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Disorder relevance at marginality and critical point shift

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Abstract. Recently the renormalization group predictions on the effect of disorder on pinning models have been put on mathematical grounds. The picture is particularly complete if the disorder is *relevant* or *irrelevant* in the Harris criterion sense: the question addressed is whether quenched disorder leads to a critical behavior which is different from the one observed in the pure, i.e. annealed, system. The Harris criterion prediction is based on the sign of the specific heat exponent of the pure system, but it yields no prediction in the case of vanishing exponent. This case is called *marginal*, and the physical literature is divided on what one should observe for marginal disorder, notably there is no agreement on whether a small amount of disorder leads or not to a difference between the critical point of the quenched system and the one for the pure system. In [*Comm. Pure Appl. Math.* **63** (2010) 233–265] we have proven that the two critical points differ at marginality of at least $\exp(-c/\beta^4)$, where $c > 0$ and β^2 is the disorder variance, for $\beta \in (0, 1)$ and Gaussian IID disorder. The purpose of this paper is to improve such a result: we establish in particular that the $\exp(-c/\beta^4)$ lower bound on the shift can be replaced by $\exp(-c(b)/\beta^b)$, $c(b) > 0$ for $b > 2$ ($b = 2$ is the known upper bound and it is the result claimed in [*J. Stat. Phys.* **66** (1992) 1189–1213]), and we deal with very general distribution of the IID disorder variables. The proof relies on coarse graining estimates and on a fractional moment change of measure argument based on multi-body potential modifications of the law of the disorder.

Résumé. Récemment, les prédictions issues des méthodes de groupe de renormalisation concernant l'influence du désordre pour les modèles d'accrochage ont été rendus rigoureuses mathématiquement. La description du phénomène est particulièrement complète dans le cas où le désordre est *pertinent* ou *non-pertinent* au sens du critère de Harris: on étudie si le désordre gelé engendre un comportement critique différent de celui que l'on observe pour le système pur, i.e. moyenné. Le critère de Harris se base sur le signe de l'exposant de la chaleur spécifique du système pur pour déterminer l'influence du désordre, mais ne prédit rien dans le cas où cet exposant vaut zéro. Ce cas est dit *marginal* et il n'y a pas de consensus dans la littérature physique sur ce que l'on devrait observer pour le système désordonné marginal; en particulier, il y a une controverse pour déterminer si un désordre de faible amplitude engendre ou non un déplacement du point critique du système avec désordre gelé par rapport à celui du système pur. Dans [*Comm. Pure Appl. Math.* **63** (2010) 233–265], nous avons démontré que, dans le cas marginal, la différence entre les deux points critiques est au moins d'ordre $\exp(-c/\beta^4)$, où $c > 0$ et β^2 est la variance du désordre, pour $\beta \in (0, 1)$ dans le cas d'un désordre gaussien IID. L'objectif de cet article est d'améliorer le résultat précédent: en particulier nous montrons que la borne inférieure $\exp(-c/\beta^4)$ pour le déplacement du point critique peut être remplacée par $\exp(-c(b)/\beta^b)$, $c(b) > 0$ pour tout $b > 2$ ($b = 2$ est la borne supérieure connue, et le résultat prédit dans [*J. Stat. Phys.* **66** (1992) 1189–1213]), et nous généralisons la preuve à des désordres IID très généraux. La démonstration s'appuie sur des estimées obtenues par *coarse graining*, et sur l'estimation de moments non-entiers de la fonction de partition, en modifiant la loi du désordre en y appliquant un potentiel multicorps.

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Keywords: Disordered pinning models; Harris criterion; Marginal disorder; Many-body interactions

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1. Introduction

1.1. *Relevant*, *irrelevant and marginal disorder*

The renormalization group approach to disordered statistical mechanics systems introduces a very interesting viewpoint on the role of disorder and on whether or not the critical behavior of a quenched system coincides with the critical behavior of the corresponding *pure* system. The Harris criterion [17] is based on such an approach and it may be summarized in the following way: if the specific heat exponent of the pure system is negative, then a *small* amount of disorder does not modify the critical properties of the pure system (*irrelevant disorder regime*), but if the specific heat exponent of the pure system is positive then even an arbitrarily small amount of disorder may lead to a quenched critical behavior different from the critical behavior of the pure system.

A class of disordered models on which such ideas have been applied by several authors is the one of pinning models (see, e.g., [8,11] and the extensive bibliography in [12,14]). The reason is in part due to the remarkable fact that pure pinning models are exactly solvable models for which, by tuning a parameter, one can explore all possible values of the specific heat exponent [10]. As a matter of fact, the validity of the Harris criterion for pinning models in the physical literature finds a rather general agreement. Moreover, for the pinning models the renormalization group approach goes beyond the critical properties and yields a prediction also on the location of the critical point.

Recently, the Harris criterion predictions for pinning models have been put on firm grounds in a series of papers [1,3,7,19] and some of these rigorous results go even beyond the predictions. Notably in [15] it has been shown that disorder has a smoothing effect in this class of models (a fact that is not a consequence of the Harris criterion and that did not find unanimous agreement in the previous physical literature).

However, a substantial amount of the literature on disordered pinning and Harris criterion revolves around a specific issue: what happens if the specific heat exponent is zero (i.e., at *marginality*)? This is really a controversial issue in the physical literature, started by the disagreement in the conclusions of [11] and [8]. In a nutshell, the disagreement lies on the fact that the authors of [11] predict that disorder is irrelevant at marginality and, notably, that quenched and annealed critical points coincide at small disorder, while the authors of [8] claim that disorder is relevant for arbitrarily small disorder, leading to a critical point shift of the order of $\exp(-c\beta^{-2})$ ($c > 0$) for $\beta \searrow 0$ (β^2 is the disorder variance).

