and recall the definition of the martingale $M_t^N(G)$ given in (3.6). By the proof of the tightness of X_t^N, the expectation of the quadratic variation of $M_t^N(G)$ vanishes as $N\uparrow\infty$. In particular, in view of (3.7), the measure Q is concentrated on trajectories π_t such that

$$\langle\pi_t, G\rangle = \langle\pi_0, G\rangle + \int_0^t \mathrm{d}s\,\langle\pi_s, \gamma^{-1}\Delta G\rangle$$

for all $0\le t\le T$, G in $C_0^2(\mathbb{R})$. By the uniqueness of weak solutions of the heat equation, Theorem 2.1 is proved.

We conclude this section with the

Proof of Lemma 3.1. $T_{\xi,l}G$ belongs to $\ell_1(\mathbb{Z})$ because it belongs to $\ell_\infty(\mathbb{Z})$ and vanishes outside a finite set. Fix a smooth function G in $C_0^2(\mathbb{R})$.

Consider first the sum over $x\le 0$. In this case $(T_\xi g_l)(x) = 0$ so that $(T_{\xi,l}G)(x) = (T_\xi G)(x)$. In particular,

$$\frac{1}{N}\sum_{x\le 0}\left|(T_{\xi,l}G)(x) - \gamma G\left(\frac{x}{N}\right)\right| = \frac{1}{N}\sum_{x\le 0}\left|\frac{1}{N}\sum_{y<x}\hat{\xi}_y^{-1}(\nabla_N G)\left(\frac{y}{N}\right)\right|,$$

where $\hat{\xi}_y^{-1} = \xi_y^{-1} - \gamma$. Both sums in x and y start from $-AN$, for some $A>0$, because G has compact support. Fix $\varepsilon>0$. Since G' is uniformly continuous, there exists $\delta>0$ such that $|G'(v) - G'(u)|\le\varepsilon$ if $|v-u|\le\delta$. We may therefore replace $(\nabla_N G)(y/N)$ by $G'(k\delta)$, for $k\delta\le y/N\le(k+1)\delta$, paying a price bounded by $C(G)\varepsilon$. After this replacement, the law of large numbers (2.3) ensures that the previous expression vanishes as $N\uparrow\infty$.

Similarly, for $x\ge lN$, $(T_\xi g_l)(x) = T_{\xi,g_l}$ so that $(T_{\xi,l}G)(x) = (T_\xi G)(x) - T_{\xi,G}$. Therefore,

$$(T_{\xi,l}G)(x) - \gamma G\left(\frac{x}{N}\right) = -\frac{1}{N}\sum_{y\ge x}\hat{\xi}_y^{-1}(\nabla_N G)\left(\frac{y}{N}\right)$$

and we may repeat the previous arguments to show that the sum for $x\ge lN$ vanishes as $N\uparrow\infty$.

Finally, for $0\le x<lN$, we estimate separately $(T_\xi G)(x) - \gamma G(x/N)$ and $\{T_{\xi,G}/\,T_{\xi,g_l}\}(T_\xi g_l)(x)$. The first piece is handled as before, while the second vanishes as $N\uparrow\infty$ in view of (3.2) and because $(T_\xi g_l)(x)/T_{\xi,g_l}$ is absolutely bounded by 1. This proves Lemma 3.1. □

4. Fluctuations of the empirical measure

Let $\{\xi_x\colon x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, P, \mathcal{F})$. We prove in this section a quenched non-equilibrium central limit theorem for the empirical measure. The proof relies on sharp estimates of the decay of the space–time correlation functions presented in Section 6 which requires the strong ellipticity condition: $P[\varepsilon \le \xi_0^{-1} \le \varepsilon^{-1}] = 1$ for some $\varepsilon > 0$. To stress that it is only in the estimation of the correlation functions that we need this condition, we present all other proofs under the weaker assumption that $E[\xi_0^{-6}] < \infty$. Moreover, the hypotheses of independence, identical distribution and finiteness of the sixth moment can be relaxed.

Throughout this section the index l of the operator $T_{\xi,l}$ introduced in the previous section depends on N as $l = l_N = N^{1/4}$. Recall that we denote by $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions. We may extend the operators T_ξ, $T_{\xi,l}$ to $\mathcal{S}(\mathbb{R})$:

Lemma 4.1. *Assume that $E[\xi_0^{-6}] < \infty$ and fix a function $G \in \mathcal{S}(\mathbb{R})$. There exists a subset Ω_G with total measure such that for each ξ in Ω_G $T_\xi G(x)$ is well defined and*

$$\lim_{N\to\infty} N^{1/4} \sup_{x\in\mathbb{Z}} \left| T_\xi G(x) - \gamma G\left(\frac{x}{N}\right)\right| = 0.$$

In particular, $\lim_{N\to\infty} N^{1/4} T_{\xi,G} = 0$. *Moreover,*

$$\lim_{N\to\infty} \sup_{x\in\mathbb{Z}} \left| T_{\xi,l} G(x) - \gamma G\left(\frac{x}{N}\right)\right| = 0$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| T_{\xi,l} G(x) - \gamma G\left(\frac{x}{N}\right)\right| = 0.$$

The proof of this lemma is given at the end of this section. By interpolation it follows from this result that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| T_{\xi,l} G(x)\right|^p = \int \left|\gamma G(u)\right|^p \,\mathrm{d}u \tag{4.1}$$

ξ-almost surely for all $1 \le p \le \infty$.

Recall the definition of the density field Y_t^N given just before the statement of Theorem 2.2. Denote by Z_t^N the fluctuation density field corrected by the environment:

$$Z_t^N(G) =: \frac{1}{\gamma} Y_t^N(T_{\xi,l}G) = \frac{1}{\gamma\sqrt{N}} \sum_{x\in\mathbb{Z}} (T_{\xi,l}G)(x)\left\{\eta_t^N(x) - \rho_t^{N,\xi}(x)\right\}$$

for functions G in $\mathcal{S}(\mathbb{R})$.

We prove in this section a non-equilibrium central limit theorem for the density field Z_t^N in random environment and deduce from this result the convergence of the finite dimensional distributions of the field Y_t^N defined in Section 2. Recall that we denote by $\mathcal{S}'(\mathbb{R})$ the Schwartz space of distributions. For a profile $\rho_0 : \mathbb{R} \to (0,1)$ and an environment $\xi = \{\xi_x\colon x \in \mathbb{Z}\}$ and $T > 0$, let $Q_{\rho_0}^{N,\xi}$ be the measure on $D([0,T], \mathcal{S}'(\mathbb{R}))$ induced by the process Z_t^N and the initial state $\nu_{\rho_0(\cdot)}^N$.

Proposition 4.2. *Fix a profile $\rho_0 : \mathbb{R} \to (0,1)$ with a bounded and integrable first derivative. There exists a set of environments Ω_0 with total measure such that for each ξ in Ω_0, $Q_{\rho_0}^{N,\xi}$ converges to a centered Gaussian field Z_t with covariance given by* (2.6).

The strategy of the proof of Proposition 4.2 is similar to the one adopted for the hydrodynamic limit. We prove tightness of the distributions of Z_t^N in $D([0,T],\mathcal{S}'(\mathbb{R}))$ and that all limit points of Z_t^N satisfy a martingale problem which characterizes the limiting measure.

