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# On the small time asymptotics of the two-dimensional stochastic Navier–Stokes equations

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**Abstract.** In this paper, we establish a small time large deviation principle (small time asymptotics) for the two-dimensional stochastic Navier–Stokes equations driven by multiplicative noise, which not only involves the study of the small noise, but also the investigation of the effect of the small, but highly nonlinear, unbounded drifts.

**Résumé.** Dans cet article, nous établissons un principe de grandes déviations en temps petit pour l'équation de Navier–Stokes bi-dimensionnelle stochastique conduite par un bruit multiplicatif. Celui-ci nécessite non seulement l'étude d'un bruit faible, mais aussi la compréhension des effets de dérives petites mais non bornées et non linéaires.

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*Keywords:* stochastic Navier–Stokes equation; small time asymptotics; large deviation principle

## 1. Introduction

It is well known that the two-dimensional stochastic Navier–Stokes equation with Dirichlet boundary condition describes the time evolution of an incompressible fluid and is given by

$$\begin{cases} \mathrm{d}u - \nu\triangle u\,\mathrm{d}t + (u\cdot\nabla)u\,\mathrm{d}t + \nabla p\,\mathrm{d}t = g\,\mathrm{d}t + \sigma(t,u)\,\mathrm{d}W(t), & \\ (\operatorname{div} u)(t,x) = 0 & \text{for } x\in D,\ t>0,\\ u(t,x) = 0 & \text{for } x\in\partial D, t>0,\\ u(0,x) = u_0(x) & \text{for } x\in D, \end{cases}$$

where $D$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial D$, $u(t,x)\in\mathbb{R}^2$ denotes the velocity field at time $t$ and position $x$, $p(t,x)$ denotes the pressure field, $\nu>0$ is the viscosity and $W$ is a Brownian motion on a Hilbert space. Moreover, $\sigma(t,u)\,\mathrm{d}W$ is the random force field acting on the fluid and $g$ is the deterministic part of the force.

To formulate the stochastic Navier–Stokes equations, we introduce the following standard spaces:

$$V = \{v\in H_0^1(D;\mathbb{R}^2) : \nabla\cdot v = 0 \text{ a.e., in } D\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2\,\mathrm{d}x\right)^{1/2} = \|v\|$$

and denote by $((\cdot,\cdot))$ the inner product of $V$. $H$ is the closure of $V$ in the $L^2$-norm

$$|v|_H := \left(\int_D |v|^2 \, \mathrm{d}x\right)^{1/2} = |v|.$$

The inner product on $H$ will be denoted by $(\cdot,\cdot)$.

Define the operator $A$ (Stokes operator) in $H$ by the formula

$$Au = -\nu P_H \triangle u, \quad \forall u \in H^2\big(D;\mathbb{R}^2\big) \cap V,$$

where the linear operator $P_H$ (Helmhotz–Hodge projection) is the projection operator from $L^2$ to $H$, and the nonlinear operator $B$

$$B(u,v) = P_H\big((u\cdot\nabla)v\big),$$

with the notation $B(u) = B(u,u)$. Obviously the domain of $B$ requires that $(u\cdot\nabla v)$ belongs to the space $L^2$.

By applying the operator $P_H$ to each term of the above stochastic Navier–Stokes equation (SNSE), we can rewrite the SNSE in the following abstract form:

$$\mathrm{d}u(t) + Au(t)\,\mathrm{d}t + B(u(t))\,\mathrm{d}t = f(t)\,\mathrm{d}t + \sigma\big(t,u(t)\big)\,\mathrm{d}W(t) \quad \text{in } L^2(0,T;V') \tag{1.1}$$

with the initial condition

$$u(0) = u_0 \quad \text{in } H, \tag{1.2}$$

where $W(t)$ is a Brownian motion taking values in a Hilbert space $U$, and $V'$ is the dual of $V$.

There exists a great amount of literature on the stochastic Navier–Stokes equation. Let us mention some of them. A good reference for stochastic Navier–Stokes equations driven by additive noise is the book [4] and the references therein. The existence and uniqueness of solutions for stochastic 2-D Navier–Stokes equations with multiplicative noise were obtained in [8], [13] and [14]. The ergodic properties and invariant measures of the stochastic 2-D Navier–Stokes equations were studied in [9] and [11]. The small noise large deviation of the stochastic 2-D Navier–Stokes equations was established in [14] and the large deviation of occupation measures was considered in [10].

The purpose of this paper is to study the small time asymptotics (large deviations) of the two-dimensional stochastic Navier–Stokes equations driven by multiplicative noise on $C([0,1];H)$. After the existence, uniqueness and continuity of the solution of SNSE are established, it is still generally difficult to quantify the behavior of the solution at a position $x$ and a positive time $t$. It might be relatively simple to estimate the limiting behavior of the solution in time interval $[0,t]$ as $t$ goes to zero, which describes the behavior of the velocity of the fluid at a given point in space when the time is very small. This is one of the motivations for studying the small time asymptotics. Another motivation will be to get the following Varadhan identity through the small time asymptotics:

$$\lim_{t\to 0} 2t \log P\big(u(0)\in B, u(t)\in C\big) = -d^2(B,C), \tag{1.3}$$

where $d$ is an appropriate Riemannian distance associated with the diffusion generated by the solution of the SNSE. This will be a topic of future study.

Apart from the above motivations, the small time asymptotics is also theoretically interesting, since the study involves the investigation of the small noise and the effect of the small, but highly nonlinear drift. We like to mention that the study of the small time asymptotics (large deviations) of finite dimensional diffusion processes was initiated by Varadhan in the influential work [16]. The small time asymptotics of infinite dimensional diffusion processes were studied in [1,2,7,12] and [17].

To establish the small time large deviation for the stochastic Navier–Stokes equation, as one expects, the main difficulty lies in dealing with the nonlinear term $B(u) = P_H((u\cdot\nabla)u)$ and the unbounded term $Au = -\nu P_H\triangle u$, which barely belong to $V'$. To control $B(u)$, our idea is to show that the probability that the solution stays outside an energy ball is exponentially small so that we can restrict the solution in a sufficiently large energy ball. We hope that this method used for treating the non-linear term could be useful in some other study of stochastic Navier–Stokes equations. Another key step of obtaining the small time large deviation is to prove that the law of the solution of the Navier–Stokes equation $u(\varepsilon t)$ is exponentially equivalent to the law of the solutions of the equation:

$$v^\varepsilon(t) = x + \sqrt{\varepsilon}\int_0^t \sigma\big(\varepsilon s, v^\varepsilon(s)\big)\,\mathrm{d}W(s). \tag{1.4}$$

This is done through several approximations. A remarkable martingale inequality proved by Barlow, Davis and Yor in [5] and [3], plays an important role throughout the paper. We point out that this martingale inequality is more precise than most of the B–D–G's inequalities stated in the literature because the constant on the right is $p^{1/2}$ (see (3.11)), but not $p$.

The rest of the paper is organized as follows. In Section 2, we collect some preliminaries which are frequently used in the sequel. Section 3 is the main part of the paper, where we proved the small time large deviations for stochastic Navier Stokes equation. In Section 4, we further relax the conditions on the diffusion coefficient $\sigma(\cdot)$.

## 2. Preliminaries

Identifying $H$ with its dual $H'$, we consider Eq. (1.1) in the framework of Gelfrand triple:

$$V \subset H \cong H' \subset V'.$$

In this way, we may consider $A$ as a bounded operator from $V$ into $V'$. Moreover, we also denote by $\langle\cdot,\cdot\rangle$, the duality between $V$ and $V'$. Hence, for $u = (u_i) \in V$, $w = (w_i) \in V$, we have

$$\langle Au, w\rangle = \nu\sum_{i,j}\int_D \partial_i u_j\,\partial_i w_j\,\mathrm{d}x = \nu\big((u, w)\big). \tag{2.1}$$

Introduce a trilinear form on $H \times H \times H$ by setting

$$b(u, v, w) = \sum_{i,j}^{2}\int_D u_i\partial_i v_j w_j\,\mathrm{d}x, \tag{2.2}$$

whenever the integral in (2.2) makes sense. In particular, if $u, v, w \in V$, then

$$\big\langle B(u, v), w\big\rangle = \big\langle(u\cdot\nabla v), w\big\rangle = \sum_{i,j}^{2}\int_D u_i\partial_i v_j w_j\,\mathrm{d}x = b(u, v, w).$$

By the integration by parts,

$$b(u, v, w) = -b(u, w, v), \tag{2.3}$$

therefore

$$b(u, v, v) = 0, \quad \forall u, v \in V. \tag{2.4}$$

There are some well-known estimates for $b$ (see [15], for example), which will be required in the rest of this paper and we list them here. Throughout the paper, we denote various generic positive constants by the same letter $c$. We have

$$\big|b(u, v, w)\big| \le c\|u\|\cdot\|v\|\cdot\|w\|, \tag{2.5}$$

$$\big|b(u, v, w)\big| \le c|u|\cdot\|v\|\cdot|Aw|, \tag{2.6}$$

$$\big|b(u, v, w)\big| \le c\|u\|\cdot|v|\cdot|Aw|, \tag{2.7}$$

$$\big|b(u, v, w)\big| \le 2\|u\|^{1/2}\cdot|u|^{1/2}\cdot\|w\|^{1/2}\cdot|w|^{1/2}\cdot\|v\| \tag{2.8}$$

for suitable $u, v, w$. Moreover, combining (2.3) and (2.8), we obtain a useful estimate as follows:

$$|Bu|_{V'} = \sup_{\|v\|\le 1} \big|b(u,u,v)\big| = \sup_{\|v\|\le 1} \big|b(u,v,u)\big| \le 2\|u\| \cdot |u|. \tag{2.9}$$

