

# Lipschitzian norm estimate of one-dimensional Poisson equations and applications

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**Abstract.** By direct calculus we identify explicitly the Lipschitzian norm of the solution of the Poisson equation  $-\mathcal{L}G = g$  in terms of various norms of  $g$ , where  $\mathcal{L}$  is a Sturm–Liouville operator or generator of a non-singular diffusion in an interval. This allows us to obtain the best constant in the  $L^1$ -Poincaré inequality (a little stronger than the Cheeger isoperimetric inequality) and some sharp transportation–information inequalities and concentration inequalities for empirical means. We conclude with several illustrative examples.

**Résumé.** Par un calcul direct, on identifie explicitement la norme Lipschitzienne de la solution de l'équation de Poisson  $-\mathcal{L}G = g$  en terme de différentes normes de  $g$ , où  $\mathcal{L}$  est l'opérateur de Sturm–Liouville ou le générateur d'une diffusion non singulière sur un intervalle. Ainsi, nous pouvons obtenir, d'une part la meilleure constante dans l'inégalité de Poincaré  $L^1$  (une inégalité un peu plus forte que l'inégalité isopérimétrique de Cheeger) et d'autre part certaines inégalités de transport-information et de concentration fines pour la moyenne empirique. On conclut avec des exemples illustratifs.

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## 1. Framework and introduction

Let  $I$  be an interval of  $\mathbb{R}$  so that its interior  $I^0 = (x_0, y_0)$  where  $-\infty \leq x_0 < y_0 \leq +\infty$ . Consider a Sturm–Liouville operator on  $I$ :

$$\mathcal{L} = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

with the Neumann boundary condition at  $\partial I = \{x_0, y_0\} \cap \mathbb{R}$ , where  $a, b : I \rightarrow \mathbb{R}$  are measurable and satisfy:

- (A1)  $a, b$  are locally bounded (i.e., bounded on any compact subinterval of  $I$ );
- (A2)  $a(x) > 0$ , dx-a.e. and  $1/a$  is locally dx-integrable on  $I$ .

Here dx is the Lebesgue measure. On  $I^0$ ,  $\mathcal{L}$  can be rewritten as the Feller's form

$$\mathcal{L} = \frac{1}{m'(x)} \frac{d}{dx} \left( \frac{1}{s'(x)} \frac{d}{dx} \right) = \frac{d}{dm} \frac{d}{ds}, \tag{1.1}$$

where  $m, s$  are respectively the speed and scale functions of Feller, which are absolutely continuous functions on  $I$  such that  $dx$ -a.s.

$$s'(x) = \exp\left(-\int_c^x \frac{b(u)}{a(u)} du\right) \quad \text{and} \quad m'(x) = \frac{1}{a(x)s'(x)}, \quad (1.2)$$

where  $c$  is some fixed point in  $I$ . Let  $C_0^\infty(I)$  be the space of infinitely differentiable real functions  $f$  on  $I$  with compact support and  $\mathcal{D}$  be the space of all functions  $f$  in  $C_0^\infty(I)$  such that  $f'|_{\partial I} = 0$  (i.e., satisfying the Neumann boundary condition). The operator  $\mathcal{L}$  defined on  $\mathcal{D}$  is symmetric on  $L^2(I, m)$ , where  $m$  denotes also the measure  $m'(x) dx$ . Let  $(X_t: t \geq 0)$  be the diffusion on the interval  $I$  generated by  $\mathcal{L}$  (the Neumann boundary condition corresponds to the reflection at the boundary  $\partial I$ ). See [17] for background and precise definitions.

We will assume that:

(A3) the diffusion is non-explosive and positively recurrent, i.e.,  $m(I) = \int_I m'(y) dy < +\infty$  and

$$\begin{aligned} \int_c^{y_0} s'(x) \left( \int_c^x m'(y) dy \right) dx &= +\infty \quad \text{if } y_0 \notin I, \\ \int_{x_0}^c s'(x) \left( \int_x^c m'(y) dy \right) dx &= +\infty \quad \text{if } x_0 \notin I. \end{aligned}$$

(A4) the generator  $\mathcal{L}$ , defined on  $\mathcal{D} = \{f \in C_0^\infty(I); f'|_{\partial I} = 0\}$ , is essentially self-adjoint on  $L^2(I, dm)$ , or equivalently [11,12]:

$$s \notin L^2((x_0, c], dm) \quad \text{if } x_0 \notin I; \quad \text{and} \quad s \notin L^2([c, y_0), dm) \quad \text{if } y_0 \notin I.$$

Notice that when  $a(x) = 1$  and  $I = \mathbb{R}$ , the assumptions (A3) and (A4) are automatically satisfied once if  $m(I) < +\infty$  (see [17] for (A3), [12] for (A4)).

Throughout this paper we assume that (A1)–(A4) are satisfied. In that case  $(X_t)_{t \geq 0}$  is reversible w.r.t. the probability measure  $\mu(dx) = \frac{1}{m(I)} m'(x) dx$ . Let  $(P_t)_{t \geq 0}$  be the transition semigroup of  $(X_t)_{t \geq 0}$ ,  $\mathcal{L}_2$  the generator of  $(P_t)$  on  $L^2(I, \mu)$  with domain  $\mathbb{D}(\mathcal{L}_2)$ , which is an extension of  $(\mathcal{L}, \mathcal{D})$ .

Consider the Poisson equation

$$-\mathcal{L}_2 G = g, \quad (1.3)$$

where  $g \in L^2(I, \mu)$  such that  $\mu(g) := \int_I g d\mu = 0$ . By the ergodicity of the diffusion, the solution  $G$  of the Poisson equation, if exists, is unique in  $L^2(I, \mu)$  up to the difference of some constant. In the physical interpretation of the heat diffusion,  $g$  represents the heat source,  $G$  is the equilibrium heat distribution.

The objective of this paper is to estimate

$$\|G\|_{\text{Lip}(\rho)} := \sup_{x, y \in I, x < y} \frac{|G(y) - G(x)|}{|\rho(y) - \rho(x)|} \quad (1.4)$$

in terms of various norms on the heat source  $g$ . Here  $\rho$  is some absolutely continuous function on  $I$  such that  $\rho'(x) > 0$ ,  $dx$ -a.e.

Let  $\lambda_1$  be the spectral gap of  $\mathcal{L}_2$ , i.e. the lowest eigenvalue or spectral point above zero of  $-\mathcal{L}_2$ . Then  $c_P := \lambda_1^{-1}$  is the best constant in the following Poincaré inequality

$$\text{Var}_\mu(f) \leq c_P \int_I a(x) f'(x)^2 d\mu(x), \quad f \in \mathcal{D}, \quad (1.5)$$

where  $\text{Var}_\mu(f) := \mu(f^2) - (\mu(f))^2$  is the variance of  $f$  w.r.t.  $\mu$  and  $\mu(f) := \int_I f d\mu$ . The importance of the spectral gap is that it describes the exponential convergence rate:

$$\|P_t f - \mu(f)\|_2 \leq e^{-\lambda_1 t} \|f - \mu(f)\|_2 \quad \forall t \geq 0,$$

where  $\|\cdot\|_2$  is the  $L^2(I, \mu)$ -norm. The constant  $\lambda_1$  can be also interpreted by means of the Poisson equation:

$$\|G - \mu(G)\|_2 \leq c_P \|g\|_2 \quad \text{or} \quad \int_I a(x)G'(x)^2 d\mu(x) \leq c_P \|g\|_2^2.$$

Those physical interpretations explain why the study of  $\lambda_1$  or  $c_P$  is of fundamental importance. Since the study on  $\lambda_1$  is of a very long history, it is not possible for us to describe even the main line, the reader is referred to the books [9,24] for bibliographies. For the stronger log-Sobolev inequality, the first characterization was due to Bobkov–Götze [3], see [2,9] for further improvements of constant.

Our initial motivation was to understand Chen’s variational formula for  $\lambda_1$  [8]:

$$c_P = \inf_{\rho} \sup_{x \in I} \frac{s'(x)}{\rho'(x)} \int_x^{y_0} [\rho(t) - \mu(\rho)]m'(t) dt, \tag{1.6}$$

where  $\rho$  runs over all  $C^1(I)$  functions with  $\rho' > 0$ , in  $L^2(I, \mu)$ . Notice that no variational formula is known for the best log-Sobolev constant on the real line. But our main motivation comes from some concentration inequalities for the empirical mean  $(1/t) \int_0^t g(X_s) ds$ , which are immediate consequences of the estimate on  $\|G\|_{\text{Lip}(\rho)}$  via the forward–backward martingale decomposition or transportation–information inequalities developed in [15], see also [18].