We start proving tightness. For a function G in $\mathcal{S}(\mathbb{R})$, consider the martingale $M_t^N(G)$ defined by

$$M_t^N(G) = Z_t^N(G) - Z_0^N(G) - \int_0^t \gamma_1^N(s,G)\,\mathrm{d}s, \tag{4.2}$$

where

$$\gamma_1^N(s,G) = Y_s^N\big(\gamma^{-1}\Delta_N G\big) - \frac{T_{\xi,G}}{T_{\xi,g}} Y_s^N\big(\gamma^{-1}\Delta_N g_l\big).$$

The quadratic variation $\langle M^N(G)\rangle_t$ of this martingale is equal to $\int_0^t \gamma_2^N(s,G)\,\mathrm{d}s$, with $\gamma_2^N(s,G)$ given by

$$\frac{1}{\gamma^2 N}\sum_{x\in\mathbb{Z}} \xi_x^{-1}\left\{(\nabla_N G)\left(\frac{x}{N}\right) - \frac{T_{\xi,G}}{T_{\xi,g}}(\nabla_N g_l)\left(\frac{x}{N}\right)\right\}^2 \big(\eta_s^N(x+1) - \eta_s^N(x)\big)^2.$$

In view of Mitoma's criterion for the relative compactness of a sequence of measures in $D([0,T],\mathcal{S}'(\mathbb{R}))$ (cf. [5,7, 15]), to show that the process Z_t^N is tight, it is enough to prove that

$$\sup_N \sup_{0\le t\le T} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[Z_t^N(G)^2\big] < \infty, \qquad \sup_N \sup_{0\le t\le T} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[\gamma_i^N(t,G)^2\big] < \infty \tag{4.3}$$

for $i = 1, 2$ and a dense family of functions G in $\mathcal{S}(\mathbb{R})$. Moreover, to show that all limit points of the sequence Z_t^N are concentrated on $C([0,T],\mathcal{S}'(\mathbb{R}))$, we need to check that for each function G in $\mathcal{S}(\mathbb{R})$ there exists a sequence $\delta_N = \delta(t,G,N)$, vanishing as $N\uparrow\infty$, such that

$$\lim_{N\to\infty} \mathbb{P}_{\nu^N_{\rho_0(\cdot)}}\Big[\sup_{0\le t\le T}\big|Z_t^N(G) - Z_{t-}^N(G)\big| \ge \delta_N\Big] = 0. \tag{4.4}$$

To prove (4.3), consider a countable dense subset of functions $\mathcal{S}_0(\mathbb{R}) = \{G_k\colon k\ge 1\}$ in $\mathcal{S}(\mathbb{R})$. Let $\Omega_0 = \bigcap_{k\ge 1}\{\Omega_{G_k}\cap\Omega_{(G_k')^2}\cap\Omega_{\Delta G_k}\}$, where Ω_G are the total measure sets introduced in Lemma 4.1. Fix an environment ξ in Ω_0 and a function G in the class $\{G_k\colon k\ge 1\}$. By Theorem 6.1,

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[Z_t^N(G)^2\big] \le \frac{1}{\gamma^2 N}\sum_{x\in\mathbb{Z}}(T_{\xi,l}G)(x)^2 + \frac{C_1}{\gamma^2}\left(\frac{1}{N}\sum_{x\in\mathbb{Z}}\big|(T_{\xi,l}G)(x)\big|\right)^2$$

for some finite constant C_1 depending only on ε, ρ_0 and T. Since ξ belongs to Ω_0, by (4.1), as $N\uparrow\infty$, these expressions converge to finite expressions.

On the other hand, by definition of $\gamma_1^N(t,G)$,

$$\frac{\gamma^2}{2}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[\gamma_1^N(t,G)^2\big] \le \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[Y_t^N(\Delta_N G)^2\big] + \left\{\frac{T_{\xi,G}}{T_{\xi,g}}\right\}^2 E_{\nu^N_{\rho_0(\cdot)}}\big[Y_t^N(\Delta_N g_l)^2\big].$$

The first term is handled in the same way as $Z_t^N(G)$. The second term is also simple to estimate, since

$$Y_t^N(\Delta_N g_l)^2 = \frac{N}{l_N^2}\big\{\big[\eta_t^N(0) - \rho_t^N(0)\big] - \big[\eta_t^N(lN) - \rho_t^N(lN)\big]\big\}^2,$$

and since, by Lemma 4.1, $(T_{\xi,G}/T_{\xi,g})^2 N l_N^{-2}$ vanishes as $N\uparrow\infty$ for all ξ in Ω_0.

Finally, by definition of $\gamma_2^N(t,G)$,

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[\gamma_2^N(t,G)^2\big] \le \frac{2}{\gamma^2 N}\sum_{x\in\mathbb{Z}}\xi_x^{-1}\nabla_N G\left(\frac{x}{N}\right)^2 + \frac{2}{N l_N^2\gamma^2}\left(\frac{T_{\xi,G}}{T_{\xi,g}}\right)^2 \sum_{0\le x<lN}\xi_x^{-1}.$$

The first term converges to a finite constant as $N \uparrow \infty$, while the second term vanishes for ξ in Ω_0.

Since condition (4.4) follows from the fact that no more than one particle jumps at each time, the previous estimates show that for each environment ξ in Ω_0, the sequence $Q^{N,\xi}_{\rho_0}$ is tight and that each limiting point is concentrated on $C([0,T], \mathcal{S}'(\mathbb{R}))$.

We consider now the question of uniqueness of limit points. Fix ξ in Ω_0, let Q^ξ be a limit point of $Q^{N,\xi}_{\rho_0}$ and assume without loss of generality that $Q^{N,\xi}_{\rho_0}$ converges to Q^ξ. Let $\mathfrak{A}$, $\mathfrak{B}_t$, $t \ge 0$, stand for the operators $\gamma^{-1}\Delta$, $\sqrt{2\gamma^{-1}\chi(\rho(t,u))}\nabla$, respectively, where ρ is the solution of the heat equation (2.1) and χ is the compressibility given by $\chi(\alpha) = \alpha(1-\alpha)$.

According to the Holley–Stroock [7] theory of generalized Ornstein–Uhlenbeck processes and to Stroock and Varadhan [21], there exists a unique process Z_t in $C([0,+\infty), \mathcal{S}'(\mathbb{R}))$ with the following two properties: Z_0 is a centered Gaussian field with covariance given by

$$E\big[Z_0(G)Z_0(H)\big] = \int_{\mathbb{R}} G(u)H(u)\chi\big(\rho_0(u)\big)\,\mathrm{d}u \tag{4.5}$$

for all G, H in $\mathcal{S}(\mathbb{R})$. Moreover, the processes $M_t(G)$, $m_t(G)$ defined by

$$Z_t(G) - Z_0(G) - \int_0^t Z_s(\mathfrak{A}G)\,\mathrm{d}s \quad \text{and} \quad \big(M_t(G)\big)^2 - \int_0^t \|\mathfrak{B}_s G\|^2\,\mathrm{d}s \tag{4.6}$$

are martingales with respect to the canonical filtration $\{\mathcal{F}_s\colon s \ge 0\}$ for all G in $\mathcal{S}(\mathbb{R})$. Of course, it is enough to check these conditions for a dense family of functions in $\mathcal{S}(\mathbb{R})$.

Recall the definition of the class $\mathcal{S}_0(\mathbb{R})$ and fix a function G in $\mathcal{S}_0(\mathbb{R})$. An elementary computation of the characteristic function $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\exp\{\mathrm{i}\theta Z^N_0(G)\}]$ shows that Z^N_0 converges to a centered Gaussian field with covariances given by (4.5).

Recall from (4.2) the definition of the martingale $M^N_t(G)$ and fix a bounded function U in $\mathcal{F}_s$. To prove that $M_t(G)$ is a martingale, it is enough to show that

$$\lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\big[M^N_t(G)U\big] = E\big[M_t(G)U\big] \tag{4.7}$$

for all $t \ge s$.

Since $Z^N_t(G)$ is bounded in $L^2(\mathbb{P}_{\nu^N_{\rho_0(\cdot)}})$, (4.7) holds with $Z^N_t(G) - Z^N_0(G)$, $Z_t(G) - Z_0(G)$ in place of $M^N_t(G)$, $M_t(G)$. By the Schwarz inequality and a previous estimate,

$$\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\bigg[\bigg(\frac{T_{\xi,G}}{T_{\xi,g}}\int_0^t Y^N_s(\Delta_N g_l)\,\mathrm{d}s\bigg)^2\bigg] \le \frac{Ct^2N}{l_N^2}\bigg(\frac{T_{\xi,G}}{T_{\xi,g}}\bigg)^2$$

vanishes as $N \uparrow \infty$ for all ξ in Ω_0. On the other hand, by the Schwarz inequality, Theorem 6.1 and Lemma 4.1,

$$\begin{aligned}\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\bigg[\bigg(\int_0^t \big\{Y^N_s(\Delta_N G) - Z^N_s(\Delta_N G)\big\}\,\mathrm{d}s\bigg)^2\bigg] &\le C_1 t^{5/2}\bigg(\frac{1}{N}\sum_{x\in\mathbb{Z}}\bigg|\Delta_N G\bigg(\frac{x}{N}\bigg) - \gamma^{-1}(T_{\xi,l}\Delta_N G)(x)\bigg|\bigg)^2 \\ &\quad + t^2\frac{1}{N}\sum_{x\in\mathbb{Z}}\bigg\{\Delta_N G\bigg(\frac{x}{N}\bigg) - \gamma^{-1}(T_{\xi,l}\Delta_N G)(x)\bigg\}^2\end{aligned}$$

vanishes as $N \uparrow \infty$ for all ξ in Ω_0. Replacing $Y^N_s(\Delta_N G)$ by $Z^N_s(\Delta G)$ and recalling all previous estimates, we deduce that

$$\lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\bigg[U\int_0^t \gamma^N_1(s,G)\,\mathrm{d}s\bigg] = \lim_{N\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}}\bigg[U\int_0^t \gamma^{-1}Z^N_s(\Delta G)\,\mathrm{d}s\bigg] = E\bigg[U\int_0^t Z_s(\mathfrak{A}G)\,\mathrm{d}s\bigg]$$

because $Z^N_s(\Delta G)$ is bounded in L^2. This concludes the proof of (4.7).