## 3. Small time asymptotics

First, we introduce the precise assumptions on $\sigma$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\mathcal{F}_t, t \ge 0$ that satisfies the usual conditions. Let $W(\cdot)$ be a $H$-valued Brownian motion on $(\Omega, \mathcal{F}, P)$ with the covariance operator $Q$, which is a positive, symmetric, trace class operator on $H$. Let $L_Q$ denote the class of linear operators $T$ such that $TQ^{1/2}$ is a Hilbert–Schmidt operator from $H$ to $H$. $L_Q$ is endowed with the norm $|T|^2_{L_Q} = \mathrm{Tr}(TQT^*)$, where $\mathrm{Tr}(TQT^*)$ denotes the trace of operator $TQT^*$. Let $L_Q^V$ denote the class of linear operators $\tilde{T}$ such that $\tilde{T}Q^{1/2}$ is a Hilbert–Schmidt operator from $H$ to $V$, endowed with the norm $|\tilde{T}|^2_{L_Q^V} = \mathrm{Tr}(\tilde{T}Q\tilde{T}^*)$. Introduce:

- (A.1) $E|u_0|^4 < +\infty,\ f \in L^4(0,T;V')$.
- (A.2) There exists a constant $L$ such that $|\sigma(t,u)|^2_{L_Q} \le L(1+|u|^2)$, for all $t \in (0,T)$, and all $u \in H$.
- (A.3) There exists a constant $\hat{L}$ such that for $|\sigma(t,u)|^2_{L_Q^V} \le \hat{L}(1+\|u\|^2)$, for all $t \in (0,T)$, and all $u \in V$.
- (A.4) There exists a constant $K$ such that $|\sigma(t,u)-\sigma(t,v)|^2_{L_Q} \le K|u-v|^2$, for all $t \in (0,T)$, and all $u, v \in H$.
- (A.5) There exists a constant $\hat{K}$ such that $|\sigma(t,u)-\sigma(t,v)|^2_{L_Q^V} \le \hat{K}\|u-v\|^2$, for all $t \in (0,T)$ and all $u, v \in V$.

It is known that (see, for example, [14]), under the assumptions (A.1), (A.2) and (A.4), the SNSE

$$\begin{cases} \mathrm{d}u(t) + Au(t)\,\mathrm{d}t + B\big(u(t)\big)\,\mathrm{d}t = f(t)\,\mathrm{d}t + \sigma\big(t,u(t)\big)\,\mathrm{d}W(t), \\ u(0) = x \in H \end{cases} \tag{3.1}$$

has a unique strong solution $u \in L^2(\Omega; C([0,T];H)) \cap L^2(\Omega \times [0,T]; V)$, i.e.,

$$u(t) = x - \int_0^t Au(s)\,\mathrm{d}s - \int_0^t B\big(u(s)\big)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s + \int_0^t \sigma\big(s,u(s)\big)\,\mathrm{d}W(s). \tag{3.2}$$

Let $\varepsilon > 0$, by the scaling property of the Brownian motion, it is easy to see that $u(\varepsilon t)$ coincides in law with the solution of the following equation:

$$u^\varepsilon(t) = x - \varepsilon\int_0^t Au^\varepsilon(s)\,\mathrm{d}s - \varepsilon\int_0^t B\big(u^\varepsilon(s)\big)\,\mathrm{d}s + \varepsilon\int_0^t f(\varepsilon s)\,\mathrm{d}s + \sqrt{\varepsilon}\int_0^t \sigma\big(\varepsilon s, u^\varepsilon(s)\big)\,\mathrm{d}W(s). \tag{3.3}$$

Let $\mu_x^\varepsilon$ be the law of $u^\varepsilon(\cdot)$ on $C([0,1];H)$. Define a functional $I(g)$ on $C([0,1];H)$ by

$$I(g) = \inf_{h \in \Gamma_g} \left\{ \frac{1}{2}\int_0^1 \big|\dot{h}(t)\big|^2_{H_0}\,\mathrm{d}t \right\},$$

where $H_0 := Q^{1/2}H$ endowed with the norm $|h|^2_{H_0} = |Q^{-1/2}h|^2 (h \in H_0)$, and

$$\Gamma_g = \left\{ h \in C\big([0,T];H\big)\colon\ h(\cdot) \text{ is absolutely continuous and such that} \right.$$
$$\left. g(t) = x + \int_0^t \sigma\big(s,g(s)\big)\dot{h}(s)\,\mathrm{d}s, 0 \le t \le 1 \right\}.$$

**Theorem 3.1.** *$\mu_x^\varepsilon$ satisfies a large deviation principle with the rate function $I(\cdot)$, that is,*

(i) *For any closed subset $F \subset C([0,1]; H)$,*

$$\limsup_{\varepsilon\to 0, x_n\to x} \varepsilon \log \mu_{x_n}^\varepsilon(F) \le -\inf_{g\in F}\bigl(I(g)\bigr).$$

(ii) *For any open subset $G \subset C([0,1]; H)$,*

$$\liminf_{\varepsilon\to 0, x_n\to x} \varepsilon \log \mu_{x_n}^\varepsilon(G) \ge -\inf_{g\in G}\bigl(I(g)\bigr).$$

**Proof.** Let $v^\varepsilon(\cdot)$ be the solution of the stochastic equation

$$v^\varepsilon(t) = x + \sqrt{\varepsilon}\int_0^t \sigma\bigl(\varepsilon s, v^\varepsilon(s)\bigr)\,\mathrm{d}W(s), \tag{3.4}$$

and $\nu^\varepsilon$ be the law of $v^\varepsilon(\cdot)$ on the $C([0,1]; H)$. Then by [4], we know that $\nu^\varepsilon$ satisfies a large deviation principle with the rate function $I(\cdot)$. Our main task is to show that two families of the probability measures $\mu^\varepsilon$ and $\nu^\varepsilon$ are exponentially equivalent, that is, for any $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Bigl(\sup_{0\le t\le 1}\bigl|u^\varepsilon(t) - v^\varepsilon(t)\bigr|^2 > \delta\Bigr) = -\infty. \tag{3.5}$$

Then Theorem 3.1 follows from (3.5) and Theorem 4.2.13 in [6] for $x_n = x$. Slight modifications of the proof yields the general case. □

Because of the non-linear form $B(\cdot,\cdot,\cdot)$, and the unbounded operator $A$, the proof of (3.5) is quite involved. We split it into several lemmas. The following result is an estimate of the probability that the solution of (3.3) leaves an energy ball.

**Lemma 3.1.**

$$\lim_{M\to\infty}\sup_{0<\varepsilon\le 1} \varepsilon \log P\Bigl(\bigl(|u^\varepsilon|_V^H(1)\bigr)^2 > M\Bigr) = -\infty, \tag{3.6}$$

*where* $(|u^\varepsilon|_V^H(1))^2 = \sup_{0\le t\le 1}|u^\varepsilon(t)|^2 + \varepsilon\nu\int_0^1 \|u^\varepsilon(t)\|^2\,\mathrm{d}t$.

**Proof.** Since $\langle B(u^\varepsilon(s)), u^\varepsilon(s)\rangle = 0$, applying Itô's formula , we get

$$\begin{aligned}\bigl|u^\varepsilon(t)\bigr|^2 &= |x|^2 - 2\varepsilon\int_0^t \langle u^\varepsilon(s), Au^\varepsilon(s)\rangle\,\mathrm{d}s + 2\varepsilon\int_0^t \bigl\langle u^\varepsilon(s), f(\varepsilon s)\bigr\rangle\,\mathrm{d}s \\ &\quad + 2\sqrt{\varepsilon}\int_0^t \Bigl(u^\varepsilon(s), \sigma\bigl(\varepsilon s, u^\varepsilon(s)\bigr)\,\mathrm{d}W(s)\Bigr) + \varepsilon\int_0^t \bigl|\sigma\bigl(\varepsilon s, u^\varepsilon(s)\bigr)\bigr|_{L_Q}^2\,\mathrm{d}s,\end{aligned}$$

that is,

$$\begin{aligned}\bigl|u^\varepsilon(t)\bigr|^2 + 2\varepsilon\nu\int_0^t \bigl\|u^\varepsilon(s)\bigr\|^2\,\mathrm{d}s &= |x|^2 + 2\varepsilon\int_0^t \bigl\langle u^\varepsilon(s), f(\varepsilon s)\bigr\rangle\,\mathrm{d}s + 2\sqrt{\varepsilon}\int_0^t \Bigl(u^\varepsilon(s), \sigma\bigl(\varepsilon s, u^\varepsilon(s)\bigr)\,\mathrm{d}W(s)\Bigr) \\ &\quad + \varepsilon\int_0^t \bigl|\sigma\bigl(\varepsilon s, u^\varepsilon(s)\bigr)\bigr|_{L_Q}^2\,\mathrm{d}s \\ &= |x|^2 + l_1 + l_2 + l_3.\end{aligned} \tag{3.7}$$