Our method for estimate of  $\|G\|_{\text{Lip}(\rho)}$  is direct: the solution of the Poisson equation (1.3) can be solved explicitly (unlike the corresponding heat equation), only some further (easy) control is needed for completing the job. Besides those motivations, the estimation of  $G'$  is physically meaningful: in the heat diffusion problem, in presence of the heat source  $g$  with  $\mu(g) = 0$ ,  $G$  represents the equilibrium heat distribution; an estimate on  $|G'|$  allows us to control the variation of the equilibrium heat distribution.

This paper is organized as follows. In the next section, we state the main results and present several applications in concentration inequalities and transportation–information inequalities,  $L^1$ -Poincaré inequality (a little stronger than the Cheeger isoperimetric inequality), and provide several examples to illustrate the results. In Section 3 the proof of the main result is given.

## 2. Main results and applications

### 2.1. Main results

Given an absolutely continuous function  $\rho : I \rightarrow \mathbb{R}$  such that  $\rho' > 0$ , dx-a.e., let  $d_\rho(x, y) = |\rho(x) - \rho(y)|$  be the metric on  $I$  associated with  $\rho$ . If the Lipschitzian norm  $\|f\|_{\text{Lip}(\rho)}$  of  $f$  w.r.t.  $d_\rho$  defined in (1.4) is finite, we say that  $f$  is  $\rho$ -Lipschitzian. Let  $L_0^2(I, \mu) := \{f \in L^2(I, \mu); \mu(f) = 0\}$ .

Now, we can state the main result in this paper.

**Theorem 2.1.** *Assume (A1)–(A4) and let  $\rho, \rho_1, \rho_2$  be absolutely continuous functions on  $I$  such that  $\rho, \rho_k \in L^2(I, \mu)$ ,  $\rho', \rho'_k > 0$ , dx-a.e.*

(i) *If*

$$c_{\text{Lip}}(\rho_1, \rho_2) := \text{ess sup}_{x \in I} \frac{s'(x)}{\rho'_2(x)} \int_x^{y_0} [\rho_1(t) - \mu(\rho_1)]m'(t) dt < +\infty, \tag{2.1}$$

*then for any  $\rho_1$ -Lipschitzian function  $g \in L_0^2(I, \mu)$ , there is a unique solution  $G$  with  $\mu(G) = 0$  belonging to the domain  $\mathbb{D}(\mathcal{L}_2)$  of the Poisson equation (1.3). Moreover,  $G$  (or one dx-version of it) is  $\rho_2$ -Lipschitzian and satisfies*

$$\|G\|_{\text{Lip}(\rho_2)} \leq c_{\text{Lip}}(\rho_1, \rho_2) \|g\|_{\text{Lip}(\rho_1)}. \tag{2.2}$$

*Furthermore, this inequality (2.2) becomes equality for  $g = \rho_1 - \mu(\rho_1)$ .*

(ii) Let  $\varphi : I \rightarrow \mathbb{R}^+$  be a nonnegative function in  $L^2(I, \mu)$ . If

$$c(\varphi, \rho) := \operatorname{ess\,sup}_{x \in I} \frac{s'(x)}{\rho'(x)} m(I) \left( \mu(I_x^+) \int_{I_x^-} \varphi \, d\mu + \mu(I_x^-) \int_{I_x^+} \varphi \, d\mu \right) < +\infty, \quad (2.3)$$

where  $I_x^+ = \{y \in I; y \geq x\}$ ,  $I_x^- = \{y \in I; y < x\}$ , then for any function  $g \in L^2(I, \mu)$  such that  $|g| \leq \varphi$ , there is a unique solution  $G$  with  $\mu(G) = 0$  to the Poisson equation  $-\mathcal{L}_2 G = g - \mu(g)$ . Moreover,  $G$  (or one  $dx$ -version of it) is  $\rho$ -Lipschitzian and satisfies

$$\sup_{g: |g| \leq \varphi} \|G\|_{\operatorname{Lip}(\rho)} = c(\varphi, \rho). \quad (2.4)$$

Its proof is postponed to Section 3.

**Remark 2.2.** Let  $C_{\operatorname{Lip}(\rho),0}$  be the Banach space of all  $\rho$ -Lipschitzian functions  $g$  with  $\mu(g) = 0$  equipped with norm  $\|\cdot\|_{\operatorname{Lip}(\rho)}$ . Part (i) above says that the Poisson operator  $(-\mathcal{L}_2)^{-1} : C_{\operatorname{Lip}(\rho_1),0} \rightarrow C_{\operatorname{Lip}(\rho_2),0}$  is bounded and

$$\|(-\mathcal{L}_2)^{-1}\|_{C_{\operatorname{Lip}(\rho_1),0} \rightarrow C_{\operatorname{Lip}(\rho_2),0}} = c_{\operatorname{Lip}}(\rho_1, \rho_2). \quad (2.5)$$

Since  $\mathcal{L}_2$  is self-adjoint on  $L_0^2(I, \mu)$ , a general functional analysis result (see [25], Proposition 2.9) says that

$$\|(-\mathcal{L}_2)^{-1}\|_{L_0^2(I, \mu)} \leq \|(-\mathcal{L}_2)^{-1}\|_{C_{\operatorname{Lip}(\rho),0} \rightarrow C_{\operatorname{Lip}(\rho),0}}.$$

But the left-hand side is exactly the Poincaré constant  $c_P$ , so we get

$$c_P \leq \|(-\mathcal{L}_2)^{-1}\|_{C_{\operatorname{Lip}(\rho),0} \rightarrow C_{\operatorname{Lip}(\rho),0}} = c_{\operatorname{Lip}}(\rho, \rho)$$

which is exactly the ‘ $\leq$ ’ part in (1.6). We now outline the idea of Chen for the converse inequality in (1.6). If the eigenfunction  $\rho$  associated with  $\lambda_1 = 1/c_P$  exists, i.e.  $-\mathcal{L}_2 \rho = \lambda_1 \rho$ , it must be strictly monotone (see [9]) and then could be assumed to be increasing, and  $\rho'$  is given by (3.2) with  $C = 0$  and  $g = \lambda_1 \rho$  (see Section 3 for the reason why  $C = 0$ ), i.e.

$$\rho'(x) = \lambda_1 s'(x) \int_x^{y_0} [\rho(t) - \mu(\rho)] m'(t) \, dt, \quad dx\text{-a.s.},$$

where the ‘ $\geq$ ’ part in (1.6) follows. When  $\lambda_1$  has no eigenfunction, Chen proved the converse inequality by using a sequence of increasing functions  $\rho \in L_0^2(I, \mu)$  approximating this virtual eigenfunction.

That is our interpretation to Chen’s variational formula (1.6).

**Remark 2.3.** Let  $\|g\|_\varphi$  be the largest constant  $c$  such that  $|g(x)| \leq c\varphi(x)$  over  $I$  and  $b_\varphi \mathcal{B}$  be the Banach space of those measurable functions  $g$  such that its norm  $\|g\|_\varphi$  is finite. Let  $Pg = g - \mu(g) : L^2(I, \mu) \rightarrow L_0^2(I, \mu)$ , the orthogonal projection. Part (ii) above means that  $(-\mathcal{L})^{-1}P$  is bounded from  $b_\varphi \mathcal{B}$  to  $C_{\operatorname{Lip}(\rho),0}$  and its norm is exactly  $c(\varphi, \rho)$ .

## 2.2. Applications to transportation–information inequalities and concentration inequalities

For any probability measure  $\nu$  on  $I$ , say  $\nu \in \mathcal{M}_1(I)$ , the Wasserstein distance between  $\nu$  and  $\mu$  w.r.t. a given metric  $d$  on  $I$  is defined by

$$W_{1,d}(\nu, \mu) = \inf_{\pi} \iint_{I^2} d(x, y) \pi(dx, dy),$$

where  $\pi$  runs over all couplings of  $\nu, \mu$ , i.e. all probability measures  $\pi$  on  $I^2$  with the first and second marginal distributions  $\nu, \mu$ , respectively. When  $d$  is the trivial metric ( $d(x, y) = 1_{x \neq y}$ ),  $2W_{1,d}(\mu, \nu) = \|\mu - \nu\|_{\operatorname{TV}} := \sup_{|f| \leq 1} |(\mu - \nu)(f)|$ , the total variation of  $\mu - \nu$ .