Recently we have been able to prove that, at marginality, there is a shift of the critical point induced by the presence of disorder [14], at least for Gaussian disorder. We have actually proven that the shift is at least $\exp(-c\beta^{-4})$. The purpose of the present work is to go beyond [14] in three aspects:

(1) We want to deal with rather general disorder variables: we are going to assume only that the exponential moments are finite.
(2) We are going to improve the bound $\exp(-c\beta^{-b})$, $b = 4$, on the critical point shift, to $b = 2 + \varepsilon$ ($\varepsilon > 0$ arbitrarily small, and $c = c(b)$).
(3) We will prove our results for a wide class of pinning models. Pinning models are based on discrete renewal processes, characterized by an inter-arrival distribution which has power-law decay (the exponent in the power law parametrizes the model and varying such parameter one explores the different types of critical behaviors we mentioned before). The general pinning model is obtained by relaxing the power law decay to regularly varying decay, that is (in particular) we allow *logarithmic correction* to power-law decay. This, in a sense, allows zooming into the marginal case and makes clearer the interplay between the underlying renewal and the disorder variables.

1.2. *The framework and some basic facts*

In mathematical terms, disordered pinning models are one-dimensional Gibbs measures with random one-body potentials and reference measure given by the law of a renewal process. Namely, pinning models are built starting from a (non-delayed, discrete) renewal process $\tau = \{\tau_n\}_{n=0,1,\ldots}$, that is a sequence of random variables such that $\tau_0 = 0$ and $\{\tau_{j+1} - \tau_j\}_{j=0,1,\ldots}$ are independent and identically distributed with common law (called *inter-arrival distribution*) concentrated on $\mathbb{N} := \{1, 2, \ldots\}$ (the law of τ is denoted by $\mathbf{P}$): we will actually assume that such a distribution is regularly varying of exponent $1 + \alpha$, i.e.

$$K(n) := \mathbf{P}(\tau_1 = n) = \frac{L(n)}{n^{1+\alpha}} \quad \text{for } n = 1, 2, \ldots, \tag{1.1}$$

where $\alpha \geq 0$ and $L(\cdot)$ is a slowly varying function, that is $L:(0,\infty) \to (0,\infty)$ is measurable and it satisfies $\lim_{x\to\infty} L(cx)/L(x) = 1$ for every $c > 0$. There is actually no loss of generality in assuming $L(\cdot)$ smooth and we will do so (we refer to [4] for properties of slowly varying functions).

Remark 1.1. *Examples of slowly varying functions include* logarithmic slowly varying functions (*this is probably not a standard terminology, but it will come handy*), *that is the positive measurable functions that behave like* $\mathtt{a}(\log(x))^{\mathtt{b}}$ *as* $x \to \infty$, *with* $\mathtt{a} > 0$ *and* $\mathtt{b} \in \mathbb{R}$. *These functions are just a particular class of slowly varying functions, but it is already rich enough to appreciate the results we are going to present. Moreover, we will say that* $L(\cdot)$ *is trivial if* $\lim_{x\to\infty} L(x) = c \in (0,\infty)$. *The general statements about slowly varying function that we are going to use can be verified in an elementary way for logarithmic slowly varying functions*; *readers who feel uneasy with the general theory may safely focus on this restricted class.*

Without loss of generality we assume that $\sum_{n\in\mathbb{N}} K(n) = 1$ (actually, we have implicitly done so when we have introduced τ). This does not look at all like an innocuous assumption at first, because it means that τ is *persistent*, namely $\tau_j < \infty$ for every j, while if $\sum_n K(n) < 1$ then τ is *terminating*, that is $|\{j\colon \tau_j < \infty\}| < \infty$ a.s. It is however really a harmless assumption, as explained in detail in [12], Chapter 1, and recalled in the caption of Fig. 1.

The disordered potentials are introduced by means of the IID sequence $\{\omega_n\}_{n=1,2,\ldots}$ of random variables (the *charges*) such that $\mathrm{M}(t) := \mathbb{E}[\exp(t\omega_1)] < \infty$ for every t. Without loss of generality we may and do assume that $\mathbb{E}[\omega_1] = 0$ and $\mathrm{var}_{\mathbb{P}}(\omega_1) = 1$.

The model we are going to focus on is defined by the sequence of probability measures $\mathbf{P}_{N,\omega,\beta,h} = \mathbf{P}_{N,\omega}$, indexed by $N \in \mathbb{N}$, defined by

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(\tau) := \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} \big(\beta\omega_n + h - \log \mathrm{M}(\beta)\big)\delta_n\right)\delta_N, \tag{1.2}$$

where $\beta \geq 0$, $h \in \mathbb{R}$, δ_n is the indicator function that $n = \tau_j$ for some j and $Z_{N,\omega}$ is the partition function, that is the normalization constant. It is practical to look at τ as a random subset of $\{0\} \cup \mathbb{N}$, so that, for example, $\delta_n = \mathbb{1}_{n\in\tau}$.