To prove (4.7) with $m_t(G)$, $m_t^N(G) = M_t^N(G)^2 - \langle M^N(G)\rangle_t$ in place of $M_t(G)$, $M_t^N(G)$, observe first that $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[M_t^N(G)^4]$ is bounded uniformly in N in view of Theorem 6.1, so that $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[UM_t^N(G)^2]$ converges to $E[UM_t(G)^2]$.

To show that $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[U\langle M^N(G)\rangle_t]$ converges to $E[U\int_0^t \|\mathfrak{B}_s G\|^2\,\mathrm{d}s]$, notice that $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[\langle M^N(G)\rangle_t^2]$ is bounded uniformly in N, for all ξ in Ω_0, and that $\langle M^N(G)\rangle_t$ can be written as

$$\int_0^t \mathrm{d}s\,\frac{1}{\gamma^2 N}\sum_{x\in\mathbb{Z}}\xi_x^{-1}(\nabla_N G)(x)^2\big(\eta_s^N(x+1)-\eta_s^N(x)\big)^2$$

plus a remainder which vanishes as $N\uparrow\infty$ for all ξ in Ω_0. By Theorem 6.1 and the Schwarz inequality,

$$\int_0^t \mathrm{d}s\,\frac{1}{N}\sum_{x\in\mathbb{Z}}\xi_x^{-1}(\nabla_N G)\left(\frac{x}{N}\right)^2\big\{\eta_s^N(x)-\rho_s^N(x)\big\}$$

vanishes in $L^2(\mathbb{P}_{\nu^N_{\rho_0(\cdot)}})$, as well as the same expression with $\bar\eta_s^N(x)\bar\eta_s^N(x+1)$ in place of $\bar\eta_s^N(x) = \eta_s^N(x)-\rho_s^N(x)$. The penultimate integral is thus equal to

$$\int_0^t \mathrm{d}s\,\frac{1}{\gamma^2 N}\sum_{x\in\mathbb{Z}}\xi_x^{-1}(\nabla_N G)\left(\frac{x}{N}\right)^2\big\{\rho_s^N(x+1)+\rho_s^N(x)-2\rho_s^N(x+1)\rho_s^N(x)\big\}$$

plus a remainder which vanishes in $L^2(\mathbb{P}_{\nu^N_{\rho_0(\cdot)}})$. This shows that $\mathbb{E}_{\nu^N_{\rho_0(\cdot)}}[U\langle M^N(G)\rangle_t]$ converges to $E[U\int_0^t \|\mathfrak{B}_s G\|^2\,\mathrm{d}s]$ and concludes the proof of uniqueness.

Proof of Theorem 2.2. By Lemma 4.1 and Theorem 6.1, for each $t\ge 0$ and G in $\mathcal{S}(\mathbb{R})$, $Z_t^N(G) - Y_t^N(G)$ vanishes in L^2 ξ-almost surely as $N\uparrow\infty$. In particular, we may deduce from the central limit theorem for Z_t^N the convergence of the finite dimensional distributions of Y_t^N. □

We conclude this section with the following proof.

Proof of Lemma 4.1. Fix G in $\mathcal{S}(\mathbb{R})$ and recall that $\hat\xi_x^{-1} = \xi_x^{-1} - \gamma$. For N fixed, $T_\xi G(x)$ is well defined because $\sum_{-k\le x\le k}\xi_x^{-1}(\nabla_N G)(x/N)$ is a Cauchy sequence in $L^2(P)$. Observe that, by the Mean Value Theorem, for any function $G\in\mathcal{S}(\mathbb{R})$, there exists a constant $C_0(G)$ such that

$$\left|(\nabla_N G)\left(\frac{x}{N}\right)\right| \le \frac{C_0(G)}{1+(x/N)^4}.$$

By Doob's inequality, for each $x<y$, $A>0$,

$$P\left[\max_{x<z\le y}\left|\frac{1}{N}\sum_{w=x+1}^{z}\hat\xi_w^{-1}(\nabla_N G)\left(\frac{w}{N}\right)\right| > A\right] \le \frac{C_0(G)E[\xi_x^{-6}]}{A^6 N^3}$$

for some finite constant C_0 depending on G. Take $A = N^{-\{(1/4)+\epsilon\}}$ for some $0<\epsilon<1/12$, estimate the right-hand side by $C_0(G)E[\xi_x^{-6}]N^{6\epsilon-3/2}$ and let $y\uparrow\infty$ to conclude by Borel–Cantelli that

$$N^{1/4}\max_{x<z}\left|\frac{1}{N}\sum_{w=x+1}^{z}\hat\xi_w^{-1}(\nabla_N G)\left(\frac{w}{N}\right)\right|$$

vanishes, as $N\uparrow\infty$, almost surely. This proves the first statement of the lemma.

Since

$$T_{\xi,G} = \frac{1}{N}\sum_{x\in\mathbb{Z}} \xi_x^{-1}(\nabla_N G)\left(\frac{x}{N}\right) = \frac{1}{N}\sum_{x\in\mathbb{Z}} \hat{\xi}_x^{-1}(\nabla_N G)\left(\frac{x}{N}\right),$$

$T_{\xi,G}$ is absolutely bounded by $\sup_{x\in\mathbb{Z}} |T_\xi G(x) - \gamma G(x/N)|$ and the second claim of the lemma follows from the first one.

To prove the last two statements, notice that $(T_{\xi,l}G)(x) - \gamma G(x/N)$ is absolutely bounded by $R^N_{\xi,G}(x) + |T_{\xi,G}|\mathbf{1}\{0 < x < l_N N\}$, where $R^N_{\xi,G}(x)$ is equal to

$$\frac{1}{N}\left|\sum_{y<x} \hat{\xi}_y^{-1}\nabla_N G\left(\frac{y}{N}\right)\right| \quad \text{for } x \le 0, \qquad \frac{1}{N}\left|\sum_{y\ge x} \hat{\xi}_y^{-1}\nabla_N G\left(\frac{y}{N}\right)\right| \quad \text{for } x > 0.$$

In particular, by the first part of the proof and since $l_N = N^{1/4}$, $\sup_{x\in\mathbb{Z}} |T_{\xi,l}G(x) - \gamma G(x/N)|$, $l_N T_{\xi,G}$ vanishes, as $N \uparrow \infty$, ξ-almost surely. On the other hand, by the Tchebychev and Hölder inequality,

$$P\left[\frac{1}{N}\sum_{x\in\mathbb{Z}} R^N_{\xi,G}(x) > A\right] \le \frac{C_0}{A^4 N^4}\sum_{x\in\mathbb{Z}}\left(1 + |x|^{3(1+\epsilon)}\right)E\left[R^N_{\xi,G}(x)^4\right]$$

for some $\epsilon > 0$ and some finite constant $C_0 = C_0(\epsilon)$. Since G belongs to $\mathcal{S}(\mathbb{R})$, the previous expectation is less than or equal to $C_0 E[\xi_0^{-4}]N^{-2}F_G(x/N)$ for some rapidly decreasing positive function F_G. The left-hand side is thus bounded above by $C_0(\epsilon)E[\xi_0^{-4}]N^{3\epsilon-2}A^{-4}$. Choosing $0 < \epsilon < 1/7$, $A = N^{-\epsilon}$ we conclude the proof of the last statement of the lemma with a Borel–Cantelli argument. □

5. Central limit theorem for a tagged particle

We prove in this section Theorem 2.3. Unless otherwise stated, we assume throughout this section that ρ_0 is an initial condition with first derivative in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and second derivative in $L^\infty(\mathbb{R})$, and that the environment satisfies the assumptions of the previous section. The proof follows closely the approach presented in [8]. We omit therefore some details.