Under (A1)–(A4), the Dirichlet form  $(\mathcal{E}, \mathbb{D}(\mathcal{E}))$  associated with the transition semigroup  $(P_t)$  of  $(X_t)$  is given by

$$\begin{aligned} \mathbb{D}(\mathcal{E}) &= \mathbb{D}(\sqrt{-\mathcal{L}_2}) = \left\{ f \in L^2(I, \mu) \cap \mathcal{AC}(I), \int_I a(x) f'(x)^2 d\mu(x) < +\infty \right\}, \\ \mathcal{E}(f, f) &:= \int_I a(x) f'(x)^2 d\mu(x), \quad f \in \mathbb{D}(\mathcal{E}). \end{aligned}$$

For  $f, g \in \mathbb{D}(\mathcal{E})$ , let  $\Gamma(f, g) = af'g'$  be the *carré-du-champs* operator. The Fisher–Donsker–Varadhan information of  $\nu$  w.r.t.  $\mu$  is defined by

$$I(\nu|\mu) = \begin{cases} \mathcal{E}\left(\sqrt{\frac{d\nu}{d\mu}}, \sqrt{\frac{d\nu}{d\mu}}\right), & \text{if } \nu \ll \mu \text{ and } \sqrt{\frac{d\nu}{d\mu}} \in \mathbb{D}(\mathcal{E}), \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.6)$$

Recall that for  $\rho_0(x) = \int_c^x \frac{1}{\sqrt{a(y)}} dy$  the associated metric  $d_{\rho_0}(x, y) = |\rho_0(y) - \rho_0(x)|$  is the intrinsic metric of the diffusion  $(X_t)$ .

**Corollary 2.4.** *Assume (A1)–(A4). Let  $\rho \in \mathcal{AC}(I) \cap L^2(I, \mu)$  so that  $\rho'(x) > 0$ , dx-a.e. and*

$$c_\rho = \operatorname{ess\,sup}_{x \in I} s'(x) \sqrt{a(x)} \int_x^{y_0} [\rho(y) - \mu(\rho)] m'(y) dy < +\infty. \quad (2.7)$$

Then for all  $\nu \in \mathcal{M}_1(I)$

$$(W_{1, d_\rho}(\nu, \mu))^2 \leq 4c_\rho^2 I(\nu|\mu), \quad (2.8)$$

or equivalently for every  $\rho$ -Lipschitzian function  $g$  on  $I$ , we have for any initial measure  $\nu \ll \mu$  and  $t, r > 0$ ,

$$\mathbb{P}_\nu \left( \frac{1}{t} \int_0^t g(X_s) ds > \mu(g) + r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(I, \mu)} \exp \left( - \frac{tr^2}{4c_\rho^2 \|g\|_{\operatorname{Lip}(\rho)}^2} \right). \quad (2.9)$$

**Proof.** Remark that  $c_\rho = c_{\operatorname{Lip}(\rho, \rho_0)}$ , the constant given in (2.1). The equivalence between the transportation–information inequality (2.8) and the Gaussian concentration inequality (2.9) is due to Guillin et al. [15], Theorem 2.4.

By Kantorovitch–Rubinstein dual equality, (2.8) is equivalent to: if  $\|g\|_{\operatorname{Lip}(\rho)} \leq 1$ ,

$$\left( \int_I g d(\nu - \mu) \right)^2 \leq 4c_\rho^2 I(\nu|\mu) \quad \forall \nu \in \mathcal{M}_1(I).$$

We may assume that  $I(\nu|\mu) < +\infty$ , i.e.,  $\nu = h^2\mu$  with  $h \in \mathbb{D}(\mathcal{E})$ . Let  $G$  be the solution of  $-\mathcal{L}_2 G = g - \mu(g)$  with  $\mu(G) = 0$  (its existence and uniqueness is assured by Theorem 2.1(i)). Notice that with  $f = h^2$  ( $h \geq 0$ ),

$$\begin{aligned} \int_I g d(\nu - \mu) &= \langle -\mathcal{L}_2 G, f \rangle = \mathcal{E}(G, f) = \int_I a(x) G'(x) f'(x) d\mu(x) \\ &\leq \operatorname{ess\,sup}_{x \in I} [\sqrt{a(x)} |G'(x)|] \int_I \sqrt{a(x)} |f'(x)| d\mu(x) \\ &\leq 2c_\rho \sqrt{\mu(h^2) \mu[ah'^2]} = 2c_\rho \sqrt{I(\nu|\mu)}, \end{aligned} \quad (2.10)$$

where the last inequality follows by Theorem 2.1(i) and Cauchy–Schwarz inequality, for  $\operatorname{ess\,sup}_{x \in I} \sqrt{a(x)} |G'(x)| = \|G\|_{\operatorname{Lip}(\rho_0)}$ .  $\square$

**Remark 2.5 (Proposed by the referee).** *If the observable  $g$  is fixed and absolutely continuous, the best choice of  $\rho$  for the Gaussian concentration inequality (2.9) is  $\tilde{\rho}$  such that  $\tilde{\rho}' = |g'|$  by Lemma 3.2 in Section 3 (though such  $\tilde{\rho}$  is not strictly increasing, but Theorem 2.1 is still valid as seen for its proof).*

**Remark 2.6.** The second inequality in (2.10) can be read as

$$W_{1,\rho}(f\mu, \mu) \leq c_\rho \int_I \sqrt{\Gamma(f, f)} \, d\mu \leq 2c_\rho \sqrt{I(f\mu|\mu)}.$$

Repeating the argument above but using part (ii) of Theorem 2.1, we get (2.11) below.

**Corollary 2.7.** Assume (A1)–(A4). Let  $0 \leq \varphi \in L^2(I, \mu)$  such that  $c(\varphi, \rho_0) < +\infty$ . Then for all  $\nu = f\mu \in \mathcal{M}_1(I)$ ,

$$\|\varphi(\nu - \mu)\|_{\text{TV}} \leq c(\varphi, \rho_0) \int_I \sqrt{\Gamma(f, f)} \, d\mu \leq 2c(\varphi, \rho_0) \sqrt{I(\nu|\mu)}. \tag{2.11}$$

Or equivalently for every  $g : I \rightarrow \mathbb{R}$  such that  $|g(x) - g(y)| \leq \beta_\varphi(x, y) := [\varphi(x) + \varphi(y)]1_{x \neq y}$  (i.e.,  $\|g\|_{\text{Lip}(\beta_\varphi)} \leq 1$ ), we have for any initial measure  $\nu \ll \mu$  and  $t, r > 0$ ,

$$\mathbb{P}_\nu \left( \frac{1}{t} \int_0^t g(X_s) \, ds > \mu(g) + r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(I, \mu)} \exp \left( -\frac{tr^2}{4c(\varphi, \rho_0)^2} \right). \tag{2.12}$$

The Gaussian concentration inequality (2.12) follows from (2.11) by [15], Theorem 2.4, and the fact that  $\|\varphi(\nu - \mu)\|_{\text{TV}} = \sup_{g: \|g\|_{\text{Lip}(\beta_\varphi)} \leq 1} \int_I g \, d(\nu - \mu)$  (cf. [14]). Notice that  $\beta_\varphi$  is a metric once  $\varphi$  is positive (and a pseudo-metric satisfying the triangular inequality in the general case).

**Remark 2.8.** When  $\varphi = 1$  in (2.11), the constant  $c(\varphi, \rho_0)$  becomes

$$c_\delta := 2 \operatorname{ess\,sup}_{x \in I} \sqrt{a(x)} s'(x) m(I) \mu(I_x^+) \mu(I_x^-), \tag{2.13}$$

where  $I_x^+, I_x^-$  are given in Theorem 2.1; and the inequality (2.11) becomes: for every  $\mu$ -probability density  $f \in \mathcal{AC}(I)$

$$\int_I |f - 1| \, d\mu \leq c_\delta \int_I \sqrt{\Gamma(f, f)} \, d\mu \leq 2c_\delta \sqrt{I(f\mu|\mu)}. \tag{2.14}$$

It was proved by Guillin et al. [15], Theorem 3.1, that if the Poincaré inequality holds, then

$$\int_I |f - 1| \, d\mu \leq \sqrt{2c_G I(\nu|\mu)}$$

with the best constant  $c_G \leq 2c_P$  (the index  $G$  is referred to the equivalent Gaussian concentration inequality); and conversely if the last inequality holds, then  $c_P \leq 2c_G$ .