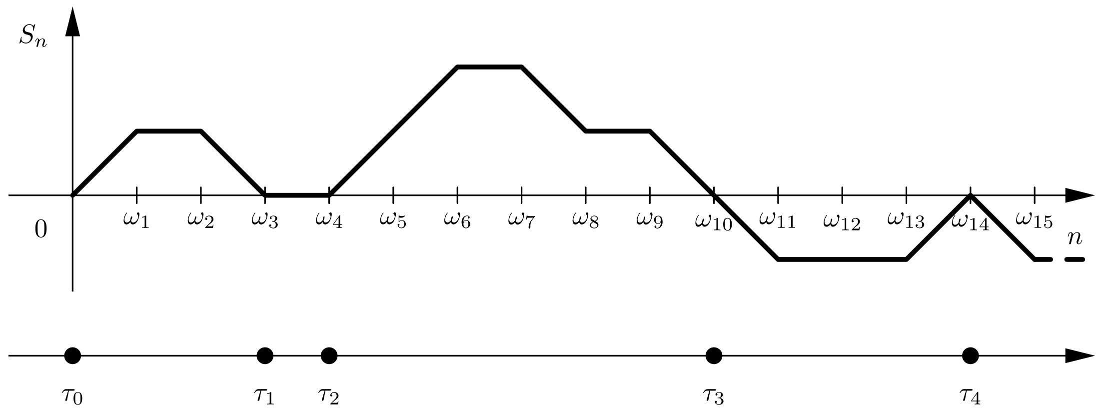

Fig. 1. A symmetric random walk trajectory with increments taking values in $\{-1, 0, +1\}$ is represented as a directed random walk. On the x-axis, the *defect line*, there are quenched charges ω that are collected by the walk when it hits the charge location. The energy of a trajectory just depends on the underlying renewal process τ. For the case in the figure, $K(n) := \mathbf{P}(\tau_1 = n) \sim \mathit{const}\, n^{-3/2}$ for $n \to \infty$ (e.g., [12], Appendix A.6). Moreover, the walk is recurrent, so $\sum_n K(n) = 1$. There is however another interpretation of the model: the charges may be thought of as sticking to S, not viewed this time as a directed walk. If the walk hits the origin at time n, the energy is incremented by $(\beta\omega_n + h - \log \mathrm{M}(\beta))$. This interpretation is particularly interesting for a three-dimensional symmetric walk in $\mathbb{Z}^3$: the walk may be interpreted as a polymer in $d = 3$, carrying charges on each monomer, and the monomers interact with a point in space (the origin) via a charge-dependent potential. Also in this case $K(n) \sim \mathit{const}\, n^{-3/2}$, but the walk is transient so that $\sum_n K(n) < 1$ (e.g., [12], Appendix A.6). It is rather easy to see that any model based on a terminating renewal with inter-arrival distribution $K(\cdot)$ can be mapped to a model based on the persistent renewal with inter-arrival distribution $K(\cdot)/\sum_n K(n)$ at the expense of changing h to $h + \log \sum_n K(n)$. For much more detailed accounts on the (very many!) models that can be directly mapped to pinning models we refer to [10,12].

Remark 1.2. *We have chosen* $\mathrm{M}(t) < \infty$ *for every* t *only for ease of exposition. The results we present directly generalize to the case in which* $\mathrm{M}(t_0) + \mathrm{M}(-t_0) < \infty$ *for a* $t_0 > 0$. *In this case it suffices to look at the system only for* $\beta \in [0, t_0)$.

Three comments on (1.2) are in order:

(1) We have introduced the model in a very general set-up which is, possibly, not too intuitive, but it allows a unified approach to a large class of models [10,12]. It may be useful at this stage to look at Fig. 1 that illustrates the random walk pinning model.
(2) The presence of $-\log \mathrm{M}(\beta)$ in the exponent is just a parametrization of the problem that comes particularly handy and it can be absorbed by redefining h.
(3) The presence of δ_N in the right-hand side means that we are looking only at trajectories that are *pinned* at the endpoint of the system. This is just a boundary condition and we may as well remove δ_N for the purpose of the results that we are going to state, since it is well known, for example, that the free energy of this system is independent of the boundary condition (e.g., [12], Chapter 4). Nonetheless, at a technical level it is more practical to work with the system pinned at the endpoint.

The (Laplace) asymptotic behavior of $Z_{N,\omega}$ shows a phase transition. In fact, if we define the free energy as

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega}, \tag{1.3}$$

where the limit exists since the sequence $\{\mathbb{E}\log Z_{N,\omega}\}_N$ is super-additive (see, e.g., [12], Chapter 4, where it is also proven that $\mathrm{F}(\beta, h)$ coincides with the $\mathbb{P}(\mathrm{d}\omega)$-almost sure limit of $(1/N)\log Z_{N,\omega}$, so that $\mathrm{F}(\beta, h)$ is effectively the *quenched* free energy), then it is easy to see that $\mathrm{F}(\beta, h) \geq 0$: in fact,

$$\begin{aligned}\mathrm{F}(\beta, h) &\geq \limsup_{N\to\infty} \frac{1}{N} \mathbb{E} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} \big(\beta\omega_n + h - \log \mathrm{M}(\beta)\big)\delta_n\right) \mathbb{1}_{\tau_1 = N}\right] \\ &= \lim_{N\to\infty} \frac{1}{N} \Big(\big(h - \log \mathrm{M}(\beta)\big) + \log \mathbf{P}(\tau_1 = N)\Big) = 0.\end{aligned} \tag{1.4}$$

The transition we are after is captured by setting

$$h_c(\beta) := \sup\big\{h\colon\ \mathrm{F}(\beta, h) = 0\big\} = \inf\big\{h\colon\ \mathrm{F}(\beta, h) > 0\big\}, \tag{1.5}$$

where the equality is a direct consequence of the fact that $\mathrm{F}(\beta, \cdot)$ is non-decreasing (let us point out also that the free energy is a continuous function of both arguments, as it follows from standard convexity arguments). We have the bounds (see point (2) just below for the proof)

$$\mathrm{F}\big(0, h - \log \mathrm{M}(\beta)\big) \leq \mathrm{F}(\beta, h) \leq \mathrm{F}(0, h), \tag{1.6}$$

which directly imply

$$h_c(0) \leq h_c(\beta) \leq h_c(0) + \log \mathrm{M}(\beta). \tag{1.7}$$

Two important observations are:

(1) The bounds in (1.6) are given in terms of $\mathrm{F}(0, \cdot)$, that is the free energy of the non-disordered system, which can be solved analytically (e.g., [10,12]). In particular $h_c(0) = 0$ for every α and every choice of $L(\cdot)$ (in fact $h_c(0) = -\log \sum_n K(n)$ and we are assuming that τ is persistent). We will keep in our formulae $h_c(0)$ both because we think that it makes them more readable and because they happen to be true also if τ were a terminating renewal.
(2) The upper bound in (1.6), that entails the lower bound in (1.7), follows directly from the standard *annealed bound*, that is $\mathbb{E}\log Z_{N,\omega} \leq \log \mathbb{E} Z_{N,\omega}$, and by observing that the *annealed partition function* $\mathbb{E} Z_{N,\omega}$ coincides

with the partition function of the quenched model with $\beta = 0$, that is simply the non-disordered case (of course, the presence of the term $-\log \mathrm{M}(\beta)$ in (1.2) finds here its motivation). The lower bound in (1.6), entailing the upper bound in (1.7), follows by a convexity argument too (see [12], Chapter 5).