We first consider the current through a bond. For each x in $\mathbb{Z}$, denote by $J^N_{x,x+1}(t)$ the current over the bond $\{x, x+1\}$ in the time interval $[0, t]$. This is the total number of particles which jumped from x to $x+1$ minus the total number of particles which jumped from $x+1$ to x in the time interval $[0, t]$.

The current $J^N_{x,x+1}(t)$ can be related to the occupation variables $\eta_t(x)$ through the formula

$$J^N_{x-1,x}(t) - J^N_{x,x+1}(t) = \eta_t(x) - \eta_0(x). \tag{5.1}$$

The first result states a law of large numbers for the current through a bond assuming that the environment satisfies condition (2.3).

Proposition 5.1. *Consider a sequence $\{\xi_x\colon x \in \mathbb{Z}\}$ satisfying* (2.3) *and a profile $\rho_0 : \mathbb{R}_+ \to [0, 1]$ satisfying the assumptions stated at the beginning of this section. For every $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}^N_{\nu^N_{\rho_0(\cdot)}}\left[\left|\frac{J^N_{0,1}(t)}{N} + \int_0^t \gamma^{-1}(\partial_u\rho)(s, 0)\,\mathrm{d}s\right| > \delta\right] = 0,$$

where $\rho(t, u)$ is the solution of (2.1).

Proof. Fix $a > 0$. Identity (5.1) and a summation by parts give that

$$\frac{1}{aN^2}\sum_{x=1}^{aN} \xi_x^{-1}\left\{J^N_{x,x+1}(t) - J^N_{0,1}(t)\right\} = \frac{1}{aN^2}\sum_{x=1}^{Na}\left\{\eta^N_0(x) - \eta^N_t(x)\right\}\sum_{k=x}^{aN} \xi_k^{-1}. \tag{5.2}$$

Since the right-hand side is of order a and we will send a to 0 at the end of the proof, the law of large numbers for $J_{0,1}^N(t)/N$ follows from a law of large numbers for $a^{-1}N^{-2}\sum_{x=1}^{Na}\xi_x^{-1}J_{x,x+1}^N(t)$ for each a fixed. We may rewrite this latter expression as

$$\frac{1}{aN^2}\sum_{x=1}^{aN}\xi_x^{-1}M_{x,x+1}^N(t)+\frac{1}{a}\int_0^t\left\{\eta_s^N(1)-\eta_s^N(aN+1)\right\}\mathrm{d}s, \tag{5.3}$$

where

$$M_{x,x+1}^N(t)=:J_{x,x+1}^N(t)-N^2\int_0^t\xi_x\left\{\eta_s^N(x)-\eta_s^N(x+1)\right\}\mathrm{d}s,$$

x in $\mathbb{Z}$, are orthogonal martingales with quadratic variation $\langle M_{x,x+1}^N\rangle_t$ given by

$$N^2\int_0^t\xi_x\left\{\eta_s^N(x)-\eta_s^N(x+1)\right\}^2\mathrm{d}s.$$

In view of (2.3) and of the explicit expression for the quadratic variation of the orthogonal martingales $M_{x,x+1}^N(t)$, the first term in (5.3) vanishes in $L_2(\mathbb{P}_{\nu_{\rho_0(\cdot)}^N}^N)$ as $N\uparrow\infty$. On the other hand, by Lemma 6.5, the variance of the second term in (5.3) vanishes as $N\uparrow\infty$. Its expectation is equal to

$$\frac{1}{a}\int_0^t\left\{\rho_s^{N,\xi}(1)-\rho_s^{N,\xi}(aN+1)\right\}\mathrm{d}s.$$

By Lemma 6.6, this integral converges to $a^{-1}\int_0^t\{\rho_s(0)-\rho_s(a)\}\,\mathrm{d}s$, where ρ is the solution of (2.1). It remains to let $a\downarrow 0$ to conclude the proof. □

We prove now a quenched non-equilibrium central limit theorem for the current. Let $\bar{J}_{x,x+1}^N(t)=J_{x,x+1}^N(t)-E_{\nu_{\rho_0(\cdot)}^N}[J_{x,x+1}^N(t)]$.

Proposition 5.2. *There exists a total measure set $\Omega_0\subset\Omega$ with the following property. For each ξ in Ω_0, each $k\ge 1$ and each $0\le t_1<\cdots<t_k$, the random vector $N^{-1/2}(\bar{J}_{-1,0}^N(t_1),\dots,\bar{J}_{-1,0}^N(t_k))$ converges in law to a Gaussian vector $(J_{t_1},\dots,J_{t_k})$ with covariances given by*

$$\begin{aligned}E[J_sJ_t]&=\int_{-\infty}^0\mathrm{d}v\,P[B_s\le v]P[B_t\le v]\chi\bigl(\rho_0(v)\bigr)\\&\quad+\int_0^\infty\mathrm{d}v\,P[B_s\ge v]P[B_t\ge v]\chi\bigl(\rho_0(v)\bigr)\\&\quad+2\gamma^{-1}\int_0^s\mathrm{d}r\int_{-\infty}^\infty\mathrm{d}v\,p_{t-r}(0,v)p_{s-r}(0,v)\chi\bigl(\rho(r,v)\bigr)\end{aligned}$$

provided $s\le t$. In this formula, $B_t=B_{t/\gamma}^0$, where B_t^0 is a standard Brownian motion starting from the origin, and $p_t(v,w)$ is the kernel of B_t.

Proof. The proof of this proposition is similar to the one of Theorem 2.3 in [8]. Some details are therefore omitted.

Let $H_0(u)=\mathbf{1}\{u\ge 0\}$ and define the sequence $\{G_n\colon n\ge 1\}$ of approximations of H_0 by

$$G_n(u)=\left\{1-\left(\frac{u}{n}\right)\right\}^+\mathbf{1}\{u\ge 0\}.$$

We claim that for every $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{E}_{\nu^N_{\rho_0(\cdot)}} \big[N^{-1/2}\overline{J}_{-1,0}(t) - Y^N_t(G_n) + Y^N_0(G_n)\big]^2 = 0 \tag{5.4}$$

uniformly in N. The proof of (5.4) relies on the estimates of the two point space–time correlation functions, presented in Lemma 6.5, and follows closely the proof of Proposition 3.1 in [8]. We leave the details to the reader.

Fix $t \geq 0$ and $n \geq 1$. By approximating G_n in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by a sequence $\{H_{n,k}\colon k \geq 1\}$ of smooth functions with compact support, recalling Theorem 2.2, we show that $Y^N_t(G_n)$ converges in law to a Gaussian variable denoted by $Y_t(G_n)$.

By (5.4), $\{Y^N_t(G_n) - Y^N_0(G_n)\colon n \geq 1\}$ is a Cauchy sequence uniformly in N. In particular, $Y_t(G_n) - Y_0(G_n)$ is a Cauchy sequence and converges to a Gaussian limit denoted by $Y_t(H_0) - Y_0(H_0)$. Therefore, by (5.4), $N^{-1/2}\overline{J}_{-1,0}(t)$ converges in law to $Y_t(H_0) - Y_0(H_0)$.