**Remark 2.9.** Gozlan [13] has established some connections between Talagrand’s transportation–entropy inequalities and weighted Poincaré inequalities, see also [23].

The concentration inequalities (2.9) and (2.12) do not contain the asymptotic variance of  $g$ :

$$\sigma^2(g) := \lim_{t \rightarrow \infty} \frac{1}{t} \operatorname{Var}_{\mathbb{P}_\mu} \left( \int_0^t g(X_s) \, ds \right)$$

which plays a fundamental role in the central limit theorem (then in statistical applications). This is provided in the following Bernstein’s type concentration inequality.

**Corollary 2.10.** Assume (A1)–(A4). Suppose that the constant  $c_\delta$  in (2.13) is finite.

(i) If the constant  $c_\rho$  in (2.7) is finite, then for every  $\rho$ -Lipschitzian function  $g$  with  $\|g\|_{\text{Lip}(\rho)} \leq 1$ , we have for any initial measure  $\nu \ll \mu$  and  $t, x > 0$ ,

$$\mathbb{P}_\nu \left( \frac{1}{t} \int_0^t g(X_s) ds > \mu(g) + \sqrt{(2\sigma^2(g) + 4c_\rho^2 \min\{1, c_\delta \sqrt{x}\})x} \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(I, \mu)} e^{-tx}.$$

(ii) If the constant  $c(\varphi, \rho_0)$  in (2.3) is finite, then for every measurable function  $g$  such that  $|g(x) - g(y)| \leq \varphi(x) + \varphi(y)$ , the inequality in (i) holds with  $c_\rho$  replaced by  $c(\varphi, \rho_0)$ .

**Proof.** Our proof below follows [16].

(i) We may and will assume that  $\|g\|_{\text{Lip}(\rho)} \leq 1$ . Let  $G$  be the solution of  $-\mathcal{L}G = g - \mu(g)$ . Notice that  $\sigma^2(g) = 2\langle G, g \rangle_\mu = 2\mathcal{E}(G, G)$ .

We have for  $\nu = h^2\mu$  with  $I(\nu|\mu) < +\infty$ ,

$$\int_I g d(\nu - \mu) = 2 \int_I aG'hh' d\mu(x) \leq 2\sqrt{\int_I aG^2h^2 d\mu} \cdot I(\nu|\mu).$$

Since  $0 \leq aG^2 \leq \|G\|_{\text{Lip}(\rho_0)}^2 \leq c_\rho^2$  by Theorem 2.1(i), using the fact that  $\int_I F d(\nu - \mu) \leq \frac{1}{2}\|\nu - \mu\|_{\text{TV}}$  for  $F$  verifying  $|F(x) - F(y)| \leq 1$ , we have by (2.14),

$$\int_I aG^2h^2 d\mu \leq \int_I aG^2 d\mu + \frac{c_\rho^2}{2} \int_I |h^2 - 1| d\mu \leq \frac{\sigma^2(g)}{2} + c_\rho^2 \min\{1, c_\delta \sqrt{I(\nu|\mu)}\}.$$

Plugging it into the previous inequality (for  $\pm g$ ), we obtain

$$\left( \int_I g d(\nu - \mu) \right)^2 \leq (2\sigma^2(g) + 4c_\rho^2 \min\{1, c_\delta \sqrt{I(\nu|\mu)}\})I(\nu|\mu) \quad \forall \nu.$$

This is equivalent to the desired concentration inequality by [15], Theorem 2.4.

(ii) The same argument as above (but using part (ii) of Theorem 2.1 instead of part (i)), we have  $\forall g$  such that  $|g| \leq \varphi$

$$\left( \int_I g d(\nu - \mu) \right)^2 \leq (2\sigma^2(g) + 4c(\varphi, \rho_0)^2 \min\{1, c_\delta \sqrt{I(\nu|\mu)}\})I(\nu|\mu) \quad \forall \nu.$$

This leads to the desired concentration inequality again by [15], Theorem 2.4. □

### 2.3. $L^1$ -Poincaré inequality and Cheeger’s isoperimetric inequality

The Poincaré inequality has a  $L^1$  counterpart related to Cheeger’s isoperimetric inequality. Namely, let  $c_{P,1}$  be the best constant such that the following  $L^1$ -Poincaré inequality holds: for any  $f \in \mathcal{AC}(I) \cap L^1(I, \mu)$

$$\int_I |f - \mu(f)| d\mu \leq c_{P,1} \int_I \sqrt{a(x)} |f'| d\mu, \tag{2.15}$$

where  $\mathcal{AC}(I)$  is the space of all absolutely continuous functions on  $I$ . Theorem 2.1 allows us to identify the best constant  $c_{P,1}$  in the  $L^1$ -Poincaré inequality (2.15).

**Theorem 2.11.** Assume (A1)–(A4). The best constant  $c_{P,1}$  in the  $L^1$ -Poincaré inequality (2.15) is finite if and only if  $c_\delta$  given in (2.13) is finite. In this case  $c_{P,1} = c_\delta$ .

**Proof.** At first  $c_{P,1} \leq c_\delta$ , by (2.14) (the passage from  $\mu$ -density  $f$  to general  $f$  in (2.15) is easy). For the converse inequality, we may assume that  $c_{P,1} < +\infty$ . In that case for any  $g \in b\mathcal{B}$  such that  $|g| \leq 1$ ,  $G = (-\mathcal{L}_2)^{-1}(g - \mu(g))$  exists (because the Poincaré inequality holds by Cheeger's inequality). We have for any  $\mu$ -probability density  $f \in \mathcal{AC}(I)$ ,

$$\begin{aligned} \int_I a(x)G'(x)f'(x) d\mu(x) &= \langle -\mathcal{L}_2 G, f \rangle_\mu = \langle g, f - 1 \rangle_\mu \\ &\leq \int_I |f - 1| d\mu \leq c_{P,1} \int_I \sqrt{a(x)} |f'(x)| d\mu(x). \end{aligned}$$

That implies  $\|G\|_{\text{Lip}(\rho_0)} = \text{ess sup}_{x \in I} \sqrt{a(x)} |G'(x)| \leq c_{P,1}$  (however this elementary fact do no longer work in the multi-dimensional case). Hence  $c_\delta \leq c_{P,1}$  by Theorem 2.1(ii).  $\square$

Let us discuss now some connections between (2.15) and isoperimetric inequalities. Consider the intrinsic metric  $d_{\rho_0}$  associated with the diffusion where  $\rho_0(x) = \int_c^x \frac{1}{\sqrt{a(y)}} dy$ , and the corresponding isoperimetric function

$$I_\mu(p) := \inf\{\mu_\partial(\partial A); \mu(A) = p\}, \quad p \in (0, 1).$$

Here  $\partial A$  is the boundary of  $A$  and the surface measure  $\mu_s$  of  $A$  is defined by  $\mu_s(\partial A) = \liminf_{\varepsilon \rightarrow 0_+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}$  and  $A_\varepsilon = \{x \in I, \text{ such that } d_{\rho_0}(x, A) \leq \varepsilon\}$ , the  $\varepsilon$ -neighborhood of  $A$ .

**Remark 2.12.** Let  $c_{\text{cheeger}}$  be Cheeger's isoperimetric constant of  $\mu$  w.r.t. the intrinsic metric  $d_{\rho_0}$ , i.e. the best constant in the following Cheeger isoperimetric inequality

$$\min(\mu(A), 1 - \mu(A)) \leq c_{\text{cheeger}} \mu_s(\partial A)$$

for all measurable subsets  $A \subset I$ , or equivalently  $I_\mu(p) \geq \frac{1}{c_{\text{cheeger}}} \min\{p, 1 - p\}$ . It is well known (cf. [4,20]) that  $c_{\text{cheeger}}$  is also the best constant in the functional version of Cheeger's isoperimetric inequality below: for any  $f \in \mathcal{AC}(I) \cap L^1(I, \mu)$

$$\int_I |f - m_\mu(f)| d\mu \leq c_{\text{cheeger}} \int_I \sqrt{a(x)} |f'| d\mu, \quad (2.16)$$

where  $m_\mu(f)$  is a median of  $f$  w.r.t.  $\mu$  (via Co-Area formula). Since

$$\frac{1}{2} \mu(|f - \mu(f)|) \leq \mu(|f - m_\mu(f)|) \leq \mu(|f - \mu(f)|)$$

we have

$$\frac{1}{2} c_{P,1} \leq c_{\text{cheeger}} \leq c_{P,1}. \quad (2.17)$$

The two inequalities above are both sharp as seen for the examples later. An important result of Bobkov–Houdré [5], Theorem 1.3, says that

$$\begin{aligned} c_{\text{cheeger}} &= \text{ess sup}_{x \in I} \frac{m(I) \min\{\mu(I_x^+), \mu(I_x^-)\}}{m'(x) \sqrt{a(x)}} \\ &= \text{ess sup}_{x \in I} m(I) \sqrt{a(x)} s'(x) \min\{\mu(I_x^+), \mu(I_x^-)\} \end{aligned} \quad (2.18)$$

or say roughly, the extreme set for  $c_{\text{cheeger}}$  is a semi-interval  $I_x^+$ . In recent years, the best constant  $c_{\text{cheeger}}$  (in multi-dimensional case) has been extensively investigated, see [1,5,7,20,22,26] and relevant references therein.