Remark 1.3. *It is rather easy (just take the derivative of the free energy with respect to h) to realize that the phase transition we have outlined in this model is a localization transition: when $h < h_c(\beta)$, for N large, the random set τ is* almost empty, *while when $h > h_c(\beta)$ it is of size const N (in fact const $= \partial_h \mathrm{F}(\beta, h)$). Very sharp results have been obtained on this issue: we refer to* [12], *Chapters* 7 *and* 8, *and references therein.*

1.3. *The Harris criterion*

We can now make precise the Harris criterion predictions mentioned in Section 1.1. As we have seen, in our case the pure (or *annealed*) model is just the non-disordered model, and the latter is exactly solvable, so that the critical behavior is fully understood, notably [12], Chapter 2,

$$\lim_{a \searrow 0} \frac{\log \mathrm{F}(0, h_c(0) + a)}{\log a} = \max\left(1, \frac{1}{\alpha}\right) =: \nu_{\mathrm{pure}}. \tag{1.8}$$

The specific heat exponent of the pure model (that is the critical exponent associated to $1/\partial_h^2 \mathrm{F}(0, h)$) is computed analogously and it is equal to $2 - \nu_{\mathrm{pure}}$. Therefore the Harris criterion predicts *disorder relevance* for $\alpha > 1/2$ ($2 - \nu_{\mathrm{pure}} > 0$) and *disorder irrelevance* for $\alpha < 1/2$ ($2 - \nu_{\mathrm{pure}} < 0$) at least for β below a threshold, with $\alpha = 1/2$ as marginal case. So, what one expects is that $\nu_{\mathrm{pure}} = \nu_{\mathrm{quenched}}$ (with obvious definition of the latter) if $\alpha < 1/2$ for β *not too large* and $\nu_{\mathrm{pure}} \neq \nu_{\mathrm{quenched}}$ if $\alpha > 1/2$ (for every $\beta > 0$).

While a priori the Harris criterion attacks the issue of critical behavior, it turns out that a Harris-like approach in the pinning context [8,11] yields information also on $h_c(\beta)$, namely that $h_c(\beta) = h_c(0)$ if $\alpha < 1/2$ and β again not too large, while $h_c(\beta) > h_c(0)$ as soon as $\beta > 0$. For the sequel it is important to recall some aspects of the approaches in [8,11].

The main focus of [8,11] is on the case $\alpha = 1/2$ and trivial $L(\cdot)$. In fact they focus on the interface wetting problem in two dimensions, that boils down to directed random walk pinning in $(1 + 1)$-dimensions. In this framework the conclusions of the two papers differ: [11] stands for $h_c(\beta) = h_c(0)$ for β small, while in [8] one finds an argument in favor of

$$h_c(\beta) - h_c(0) \approx \exp\bigl(-c\beta^{-2}\bigr), \tag{1.9}$$

as $\beta \searrow 0$ (with $c > 0$ an explicit constant).

We will not go into the details of these arguments, but we wish to point out why, in these arguments, $\alpha = 1/2$ plays such a singular role:

(1) In the approach of [11] an expansion of the free energy to all orders in the variance of $\exp(\beta\omega_1 - \log \mathrm{M}(\beta))$, that is $(\mathrm{M}(2\beta)/\mathrm{M}^2(\beta)) - 1 \overset{\beta \searrow 0}{\sim} \beta^2$, is performed. In particular (in the Gaussian case)

$$\mathrm{F}\bigl(\beta, h_c(0) + a\bigr) = \mathrm{F}\bigl(0, h_c(0) + a\bigr) - \frac{1}{2}\bigl(\exp\bigl(\beta^2\bigr) - 1\bigr)\bigl(\partial_a \mathrm{F}\bigl(0, h_c(0) + a\bigr)\bigr)^2 + \cdots \tag{1.10}$$

and, when $L(\cdot)$ is trivial, $\partial_a \mathrm{F}(0, h_c(0) + a)$ behaves like (a constant times) $a^{(1-\alpha)/\alpha}$ for $\alpha \in (0, 1)$ (this is detailed, for example, in [13]) and like a constant for $\alpha > 1$. This suggests that the expansion (1.10) cannot work for $\alpha > 1/2$, because the second-order term, for $a \searrow 0$, becomes larger than the first order term ($a^{\max(1/\alpha, 1)}$). The borderline case is $\alpha = 1/2$, and trust in such an expansion for $\alpha = 1/2$ may follow from the fact that β can be chosen small. In conclusion, an argument along the lines of [11] predicts disorder relevance if and only if $\alpha > 1/2$ (if $L(\cdot)$ is trivial).

(2) The approach of [8] instead is based on the analysis of $\mathrm{var}_{\mathbb{P}}(Z_{N,\omega})$ at the pure critical point $h_c(0)$. This directly leads to studying the random set $\widetilde{\tau} := \tau \cap \tau'$ (it appears in the computation in a very natural way, we call it *intersection renewal*), with τ' an independent copy of τ (note that $\widetilde{\tau}$ is still a renewal process): in physical terms,

one is looking at the *two-replica system*. It turns out that, even if we have assumed τ persistent, $\widetilde{\tau}$ may not be: in fact, if $L(\cdot)$ is trivial, then $\widetilde{\tau}$ is persistent if and only if $\alpha \geq 1/2$ (see just below for a proof of this fact). And [8] predicts disorder relevance if and only if $\alpha \geq 1/2$.

Some aspects of these two approaches were made rigorous mathematically: The expansion of the free energy (1.10) was proved to hold for $\alpha < 1/2$ in [16], and the second moment analysis of [8] was used to prove *disorder irrelevance* in [1,19], making it difficult to choose between the predictions.