For $l_1$, we have

$$|l_1| \le 2\varepsilon\int_0^t \bigl\|u^\varepsilon(s)\bigr\|\cdot\bigl|f(\varepsilon s)\bigr|_{V'}\,\mathrm{d}s \le \varepsilon\nu\int_0^t \bigl\|u^\varepsilon(s)\bigr\|^2\,\mathrm{d}s + \frac{1}{\nu}|f|_{L^2(0,\varepsilon t; V')}^2. \tag{3.8}$$

In view of (A.2),

$$|I_3| \le \varepsilon \cdot L \int_0^t \left(1 + \left|u^\varepsilon(s)\right|^2\right) \mathrm{d}s. \tag{3.9}$$

Putting (3.7), (3.8) and (3.9) together, it follows that

$$\begin{aligned}\left|u^\varepsilon(t)\right|^2 + \varepsilon\nu \int_0^t \left\|u^\varepsilon\right\|^2 \mathrm{d}s \le \left(|x|^2 + \varepsilon L t + \frac{1}{\nu}|f|^2_{L^2(0,\ \varepsilon t;\ V')}\right) + \varepsilon L \int_0^t \left|u^\varepsilon(s)\right|^2 \mathrm{d}s \\ + 2\sqrt{\varepsilon}\left|\int_0^t \left(u^\varepsilon(s), \sigma\left(\varepsilon s, u^\varepsilon(s)\right) \mathrm{d}W(s)\right)\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\left|u^\varepsilon\right|_V^H(T)\right)^2 \le 2\left(|x|^2 + \varepsilon L T + \frac{1}{\nu}|f|^2_{L^2(0,\varepsilon T; V')}\right) + 2\varepsilon L \int_0^T \left(\left|u^\varepsilon(s)\right|_V^H(s)\right)^2 \mathrm{d}s \\ + 4\sqrt{\varepsilon} \sup_{0\le t\le T}\left|\int_0^t \left(u^\varepsilon(s), \sigma\left(\varepsilon s, u^\varepsilon(s)\right) \mathrm{d}W(s)\right)\right|.\end{aligned}$$

Hence, for $p \ge 2$, we have,

$$\begin{aligned}\left(E\left(\left|u^\varepsilon\right|_V^H(T)\right)^{2p}\right)^{1/p} \le 2\left(|x|^2 + \varepsilon L T + \frac{1}{\nu}|f|^2_{L^2(0,\varepsilon T; V')}\right) + 2\varepsilon L\left(E\left(\int_0^T \left(\left|u^\varepsilon\right|_V^H(s)\right)^2 \mathrm{d}s\right)^p\right)^{1/p} \\ + 4\sqrt{\varepsilon}\left(E\left(\sup_{0\le t\le T}\left|\int_0^t \left(u^\varepsilon(s), \sigma\left(\varepsilon s, u^\varepsilon(s)\right) \mathrm{d}W(s)\right)\right|^p\right)\right)^{1/p}.\end{aligned} \tag{3.10}$$

To estimate the stochastic integral term, we will use the following remarkable result from [3] and [5] that there exists a universal constant $c$ such that, for any $p \ge 2$ and for any continuous martingale $(M_t)$ with $M_0 = 0$, one has

$$\left\|M_t^*\right\|_p \le c p^{1/2} \left\|\langle M\rangle_t^{1/2}\right\|_p, \tag{3.11}$$

where $M_t^* = \sup_{0\le s\le t}|M_s|$ and $\|\cdot\|_p$ stands for the $L^P$-norm.

Using this result, we have

$$\begin{aligned}&4\sqrt{\varepsilon}\left(E\left(\sup_{0\le t\le T}\int_0^t \left(u^\varepsilon(s), \sigma\left(\varepsilon s, u^\varepsilon(s)\right) \mathrm{d}W(s)\right)\right)^p\right)^{1/p} \\ &\quad\le 4c\sqrt{p\varepsilon}\left(E\left(\int_0^T \left|u^\varepsilon(s)\right|^2 \cdot \left|\sigma\left(\varepsilon s, u^\varepsilon(s)\right)\right|^2_{L_Q} \mathrm{d}s\right)^{p/2}\right)^{1/p} \\ &\quad\le 4c\sqrt{p\varepsilon}\left(E\left(\int_0^T \left|u^\varepsilon(s)\right|^2\left(1 + \left|u^\varepsilon(s)\right|^2\right) \mathrm{d}s\right)^{p/2}\right)^{1/p} \\ &\quad\le 4c\sqrt{p\varepsilon}\left[\left(E\left(\int_0^T \left(1 + \left|u^\varepsilon(s)\right|^2\right)^2 \mathrm{d}s\right)^{p/2}\right)^{2/p}\right]^{1/2} \\ &\quad\le 4c\sqrt{p\varepsilon}\left[\left(E\left(\int_0^T \left(1 + \left|u^\varepsilon(s)\right|^4\right) \mathrm{d}s\right)^{p/2}\right)^{2/p}\right]^{1/2} \\ &\quad\le 4c\sqrt{p\varepsilon}\left[\int_0^T 1 + \left(E\left|u^\varepsilon(s)\right|^{2p}\right)^{2/p} \mathrm{d}s\right]^{1/2},\end{aligned} \tag{3.12}$$

where $c$ is a constant which may change from line to line. On the other hand,

$$2\varepsilon L\left(E\left(\int_0^T \left(|u^\varepsilon|_V^H(s)\right)^2 \mathrm{d}s\right)^p\right)^{1/p} \le 2\varepsilon L\int_0^T \left(E\left(|u^\varepsilon|_V^H(s)\right)^{2p}\right)^{1/p}\mathrm{d}s. \tag{3.13}$$

Combining (3.10), (3.12) and (3.13), we arrive at

$$\begin{aligned}&\left(E\left(|u^\varepsilon|_V^H(T)\right)^{2p}\right)^{2/p}\\&\quad\le 8\left(|x|^2+\varepsilon LT+\frac{1}{\nu}|f|^2_{L^2(0,\ \varepsilon T;\ V')}\right)^2+8\varepsilon^2L^2T\int_0^T\left(E\left(|u^\varepsilon|_V^H(s)\right)^{2p}\right)^{2/p}\mathrm{d}s\\&\qquad+32c^2p\varepsilon T+32c^2p\varepsilon\int_0^T\left(E\left(|u^\varepsilon|_V^H(s)\right)^{2p}\right)^{2/p}\mathrm{d}s.\end{aligned} \tag{3.14}$$

Applying the Gronwall's inequality, we obtain

$$\begin{aligned}&\left(E\left(|u^\varepsilon|_V^H(1)\right)^{2p}\right)^{2/p}\\&\quad\le\left[8\left(|x|^2+\varepsilon L+\frac{1}{\nu}|f|^2_{L^2(0,\varepsilon;V')}\right)^2+32c^2p\varepsilon\right]\cdot\exp\left(8\varepsilon^2L^2+32c^2p\varepsilon\right).\end{aligned} \tag{3.15}$$

Since $P((|u^\varepsilon|_V^H(1))^2>M)\le M^{-p}E(|u^\varepsilon|_V^H(1))^{2p}$, let $p=\frac{1}{\varepsilon}$ in (3.15) to get

$$\begin{aligned}&\varepsilon\log P\left(\left(|u^\varepsilon|_V^H(1)\right)^2>M\right)\\&\quad\le-\log M+\log\left(E\left(|u^\varepsilon|_V^H(1)\right)^{2p}\right)^{1/p}\\&\quad\le-\log M+\log\sqrt{\left[8\left(|x|^2+\varepsilon L+\frac{1}{\nu}|f|^2_{L^2(0,\ \varepsilon;V')}\right)^2+32c^2\right]}+4\varepsilon^2L^2+16c^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{0<\varepsilon\le1}\varepsilon\log P\left(\left(|u^\varepsilon|_V^H(1)\right)^2>M\right)\\&\quad\le-\log M+\log\sqrt{\left[8\left(|x|^2+L+\frac{1}{\nu}|f|^2_{L^2(0,1;\ V')}\right)^2+32c^2\right]}+16c^2+4L^2.\end{aligned}$$

Letting $M\to\infty$ on both side of the above inequality, we finish the proof. □

Since $V$ is dense in $H$, there exists a sequence $\{x_n\}_{n=1}^\infty\subset V$ such that

$$\lim_{n\to+\infty}|x_n-x|=0.$$

Let $u_n^\varepsilon(\cdot)$ be the solution of (3.3) with the initial value $x_n$. From the proof of Lemma 3.1, it follows that

$$\lim_{M\to+\infty}\sup_n\sup_{0<\varepsilon\le1}\varepsilon\log P\left(\left(|u_n^\varepsilon|_V^H(1)\right)^2>M\right)=-\infty. \tag{3.16}$$

Let $v_n^\varepsilon(\cdot)$ be the solution of (3.4) with the initial value $x_n$, and we have the following result.