**Remark 2.13 (Proposed by the referee).** By Bobkov–Houdré [4], Theorem 1.2, the  $L^1$ -Poincaré inequality (2.15) is equivalent to the following isoperimetric inequality associated with  $d_{\rho_0}$ :

$$2\mu(A)\mu(A^c) \leq c_{P,1}\mu_\delta(\partial A) \tag{2.19}$$

for any measurable subset  $A$  of  $I$ , or equivalently  $I_\mu(p) \geq \frac{2}{c_{P,1}}p(1-p)$ ,  $p \in (0, 1)$ . That equivalence holds on a general metric space.

Notice that if  $a(x)$  is continuous and positive,  $c_\delta$  is just the best constant in (2.19) (in place of  $c_{P,1}$ ) for  $A$  varying over  $I_x^+$ ,  $x \in I$ .

When  $a(x) = 1$  and  $\mu$  is log-concave (i.e.,  $\mu = f \, dx$  with  $\log f$  concave), Bobkov–Houdré [4], Corollary 13.8, showed that the optimal set for  $I_\mu(p)$  is  $I_x^+$  with  $\mu(I_x^+) = p$  for every  $p \in (0, 1)$ , and then  $c_{P,1} = c_\delta$ .

The referee indicates another approach for Theorem 2.11 even for general  $\mu$  not necessarily log-concave, when  $a(x) = 1$ . The idea goes as follows. At first notice that  $p(1-p)$  is the isoperimetric function  $I_\nu(p)$  of the logistic distribution  $\nu$ :  $\nu(-\infty, x] = (1 + e^{-x})^{-1}$ . Following the proof of Bobkov–Houdré [5], proof of Theorem 1.3, if  $c_\delta < +\infty$ , the increasing mapping  $U : \mathbb{R} \rightarrow I$  pushing forward  $\nu$  to  $\mu$  must be Lipschitzian and  $\|U\|_{\text{Lip}} = 2c_\delta$ . Then one sees that the best constant  $c_{P,1}$  in (2.19) for  $\mu$  is just  $\|U\|_{\text{Lip}}/2 = c_\delta$ .

Let us remark finally that the  $L^1$ -Poincaré inequality (2.15) is equivalent to the following concentration inequality ([4], Theorem 2.1, pp. 20–21):

$$\mu(A_\varepsilon) \geq \frac{p}{p + (1-p)\exp(-2\varepsilon/c_{P,1})}, \quad \mu(A) = p \in (0, 1), \varepsilon > 0.$$

#### 2.4. A qualitative description for the boundedness of the Poisson operator

For  $g \in L^2_0(I, \mu)$ , the solution  $G$  with  $\mu(G) = 0$  of the Poisson equation  $-\mathcal{L}_2G = g$ , if exists, will be denoted by  $(-\mathcal{L})^{-1}g$ . One may think naturally that when  $\varphi$  is bounded but tends to zero at the boundary  $\partial I$ , the Lipschitzian norm  $c(\varphi, \rho_0)$  may be finite even if  $c_\delta = +\infty$ . The same picture might appear in one’s mind for  $c_{\text{Lip}}(\rho, \rho_0)$  when  $\rho'$  tends to 0 at the boundary  $\partial I$ . However this is not the case.

**Proposition 2.14.** Assume (A1)–(A4). Let  $\rho, \varphi$  be as in Theorem 2.1, but moreover bounded and  $\varphi > 0$ . Let  $\rho_0(x) = \int_c^x \frac{1}{\sqrt{a(y)}} \, dy$ . Consider the following properties:

- (i)  $c_\rho = c_{\text{Lip}}(\rho, \rho_0) = \|(-\mathcal{L}_2)^{-1}\|_{C_{\text{Lip}}(\rho,0) \rightarrow C_{\text{Lip}}(\rho_0,0)} < +\infty$ .
- (ii)  $c(\varphi, \rho_0) = \sup_{g:|g|\leq\varphi} \|(-\mathcal{L}_2)^{-1}(g - \mu(g))\|_{\text{Lip}(\rho_0)} < +\infty$ .
- (iii)  $c_\delta = \sup_{|g|\leq 1} \|(-\mathcal{L}_2)^{-1}(g - \mu(g))\|_{\text{Lip}(\rho_0)} < +\infty$ .
- (iv) The  $L^1$ -Poincaré inequality (2.15) holds, i.e.,  $c_{P,1} < +\infty$ .
- (v) The transportation–information inequality below holds: there is some finite best constant  $c_G > 0$  such that for all  $\nu = f\mu \in \mathcal{M}_1(I)$ ,

$$\int_I |f - 1| \, d\mu \leq \sqrt{2c_G I(\nu|\mu)}.$$

- (vi) The Poincaré inequality (1.5) holds, i.e.,  $c_P < +\infty$ .

Then

- (a) the properties (i)–(iv) are equivalent.
- (b) (iv)  $\Rightarrow$  (v)  $\Leftrightarrow$  (vi).
- (c) If  $a(x) = 1$  and  $b' \leq K$  (i.e., the Bakry–Emery curvature is bounded from below by  $-K$ ), (vi)  $\Rightarrow$  (iv) and then (i)–(vi) are all equivalent.

**Proof.** (a) Equivalence between (i), (ii) and (iii). It is enough to regard the behavior at the boundary of the functions appearing in the definitions of  $c_\rho = c_{\text{Lip}}(\rho, \rho_0)$ ,  $c_\delta$  and  $c(\varphi, \rho_0)$ . For instance, if  $y_0 \notin I$ , for  $x$  close to  $y_0$ , say  $x \geq z > c$ ,

we have

$$\begin{aligned} (\rho(z) - \mu(\rho))\mu(I_x^+) &\leq \int_x^{y_0} (\rho(y) - \mu(\rho)) d\mu(y) \leq (\rho(y_0) - \mu(\rho))\mu(I_x^+), \\ \mu(I_z^-)\mu(I_x^+) &\leq \mu(I_x^+)\mu(I_x^-) \leq \mu(I_x^+), \\ \mu(1_{I_z^-}\varphi)\mu(I_x^+) &\leq \mu(I_x^+)\mu(1_{I_x^-}\varphi) + \mu(I_x^-)\mu(1_{I_x^+}\varphi) \leq 2\|\varphi\|_\infty\mu(I_x^+). \end{aligned}$$

Hence the supremums over  $[c, y_0]$  of the functions appearing in the definitions of  $c_\rho$ ,  $c_\delta$  and  $c(\varphi, \rho_0)$  are simultaneously finite or infinite. The same argument works when  $x_0 \notin I$ . That completes the proof of the equivalence between (i), (ii) and (iii).

(iii)  $\Leftrightarrow$  (iv). That is contained in Theorem 2.11:  $c_\delta = c_{P,1}$ .

(b) (iv)  $\Rightarrow$  (v). Since  $\int_I \sqrt{a(x)}|f'| d\mu \leq 2\sqrt{T(f\mu|\mu)}$ , we have  $c_G \leq 2c_{P,1}^2$ .

(v)  $\Leftrightarrow$  (vi). This is noticed in Remark 2.8:  $c_P/2 \leq c_G \leq 2c_P$ .