We can actually find in the physical literature a number of authors standing for one or the other of the two predictions in the marginal case $\alpha = 1/2$ (the reader can find a detailed review of the literature in [14]). But we would like to go a step farther and we point out that, by generalizing naively the approach in [8], one is tempted to conjecture disorder relevance (at arbitrarily small β) if and only if the intersection renewal is recurrent. Let us make this condition explicit: while one does not have direct access to the inter-arrival distribution of $\widetilde{\tau}$, it is straightforward, by independence, to write the renewal function of $\widetilde{\tau}$:

$$\mathbf{P}(n \in \widetilde{\tau}) = \mathbf{P}(n \in \tau)^2. \tag{1.11}$$

It is then sufficient to use the basic (and general) renewal process formula $\sum_n \mathbf{P}(n \in \widetilde{\tau}) = (1 - \sum_n \mathbf{P}(\widetilde{\tau}_1 = n))^{-1}$ to realize that $\widetilde{\tau}$ is persistent if and only if $\sum_n \mathbf{P}(n \in \widetilde{\tau}) = \infty$. Since under our assumptions for $\alpha \in (0, 1)$ [9], Theorem B,

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{\alpha \sin(\pi\alpha)}{\pi} \frac{1}{n^{1-\alpha} L(n)}, \tag{1.12}$$

we easily see that the intersection renewal $\widetilde{\tau}$ is persistent for $\alpha > 1/2$ and terminating if $\alpha < 1/2$ (the case $\alpha = 0$ can be treated too [4], and $\widetilde{\tau}$ is terminating). In the $\alpha = 1/2$ case the argument we have just outlined yields

$$\tau \cap \tau' \text{ is persistent} \quad \Longleftrightarrow \quad \sum_n \frac{1}{n L(n)^2} = \infty. \tag{1.13}$$

Roughly, this is telling us that the intersection renewal $\widetilde{\tau}$ is persistent up to a slowly varying function $L(x)$ diverging *slightly less* than $(\log x)^{1/2}$. In particular, as we have already pointed out, if $L(\cdot)$ is trivial, $\widetilde{\tau}$ is persistent.

Let us remark that the expansion (1.10) has been actually made rigorous in [16], but only under the assumption that the intersection renewal $\widetilde{\tau}$ is terminating (that is, $\mathtt{b} > 1/2$ for logarithmic slowly varying functions).

Remark 1.4. *In view of the argument we have just outlined, we introduce the increasing function* $\widetilde{L} : (0, \infty) \to (0, \infty)$ *defined as*

$$\widetilde{L}(x) := \int_0^x \frac{1}{(1+y)L(y)^2}\,\mathrm{d}y, \tag{1.14}$$

that is going to play a central role from now on. Let us point out that, by [4], *Theorem* 1.5.9a, $\widetilde{L}(\cdot)$ *is a slowly varying function which has the property*

$$\lim_{x\to\infty} \widetilde{L}(x) L(x)^2 = +\infty, \tag{1.15}$$

which is a non-trivial statement when $L(\cdot)$ *does not diverge at infinity. Of course we are most interested in the fact that, when* $\alpha = 1/2$, $\widetilde{L}(x)$ *diverges as* $x \to \infty$ *if and only if the intersection renewal* $\widetilde{\tau}$ *is recurrent* (cf. (1.13)). *For completeness we point out that* $\widetilde{L}(\cdot)$ *is a special type of slowly varying function* (*a* den Haan function [4], *Chapter* 3), *but we will not exploit the further regularity properties stemming out of this observation.*

1.4. *Review of the rigorous results*

Much mathematical work has been done on disordered pinning models recently. Let us start with a quick review of the $\alpha \neq 1/2$ case:

- If $\alpha > 1/2$ disorder relevance is established. The positivity of $h_c(\beta) - h_c(0)$ (with precise asymptotic estimates as $\beta \searrow 0$) is proven [2,7]. It has been also shown that disorder has a smoothing effect on the transition and the quenched free energy critical exponent differs from the annealed one [15].
- If $\alpha < 1/2$ disorder irrelevance is established, along with a number of sharp results saying in particular that, if β is not too large, $h_c(\beta) = h_c(0)$ and that the free energy critical behavior coincides in the quenched and annealed framework [1,3,16,19].

In the case $\alpha = 1/2$ results are less complete. Particularly relevant for the sequel are the next two results that we state as theorems. The first one is taken from [1] (see also [19]) and uses the auxiliary function $a_0(\cdot)$ defined by

$$a_0(\beta) := C_1 L\big(\widetilde{L}^{-1}\big(C_2/\beta^2\big)\big)\big/\big(\widetilde{L}^{-1}\big(C_2/\beta^2\big)\big)^{1/2} \quad \text{with } C_1 > 0 \text{ and } C_2 > 0, \tag{1.16}$$

if $\lim_{x\to\infty} \widetilde{L}(x) = \infty$, and $a_0(\cdot) \equiv 0$, otherwise.

Theorem 1.5. *Fix $\omega_1 \sim \mathcal{N}(0,1)$, $\alpha = 1/2$ and choose a slowly varying function $L(\cdot)$. Then there exists $\beta_0 > 0$ and $a_1 > 0$ such that for every $\varepsilon > 0$ there exist C_1 and $C_2 > 0$ such that*

$$1 - \varepsilon \le \frac{\mathrm{F}(\beta, a)}{\mathrm{F}(0, a)} \le 1 \quad \text{for } a > a_0(\beta), a \le a_1 \text{ and } \beta \le \beta_0. \tag{1.17}$$

This implies for $\beta \le \beta_0$

$$h_c(\beta) - h_c(0) \le a_0(\beta). \tag{1.18}$$

It is worth pointing out that Theorem 1.5 yields an upper bound matching (1.9) when $L(\cdot)$ is trivial.
The next result addresses instead the lower bound on $h_c(\beta) - h_c(0)$ and it is taken from [14].