**Lemma 3.2.** *For any $n\in\mathbb{Z}^+$,*

$$\lim_{M\to\infty}\sup_{0<\varepsilon\le1}\varepsilon\log P\left(\sup_{0\le t\le1}\left\|v_n^\varepsilon(t)\right\|^2>M\right)=-\infty.$$

**Proof.** Applying Itô's formula to $\|v_n^\varepsilon(t)\|^2$, one obtains

$$\|v_n^\varepsilon(t)\|^2 = \|x_n\|^2 + 2\sqrt{\varepsilon}\int_0^t \big(\big(v_n^\varepsilon(s), \sigma\big(\varepsilon s, v_n^\varepsilon(s)\big)\,\mathrm{d}W(s)\big)\big) + \varepsilon\int_0^t \big|\sigma\big(\varepsilon s, v_n^\varepsilon(s)\big)\big|^2_{L_Q^V}\,\mathrm{d}s.$$

By (A.3) and inequality (3.11), we have

$$\begin{aligned}\Big(E\Big[\sup_{0\le t\le r}\|v_n^\varepsilon(t)\|^{2p}\Big]\Big)^{2/p} &\le 2\|x_n\|^4 + 8c\varepsilon p\bigg(E\bigg[\int_0^r \|v_n^\varepsilon(s)\|^2\big|\sigma\big(\varepsilon s, v_n^\varepsilon(s)\big)\big|^2_{L_Q^V}\,\mathrm{d}s\bigg]^{p/2}\bigg)^{2/p} \\ &\quad + 4\varepsilon^2\hat{L}^2 r\bigg(r + \int_0^r \Big(E\Big[\sup_{0\le l\le s}\|v_n^\varepsilon(l)\|^{2p}\Big]\Big)^{2/p}\,\mathrm{d}s\bigg) \\ &\le 2\|x_n\|^4 + 16c\varepsilon p\hat{L}\bigg(r + \int_0^r \Big(E\Big[\sup_{0\le l\le s}\|v_n^\varepsilon(l)\|^{2p}\Big]\Big)^{2/p}\,\mathrm{d}s\bigg) \\ &\quad + 4\varepsilon^2\hat{L}^2 r\bigg(r + \int_0^r \Big(E\Big[\sup_{0\le l\le s}\|v_n^\varepsilon(l)\|^{2p}\Big]\Big)^{2/p}\,\mathrm{d}s\bigg).\end{aligned} \tag{3.17}$$

By Gronwall's inequality,

$$\Big(E\Big[\sup_{0\le t\le 1}\|v_n^\varepsilon(t)\|^{2p}\Big]\Big)^{2/p} \le \big(2\|x_n\|^4 + 16c\varepsilon p\hat{L} + 4\varepsilon^2\hat{L}^2\big)\mathrm{e}^{16c\varepsilon p\hat{L} + 4\varepsilon^2\hat{L}^2}. \tag{3.18}$$

The rest of the proof is the same as that of Lemma 3.1. □

**Lemma 3.3.** *For any* $\delta > 0$,

$$\lim_{n\to+\infty}\sup_{0<\varepsilon\le 1}\varepsilon\log P\Big(\sup_{0\le t\le 1}\big|u^\varepsilon(t) - u_n^\varepsilon(t)\big|^2 > \delta\Big) = -\infty. \tag{3.19}$$

**Proof.** For $M > 0$, define a stopping time

$$\tau_{\varepsilon,M} = \inf\bigg\{t\colon \varepsilon\nu\int_0^t \big\|u^\varepsilon(r)\big\|^2\,\mathrm{d}r > M,\ \text{or}\ \big|u^\varepsilon(t)\big|^2 > M\bigg\}.$$

Clearly,

$$\begin{aligned}P\Big(\sup_{0\le t\le 1}\big|u^\varepsilon(t) - u_n^\varepsilon(t)\big|^2 > \delta, \big(\big|u^\varepsilon\big|_V^H(1)\big)^2 \le M\Big) &\le P\Big(\sup_{0\le t\le 1}\big|u^\varepsilon(t) - u_n^\varepsilon(t)\big|^2 > \delta, \tau_{\varepsilon,M} \ge 1\Big) \\ &\le P\Big(\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}}\big|u^\varepsilon(t) - u_n^\varepsilon(t)\big|^2 > \delta\Big).\end{aligned} \tag{3.20}$$

Let $k$ be a positive constant. Applying Itô's formula to $\mathrm{e}^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|u^\varepsilon(s)\|^2\,\mathrm{d}s}|u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(t\wedge\tau_{\varepsilon,M})|^2$, we get

$$\begin{aligned}&\mathrm{e}^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|u^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u^\varepsilon(t\wedge\tau_{\varepsilon,M}) - u_n^\varepsilon(t\wedge\tau_{\varepsilon,M})\big|^2 + 2\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s) - u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s \\ &\quad = |x - x_n|^2 - k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s)\big\|^2\big|u^\varepsilon(s) - u_n^\varepsilon(s)\big|^2\,\mathrm{d}s \\ &\qquad - 2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big(b\big(u^\varepsilon(s), u^\varepsilon(s), u^\varepsilon(s) - u_n^\varepsilon(s)\big) \\ &\qquad - b\big(u_n^\varepsilon(s), u_n^\varepsilon(s), u^\varepsilon(s) - u_n^\varepsilon(s)\big)\big)\,\mathrm{d}s\end{aligned}$$

$$+\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}} \mathrm{e}^{-k\varepsilon\int_0^s \|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|\sigma\big(\varepsilon s,u^\varepsilon(s)\big)-\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)\big|_{L_Q}^2\,\mathrm{d}s$$
$$+2\sqrt{\varepsilon}\int_0^{t\wedge\tau_{\varepsilon,M}} \mathrm{e}^{-k\varepsilon\int_0^s \|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big(u^\varepsilon(s)-u_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u^\varepsilon(s)\big)-\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big). \tag{3.21}$$

Notice that by (2.4), we have

$$b\big(u_n^\varepsilon(t),u_n^\varepsilon(t),u^\varepsilon(t)-u_n^\varepsilon(t)\big)=b\big(u_n^\varepsilon(t),u^\varepsilon(t),u^\varepsilon(t)-u_n^\varepsilon(t)\big),$$

hence by (2.9),

$$\begin{aligned}&\big|b\big(u^\varepsilon(t),u^\varepsilon(t),u^\varepsilon(t)-u_n^\varepsilon(t)\big)-b\big(u_n^\varepsilon(t),u_n^\varepsilon(t),u^\varepsilon(t)-u_n^\varepsilon(t)\big)\big|\\&\quad=\big|b\big(u^\varepsilon(t)-u_n^\varepsilon(t),u^\varepsilon(t),u^\varepsilon(t)-u_n^\varepsilon(t)\big)\big|\\&\quad\le 2\big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|\cdot\big\|u^\varepsilon(t)\big\|\cdot\big\|u^\varepsilon(t)-u_n^\varepsilon(t)\big\|.\end{aligned}$$

Therefore, by (3.21),

$$\begin{aligned}&\mathrm{e}^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\|u^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u^\varepsilon(t\wedge\tau_{\varepsilon,M})-u_n^\varepsilon(t\wedge\tau_{\varepsilon,M})\big|^2+2\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s)-u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\\
&\quad\le|x-x_n|^2-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s)\big\|^2\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\,\mathrm{d}s\\
&\qquad+4\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|\cdot\big\|u^\varepsilon(s)\big\|\cdot\big\|u^\varepsilon(s)-u_n^\varepsilon(s)\big\|\,\mathrm{d}s\\
&\qquad+\varepsilon K\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big(u^\varepsilon(s)-u_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u^\varepsilon(s)\big)-\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big)\bigg|\\
&\quad\le|x-x_n|^2-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s)\big\|^2\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\,\mathrm{d}s\\
&\qquad+\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u^\varepsilon(s)-u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\\
&\qquad+\frac{4\varepsilon}{\nu}\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\cdot\big\|u^\varepsilon(s)\big\|^2\,\mathrm{d}s\\
&\qquad+\varepsilon K\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big(u^\varepsilon(s)-u_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u^\varepsilon(s)\big)-\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big)\bigg|.\end{aligned}\tag{3.22}$$

Choosing $k>\frac{4}{\nu}$ and using (3.11), we have,

$$\begin{aligned}&\Big(E\Big[\sup_{0\le s\le t\wedge\tau_{\varepsilon,M}}\big(\mathrm{e}^{-k\int_0^s\|u^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u^\varepsilon(s)-u_n^\varepsilon(s)\big|^2\big)\Big]^p\Big)^{2/p}\\
&\quad\le 2|x-x_n|^4+2\varepsilon^2K^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-k\varepsilon\int_0^r\|u^\varepsilon(l)\|^2\,\mathrm{d}l}\big|u^\varepsilon(r)-u_n^\varepsilon(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s\\
&\qquad+8c\varepsilon pK^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}}\mathrm{e}^{-2k\varepsilon\int_0^r\|u^\varepsilon(l)\|^2\,\mathrm{d}l}\big|u^\varepsilon(r)-u_n^\varepsilon(r)\big|^4\Big)^{p/2}\Big]\Big)^{2/p}\,\mathrm{d}s\end{aligned}$$

$$\le 2|x-x_n|^4 + 2\varepsilon^2 K^2 \int_0^t \Big( E\Big[\Big( \sup_{0\le r\le s\wedge\tau_{\varepsilon,M}} \mathrm{e}^{-k\varepsilon\int_0^r \|u^\varepsilon(l)\|^2\,\mathrm{d}l} \big|u^\varepsilon(r)-u_n^\varepsilon(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s$$

$$+ 8c\varepsilon pK^2 \int_0^t \Big( E\Big[\Big( \sup_{0\le r\le s\wedge\tau_{\varepsilon,M}} \mathrm{e}^{-k\varepsilon\int_0^r \|u^\varepsilon(l)\|^2\,\mathrm{d}l} \big|u^\varepsilon(r)-u_n^\varepsilon(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s. \tag{3.23}$$