(c) (vi)  $\Rightarrow$  (iv). This converse of the Cheeger’s inequality is known in the actual lower bounded Bakry–Emery’s curvature case see Buser [7] and Ledoux [21], Theorem 5.2 (otherwise there are counter-examples).  $\square$

### 2.5. Several examples

**Example 2.15 (Gaussian measure).** Let  $I = \mathbb{R}$ ,  $a(x) = 1$  and  $b(x) = -x/\sigma^2$  where  $\sigma > 0$ . Then  $m'(x) = e^{-x^2/2\sigma^2}$  and  $\mu = \mathcal{N}(0, \sigma^2)$ , the centered Gaussian law with variance  $\sigma^2$ . For  $\rho_0(x) = x$ , we see that

$$c_{\text{Lip}}(\rho_0, \rho_0) = c_{\rho_0} = \sup_{x \in \mathbb{R}} e^{x^2/2\sigma^2} \int_x^\infty ye^{-y^2/2\sigma^2} dy = \sigma^2.$$

By Remark 2.2,  $c_P \leq c_{\rho_0} = \sigma^2$  which is in reality an equality as well known [20]. The transportation inequality (2.8) becomes equality for  $\nu = \mathcal{N}(m, \sigma^2)$ .

By calculus we identify the constant  $c_\delta$  in (2.13) as

$$c_\delta = 2 \sup_{x \in \mathbb{R}} e^{x^2/2\sigma^2} \sqrt{2\pi}\sigma \mu([x, +\infty))\mu((-\infty, x)) = \sqrt{\frac{\pi}{2}}\sigma.$$

On the other side,  $c_{\text{cheeger}} \geq \sqrt{\frac{\pi}{2}}\sigma$  as seen for  $A = \mathbb{R}^+$ . Then by (2.17) and Theorem 2.11,  $c_{\text{cheeger}} = c_\delta = c_{P,1}$ .

**Example 2.16 (Uniform distribution).** Let  $I = [-D/2, D/2]$  where  $D > 0$ ,  $a(x) = 1$  and  $b(x) = 0$ . The unique invariant probability measure  $\mu$  is the uniform measure on  $I$ . Since  $m'(x) = 1 = s'(x)$ , we have

$$c_{\rho_0} = c_{\text{Lip}}(\rho_0, \rho_0) = \sup_{x \in [-D/2, D/2]} \int_x^{D/2} y dy = \frac{D^2}{8}$$

and the constant  $c_\delta = c(\varphi, \rho_0)$  with  $\varphi = 1$  is given by

$$c_\delta = \sup_{x \in [-D/2, D/2]} 2D\mu([-D/2, x])\mu([x, D/2]) = \frac{D}{2}.$$

As  $c_{\text{cheeger}} \geq D/2$  (as seen for  $A = [0, D/2]$ ), we have  $c_{\text{cheeger}} = D/2 = c_\delta = c_{P,1}$  by (2.17) and Theorem 2.11.

**Example 2.17 (Exponential measure on  $\mathbb{R}^+$ ).** Let  $I = \mathbb{R}^+ = [0, +\infty)$ ,  $a(x) = 1$  and  $b(x) = -\lambda$  where  $\lambda > 0$ . Then  $m'(x) = e^{-\lambda x} = 1/s'(x)$ ,  $\rho_0(x) = x$  and  $\mu$  is the exponential distribution with parameter  $\lambda$ . It is easy to see that  $c_{\rho_0} = c_{\text{Lip}}(\rho_0, \rho_0) = +\infty$ : no spectral gap in the  $\rho_0$ -Lipschitzian norm. In fact the transportation–information inequality (2.8) is false for  $\rho = \rho_0$ . By Theorem 2.11

$$c_{P,1} = c_\delta = 2 \sup_{x \geq 0} \frac{1}{\lambda} e^{\lambda x} \mu(0, x)\mu(x, +\infty) = 2 \sup_{x \geq 0} \frac{1}{\lambda} \mu(0, x) = \frac{2}{\lambda}.$$

However  $c_{cheeger} = \frac{1}{\lambda}$  by Bobkov–Houdré [5], which together with the Gaussian measure above shows that the two inequalities in (2.17) are both sharp (as promised). We have also the transportation–information inequality (2.14), which is read as

$$\|v - \mu\|_{TV} \leq \frac{4}{\lambda} \sqrt{I(v|\mu)} \quad \forall v.$$

It is sharp. Indeed let  $v$  be the exponential law with parameter  $\tilde{\lambda} \in (0, \lambda)$ . We have  $I(v|\mu) = (\lambda - \tilde{\lambda})^2/4$ , and the right-hand side above is given by  $2(1 - x)$  where  $x = \tilde{\lambda}/\lambda$ . The left-hand side above is given by  $2(x^{x/(1-x)} - x^{1/(1-x)})$ . Then the inequality above for such  $v$  says

$$x^{x/(1-x)} - x^{1/(1-x)} \leq 1 - x, \quad 0 < x < 1,$$

which is sharp as  $x \rightarrow 0$ .

For this model it is well known that  $c_P = 4/\lambda^2$  [20]. The inequality above is same as provided by [15], Theorem 3.1 (from the Poincaré inequality).

**Example 2.18 (Log-concave measure on  $\mathbb{R}$ ).** Let  $I = \mathbb{R}$ ,  $a(x) = 1$  and  $b(x) = -V'(x)$  where  $V$  is  $C^2$ , strictly convex on  $\mathbb{R}$  such that  $V(0) = 0$  and  $\int_{\mathbb{R}} e^{-V} dx < +\infty$ . Then  $m'(x) = e^{-V(x)}$  and  $s'(x) = e^{V(x)}$  and  $\rho_0(x) = x$ . Let  $\rho(x) = V'(x)$ , which is  $\mu$ -integrable and  $\mu(\rho) = 0$ . We have

$$c_\rho = c_{Lip}(\rho, \rho_0) = \sup_{x \in \mathbb{R}} e^{V(x)} \int_x^{+\infty} V'(y) e^{-V(y)} dy = \sup_{x \geq 0} e^{V(x)} e^{-V(x)} = 1.$$

Thus assuming  $\int V'^2 e^{-V} dx < +\infty$ , we have the transportation–information inequality (2.8) and the Gaussian concentration inequality (2.9). For instance, for any  $g \in C^1(\mathbb{R})$  such that  $|g'| \leq V''$  we have for any initial measure  $\nu \ll \mu$  and  $t, r > 0$ ,

$$\mathbb{P}_\nu \left( \frac{1}{t} \int_0^t g(X_s) ds > \mu(g) + r \right) \leq \left\| \frac{d\nu}{d\mu} \right\|_{L^2(I, \mu)} \exp\left(-\frac{tr^2}{4}\right). \tag{2.20}$$

Furthermore, for any nonnegative  $\varphi \leq M(1 + |V'|)$ , it is easy to see that  $c(\varphi, \rho_0) < +\infty$ , then the transportation–information inequality (2.11) holds.

In comparison recall the Lyapunov function criterion in [15], Theorem 5.1, for (2.11): for some  $0 \leq U \in C^2$ ,  $-U'' + V'U' + |U'|^2 \geq c\varphi^2 - K$  for some two positive constants  $c, K$  (which does not require the convexity of  $V$ ).

It will be very interesting to generalize it to log-concave measures on multi-dimensional spaces  $\mathbb{R}^d$ . See Bobkov–Ledoux [6] for some results in this direction.

**Example 2.19 (Jacobi diffusion).** Let  $I = ]0, 1[$ ,  $a(x) = x(1 - x)$  and  $b(x) = -x + 1/2$ , then  $\mu(x) = 1/(\pi\sqrt{x(1-x)})$  [10]. For  $\rho_0(x) = \frac{\pi}{2} + \text{Arcsin}(2x - 1)$ , we see that

$$c_{Lip}(\rho_0, \rho_0) = c_{\rho_0} = \sup_{x \in ]0, 1[} \left( \frac{\pi^2}{8} - \frac{1}{2} \text{Arcsin}^2(2x - 1) \right) = \frac{\pi^2}{8}.$$

By calculus we identify the constant  $c_\delta$  in (2.13) as

$$c_\delta = \frac{2}{\pi} \sup_{x \in ]0, 1[} \left( \frac{\pi^2}{4} - \text{Arcsin}^2(2x - 1) \right) = \frac{\pi}{2}.$$

Using (2.18), see Bobkov–Houdré [5], we obtain  $c_{cheeger} = \frac{\pi}{2}$ , so we have  $c_{cheeger} = c_{P,1} = c_\delta = \frac{\pi}{2}$  by Theorem 2.11.