Theorem 1.6. *Fix $\omega_1 \sim \mathcal{N}(0,1)$ and $\alpha = 1/2$. If $L(\cdot)$ is trivial, then $h_c(\beta) - h_c(0) > 0$ for every $\beta > 0$ and there exists $C > 0$ such that*

$$h_c(\beta) - h_c(0) \ge \exp\big(-C/\beta^4\big) \tag{1.19}$$

for $\beta \le 1$.

It should be pointed out that [14] has been worked out for trivial $L(\cdot)$, addressing thus precisely the controversial issue in the physical literature. The case of $\lim_{x\to\infty} L(x) = 0$ has been treated [2] (see [7] for a weaker result) where $h_c(\beta) - h_c(0) > 0$ has been established with an explicit but not optimal bound. We point out also that a result analogous to Theorem 1.6 has been proven for a hierarchical version of the pinning model [18] (see [14] for the case of the hierarchical model proposed in [8]).

The understanding of the marginal case is therefore still partial and the following problems are clearly open:

(1) What is really the behavior of $h_c(\beta) - h_c(0)$ in the marginal case? In particular, for $L(\cdot)$ trivial, is (1.9) correct?
(2) Going beyond the case of $L(\cdot)$ trivial: is the two-replica condition (1.13) equivalent to disorder relevance for small β?
(3) What about non-Gaussian disorder? It should be pointed out that a part of the literature focuses on Gaussian disorder, notably Theorem 1.5, but this choice appears to have been made in order to have more concise proofs (for example, the results in [7] are given for very general disorder distribution). Theorem 1.6 instead exploits a technique that is more inherently Gaussian and generalizing the approach in [14] to non-Gaussian disorder is not straightforward.

As we explain in the next subsection, in this paper we will give *almost* complete answers to questions (1)–(3). In addition we will prove a monotonicity result for the phase diagram of pinning model which holds in great generality.

1.5. *The main result*

Our main result requires the existence of $\varepsilon \in (0, 1/2]$ such that

$$L(x) = \mathrm{o}\big(\big(\log(x)\big)^{(1/2)-\varepsilon}\big) \quad \text{as } x \to \infty, \tag{1.20}$$

that is $\lim_{x\to\infty} L(x)(\log(x))^{-(1/2)+\varepsilon} = 0$. Of course, if $L(\cdot)$ vanishes at infinity, (1.20) holds with $\varepsilon = 1/2$. Going back to the slowly varying function $\widetilde{L}(\cdot)$, cf. Remark 1.4, we note that, under assumption (1.20), we have

$$\widetilde{L}(x) \overset{x\to\infty}{\gg} \int_2^x \frac{1}{y(\log y)^{1-2\varepsilon}}\,\mathrm{d}y = \frac{1}{2\varepsilon}(\log x)^{2\varepsilon} - \frac{1}{2\varepsilon}(\log 2)^{2\varepsilon}. \tag{1.21}$$

Therefore, under assumption (1.20), we have that if $q > (2\varepsilon)^{-1}$ then

$$\lim_{x\to\infty} \frac{\widetilde{L}(x)}{L(x)^{2/(q-1)}} = \infty, \tag{1.22}$$

which guarantees that given $q > (2\varepsilon)^{-1}$ (actually, in the sequel $q \in \mathbb{N}$) and $A > 0$,

$$\Delta(\beta; q, A) := \big(\inf\big\{n \in \mathbb{N}\colon\ \widetilde{L}(n)/L(n)^{2/(q-1)} \geq A\beta^{-2q/(q-1)}\big\}\big)^{-1} \tag{1.23}$$

is greater than 0 for every $\beta > 0$.

Our main result is the following theorem.

Theorem 1.7. *Let us assume that $\alpha = 1/2$ and that (1.20) holds for some $\varepsilon \in (0, 1/2]$. For every β_0 and every integer $q > (2\varepsilon)^{-1}$ there exists $A > 0$ such that*

$$h_c(\beta) - h_c(0) \geq \Delta(\beta; q, A) > 0 \tag{1.24}$$

for every $\beta \leq \beta_0$.

The result may be more directly appreciated in the particular case of $L(\cdot)$ of logarithmic type, cf. Remark 1.1, with $\mathtt{b} < 1/2$, so that (1.20) holds with $\varepsilon < \min((1/2) - \mathtt{b}, 1/2)$. By explicit integration we see that $\widetilde{L}(x) \sim (\mathtt{a}^2(1 - 2\mathtt{b}))^{-1}(\log(x))^{1-2\mathtt{b}}$ so that

$$\frac{\widetilde{L}(x)}{L(x)^{2/(q-1)}} \sim \frac{\mathtt{a}^{-2q/(q-1)}}{(1-2\mathtt{b})}\big(\log(x)\big)^{1-2\mathtt{b}q(q-1)^{-1}} \tag{1.25}$$

and in this case

$$\Delta(\beta; q, A) \overset{\beta\searrow 0}{\sim} \exp\big(-c(\mathtt{b}, A, q)\beta^{-b}\big), \tag{1.26}$$

where $c(\mathtt{b}, A, q) := ((1-2\mathtt{b})\mathtt{a}^{2q/(q-1)}A)^{1/C}$ and $b := 2q/((q-1)C)$ with $C := 1 - 2\mathtt{b}q(q-1)^{-1}$. In short, by choosing q large the exponent $b > 2/(1-2\mathtt{b})$ becomes arbitrarily close to $2/(1-2\mathtt{b})$, at the expense of course of a large constant $c(\mathtt{b}, A, q)$, since A will have to be chosen sufficiently large.

We sum up these steps into the following simplified version of Theorem 1.7.