The same argument shows that any vector $N^{-1/2}(\overline{J}_{-1,0}(t_1), \dots, \overline{J}_{-1,0}(t_k))$ converges in law to $(Y_{t_1}(H_0) - Y_0(H_0), \dots, Y_{t_k}(H_0) - Y_0(H_0))$. The covariances can be computed since by (2.6)

$$\begin{aligned}
&E\big[\{Y_t(H_0) - Y_0(H_0)\}\{Y_s(H_0) - Y_0(H_0)\}\big] \\
&\quad = \lim_{n\to\infty} E\big[\{Y_t(G_n) - Y_0(G_n)\}\{Y_s(G_n) - Y_0(G_n)\}\big] \\
&\quad = \lim_{n\to\infty} \bigg\{\int_{\mathbb{R}} \{(T_t G_n)(T_s G_n) + G_n^2 - (T_t G_n)G_n - (T_s G_n)G_n\}\chi\big(\rho_0(u)\big) \\
&\qquad + 2\gamma^{-1}\int_0^s \mathrm{d}r \int_{\mathbb{R}} (\nabla T_{t-r} G_n)(\nabla T_{s-r} G_n)\chi\big(\rho(r,u)\big)\bigg\}.
\end{aligned}$$

A long but elementary computation permits us to recover the expression presented in the statement of the proposition (cf. the proof of Theorem 2.3 in [8]). □

We turn now to the behavior of a tagged particle. Let $\nu^{N,*}_{\rho_0(\cdot)}$ be the product measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin. All our previous results for the process starting from $\nu^N_{\rho_0(\cdot)}$ remain in force for the process starting from $\nu^{N,*}_{\rho_0(\cdot)}$, since we can couple both processes in such a way that they differ at most at one site at any given time.

Denote by X^N_t the position at time $t \geq 0$ of the particle initially at the origin. Since the relative ordering of particles is conserved by the dynamics, a law of large numbers for X^N_t is a consequence of the hydrodynamic limit and the law of large numbers for the current [8,13,17]. In fact, the distribution of X^N_t can be obtained from the joint distribution of the current and the empirical measure via the relation

$$\big\{X^N_t \geq n\big\} = \Bigg\{J_{-1,0}(t) \geq \sum_{x=0}^{n-1} \eta_t(x)\Bigg\} \tag{5.5}$$

for all $n \geq 0$ and a similar relation for $n \leq 0$.

Theorem 5.3. *Consider a sequence $\{\xi_x\colon x \in \mathbb{Z}\}$ satisfying (2.3) and a profile $\rho_0 : \mathbb{R}_+ \to [0, 1]$ satisfying the assumptions presented at the beginning of this section. For every $t \geq 0$, X_t/N converges in $\mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}$-probability to u_t, the solution of*

$$\int_0^{u_t} \rho(t,u)\,\mathrm{d}u = -\frac{1}{\gamma}\int_0^t (\partial_u \rho)(s,0)\,\mathrm{d}s. \tag{5.6}$$

Notice that u_t satisfies the differential equation

$$\dot{u}_t = -\frac{1}{\gamma}\frac{(\partial_u \rho)(t,u_t)}{\rho(t,u_t)}.$$

The proof of this result is similar to the one of Theorem 2.5 in [8] and left to the reader.

It remains to prove a central limit theorem for the position of the tagged particle.

Proof of Theorem 2.3. Recall the definition of u_t^N presented just before the statement of the theorem, assume that $u_t^N > 0$ and fix a in $\mathbb{R}$. By Eq. (5.5), the set $\{X_t \ge u_t^N + a\sqrt{N}\}$ is equal to the set in which

$$\overline{J}_{-1,0}(t) \ge \sum_{x=0}^{u_t^N} \overline{\eta}_t(x) + \sum_{x=1}^{a\sqrt{N}-1} \eta_t\big(x+u_t^N\big) - \left\{\mathbb{E}_{\nu^{N,*}_{\rho_0(\cdot)}}\big[J_{-1,0}(t)\big] - \sum_{x=0}^{u_t^N} \rho_t^{N,*}(x)\right\}. \tag{5.7}$$

We claim that second term on the right-hand side of this equation divided by $\sqrt{N}$ converges to its mean in L^2. Indeed, by Theorem 6.1, its variance is bounded by $C_0(\varepsilon,\rho_0)aN^{-1/2}$ for some finite constant C_0. Notice that we are taking expectations with respect to a measure, $\nu^{N,*}_{\rho_0(\cdot)}$, whose associated profile does not have a bounded first derivative. However, coupling this measure with $\nu^N_{\rho_0(\cdot)}$, in such a way that they differ at most by one particle at every time, we can still show that the variance is bounded by $C_0(\varepsilon,\rho_0)aN^{-1/2}$ as claimed. The same ideas, the linearity of Eq. (2.5) and the Nash estimate, stated in Proposition 6.2 below, show that $\rho^{N,*}$ converges uniformly on compact sets to the the solution of the heat equation (2.1) because ρ^N converges in view of Lemma 6.6.

To compute the expectation of the second term on the right hand side of (5.7), observe that the middle term in (2.7) is equal to $\mathbb{E}_{\nu^{N,*}_{\rho_0(\cdot)}}[J_{-1,0}(t)]$. By the proof of the law of large numbers for the current, this middle expression divided by N converges to $-\gamma^{-1}\int_0^t (\partial_u\rho)(s,0)\,\mathrm{d}s$. In particular, by the law of large numbers for the empirical measure and by relation (5.6), $N^{-1}u_t^N$ converges to u_t. Hence, by the uniform convergence of $\rho^{N,*}$,

$$\frac{1}{\sqrt{N}} \sum_{x=1}^{a\sqrt{N}-1} \rho_t^{N,*}\big(x+u_t^N\big)$$

converges to $a\rho(t,u_t)$ and so does in probability the second term on the right-hand side of (5.7).

By definition of u_t^N, the third term on the right-hand side is absolutely bounded by 1.

Finally, by (5.4), for fixed t, $N^{-1/2}\{\overline{J}_{-1,0}(t) - \sum_{x=0}^{u_t^N} \overline{\eta}_t(x)\}$ behaves as $Y_t^N(G_n) - Y_0^N(G_n) - Y_t^N(\mathbf{1}\{[0,u_t^N/N]\})$, as $N\uparrow\infty$, $n\uparrow\infty$. Repeating the arguments presented in the proof of Proposition 5.2, since u_t^N/N converges to u_t, we show that this latter variable converges in law to a centered Gaussian variable, denoted by W_t, and which is formally equal to $Y_t(H_{u_t}) - Y_0(H_0)$, where $H_a(u) = \mathbf{1}\{u \ge a\}$.

Up to this point we proved that

$$\lim_{N\to\infty} \mathbb{P}_{\nu^{N,*}_{\rho_0(\cdot)}}\left[\frac{X_t^N - u_t^N}{\sqrt{N}} \ge a\right] = P\big[W_t \ge a\rho(t,u_t)\big]$$

provided $u_t > 0$. Analogous arguments permit us to prove the same statement in the case $u_t = 0$, $a > 0$. By symmetry around the origin, we can recover the other cases: $u_t < 0$ and a in $\mathbb{R}$, $u_t = 0$ and $a < 0$.

Putting all these facts together, we conclude that for each fixed t, $(X_t - u_t^N)/\sqrt{N}$ converges in distribution to the Gaussian $W_t/\rho(t,u_t) = [Y_t(H_{u_t}) - Y_0(H_0)]/\rho(t,u_t)$. The same arguments show that any vector $(N^{-1/2}[X_{t_1} - u_{t_1}^N], \dots, N^{-1/2}[X_{t_k} - u_{t_k}^N])$ converges to the corresponding centered Gaussian vector. It remains to compute the covariances, which can be derived as in the proof of Proposition 5.2. Details are left to the reader. □

6. Correlation estimates

We assume throughout this section that $\{\xi_x\colon x\in\mathbb{Z}\}$ is a sequence of numbers bounded below and above: $0<\varepsilon<\xi_x<\varepsilon^{-1}$ for all x, and that the profile $\rho_0:\mathbb{R}\to[0,1]$ has bounded first derivative. Recall that $\rho_t^N(x) = \mathbb{E}_{\nu_{\rho_0^N(\cdot)}}[\eta_t(x)]$ satisfies Eq. (2.5).