**Example 2.20 (Continuous branching process).** Let  $I = ]0, +\infty[$ ,  $a(x) = 2x$  and  $b(x) = -2x + 1$ , then  $\mu(x) = \frac{1}{\sqrt{\pi}} \frac{e^{-x}}{\sqrt{x}}$ . This process arise as diffusion limits of discrete space branching process, see [19]. For  $\rho_0(x) = \sqrt{2x}$ , we see that

$$c_{\text{Lip}}(\rho_0, \rho_0) = c_{\rho_0} = \sup_{x \in \mathbb{R}^+} \left( 1 - \frac{e^x}{\sqrt{\pi}} \int_x^\infty \frac{e^{-y}}{\sqrt{y}} dy \right) = 1.$$

**Example 2.21.** See Example 1.4.2 in [24]. Let  $I = \mathbb{R}^+$ ,  $a(x) = (1+x)^\alpha$  with  $\alpha > 1$ , and  $b(x) = 0$ , then  $\mu(x) = \frac{\alpha-1}{(1+x)^\alpha}$ . For  $\alpha > 2$  and  $\rho_0(x) = \frac{2}{\alpha-2}(1 - (1+x)^{-(\alpha-2)/2})$ , we see that

$$c_{\text{Lip}}(\rho_0, \rho_0) = c_{\rho_0} = \frac{4}{(\alpha-2)(3\alpha-4)} \sup_{x \in \mathbb{R}^+} (1+x)^{-(\alpha-2)/2} (1 - (1+x)^{-(\alpha-2)/2}) = \frac{1}{(\alpha-2)(3\alpha-4)}.$$

By calculus we identify the constant  $c_\delta$  in (2.13) as

$$c_\delta = \frac{2}{\alpha-1} \sup_{x \in \mathbb{R}^+} (1+x)^{-(\alpha-2)/2} (1 - (1+x)^{-(\alpha-1)}) = \frac{4}{3\alpha-4} \left( \frac{\alpha-2}{3\alpha-4} \right)^{(\alpha-2)/(2(\alpha-1))}.$$

By Theorem 2.11, we have  $c_{P,1} = c_\delta$ . However, using (2.18), we obtain  $c_{\text{cheeger}} = \frac{1}{\alpha-1} \left( \frac{1}{2} \right)^{(\alpha-2)/(2(\alpha-1))}$ .

### 3. Proof of Theorem 2.1

#### 3.1. Several lemmas

Let  $\mathcal{L}^*$  be the adjoint operator of  $(\mathcal{L}, \mathcal{D})$  in  $L^2(I, m)$ , more precisely a function  $f$  in  $L^2(I, m)$  belongs to the domain of definition  $\mathbb{D}_2(\mathcal{L}^*)$  of  $\mathcal{L}^*$  if there is  $g \in L^2(I, m)$  such that  $\langle f, \mathcal{L}h \rangle_m = \langle g, h \rangle_m$  for all  $h \in \mathcal{D}$ , in such case  $\mathcal{L}^* f = g$ . Here  $\langle \cdot, \cdot \rangle_m$  is the inner product on  $L^2(I, m)$ .

We want to understand the Poisson equation (1.3) as an ordinary differential equation. That is the purpose of the following lemma.

**Lemma 3.1.** Assume (A1) and (A2). For a given  $f \in L^2(I, m)$ ,  $f \in \mathbb{D}_2(\mathcal{L}^*)$  if and only if

- (i)  $f$  admits a dx-version  $\tilde{f}$  such that  $\tilde{f} \in C^1(I)$ ,  $\tilde{f}'|_{\partial I} = 0$ , and  $\tilde{f}' \in \mathcal{AC}(I)$ ;
- (ii)  $a\tilde{f}'' + b\tilde{f}' \in L^2(I, m)$ .

In that case  $\mathcal{L}^* f = a\tilde{f}'' + b\tilde{f}'$ .

**Proof.** This follows by integration by parts argument and the distribution theory, as in [12], Appendix C, Theorem 2.7, or [27], Lemma 4.5. So we omit the details.  $\square$

Since  $\mathcal{L}_2$  is an extension of  $(\mathcal{L}, \mathcal{D})$ , then  $\mathcal{L}^*$  is an extension of  $\mathcal{L}_2^* = \mathcal{L}_2$  (because the generator of a symmetric strongly continuous semigroup is always self-adjoint). Of course under (A4),  $\mathcal{L}^* = \mathcal{L}_2$ . Then in our framework (i.e., (A1)–(A4) are satisfied), solving the Poisson equation (1.3) is equivalent to check  $G \in C^1(I) \cap L_0^2(I, \mu)$  such that  $G' \in \mathcal{AC}(I)$  and  $G'|_{\partial I} = 0$  and

$$-(aG'' + bG') = g. \tag{3.1}$$

This is a first-order differential equation for  $G'$ . It can be easily solved as

$$G'(x) = s'(x) \left[ C + \int_x^{y_0} g(y)m'(y) dy \right] \tag{3.2}$$

for some constant  $C$  (to be determined).

**Lemma 3.2.** Let  $\rho$  be as in Theorem 2.1 and  $g : I \rightarrow \mathbb{R}$  be  $\rho$ -Lipschitzian with  $\mu(g) = 0$ . Then for all  $x \in I$ ,

$$\int_x^{y_0} g(t)m'(t) dt \leq \|g\|_{\text{Lip}(\rho)} \int_x^{y_0} [\rho(t) - \mu(\rho)]m'(t) dt.$$

**Proof.** Without loss of generality, we may suppose that  $\|g\|_{\text{Lip}(\rho)} = 1$  and  $m(I) = 1$  and  $\mu(\rho) = 0$ . Letting  $m(x) := \int_{x_0}^x m'(t) dt$  and  $\tilde{g} = g \circ m^{-1}$ ,  $\tilde{\rho} = \rho \circ m^{-1}$ , we have

$$\int_x^{y_0} g(t)m'(t) dt = \int_{m(x)}^1 \tilde{g}(u) du \quad \text{and} \quad \int_x^{y_0} \rho(t)m'(t) dt = \int_{m(x)}^1 \tilde{\rho}(u) du.$$

As  $\|\tilde{g}\|_{\text{Lip}(\tilde{\rho})} = \|g\|_{\text{Lip}(\rho)}$ , we have only to prove that for all  $x \in [0, 1]$

$$h(x) = \int_x^1 \tilde{\rho}(s) ds - \int_x^1 \tilde{g}(s) ds \geq 0.$$

Since  $h(0) = h(1) = 0$  and  $h'$  is absolutely continuous on  $[0, 1]$  and for  $dx$ -a.s.  $x \in [0, 1]$

$$h''(x) = -\tilde{\rho}'(x) + \tilde{g}'(x) \leq \|\tilde{g}\|_{\text{Lip}(\tilde{\rho})}\tilde{\rho}'(x) - \tilde{\rho}'(x) \leq 0.$$

So  $h$  is concave on  $[0, 1]$ . Consequently  $h(x) \geq 0$  for all  $x \in [0, 1]$ . □

**Lemma 3.3.** Let  $0 \leq \varphi \in L^2(I, \mu)$ . Then for every  $x \in I$ ,

$$\sup_{g: |g| \leq \varphi} \int_x^{y_0} [g(t) - \mu(g)]m'(t) dt = m(I) \left( \mu(I_x^+) \int_{I_x^-} \varphi d\mu + \mu(I_x^-) \int_{I_x^+} \varphi d\mu \right),$$

where  $I_x^+ = [x, y_0] \cap I$ ,  $I_x^- = [x_0, x] \cap I$ . The supremum is attained for  $g = 1_{I_x^+} \varphi - 1_{I_x^-} \varphi$ .