Applying Gronwall's inequality, one obtains,

$$\Big(E\Big[\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} \big(\mathrm{e}^{-k\varepsilon\int_0^t \|u^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2\big)^p\Big]\Big)^{2/p}$$
$$\le 2|x-x_n|^4 \mathrm{e}^{2\varepsilon^2K^2+8c\varepsilon pK^2}. \tag{3.24}$$

Hence,

$$\Big(E\Big[\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} \big(\big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2\big)^p\Big]\Big)^{2/p}$$
$$\le \Big(E\Big[\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} \big(\mathrm{e}^{-k\varepsilon\int_0^t \|u^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2\big)^p \mathrm{e}^{kp\varepsilon\int_0^{1\wedge\tau_{\varepsilon,M}} \|u^\varepsilon(s)\|^2\,\mathrm{d}s}\Big]\Big)^{2/p}$$
$$\le \mathrm{e}^{2kM}\Big(E\Big[\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} \big(\mathrm{e}^{-k\varepsilon\int_0^t \|u^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2\big)^p\Big]\Big)^{2/p}$$
$$\le 2\mathrm{e}^{2kM}|x-x_n|^4\mathrm{e}^{2\varepsilon^2K^2+8c\varepsilon pK^2}. \tag{3.25}$$

Fix $M$, and take $p=2/\varepsilon$ to get

$$\sup_{0<\varepsilon\le 1} \varepsilon\log P\Big(\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} \big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2 > \delta\Big)$$
$$\le \sup_{0\le\varepsilon\le 1} \varepsilon\log\frac{E[\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}} |u^\varepsilon(t)-u_n^\varepsilon(t)|^{2p}]}{\delta^p}$$
$$\le 2kM + 2K^2 + 16cK^2 - 2\log\delta + \log 2|x-x_n|^4$$
$$\to -\infty,\quad \text{as } n\to+\infty. \tag{3.26}$$

By Lemma 3.1, for any $R>0$, there exists a constant $M$ such that for any $\varepsilon\in(0,1]$, the following inequality holds,

$$P\big(\big(\big|u^\varepsilon\big|_V^H(1)\big)^2 > M\big) \le \mathrm{e}^{-R/\varepsilon}. \tag{3.27}$$

For such a $M$, by (3.20), (3.26), there exits a positive integer $N$, such that for any $n\ge N$,

$$\sup_{0<\varepsilon\le 1} \varepsilon\log P\Big(\sup_{0\le t\le 1} \big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2 > \delta, \big(\big|u^\varepsilon\big|_V^H(1)\big)^2 \le M\Big) \le -R. \tag{3.28}$$

Putting (3.27) and (3.28) together, one sees that there exists a positive integer $N$, such that for any $n\ge N$, $\varepsilon\in(0,1]$

$$P\Big(\sup_{0\le t\le 1} \big|u^\varepsilon(t)-u_n^\varepsilon(t)\big|^2 > \delta\Big) \le 2\mathrm{e}^{-R/\varepsilon}. \tag{3.29}$$

Since $R$ is arbitrary, the lemma follows. □

We have the following similar result for $v^\varepsilon(\cdot)$ and $v_n^\varepsilon(\cdot)$.

**Lemma 3.4.** *For any* $\delta > 0$,

$$\lim_{n\to+\infty}\sup_{0<\varepsilon\le 1}\varepsilon\log P\Big(\sup_{0\le t\le 1}\big|v^\varepsilon(t)-v_n^\varepsilon(t)\big|^2>\delta\Big)=-\infty. \tag{3.30}$$

**Proof.** The proof is similar to that of Lemma 3.3. Applying Itô's formula to $|v^\varepsilon(t)-v_n^\varepsilon(t)|^2$, and taking (3.11), (A.4) into consideration, it follows that

$$\begin{aligned}&\Big(E\Big[\sup_{0\le s\le t}\big|v^\varepsilon(s)-v_n^\varepsilon(s)\big|^{2p}\Big]\Big)^{2/p}\\&\quad\le 2|x-x_n|^4+\big(8c\varepsilon pK^2+2\varepsilon^2K^2\big)\int_0^t\Big(E\Big[\sup_{0\le r\le s}\big|v^\varepsilon(r)-v_n^\varepsilon(r)\big|^{2p}\Big]\Big)^{2/p}\,\mathrm{d}s.\end{aligned} \tag{3.31}$$

Applying Gronwall's inequality,

$$\Big(E\Big[\sup_{0\le t\le 1}\big|v^\varepsilon(t)-v_n^\varepsilon(t)\big|^{2p}\Big]\Big)^{2/p}\le 2|x-x_n|^4\mathrm{e}^{8c\varepsilon pK^2+2\varepsilon^2K^2}. \tag{3.32}$$

The rest proof is the same as that of Lemma 3.3. □

**Lemma 3.5.** *For any* $\delta > 0$, *and every positive integer* $n$,

$$\lim_{\varepsilon\to 0}\varepsilon\log P\Big(\sup_{0\le t\le 1}\big|u_n^\varepsilon(t)-v_n^\varepsilon(t)\big|^2>\delta\Big)=-\infty. \tag{3.33}$$

**Proof.** For $M > 0$, define the following stopping times:

$$\begin{aligned}\tau_{\varepsilon,M}^{1,n}&=\inf\Big\{t\ge 0:\ \varepsilon\int_0^t\big\|u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s>M,\ \text{or}\ \big|u_n^\varepsilon(t)\big|^2>M\Big\},\\\tau_{\varepsilon,M}^{2,n}&=\inf\big\{t\ge 0:\ \big\|v_n^\varepsilon(t)\big\|^2>M\big\}.\end{aligned}$$

Then we have

$$\begin{aligned}&P\Big(\sup_{0\le t\le 1}\big|u_n^\varepsilon(t)-v_n^\varepsilon(t)\big|^2>\delta,\big(\big|u_n^\varepsilon\big|_V^H(1)\big)^2\le M,\sup_{0\le t\le 1}\big\|v_n^\varepsilon(t)\big\|^2\le M\Big)\\&\quad\le P\Big(\sup_{0\le t\le 1}\big|u_n^\varepsilon(t)-v_n^\varepsilon(t)\big|^2>\delta,1\le\tau_{\varepsilon,M}^{1,n}\wedge\tau_{\varepsilon,M}^{2,n}\Big)\\&\quad\le P\Big(\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}^{1,n}\wedge\tau_{\varepsilon,M}^{2,n}}\big|u_n^\varepsilon(t)-v_n^\varepsilon(t)\big|^2>\delta\Big).\end{aligned} \tag{3.34}$$

Put $\tau_{\varepsilon,M}^n=\tau_{\varepsilon,M}^{1,n}\wedge\tau_{\varepsilon,M}^{2,n}$. Applying Itô's formula to $|u_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n)-v_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^n)|^2$, we get

$$\begin{aligned}&\big|u_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)-v_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)\big|^2+2\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\\&\quad=-2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\langle u^\varepsilon(s)-v_n^\varepsilon(s),Av_n^\varepsilon(s)\big\rangle\,\mathrm{d}s-2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}b\big(u_n^\varepsilon(s),u_n^\varepsilon(s),u_n^\varepsilon(s)-v_n^\varepsilon(s)\big)\,\mathrm{d}s\\&\qquad+2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\langle u_n^\varepsilon(s)-v_n^\varepsilon(s),f(\varepsilon s)\big\rangle\,\mathrm{d}s+\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big|\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma\big(\varepsilon s,v_n^\varepsilon(s)\big)\big|_{L_Q}^2\,\mathrm{d}s\\&\qquad+2\sqrt{\varepsilon}\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big(u_n^\varepsilon(s)-v_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma\big(\varepsilon s,v_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big)\end{aligned}$$

$$
\begin{aligned}
&= -2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle u^\varepsilon_n(s) - v^\varepsilon_n(s), Av^\varepsilon_n(s)\rangle \,\mathrm{d}s - 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} b\big(u^\varepsilon_n(s) - v^\varepsilon_n(s), u^\varepsilon_n(s), u^\varepsilon_n(s) - v^\varepsilon_n(s)\big)\,\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} b\big(v^\varepsilon_n(s), u^\varepsilon_n(s) - v^\varepsilon_n(s), v^\varepsilon_n(s)\big)\,\mathrm{d}s + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \langle u^\varepsilon_n(s) - v^\varepsilon_n(s), f(\varepsilon s)\rangle \,\mathrm{d}s \\
&\quad + \varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big|^2_{L_Q}\,\mathrm{d}s \\
&\quad + 2\sqrt{\varepsilon} \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big(u^\varepsilon_n(s) - v^\varepsilon_n(s), \big(\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big)\,\mathrm{d}W(s)\big).
\end{aligned}
$$