For $n \geq 1$, denote by $\mathcal{E}_n$ the subsets of $\mathbb{Z}$ with n points. For each $\mathbf{x}_n = \{x_1, \dots, x_n\}$ in $\mathcal{E}_n$, let

$$\varphi_t(\mathbf{x}_n) = \mathbb{E}_{\nu_{\rho_0^N(\cdot)}}\left[\prod_{i=1}^{n}\{\eta_t(x_i) - \rho_t^N(x_i)\}\right].$$

Theorem 6.1. *Fix a finite time interval* $[0, T]$ *and an initial profile* ρ_0 *with a bounded first derivative. There are constants* C_n*, depending only on* ε, ρ_0, n *and* T*, such that*

$$\sup_{\substack{\mathbf{x}_{2n}\in\mathcal{E}_{2n}\\ t\in[0,T]}} \left|\varphi_t(\mathbf{x}_{2n})\right| \leq \frac{C_{2n}}{N^n}, \qquad \sup_{\substack{\mathbf{x}_{2n+1}\in\mathcal{E}_{2n+1}\\ t\in[0,T]}} \left|\varphi_t(\mathbf{x}_{2n+1})\right| \leq \frac{C_{2n+1}\log N}{N^{n+1}}.$$

The proof of this theorem follows closely the proof of [5] for the simple exclusion process without environment. We start with a Nash estimate for the transition probability of a random walk in elliptic environment [2,3,11]. Denote by $\mathcal{L}_1$ the generator of a random walk in the bond environment ξ:

$$(\mathcal{L}_1 f)(x) = \xi_{x-1}\{f(x-1) - f(x)\} + \xi_x\{f(x+1) - f(x)\}.$$

Let $p_t^\xi(x, y)$ be the transition probability associated to the generator $\mathcal{L}_1$.

Proposition 6.2. *There exists a finite constant* $C_0(\varepsilon)$*, depending only on* ε*, such that* $p_t^\xi(x, y) \leq C_0(\varepsilon)t^{-1/2}$ *for all* x*,* y *in* $\mathbb{Z}$*,* $t \geq 0$.

The proof of Theorem 6.1 relies also on a comparison between the semigroup associated to the evolution of n exclusion particles with the semigroup associated to n independent particles. For $n \geq 1$, denote by $\mathcal{L}_n$ the generator corresponding to the evolution of n exclusion particles in the environment ξ:

$$\begin{aligned}(\mathcal{L}_n h)(\mathbf{x}_n) = {} & N^2 \sum_{i=1}^{n} \mathbf{1}\{\mathbf{x} + e_i \in \mathcal{E}_n\}\xi_{x_i}\left[h(\mathbf{x}_n + e_i) - h(\mathbf{x}_n)\right] \\ & + N^2 \sum_{i=1}^{n} \mathbf{1}\{\mathbf{x} - e_i \in \mathcal{E}_n\}\xi_{x_i-1}\left[h(\mathbf{x}_n - e_i) - h(\mathbf{x}_n)\right]\end{aligned}$$

for every function $h : \mathcal{E}_n \to \mathbb{R}$. In this formula, for $1 \leq i \leq n$, e_i stands for the ith canonical vector in $\mathbb{R}^n$ and $\mathbf{x}_n$ is understood as the vector $(x_1, \dots, x_n)$. Denote by $S_n(t)$ the semigroup associated to $\mathcal{L}_n$ and by $S_n^0(t)$ the semigroup associated to n independent particles evolving in the environment ξ.

A bounded symmetric function $f : \mathbb{Z}^2 \to \mathbb{R}$ is said to be positive definite provided

$$\sum_{x,y} f(x, y)\beta_x\beta_y \geq 0$$

for every sequence $\{\beta_x : x \in \mathbb{Z}\}$ with $\sum_x \beta_x = 0$ and $\sum_x |\beta_x| < \infty$. A bounded symmetric function $f : \mathbb{Z}^n \to \mathbb{R}$ is said to be positive definite if it is so for each pair of coordinates. From [14] we have that

Proposition 6.3. *Let* $f : \mathbb{Z}^n \to \mathbb{R}$ *be a bounded, symmetric, positive definite function. Then,*

$$S_n(t)f \leq S_n^0(t)f$$

for all $t \geq 0$.

Theorem 6.1 is based on an induction argument. Observe first that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(\mathbf{x}_n) = (\mathcal{L}_n\varphi_t)(\mathbf{x}_n) + \Gamma_t(\mathbf{x}_n), \tag{6.1}$$

where

$$\Gamma_t(\mathbf{x}_n) = 2N^2 \sum_{\substack{x\in\mathbb{Z}\\ x,x+1\in\mathbf{x}_n}} \xi_x\big[\varphi_t\big(\mathbf{x}_n^{x+1}\big) - \varphi_t\big(\mathbf{x}_n^{x}\big)\big]\big[\rho_t^N(x+1) - \rho_t^N(x)\big]$$
$$- N^2 \sum_{\substack{x\in\mathbb{Z}\\ x,x+1\in\mathbf{x}_n}} \xi_x \varphi_t\big(\mathbf{x}_n^{x,x+1}\big)\big[\rho_t^N(x+1) - \rho_t^N(x)\big]^2.$$

Here and below $\mathbf{x}_n^y$, $\mathbf{x}_n^{y,z}$ stand for the configuration $\mathbf{x}_n \setminus \{y\}$, $\mathbf{x}_n \setminus \{y,z\}$, respectively.

In view of the differential equation (6.1), we can represent $\varphi_t(\mathbf{x}_n)$ as an expectation with respect to a random walk in an environment ξ with sources at the boundary $\partial\mathcal{E}_n = \{\mathbf{x}_n \in \mathcal{E}_n; \min_{i\neq j}|x_i - x_j| = 1\}$: denote by $\mathbb{E}_{\mathbf{x}_n}$ (resp. $\mathbb{E}^0_{\mathbf{x}_n}$) the expectation with respect to n exclusion (resp. independent) particles starting at $\mathbf{x}_n$. Since $\varphi_0(\mathbf{x}_n) = 0$, we have that

$$\varphi_t(\mathbf{x}_n) = \int_0^t \mathrm{d}s\, \mathbb{E}_{\mathbf{x}_n}\big[\Gamma_s\big(\mathbf{x}_n(t-s)\big)\big]. \tag{6.2}$$

Since $\varphi_t(x) = 0$ for all x in $\mathbb{Z}$, $t \ge 0$, to start the induction argument, set $n = 2$ and remark that the first term in the definition of Γ_t vanishes. On the other hand, by (6.5) below, the derivative $\nabla_N \rho_t^N$ is uniformly bounded. Since the environment is elliptic and $\varphi_t(\phi) = 1$, $\Gamma_t(\mathbf{x}_2)$ is absolutely bounded by $C_0(\varepsilon,\rho_0)\mathbf{1}\{\mathbf{x}_2 \in \partial\mathcal{E}_2\}$ for some finite constant C_0.

The function $f:\mathbb{Z}^2 \to \mathbb{R}$ defined by $f(x,y) = 2\mathbf{1}\{x=y\} + \mathbf{1}\{|x-y|=1\}$ is bounded, symmetric and positive definite. Therefore, by Proposition 6.3, by the integral representation (6.2) of φ_t and by the previous estimate of Γ_t,

$$\varphi_t(x,y) \le C(\varepsilon,\rho_0)\int_0^t \mathbb{E}^0_{(x,y)}\big[f\big(\mathbf{x}_2(s)\big)\big]\,\mathrm{d}s.$$

It remains to apply Proposition 6.2 and to integrate in time, keeping in mind that time is sped up by N^2, to obtain that

$$\sup_{x\neq y\in\mathbb{Z}} \big|\varphi_t(x,y)\big| \le \frac{C(\varepsilon,\rho_0)\sqrt{t}}{N} \le \frac{C(\varepsilon,\rho_0,T)}{N}$$

for all $0 \le t \le T$.

To extend this estimate to $n \ge 3$, we need to exploit the non-trivial cancellations in the first term of the definition of Γ_t. For $n \ge 1$, let

$$A_t^n =: \sup_{\mathbf{x}_n\in\mathcal{E}_n} \big|\varphi_t(\mathbf{x}_n)\big|,$$
$$B_t^n =: \sup_{x\in\mathbb{Z}} \sup_{\substack{\mathbf{x}_{n-1}\in\mathcal{E}_{n-1}\\ \mathbf{x}_{n-1}\not\ni x,x+1}} \big|\varphi_t\big(\mathbf{x}_{n-1}\cup\{x\}\big) - \varphi_t\big(\mathbf{x}_{n-1}\cup\{x+1\}\big)\big|.$$

We claim that there exists a finite sequence of constants $C(\varepsilon,\rho_0,n)$, $n \ge 2$, such that

$$A_t^n \le C(\varepsilon,\rho_0,n)\int_0^t \big\{N B_s^{n-1} + A_s^{n-2}\big\}\frac{\mathrm{d}s}{N\sqrt{t-s}},$$
$$B_t^n \le C(\varepsilon,\rho_0,n)\int_0^t \big\{N B_s^{n-1} + A_s^{n-2}\big\}\frac{\mathrm{d}s}{1+N^2(t-s)}. \tag{6.3}$$

Theorem 6.1 follows from these bounds and elementary computations.