**Proof.** We may assume that  $m(I) = 1$  and then  $\mu = m$ . Fix  $x \in I$ . The functional  $\Phi(g) = \int_x^{y_0} [g(t) - \mu(g)]m'(t) dt = \text{Cov}_\mu(g, 1_{I_x^+})$  (the covariance of  $g$  and  $1_{I_x^+}$  under  $\mu$ ) is a linear functional of  $g$ . Since the closed convex hull of  $\{1_A \varphi, A \in \mathcal{B}\}$  ( $\mathcal{B}$  is the Borel  $\sigma$ -field on  $I$ , and the closure is to be understood in  $L^2(I, \mu)$ ) is  $\{g; 0 \leq g \leq \varphi\}$ , and  $\{g; |g| \leq \varphi\} = \{h_1 - h_2; 0 \leq h_1, h_2 \leq \varphi\}$ , then

$$\sup_{g: |g| \leq \varphi} \Phi(g) = \sup_{h_1: 0 \leq h_1 \leq \varphi} \Phi(h_1) - \inf_{0 \leq h_2 \leq \varphi} \Phi(h_2) = \sup_{A \in \mathcal{B}} \text{Cov}_\mu(1_A \varphi, 1_{I_x^+}) + \sup_{A \in \mathcal{B}} \text{Cov}_\mu(-1_A \varphi, 1_{I_x^+}).$$

We examine the first supremum at the right-hand side. Note that

$$\text{Cov}_\mu(1_A \varphi, 1_{I_x^+}) = \int_{A \cap I_x^+} \varphi d\mu - \mu(I_x^+) \int_A \varphi d\mu.$$

With  $A \cap I_x^+ = B$  fixed, this functional of  $A$  attains the maximum when  $A$  becomes the smallest  $B$ . Next for  $A = B$  or equivalently  $A \subset I_x^+$ , the right-hand side above equals to

$$\int_A \varphi d\mu (1 - \mu(I_x^+))$$

which attains the maximum if  $A = I_x^+$ . So we have proven that

$$\max_{A \in \mathcal{B}} \text{Cov}_\mu(1_A \varphi, 1_{I_x^+}) = \int_{I_x^+} \varphi d\mu (1 - \mu(I_x^+)) = \mu(I_x^-) \int_{I_x^+} \varphi d\mu.$$

Now we turn to the last supremum. Note  $\text{Cov}_\mu(-1_A\varphi, 1_{I_x^+}) = -\int_{A\cap I_x^+} \varphi \, d\mu + \mu(I_x^+) \int_A \varphi \, d\mu$  and

$$\begin{aligned} \max_{A\in\mathcal{B}} \left( -\int_{A\cap I_x^+} \varphi \, d\mu + \mu(I_x^+) \int_A \varphi \, d\mu \right) &= \max_{B\subset I_x^+} \max_{A: A\cap I_x^+=B} \left( -\int_{A\cap I_x^+} \varphi \, d\mu + \mu(I_x^+) \int_A \varphi \, d\mu \right) \\ &= \max_{B\subset I_x^+} \left( -\int_B \varphi \, d\mu + \mu(I_x^+) \int_{B\cup I_x^-} \varphi \, d\mu \right) \\ &= \max_{B\subset I_x^+} \left( -\mu(I_x^-) \int_B \varphi \, d\mu + \mu(I_x^+) \int_{I_x^-} \varphi \, d\mu \right). \end{aligned}$$

The last functional in  $B \subset I_x^+$  attains the maximum if  $B$  is the smallest empty set. Thus

$$\max_{A\in\mathcal{B}} \left( -\int_{A\cap I_x^+} \varphi \, d\mu + \mu(I_x^+) \int_A \varphi \, d\mu \right) = \mu(I_x^+) \int_{I_x^-} \varphi \, d\mu.$$

Summarizing the conclusions in two cases, we obtain the desired result. □

### 3.2. Proof of Theorem 2.1(i)

We separate its proof into three cases:  $y_0 \in I$ ,  $x_0 \in I$  or  $I = (x_0, y_0)$ :

*Case 1.*  $y_0 \in I$ . Let  $g$  be  $\rho_1$ -Lipschitzian such that  $\mu(g) = 0$ . By Lemma 3.1, if  $G$  is a solution of the Poisson equation (1.3),  $G \in C^1(I)$ ,  $G' \in \mathcal{AC}(I)$ , and  $G'$  is given by (3.2). Since  $G'(y_0) = 0$ , the constant  $C$  there must be zero. Now applying Lemma 3.2, we get

$$|G'(x)| \leq \|g\|_{\text{Lip}(\rho_1)} s'(x) \int_x^{y_0} [\rho_1(t) - \mu(\rho_1)] m'(t) \, dt \leq \|g\|_{\text{Lip}(\rho_1)} c_{\text{Lip}(\rho_1, \rho_2)} \rho_2'(x)$$

for dx-a.e.  $x \in I$ . This yields to (2.2).

We turn to prove the existence of solution to the Poisson equation (1.3). Let  $G$  be a primitive of

$$G'(x) = s'(x) \int_x^{y_0} g(y) m'(y) \, dy.$$

By what shown above,  $\|G\|_{\text{Lip}(\rho_2)} \leq c_{\text{Lip}(\rho_1, \rho_2)} \|g\|_{\text{Lip}(\rho_1)} < \infty$ , then  $G \in L^2(I, \mu)$  (for  $\rho_2 \in L^2(I, \mu)$ ). By Lemma 3.1 and (A4),  $G \in \mathbb{D}_2(\mathcal{L}^*) = \mathbb{D}(\mathcal{L}_2)$ . Hence  $G$  is a solution of (1.3).

Finally, for  $g = \rho_1 - \mu(\rho_1)$ , we see that  $G'(x) = s'(x) \int_x^{y_0} [\rho_1(y) - \mu(\rho_1)] m'(y) \, dy$ . Then (2.2) becomes equality for that  $g$ .

*Case 2.*  $x_0 \in I$ . Parallel to the case 1, for  $G'(x)$  is again given by (3.2) with  $C = 0$ .

*Case 3.*  $I = (x_0, y_0)$ . By the proof in case 1, we have only to show that for any solution  $G$  of (1.3),  $G'$  is given by (3.2) with  $C = 0$ .

Assume in contrary that  $C \neq 0$  in (3.2). Let  $G_0$  be a fixed primitive of  $s'(x) \int_x^{y_0} g(y) m'(y) \, dy$ . As shown above  $\|G_0\|_{\text{Lip}(\rho_2)} \leq c_{\text{Lip}(\rho_1, \rho_2)} \|g\|_{\text{Lip}(\rho_1)} < +\infty$ , then  $G_0 \in L^2(I, \mu)$  (for  $\rho_2 \in L^2(I, \mu)$ ). Therefore for some constant  $K$ ,

$$G = Cs + G_0 + K.$$

But  $s \notin L^2(I, \mu)$  by (A4), then  $G \notin L^2(I, \mu)$ , contrary to the assumption that  $G \in \mathbb{D}(\mathcal{L}_2) \subset L^2(I, \mu)$ . Thus  $C = 0$  as desired.

### 3.3. Proof of Theorem 2.1(ii)

At first notice that by (3.2), if  $-\mathcal{L}_2 G = g - \mu(g)$ , then  $G \in C^1(I)$ ,  $G' \in \mathcal{AC}(I)$  and

$$G'(x) = s'(x) \left[ C + \int_x^{y_0} [g(y) - \mu(g)] m'(y) \, dy \right]. \tag{3.3}$$

We denote by  $G'_0$  the function above when  $C = 0$ . We separate its proof into the three cases as in the proof of part (i):

*Case 1.*  $y_0 \in I$ . Fix the measurable function  $g$  on  $I$  such that  $|g| \leq \varphi$ . If  $G$  is a solution of  $-\mathcal{L}_2 G = g - \mu(g)$ , as  $G'(y_0) = 0$ ,  $C = 0$  in (3.3), i.e.,  $G' = G'_0$ . By Lemma 3.3, we have for dx-a.e.  $x \in I$ ,

$$|G'(x)| = |G'_0(x)| \leq s'(x)m(I) \left( \mu(I_x^+) \int_{I_x^-} \varphi d\mu + \mu(I_x^-) \int_{I_x^+} \varphi d\mu \right) \leq c(\varphi, \rho)\rho'(x)$$

which gives us  $\|G\|_{\text{Lip}(\rho)} \leq c(\varphi, \rho)$ . Moreover, any primitive  $G_0$  of  $G'_0$  satisfies  $\|G_0\|_{\text{Lip}(\rho)} \leq c(\varphi, \rho)$ , then  $G_0 \in L^2(I, \mu)$  (for  $\rho \in L^2(I, \mu)$ ). By Lemma 3.1 and (A4),  $G_0$  is a solution of  $-\mathcal{L}_2 G = g - \mu(g)$ .

Finally the supremum of  $\|G\|_{\text{Lip}(\rho)}$  over  $\{g; |g| \leq \varphi\}$  equals to  $c(\varphi, \rho)$ , by Lemma 3.3.

*Case 2.*  $x_0 \in I$ . Same as the proof of case 1.

*Case 3.*  $x_0, y_0 \notin I$ . As in the proof of case 3 in part (i), we have  $G'$  is given by (3.3) with  $C = 0$ . Now one can repeat the proof of case 1 to conclude.

**Remark 3.4.** For some partial extensions of the results here to multi-dimensional Riemannian manifolds case, see the second named author [26].

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