Corollary 1.8. *If $\alpha = 1/2$ and $L(\cdot)$ is of logarithmic type with $\mathtt{b} \in (-\infty, 1/2)$ (cf. Remark 1.1) then $h_c(\beta) > h_c(0)$ for every $\beta > 0$ and for every $b > 2/(1-2\mathtt{b})$ there exists $c > 0$ such that, for β sufficiently small*

$$h_c(\beta) - h_c(0) \geq \exp\big(-c\beta^{-b}\big). \tag{1.27}$$

This result of course has to be compared with the upper bound in Theorem 1.5 that for $L(\cdot)$ of logarithmic type yields for $\mathtt{b} < 1/2$

$$h_c(\beta) - h_c(0) \le \widetilde{C}_1 \beta^{-2\mathtt{b}/(1-2\mathtt{b})} \exp\bigl(-\widetilde{C}_2 \beta^{-2/(1-2\mathtt{b})}\bigr), \tag{1.28}$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are positive constants that depend (explicitly) on $\mathtt{a}$, $\mathtt{b}$ and on the two constants C_1 and C_2 of Theorem 1.5 (we stress that $\widetilde{C}_1 > 0$ and $\widetilde{C}_2 > 0$ for every $\mathtt{a} > 0$ and $\mathtt{b} < 1/2$).

The main body of the proof of Theorem 1.7 is given in the next section. In the subsequent sections a number of technical results are proven. In the last section (Section 6) we prove a general result (Proposition 6.1) for the models we are considering: the monotonicity of the free energy with respect to β. This result, already known for other disordered models, appears not to have been pointed out up to now for the pinning model. We stress that Proposition 6.1 is not used in the rest of the paper, but, as discussed in Section 6, one can find a link of some interest with our main results.

2. Coarse graining, fractional moment and measure change arguments

The purpose of this section is to reduce the proof to a number of technical statements, that are going to be proven in the next sections. In doing so, we are going to introduce the quantities and notations used in the technical statements and, at the same time, we will stress the main ideas and the novelties with respect to earlier approaches (notably, with respect to [14]).

We anticipate that the main ingredients of the proof are (like in [14]) a coarse graining procedure and a fractional moment estimate on the partition function combined with a change of measure. However:

(1) In [14] we have exploited the Gaussian character of the disorder to introduce *weak, long-range correlations* while keeping the Gaussian character of the random variables. In fact, the change of measure is given by a density that is just the exponential of a quadratic functional of ω, that is a measure change via a *2-body potential*. In order to lower the exponent 4 in the right-hand side of (1.19) we will use q-body potentials $q = 3, 4, \dots$ (this is the q appearing in Theorem 1.7). Such potentials carry with themselves a number of difficulties: for example, when the law of the disorder is Gaussian, the modified measure is not. As a matter of fact, there are even problems in defining the modified disorder variables if one modifies in a straightforward way the procedure in [14] to use q-body potentials, due to integrability issues: such problems may look absent if one deals with bounded ω variables, but they actually reappear when taking limits. The change-of-measure procedure is therefore performed by introducing q-body potentials *and* suitable cut-offs. Estimating the effect of such *q-body potential with cut-off* change of measure is at the heart of our technical estimates.
(2) The coarse-graining procedure is different from the one used in [14,20], since we have to adapt it to the new change of measure procedure. However, unlike point (1), the difference between the previous coarse graining procedure and the one we are employing now is more technical than conceptual.

2.1. *The coarse graining length*

Recall the definition (1.14) of $\widetilde{L}(\cdot)$. We are assuming (1.20), therefore $\lim_{x\to\infty} \widetilde{L}(x) = +\infty$. Chosen a value of $q \in \{2, 3, \dots\}$ (q is kept fixed throughout the proof) and a positive constant A (that is going to be chosen large) we define

$$k = k(\beta; q, A) := \inf\bigl\{n \in \mathbb{N}\colon \widetilde{L}(n)/L(n)^{2/(q-1)} \ge A\beta^{-2q/(q-1)}\bigr\}. \tag{2.1}$$

Since we are interested also in cases in which $L(\cdot)$ diverges (and possibly faster than $\widetilde{L}(\cdot)$) it is in general false that $k < \infty$. However, the assumption (1.20) guarantees that, for $q > (2\varepsilon)^{-1}$, $L(x)/L(x)^{2/(q-1)} \to \infty$ for $x \to \infty$ and therefore $k < \infty$.

Moreover, if $L(\cdot)$ is of logarithmic type (Remark 1.1) with $\mathtt{b} < 1/2$, then for $q > 1/(1 - 2\mathtt{b})$ the function $\widetilde{L}(\cdot)/L(\cdot)^{2/(q-1)}$ is (eventually) increasing.

Of course $k(\beta; q, A)$ is just $1/\Delta(\beta; q, A)$, cf. (1.23), and the reason for such a link is explained in Remark 2.5. Note by now that k is monotonic in both β and A. Since β is chosen smaller than an arbitrary fixed quantity β_0, in order to guarantee that k is large we will rather play on choosing A large.

Remark 2.1. *For the proof certain monotonicity properties will be important. Notably, we know* [4], *Section* 1.5.2, *that* $1/(\sqrt{x}L(x))$ *is asymptotic to a monotonic (decreasing) function and this directly implies that we can find a slowly varying function* $\mathbf{L}(\cdot)$ *and a constant* $c_L \in (0,1]$ *such that*

$$x \mapsto \frac{1}{\sqrt{x}\mathbf{L}(x)} \text{ is decreasing} \quad \text{and} \quad c_L\mathbf{L}(x) \le L(x) \le \mathbf{L}(x) \quad \text{for every } x \in (0,\infty). \tag{2.2}$$

Given the asymptotic behavior of the renewal function of τ *(a special case of* (1.12))

$$\mathbf{P}(n \in \tau) \overset{n\to\infty}{\sim} \frac{1}{2\pi\sqrt{n}L(n)}, \tag{2.3}$$

and the fact that $\mathbf{P}(n \in \tau) > 0$ *for every* $n \in \mathbb{N}$, *we can choose* $\mathbf{L}(\cdot)$ *and* c_L *such that we have also*

$$\frac{1}{\sqrt{n+1}\mathbf{L}(n+1)} \le \mathbf{P}(n \in \tau) \le \frac{c_L^{-1}}{\sqrt{n+1}\mathbf{L}(n+1)}, \quad n = 0, 1, 2, \dots. \tag{2.4}$$