It remains to prove the estimates (6.3). The first one is simpler and follows the same steps presented for $n = 2$. Fix $n \ge 3$ and a configuration $\mathbf{x}_n$ in $\mathcal{E}_n$. Assume that the particles are evolving according to a stirring process. By (6.2) and by definition of A_s^k, B_s^k,

$$\begin{aligned}\big|\varphi_t(\mathbf{x}_n)\big| &\le \int_0^t \mathrm{d}s\, \mathbb{E}_{\mathbf{x}_n}\big[\big|\Gamma_s\big(\mathbf{x}_n(t-s)\big)\big|\big] \\ &\le C(\varepsilon,\rho_0,n)\int_0^t \mathrm{d}s\, \mathbb{P}_{\mathbf{x}_n}\big[\mathbf{x}_n(t-s)\in\partial\mathcal{E}_n\big]\big\{NB_s^{n-1}+A_s^{n-2}\big\}\end{aligned}$$

for some finite constant $C(\varepsilon,\rho_0,n)$. Since $\mathbf{x}_n(t-s)$ belongs to the boundary of $\mathcal{E}_n$, there are at least two particles at distance one. By definition of the the stirring process, any pair of particles evolves according to a symmetric exclusion process in the environment ξ. In particular, comparing the original process with independent particles and applying the Nash estimate, we can bound the probability appearing in the last displayed formula by $C\{N^2(t-s)\}^{-1/2}$. This proves the first estimate in (6.3).

We now turn to B_t^n. Since $B_t^1=0$, fix $n\ge 2$, x in $\mathbb{Z}$ and $\mathbf{x}_{n-1}$ in $\mathcal{E}_{n-1}$ such that $x, x+1\notin \mathbf{x}_{n-1}$. Consider $n+1$ particles evolving on $\mathbb{Z}$ according to the following rules. They start from $\mathbf{x}_{n-1}$, x, $x+1$ and evolve according to a stirring process. However, when the particles starting at x and $x+1$ are at distance 1, each one jumps, independently from the other, to the site occupied by the other at the rate determined by the environment. Once these particles occupy the same site, they remain together forever. Notice that the two distinguished particles behave until they meet exactly as two independent particles.

Denote by $\mathbb{P}_{\mathbf{x}_{n-1},x,x+1}$, $\mathbb{E}_{\mathbf{x}_{n-1},x,x+1}$ the probability and the expectation corresponding to the evolution just described. Let τ be the coalescence time of the distinguished particles and let $\mathbf{x}_n(t,x)$, $\mathbf{x}_n(t,x+1)$ be the configuration at time t of the system starting from $\mathbf{x}_{n-1}\cup\{x\}$, $\mathbf{x}_{n-1}\cup\{x+1\}$, respectively. By construction, $\mathbf{x}_n(t,x)=\mathbf{x}_n(t,x+1)$ for $t\ge\tau$. In particular,

$$\begin{aligned}&\varphi_t\big(\mathbf{x}_{n-1}\cup\{x\}\big)-\varphi_t\big(\mathbf{x}_{n-1}\cup\{x+1\}\big) \\ &\quad=\int_0^t \mathrm{d}s\,\mathbb{E}_{\mathbf{x}_{n-1},x,x+1}\big[\Gamma_s\big(\mathbf{x}_n(t-s,x)\big)-\Gamma_s\big(\mathbf{x}_n(t-s,x+1)\big)\big] \\ &\quad=\int_0^t \mathrm{d}s\,\mathbb{E}_{\mathbf{x}_{n-1},x,x+1}\big[\mathbf{1}\{\tau>t-s\}\big\{\Gamma_s\big(\mathbf{x}_n(t-s,x)\big)-\Gamma_s\big(\mathbf{x}_n(t-s,x+1)\big)\big\}\big].\end{aligned}$$

By definition of A_s^k and B_s^k, this expression is less than or equal to

$$C_0\sum_{y=x,x+1}\int_0^t \mathrm{d}s\,\big\{NB_s^{n-1}+A_s^{n-2}\big\}\mathbb{P}_{\mathbf{x}_{n-1},x,x+1}\big[\tau>t-s,\mathbf{x}_n(t-s,y)\in\partial\mathcal{E}_n\big]$$

for some finite constant $C_0=C_0(\varepsilon,\rho_0,n)$. In view of the Nash estimate, replacing the indicator function $\mathbf{1}\{\tau>t-s\}$ by $\mathbf{1}\{\tau>(t-s)/2\}$ and applying the Markov property at time $(t-s)/2$, we bound the previous expression by

$$C_0\int_0^t \mathrm{d}s\,\big\{NB_s^{n-1}+A_s^{n-2}\big\}\frac{1}{\sqrt{1+N^2(t-s)}}\mathbb{P}_{\mathbf{x}_{n-1},x,x+1}\left[\tau>\frac{(t-s)}{2}\right].$$

By (6.4) below, the probability appearing in the previous formula is bounded above by $C(\varepsilon)\{1+N^2(t-s)\}^{-1/2}$. This concludes the proof of estimate (6.3) and the one of Theorem 6.1.

Let x_t be a random walk in the environment $\{\xi_x\colon x\in\mathbb{Z}\}$ starting from $x_0=0$. Denote by P the probability measure on the path space $D(\mathbb{R}_+,\mathbb{Z})$ induced by x_t. For each $a\neq 0$, let τ_a be the first time the random walk x_t reaches a:

$$\tau_a =: \inf\{t\ge 0;\, x_t=a\}.$$

Lemma 6.4. *There exists a finite constant $C_0=C_0(\varepsilon)$, depending only ε, such that*

$$P(\tau_a>t)\le\frac{C_0 a}{\sqrt{1+t}}$$

for all $t>0$.

Proof. Define the function $u : \mathbb{Z} \to \mathbb{R}$ by $u(0) = 0$, $u(x+1) - u(x) = \xi_x^{-1}$. Since the environment is elliptic, $\varepsilon \le u(x)/x \le \varepsilon^{-1}$ for all $x \neq 0$. Moreover, an elementary computation shows that $u(x_t)$ is a martingale of quadratic variation $\langle u(x)\rangle_t$ given by

$$\int_0^t \left(\xi_{x_s-1}^{-1} + \xi_{x_s}^{-1}\right) \mathrm{d}s.$$

Fix $b < 0 < a$ and set $\tau = \min\{\tau_a, \tau_b\}$. By Doob's optional sampling theorem, $E[u(x_\tau)] = 0$ and $E[u(x_\tau)^2 - \langle u(x)\rangle_\tau] = 0$. Therefore,

$$P(\tau_a < \tau_b) = \frac{-u(b)}{u(a) - u(b)}, \qquad -u(a)u(b) = E\int_0^\tau \left(\xi_{x_s-1}^{-1} + \xi_{x_s}^{-1}\right) \mathrm{d}s,$$

so that $E[\tau] \le -u(a)u(b)(2\varepsilon)^{-1}$. In particular,

$$P(\tau_a > t) \le P(\tau > t) + P(\tau_a > \tau_b) \le \frac{-u(a)u(b)}{2\varepsilon t} + \frac{u(a)}{u(a) - u(b)}.$$

Minimizing over $b < 0$ we conclude the proof of the lemma. □

The same ideas provide a bound on the coalescence time of two independent particles in the environment ξ. Fix x in $\mathbb{Z}$ and consider two independent random walks X_t, Y_t, on the environment ξ such that $X_0 = x$, $Y_0 = x + 1$. For $b > 0$, let τ^*, τ_b be the first time such that $Y_t = X_t$, $Y_t = X_t + b$, respectively.