Applying the inequality

$$
2ab \le \frac{\nu}{4}a^2 + \frac{4}{\nu}b^2, \quad a, b \ge 0,
$$

and by (2.3), (2.4), (2.9), we have,

$$
\begin{aligned}
&\big|u^\varepsilon_n\big(t\wedge\tau^n_{\varepsilon,M}\big) - v^\varepsilon_n\big(t\wedge\tau^n_{\varepsilon,M}\big)\big|^2 + 2\varepsilon\nu \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\|^2\,\mathrm{d}s \\
&\le 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|\langle u^\varepsilon_n(s) - v^\varepsilon_n(s), Av^\varepsilon_n(s)\rangle\big|\,\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big| \cdot \big\|u^\varepsilon_n(s)\big\| \cdot \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\|\,\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|v^\varepsilon_n(s)\big| \cdot \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\| \cdot \big\|v^\varepsilon_n(s)\big\|\,\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|\langle u^\varepsilon_n(s) - v^\varepsilon_n(s), f(\varepsilon s)\rangle\big|\,\mathrm{d}s \\
&\quad + \varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big|^2_{L_Q}\,\mathrm{d}s \\
&\quad + 2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau^n_{\varepsilon,M}} \big(u^\varepsilon_n(s) - v^\varepsilon_n(s), \big(\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big)\,\mathrm{d}W(s)\big)\bigg| \\
&\le 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\| \cdot \big\|v^\varepsilon_n(s)\big\|\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big| \cdot \big\|u^\varepsilon_n(s)\big\| \cdot \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\|\,\mathrm{d}s \\
&\quad + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|v^\varepsilon_n(s)\big| \cdot \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\| \cdot \big\|v^\varepsilon_n(s)\big\|\,\mathrm{d}s + 2\varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big\|u^\varepsilon_n(s) - v^\varepsilon_n(s)\big\| \cdot \big|f(\varepsilon s)\big|_{V'}\,\mathrm{d}s \\
&\quad + \varepsilon \int_0^{t\wedge\tau^n_{\varepsilon,M}} \big|\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big|^2_{L_Q}\,\mathrm{d}s \\
&\quad + 2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau^n_{\varepsilon,M}} \big(u^\varepsilon_n(s) - v^\varepsilon_n(s), \big(\sigma\big(\varepsilon s, u^\varepsilon_n(s)\big) - \sigma\big(\varepsilon s, v^\varepsilon_n(s)\big)\big)\,\mathrm{d}W(s)\big)\bigg|.
\end{aligned}
$$

Furthermore, by (A.4), we get

$$\begin{aligned}
&\big|u_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)-v_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)\big|^2+2\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\\
&\quad\le\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s+\frac{4}{\nu}\varepsilon\int_0^t\big\|v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s+\frac{4}{\nu}\varepsilon\int_0^t\big|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big|^2\big\|u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\\
&\qquad+\frac{4}{\nu}\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|v_n^\varepsilon(s)\big\|^4\,\mathrm{d}s+\frac{4}{\nu}\int_0^{\varepsilon(t\wedge\tau_{\varepsilon,M}^n)}\big|f(s)\big|_{V'}^2\,\mathrm{d}s+\varepsilon K\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big|^2\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\left|\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big(u_n^\varepsilon(s)-v_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma\big(\varepsilon s,v_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big)\right|.
\end{aligned}\tag{3.35}$$

Using the Gronwall's inequality,

$$\begin{aligned}
&\big|u_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)-v_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^n\big)\big|^2\\
&\quad\le\bigg(\frac{4}{\nu}\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s+\frac{4}{\nu}\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|v_n^\varepsilon(s)\big\|^4\,\mathrm{d}s+\frac{4}{\nu}\int_0^{\varepsilon(t\wedge\tau_{\varepsilon,M}^n)}\big|f(s)\big|_{V'}^2\,\mathrm{d}s\\
&\qquad+2\sqrt{\varepsilon}\left|\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big(u_n^\varepsilon(s)-v_n^\varepsilon(s),\big(\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma\big(\varepsilon s,v_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\big)\right|\bigg)\\
&\qquad\times\mathrm{e}^{(4/\nu)\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^n}\|u_n^\varepsilon(s)\|^2\,\mathrm{d}s+\varepsilon Kt}.
\end{aligned}\tag{3.36}$$

By the similar techniques used in the previous lemma, (3.11), and the definition of the stopping time $\tau_{\varepsilon,M}^n$, we deduce from (3.26) that

$$\begin{aligned}
&\Big(E\sup_{0\le s\le t\wedge\tau_{\varepsilon,M}^n}\big|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big|^{2p}\Big)^{2/p}\\
&\quad\le\mathrm{e}^{(8M/\nu+2\varepsilon K)}\cdot\bigg(\frac{32}{\nu^2}\varepsilon^2\bigg(E\bigg(\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|v_n^\varepsilon(s)\big\|^2\,\mathrm{d}s\bigg)^p\bigg)^{2/p}+\frac{32}{\nu^2}\varepsilon^2\bigg(E\bigg(\int_0^{t\wedge\tau_{\varepsilon,M}^n}\big\|v_n^\varepsilon(s)\big\|^4\,\mathrm{d}s\bigg)^p\bigg)^{2/p}\\
&\qquad+\frac{32}{\nu^2}\bigg(\int_0^\varepsilon\big|f(s)\big|_{V'}^2\,\mathrm{d}s\bigg)^2+8c\varepsilon pK^2\int_0^t\Big(E\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}^n}\big|u_n^\varepsilon(r)-v_n^\varepsilon(r)\big|^{2p}\Big)^{2/p}\,\mathrm{d}s\bigg)\\
&\quad\le\mathrm{e}^{(8/\nu M+2\varepsilon K)}\cdot\bigg(\frac{32}{\nu^2}\varepsilon^2M^2+\frac{32}{\nu^2}\varepsilon^2M^4+\frac{32}{\nu^2}\bigg(\int_0^\varepsilon\big|f(s)\big|_{V'}^2\,\mathrm{d}s\bigg)^2\bigg)\\
&\qquad+\mathrm{e}^{(8M/\nu+2\varepsilon K)}\cdot 8cp\varepsilon K^2\int_0^t\Big(E\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}^n}\big|u_n^\varepsilon(r)-v_n^\varepsilon(r)\big|^{2p}\Big)^{2/p}\,\mathrm{d}s.
\end{aligned}\tag{3.37}$$

Therefore, let $C_{\varepsilon,M}=\mathrm{e}^{(8/\nu M+2\varepsilon K)}\cdot 8cp\varepsilon K^2$, we get

$$\begin{aligned}
&\Big(E\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}^n}\big|u_n^\varepsilon(s)-v_n^\varepsilon(s)\big|^{2p}\Big)^{2/p}\\
&\quad\le\mathrm{e}^{(8/\nu M+2\varepsilon K)}\cdot\bigg(\frac{32}{\nu^2}\varepsilon^2M^2+\frac{32}{\nu^2}\varepsilon^2M^4+\frac{32}{\nu^2}\bigg(\int_0^\varepsilon\big|f(s)\big|_{V'}^2\,\mathrm{d}s\bigg)^2\bigg)\cdot\mathrm{e}^{C_{\varepsilon,M}}.
\end{aligned}\tag{3.38}$$

From (3.16) and Lemma 3.2, we know that, for any $R>0$, there exists a $M$ such that

$$\sup_{0<\varepsilon\le 1}\varepsilon\log P\big(\big(|u_n^\varepsilon|_V^H(1)\big)^2>M\big)\le -R,\tag{3.39}$$

$$\sup_{0<\varepsilon\le 1} \varepsilon \log P\Big(\sup_{0\le t\le 1} \|v_n^\varepsilon\|^2 > M\Big) \le -R. \tag{3.40}$$

For such a constant $M$, let $p = 2/\varepsilon$ in (3.38) to obtain

$$\begin{aligned}
&\varepsilon \log P\Big(\sup_{0\le t\le 1} \big|u_n^\varepsilon(t) - v_n^\varepsilon(t)\big|^2 > \delta, \big(\big|u_n^\varepsilon\big|_V^H(1)\big)^2 \le M, \sup_{0\le t\le 1} \|v_n^\varepsilon\|^2 \le M\Big)\\
&\quad\le \varepsilon \log P\Big(\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}^n} \big|u_n^\varepsilon(t) - v_n^\varepsilon(t)\big|^2 > \delta\Big)\\
&\quad\le \log\Big(E \sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}^n} \big|u_n^\varepsilon(t) - v_n^\varepsilon(t)\big|^{2p}\Big)^{2/p} - \log\delta^2\\
&\quad\le \frac{8}{\nu}M + 2\varepsilon K + \log\bigg(\frac{32}{\nu^2}\varepsilon^2 M^2 + \frac{32}{\nu^2}\varepsilon^2 M^4 + \frac{32}{\nu^2}\Big(\int_0^\varepsilon \big|f(s)\big|_{V'}^2\,\mathrm{d}s\Big)^2\bigg) + C_{M,\varepsilon} - \log\delta^2\\
&\quad\to -\infty, \quad \text{as } \varepsilon\to 0.
\end{aligned} \tag{3.41}$$