Recall the definition of the function u defined in the proof of Lemma 6.4. Since X_t, Y_t are independent, $M_t = u(Y_t) - u(X_t) - 1$ is a martingale. Repeating the arguments presented in the proof of Lemma 6.4, we obtain that

$$P(\tau^* > t) \le \frac{C_0}{\sqrt{1+t}} \tag{6.4}$$

for all $t > 0$ and some finite constant C_0 depending only on ε. Of course, when the time is sped up by N^2, t is replaced by tN^2.

A bound on the space–time correlations can be deduced from Theorem 6.1. For x, y in $\mathbb{Z}$ and $s \le t$, let

$$\psi_{s,t}(y; x) = \mathbb{E}_{\nu_{\rho_0^N(\cdot)}}\left[\left\{\eta_s(y) - \rho_s^N(y)\right\}\left\{\eta_t(x) - \rho_t^N(x)\right\}\right].$$

Lemma 6.5. *There exists a finite constant C_0, depending only on ε, ρ_0 such that*

$$\sup_{x,y\in\mathbb{Z}} \left|\psi_{s,t}(y; x)\right| \le \frac{C_0}{N}\left\{\sqrt{s} + \frac{1}{\sqrt{t-s}}\right\}$$

for all $0 \le s \le t$.

Proof. Fix $s \ge 0$ and y in $\mathbb{Z}$. For $t \ge s$, x in $\mathbb{Z}$, let $\psi_t(x) = \psi_{s,t}(y; x)$. Notice that ψ_t satisfies the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = \mathcal{L}_1\psi_t(x), \\ \psi_s(x) = \mathbf{1}\{x \neq y\}\varphi_s(x, y) + \mathbf{1}\{x = y\}\rho_s^N(y)\left(1 - \rho_s^N(y)\right), \end{cases}$$

where $\mathcal{L}_1$ is the generator defined at the beginning of this section. It remains to recall the Nash estimate for the semigroup and the proof of Theorem 6.1, in which we showed that $\varphi_s(x, y)$ is bounded by $C\sqrt{s}/N$. □

We conclude this section with a result on the solution of the discrete linear equation (2.5).

Lemma 6.6. *Let $\rho_0 : \mathbb{R} \to [0,1]$ be a profile whose first derivative ρ_0' belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and whose second derivative ρ_0'' belongs to $L^\infty(\mathbb{R})$. The solution ρ_t^N of Eq.* (2.5) *converges uniformly on compact sets of $\mathbb{R}_+ \times \mathbb{R}$ to the solution of* (2.1). *In particular, for all $t \geq 0$ and all function G in $C_0^1(\mathbb{R})$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} G\left(\frac{x}{N}\right) \rho_t^N(x) = \int G(u)\rho(t,u)\,\mathrm{d}u,$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} G\left(\frac{x}{N}\right) \left(\nabla_N \rho_t^N\right)(x) = \int G(u)(\partial_u \rho)(t,u)\,\mathrm{d}u,$$

where $\rho_t(u)$ is the solution of the linear heat equation (2.1).

Proof. Consider the initial condition $\rho_0^\xi = \rho_0^{N,\xi} : \mathbb{Z} \to \mathbb{R}$ defined by $\rho_0^\xi(0) = \gamma\rho_0^N(0)$, $(\nabla_N \rho_0^\xi)(x) = \xi_x^{-1}(\nabla_N \rho_0^N)(x)$. By the estimates presented at the beginning of Section 4, $\rho_0^\xi(x) - \gamma\rho_0(x)$ vanishes, as $N \uparrow \infty$, uniformly in x.

Denote by $\rho_t^\xi = \gamma\rho_t^{N,\xi}$ the solution of Eq. (2.5) with initial condition ρ_0^ξ. We claim that the sequence $\{\rho_t^{N,\xi}\colon N \geq 1\}$ is equicontinuous on each compact set of $\mathbb{R}_+ \times \mathbb{R}$. The proof relies on uniform bounds of ρ_t^ξ, $\nabla_N \rho_t^\xi$, $(\mathrm{d}/\mathrm{d}t)\rho_t^\xi$.

First of all, by the maximum principle,

$$\inf_{x\in\mathbb{Z}} \rho_0^\xi(x) \leq \inf_{x\in\mathbb{Z}} \rho_t^\xi(x) \leq \sup_{x\in\mathbb{Z}} \rho_t^\xi(x) \leq \sup_{x\in\mathbb{Z}} \rho_0^\xi(x).$$

Denote by ∇_ξ the discrete derivative defined by $(\nabla_\xi h)(x) = N\xi_x\{h(x+1) - h(x)\}$. $(\nabla_\xi \rho_t^\xi)$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_\xi \rho_t^\xi\right)(x) = N^2 \xi_x \left\{\left(\nabla_\xi \rho_t^\xi\right)(x+1) + \left(\nabla_\xi \rho_t^\xi\right)(x-1) - 2\left(\nabla_\xi \rho_t^\xi\right)(x)\right\}.$$

In particular, $\nabla_\xi \rho_t^\xi$ satisfies the maximum principle and is uniformly bounded because we assumed the initial condition to have a bounded derivative.

Let $\dot\rho_t^\xi = (\mathrm{d}/\mathrm{d}t)\rho_t^\xi$. By definition,

$$\dot\rho_t^\xi(x) = N\left\{\left(\nabla_\xi \rho_t^\xi\right)(x) - \left(\nabla_\xi \rho_t^\xi\right)(x-1)\right\} = \left(\nabla_N \nabla_\xi \rho_t^\xi\right)(x-1).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot\rho_t^\xi(x) = N^2\left\{\xi_x\left[\left(\nabla_N\nabla_\xi\rho_t^\xi\right)(x) - \left(\nabla_N\nabla_\xi\rho_t^\xi\right)(x-1)\right] - \xi_{x-1}\left[\left(\nabla_N\nabla_\xi\rho_t^\xi\right)(x-1) - \left(\nabla_N\nabla_\xi\rho_t^\xi\right)(x-2)\right]\right\},$$

$\dot\rho_t^\xi(x) = (\nabla_N \nabla_\xi \rho_t^\xi)(x-1)$ satisfies a maximum principle. By definition of ρ_0^ξ, ∇_ξ,

$$\left(\nabla_N \nabla_\xi \rho_0^\xi\right)(x-1) = N\left\{\left(\nabla_\xi \rho_0^\xi\right)(x) - \left(\nabla_\xi \rho_0^\xi\right)(x-1)\right\} = (\Delta_N \rho_0)(x).$$

In particular, $(\mathrm{d}/\mathrm{d}t)\rho_t^\xi(x)$ is uniformly bounded because we assumed the initial condition to have a bounded second derivative.

Notice that the previous bound does not hold for the initial condition ρ_0 since $\nabla_N \nabla_\xi \rho_0$ is of order N. This explains the introduction of ρ_0^ξ.

The estimates just obtained prove the equicontinuity of the sequence $\{\rho_t^{N,\xi}\colon N \geq 1\}$ on each compact set of $\mathbb{R}_+ \times \mathbb{R}$. Since every limit point is a weak solution of the heat equation, by uniqueness of weak solutions, ρ_t^ξ converges uniformly on compact sets to the solution of (2.1) with initial condition $\gamma\rho_0$.

Since $\rho_0^\xi - \gamma\rho_0$ converges uniformly to 0, by the maximum principle, ρ_t^N converges uniformly on compact sets to the solution of (2.1). This concludes the proof of the lemma. □

Let $h_t(x) = \xi_x(\nabla_N \rho_t^N)(x)$. A simple computation shows that $(\mathrm{d}/\mathrm{d}t)h_t(x) = \xi_x(\Delta_N h_t)(x)$. Hence, h_t satisfies a maximum principle and

$$\sup_{t\geq 0}\sup_{x\in\mathbb{Z}}\left|(\nabla_N \rho_t)(x)\right| \leq C(\varepsilon)\sup_{x\in\mathbb{Z}}\left|(\nabla_N \rho_0)(x)\right| \tag{6.5}$$

because $\nabla_N \rho_t^N$ is absolutely bounded above and below by $C(\varepsilon)|h_t|$.

Acknowledgments

The authors would like to thank the valuable discussions with A. Faggionato and S. R. S. Varadhan.

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