Thus, there exists a $\varepsilon_0$ such that for any $\varepsilon$ satisfying $0 < \varepsilon \le \varepsilon_0$,

$$P\Big(\sup_{0\le t\le 1} \big|u_n^\varepsilon(t) - v_n^\varepsilon(t)\big|^2 > \delta, \big(\big|u_n^\varepsilon\big|_V^H(1)\big)^2 \le M, \sup_{0\le t\le 1} \|v_n^\varepsilon\|^2 \le M\Big) \le \mathrm{e}^{-R/\varepsilon}. \tag{3.42}$$

Putting (3.39), (3.40) and (3.42) together, we see that there exists a constant $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0]$,

$$P\Big(\sup_{0\le t\le 1} \big|u_n^\varepsilon(t) - v_n^\varepsilon(t)\big|^2 > \delta\Big) \le 3\mathrm{e}^{-R/\varepsilon}.$$

Since $R$ is arbitrary, the proof is finished. □

**Proof of (3.5).** By Lemmas 3.3 and 3.4, we have for any $R > 0$, there exists a $N_0$ satisfying

$$P\Big(\sup_{0\le t\le 1} \big|u^\varepsilon(t) - u_{N_0}^\varepsilon(t)\big|^2 > \delta\Big) \le \mathrm{e}^{-R/\varepsilon} \quad \text{for any } \varepsilon \in (0,1], \tag{3.43}$$

and

$$P\Big(\sup_{0\le t\le 1} \big|v^\varepsilon(t) - v_{N_0}^\varepsilon(t)\big|^2 > \delta\Big) \le \mathrm{e}^{-R/\varepsilon} \quad \text{for any } \varepsilon \in (0,1]. \tag{3.44}$$

In view of Lemma 3.5, for such $N_0$, there exists a $\varepsilon_0$ such that for any $\varepsilon \in (0, \varepsilon_0]$

$$P\Big(\sup_{0\le t\le 1} \big|u_{N_0}^\varepsilon(t) - v_{N_0}^\varepsilon(t)\big| > \delta\Big) \le \mathrm{e}^{-R/\varepsilon}. \tag{3.45}$$

Thus, for any $\varepsilon \in (0, \varepsilon_0]$,

$$P\Big(\sup_{0\le t\le 1} \big|u^\varepsilon(t) - v^\varepsilon(t)\big| > \delta\Big) \le 3\mathrm{e}^{-R/\varepsilon}. \tag{3.46}$$

Since $R$ is arbitrary, we conclude that

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} \big|u^\varepsilon(t) - v^\varepsilon(t)\big|^2 > \delta\Big) = -\infty.$$ □

## 4. Further results

In this section, instead of assuming (A.3) and (A.5), we suppose that there exists a sequence of mappings $\{\sigma^m(\cdot)\}_{m=1}^{\infty}$ which satisfy:

(A.6) There exists a constant $C_1(m)$, which may depend on $m$, such that for all $u \in V$, $|\sigma^m(t,u)|^2_{L_Q^V} \le C_1(m)(1+\|u\|^2)$ for all $t \in [0,T]$.

(A.7) There exists a constant $C_2$ such that for all $u \in H$ and $m$, $|\sigma^m(t,u)|^2_{L_Q} \le C_2(1+|u|^2)$ for all $t \in [0,T]$.

(A.8) There exists a constant $C_3(m)$, which may depend on $m$, such that for all $u, v \in V$, $|\sigma^m(t,u)-\sigma^m(t,v)|^2_{L_Q^V} \le C_3(m)\|u-v\|^2$ for all $t \in [0,T]$.

(A.9) There exists a constant $C_4$ such that for all $u, v \in H$ and $m$, $|\sigma^m(t,u) - \sigma^m(t,v)|^2_{L_Q} \le C_4|u-v|^2$ for all $t \in [0,T]$.

(A.10) $|\sigma^m(t,u) - \sigma(t,u)|^2_{L_Q} \to 0$ as $m \to +\infty$, uniformly on the bounded subsets of $[0,\infty) \times H$.

**Theorem 4.1.** *Under the assumptions* (A.1), (A.2), (A.4) *and* (A.6)–(A.10), *the large deviation principle stated in Theorem* 3.1 *still holds for* $\mu_x^\varepsilon$.

Under the assumptions (A.6)–(A.10), the stochastic equations

$$\mathrm{d}u^{\varepsilon,m}(t) + \varepsilon A u^{\varepsilon,m}(t)\,\mathrm{d}t + \varepsilon B\big(u^{\varepsilon,m}(t)\big)\,\mathrm{d}t = \varepsilon f(\varepsilon t)\,\mathrm{d}t + \sqrt{\varepsilon}\sigma^m\big(\varepsilon t, u^{\varepsilon,m}(t)\big)\,\mathrm{d}W(t)$$

and

$$\mathrm{d}v^{\varepsilon,m}(t) = \sqrt{\varepsilon}\sigma^m\big(\varepsilon t, v^{\varepsilon,m}(t)\big)\,\mathrm{d}W(t)$$

admit unique strong solutions. We denote by $u_n^{\varepsilon,m}$ and $v_n^{\varepsilon,m}$ the solutions of the above stochastic equations respectively with the same initial value as $u_n^\varepsilon$ and $v_n^\varepsilon$ in Section 3.

In order to show Theorem 4.1, arguing as in the proof of Theorem 3.1, it is sufficient to prove:

(1) For any $n$ and $\delta > 0$,

$$\lim_{m\to\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P\Big(\sup_{0\le t\le 1} \big|u_n^\varepsilon(t) - u_n^{\varepsilon,m}(t)\big|^2 > \delta\Big) = -\infty. \tag{4.1}$$

(2) For any $n$ and $\delta > 0$,

$$\lim_{m\to\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P\Big(\sup_{0\le t\le 1} \big|v_n^\varepsilon(t) - v_n^{\varepsilon,m}(t)\big|^2 > \delta\Big) = -\infty. \tag{4.2}$$

(3) For any $n, m$ and $\delta > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log P\Big(\sup_{0\le t\le 1} \big|u_n^{\varepsilon,m}(t) - v_n^{\varepsilon,m}(t)\big|^2 > \delta\Big) = -\infty. \tag{4.3}$$

**Proof of (4.1).** Similar to the proof of Lemma 3.3, for any fixed $n$, define the stopping time

$$\tau_{\varepsilon,M}^{(n)} = \inf\bigg\{t \ge 0, \varepsilon\nu \int_0^t \big\|u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s > M,\ \text{or}\ \big|u_n^\varepsilon(t)\big|^2 > M\bigg\}.$$

Applying Itô's formula to $\mathrm{e}^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}}\|u_n^\varepsilon(s)\|^2\,\mathrm{d}s}|u_n^\varepsilon(t\wedge\tau_{\varepsilon,M}^{(n)}) - u_n^{\varepsilon,m}(t\wedge\tau_{\varepsilon,M}^{(n)})|^2$ yields

$$\begin{aligned}&\mathrm{e}^{-k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}}\|u_n^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u_n^\varepsilon\big(t\wedge\tau_{\varepsilon,M}^{(n)}\big) - u_n^{\varepsilon,m}\big(t\wedge\tau_{\varepsilon,M}^{(n)}\big)\big|^2\\ &\quad + 2\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s\|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u_n^\varepsilon(s) - u_n^{\varepsilon,m}(s)\big\|^2\,\mathrm{d}s\end{aligned}$$

$$\le -k\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u_n^\varepsilon(s)\big\|^2\big|u_n^\varepsilon(s)-u_n^{\varepsilon,m}(s)\big|^2\,\mathrm{d}s$$
$$+\varepsilon\nu\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big\|u_n^\varepsilon(s)-u_n^{\varepsilon,m}(s)\big\|^2\,\mathrm{d}s$$
$$+\frac{\varepsilon}{\nu}\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big|u_n^\varepsilon(s)-u_n^{\varepsilon,m}(s)\big|^2\cdot\big\|u_n^\varepsilon(s)\big\|^2\,\mathrm{d}s$$
$$+2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big|\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma^m\big(\varepsilon s,u_n^\varepsilon(s)\big)\big|_{L_Q}^2\,\mathrm{d}s$$
$$+2\varepsilon\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\big|\sigma^m\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma^m\big(\varepsilon s,u_n^{\varepsilon,m}(s)\big)\big|_{L_Q}^2\,\mathrm{d}s$$
$$+2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\Big(u_n^\varepsilon(s)-u_n^{\varepsilon,m}(s),\big(\sigma\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma^m\big(\varepsilon s,u_n^\varepsilon(s)\big)\big)\,\mathrm{d}W(s)\Big)\bigg|$$
$$+2\sqrt{\varepsilon}\bigg|\int_0^{t\wedge\tau_{\varepsilon,M}^{(n)}} \mathrm{e}^{-k\varepsilon\int_0^s \|u_n^\varepsilon(r)\|^2\,\mathrm{d}r}\Big(u_n^\varepsilon(s)-u_n^{\varepsilon,m}(s),\big(\sigma^m\big(\varepsilon s,u_n^\varepsilon(s)\big)-\sigma^m\big(\varepsilon s,u_n^{\varepsilon,m}(s)\big)\big)\,\mathrm{d}W(s)\Big)\bigg|. \tag{4.4}$$

Taking $k>\frac{1}{\nu}$ and using (3.11), we obtain

$$\Big(E\Big[\Big(\sup_{0\le r\le t\wedge\tau_{\varepsilon,M}^{(n)}}\mathrm{e}^{-k\varepsilon\int_0^r \|u_n^\varepsilon(s)\|^2\,\mathrm{d}s}|u_n^\varepsilon(r)-u_n^{\varepsilon,m}(r)|^2\Big)^p\Big]\Big)^{2/p}$$
$$\le 8\varepsilon^2\int_0^t\big|\sigma\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)-\sigma^m\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)\big|_{L_{Q,\infty}}^2\,\mathrm{d}s$$
$$+8\varepsilon^2C_4^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}^{(n)}}\mathrm{e}^{-k\varepsilon\int_0^r \|u_n^\varepsilon(l)\|^2\,\mathrm{d}l}\big|u_n^\varepsilon(r)-u_n^{\varepsilon,m}(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s$$
$$+4c\varepsilon pC_4^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}^{(n)}}\mathrm{e}^{-k\varepsilon\int_0^r \|u_n^\varepsilon(l)\|^2\,\mathrm{d}l}\big|u_n^\varepsilon(r)-u_n^{\varepsilon,m}(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s$$
$$+4c\varepsilon pC_4^2\int_0^t\big|\sigma\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)-\sigma^m\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)\big|_{L_{Q,\infty}}^2\,\mathrm{d}s$$
$$+8c\varepsilon pC_4^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\tau_{\varepsilon,M}^{(n)}}\mathrm{e}^{-k\varepsilon\int_0^r \|u_n^\varepsilon(l)\|^2\,\mathrm{d}l}\big|u_n^\varepsilon(r)-u_n^{\varepsilon,m}(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s, \tag{4.5}$$

where $|\cdot|_{L_{Q,\infty}}$ denotes the essential upper bound of $|\cdot|_{L_Q}$ with respect to $\omega$.

By Gronwall's inequality, we deduce that

$$\Big(E\Big[\Big(\sup_{0\le t\le 1\wedge\tau_{\varepsilon,M}^{(n)}}\mathrm{e}^{-k\varepsilon\int_0^t \|u_n^\varepsilon(s)\|^2\,\mathrm{d}s}\big|u_n^\varepsilon(t)-u_n^{\varepsilon,m}(t)\big|^2\Big)^p\Big]\Big)^{2/p}$$
$$\le\big(8\varepsilon^2+4c\varepsilon pC_4^2\big)\int_0^1\big|\sigma\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)-\sigma^m\big(\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big),u_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big)\big|_{L_{Q,\infty}}^2\,\mathrm{d}s$$
$$.\qquad\times\,\mathrm{e}^{8\varepsilon^2C_4^2+12c\varepsilon}pC_4^2 \tag{4.6}$$

Therefore,

$$\Big(E\Big[\Big(\sup_{0\le t\le 1\wedge\tau^{(n)}_{\varepsilon,M}}\big|u^\varepsilon_n(t)-u^{\varepsilon,m}_n(t)\big|^2\Big)^p\Big]\Big)^{2/p}$$
$$\le \mathrm{e}^{2kM}\big(8\varepsilon^2+4c\varepsilon pC_4^2\big)\int_0^1\big|\sigma\big(\varepsilon\big(s\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n\big(s\wedge\tau^{(n)}_{\varepsilon,M}\big)\big)-\sigma^m\big(\varepsilon\big(s\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n\big(s\wedge\tau^{(n)}_{\varepsilon,M}\big)\big)\big|^2_{L_{Q,\infty}}\,\mathrm{d}s$$
$$\times\,\mathrm{e}^{8\varepsilon^2C_4^2+12c\varepsilon pC_4^2}. \tag{4.7}$$

Taking $p=2/\varepsilon$, one obtains

$$\sup_{0<\varepsilon\le 1}\varepsilon\log P\Big(\sup_{0\le t\le 1\wedge\tau^{(n)}_{\varepsilon,M}}\big|u^\varepsilon_n(t)-u^{\varepsilon,m}_n(t)\big|^2>\delta\Big)$$
$$\le \sup_{0<\varepsilon\le 1}\log\Big(E\Big(\sup_{0\le t\le 1\wedge\tau^{(n)}_{\varepsilon,M}}\big|u^\varepsilon_n(t)-u^{\varepsilon,m}_n(t)\big|^2\Big)^p\Big)^{2/p}-2\log\delta$$
$$\le\big(2kM+8C_4^2+12cC_4^2\big)+\log\big(8+4cC_4^2\big)-2\log\delta$$
$$+\log\sup_{0\le t\le 1}\sup_{0<\varepsilon\le 1}\big|\sigma\big(\varepsilon\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n(t\wedge\tau^{(n)}_{\varepsilon,M})\big)-\sigma^m\big(\varepsilon\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big)\big)\big|^2_{L_{Q,\infty}}. \tag{4.8}$$

Since

$$\sup_{0\le t\le 1}\sup_{0<\varepsilon\le 1}\big|u^\varepsilon_n\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big)\big|\le M,\quad P\text{-a.s},$$

hence, by (A.10), it follows that

$$\lim_{m\to+\infty}\sup_{0\le t\le 1,\,0<\varepsilon\le 1}\big|\sigma\big(\varepsilon\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big)\big)-\sigma^m\big(\varepsilon\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big),u^\varepsilon_n\big(t\wedge\tau^{(n)}_{\varepsilon,M}\big)\big)\big|^2_{L_{Q,\infty}}=0.$$

Therefore, since $M$ is arbitrary, by the same argument as in the proof of Lemma 3.3, (4.1) follows. □

**Proof of (4.2).** The proof is similar to that of (3.32). Notice that by the same proof of Lemma 3.1, we have for any $n$

$$\lim_{M\to+\infty}\sup_{0<\varepsilon\le 1}P\Big(\sup_{0\le t\le 1}\big|v^\varepsilon_n(t)\big|^2>M\Big)=-\infty. \tag{4.9}$$

Define, for $M>0$,

$$\bar\tau^{(n)}_{\varepsilon,M}=\inf\big\{t\ge 0:\ \big|v^\varepsilon_n(t)\big|^2>M\big\}.$$

Similar to (4.5), we have

$$\Big(E\Big[\sup_{0\le s\le t\wedge\bar\tau^{(n)}_{\varepsilon,M}}\big|v^\varepsilon_n(s)-v^{\varepsilon,m}_n(s)\big|^{2p}\Big]\Big)^{2/p}$$
$$\le 8\varepsilon^2\int_0^t\big|\sigma\big(\varepsilon\big(s\wedge\bar\tau^{(n)}_{\varepsilon,M}\big),v^\varepsilon_n\big(s\wedge\bar\tau^{(n)}_{\varepsilon,M}\big)\big)-\sigma^m\big(\varepsilon\big(s\wedge\bar\tau^{(n)}_{\varepsilon,M}\big),v^\varepsilon_n\big(s\wedge\bar\tau^{(n)}_{\varepsilon,M}\big)\big)\big|^2_{L_{Q,\infty}}\,\mathrm{d}s$$
$$+8\varepsilon^2C_4^2\int_0^t\Big(E\Big[\Big(\sup_{0\le r\le s\wedge\bar\tau^{(n)}_{\varepsilon,M}}\big|v^\varepsilon_n(r)-v^{\varepsilon,m}_n(r)\big|^2\Big)^p\Big]\Big)^{2/p}\,\mathrm{d}s$$

$$+ 4c\varepsilon p C_4^2 \int_0^t \Big( E\Big[\Big( \sup_{0\le r\le s\wedge\bar\tau_{\varepsilon,M}^{(n)}} \big|v_n^\varepsilon(r) - v_n^{\varepsilon,m}(r)\big|^2\Big)^p\Big]\Big)^{2/p} \,\mathrm{d}s$$

$$+ 4c\varepsilon p C_4^2 \int_0^t \big|\sigma\big(\varepsilon\big(s\wedge\bar\tau_{\varepsilon,M}^{(n)}\big), v_n^\varepsilon\big(s\wedge\tau_{\varepsilon,M}^{(n)}\big)\big) - \sigma^m\big(\varepsilon\big(s\wedge\bar\tau_{\varepsilon,M}^{(n)}\big), v_n^\varepsilon\big(s\wedge\bar\tau_{\varepsilon,M}^{(n)}\big)\big)\big|_{L_{Q,\infty}}^2 \,\mathrm{d}s$$

$$+ 8c\varepsilon p C_4^2 \int_0^t \Big( E\Big[\Big( \sup_{0\le r\le s\wedge\bar\tau_{\varepsilon,M}^{(n)}} \big|v_n^\varepsilon(r) - v_n^{\varepsilon,m}(r)\big|^2\Big)^p\Big]\Big)^{2/p} \,\mathrm{d}s. \tag{4.10}$$

The rest of the proof is the same as that of (4.1). □

**Proof of (4.3).** The procedure of the proof is exactly the same as that of Lemma 3.5. Just in this situation, we need to show, for any $n, m$

$$\lim_{M\to+\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P\big(\big(\big|u_n^{\varepsilon,m}\big|_V^H(1)\big)^2 > M\big) = -\infty,$$

and

$$\lim_{M\to+\infty} \sup_{0<\varepsilon\le 1} \varepsilon \log P\Big(\sup_{0\le t\le 1} \big\|v_n^{\varepsilon,m}(t)\big\|^2 > M\Big) = -\infty.$$

The above results can be obtained as in the proof of Lemmas 3.1 and 3.2. □

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