It is natural to choose $\mathbf{L}(\cdot)$ *such that* $\lim_{x\to\infty}\mathbf{L}(x)/L(x) \in [1, 1/c_L)$ *exists, and we will do so. For later convenience we set*

$$R_{1/2}(x) := \frac{1}{\sqrt{x+1}\mathbf{L}(x+1)}. \tag{2.5}$$

2.2. *The coarse graining procedure and the fractional moment bound*

Let us start by introducing for $0 \le M < N$ the notation

$$Z_{M,N} = Z_{M,N,\omega} := \mathbf{E}\left[e^{\sum_{n=M+1}^{N}(\beta\omega_n + h - \log \mathrm{M}(\beta))\delta_n}\delta_N \,|\, \delta_M = 1\right], \tag{2.6}$$

and $Z_{M,M} := 1$ (of course $Z_{N,\omega} = Z_{0,N}$). We consider without loss of generality a system of size proportional to k, that is $N = km$ with $m \in \mathbb{N}$. For $\mathcal{I} \subset \{1,\dots,m\}$ we define

$$\widehat{Z}_\omega^{\mathcal{I}} := \mathbf{E}\left[e^{\sum_{n=1}^{N}(\beta\omega_n + h - \log \mathrm{M}(\beta))\delta_n}\delta_N \mathbb{1}_{E_{\mathcal{I}}}(\tau)\right], \tag{2.7}$$

where $E_{\mathcal{I}} := \{\tau \cap (\bigcup_{i\in\mathcal{I}} B_i) = \tau \setminus \{0\}\}$, and

$$B_i := \{(i-1)k+1, \dots, ik\}, \tag{2.8}$$

that is $E_{\mathcal{I}}$ is the event that the renewal τ intersects the blocks $(B_i)_{i\in\mathcal{I}}$ and only these blocks over $\{1,\dots,N\}$. It follows from this definition that

$$Z_{N,\omega} = \sum_{\mathcal{I}\subset\{1,\dots,m\}} \widehat{Z}_\omega^{\mathcal{I}}. \tag{2.9}$$

Note that $\widehat{Z}_\omega^{\mathcal{I}} = 0$ if $m \notin \mathcal{I}$. Therefore in the following we will always assume $m \in \mathcal{I}$. For $\mathcal{I} = \{i_1,\dots,i_l\}$, $(i_1 < \dots < i_l,\ i_l = m)$, one can express $\widehat{Z}_\omega^{\mathcal{I}}$ in the following way:

$$\widehat{Z}_\omega^{\mathcal{I}} = \sum_{\substack{d_1, f_1 \in B_{i_1} \\ d_1 \le f_1}} \sum_{\substack{d_2, f_2 \in B_{i_2} \\ d_2 \le f_2}} \cdots \sum_{d_l \in B_{i_l}} K(d_1) z_{d_1} Z_{d_1,f_1} K(d_2 - f_1) z_{d_2} Z_{d_2,f_2} \cdots K(d_l - f_{l-1}) z_{d_l} Z_{d_l,N}, \tag{2.10}$$

with $z_n := \exp(\beta\omega_n + h - \log \mathrm{M}(\beta))$. Let us fix a value of $\gamma \in (0,1)$ (we actually choose $\gamma = 6/7$, but we will keep writing it as γ). Using the inequality $(\sum a_i)^\gamma \le \sum a_i^\gamma$ (which is valid for $a_i \ge 0$ and an arbitrary collection of indexes) we get

$$\mathbb{E}\left[Z_{N,\omega}^\gamma\right] \le \sum_{\mathcal{I}\subset\{1,\dots,m\}} \mathbb{E}\left[(\widehat{Z}_\omega^{\mathcal{I}})^\gamma\right]. \tag{2.11}$$

An elementary, but crucial, observation is that

$$\mathrm{F}(\beta, h) = \lim_{N\to\infty} \frac{1}{\gamma N} \mathbb{E} \log Z^{\gamma}_{N,\omega} \le \liminf_{N\to\infty} \frac{1}{\gamma N} \log \mathbb{E} Z^{\gamma}_{N,\omega}, \tag{2.12}$$

so that if we can prove that $\limsup_N \mathbb{E} Z^{\gamma}_{N,\omega} < \infty$ for $h = h_c(0) + \Delta(\beta; q, A)$ we are done.

2.3. *The change of measure*

We introduce

$$X_j := \sum_{\underline{i}\in B_j^q} V_k(\underline{i})\omega_{\underline{i}}, \tag{2.13}$$

where B_j^q is the Cartesian product of B_j with itself q times and $\omega_{\underline{i}} = \prod_{a=1}^q \omega_{i_a}$. The *potential* $V_k(\cdot)$ plays a crucial role for the sequel: we define it and discuss some of its properties in the next remark.

Remark 2.2. *The potential V is best introduced if we define the sorting operator $\mathtt{s}(\cdot)$: if $\underline{i} \in \mathbb{R}^q$ $(q = 2, 3, \ldots)$, $\mathtt{s}(\underline{i}) \in \mathbb{R}^q$ is the non-decreasing rearrangement of the entries of $\underline{i}$. We introduce then*

$$U(\underline{i}) := \prod_{a=2}^{q} R_{1/2}\big(\mathtt{s}(\underline{i})_a - \mathtt{s}(\underline{i})_{a-1}\big), \tag{2.14}$$

The potential V is defined by renormalizing U and by setting to zero the diagonal terms:

$$V_k(\underline{i}) := \frac{1}{(q!)^{1/2}k^{1/2}\widetilde{\mathbf{L}}(k)^{(q-1)/2}} U(\underline{i}) \mathbb{1}_{\{i_a \neq i_b \text{ for every } a,b\}}, \tag{2.15}$$

where $\widetilde{\mathbf{L}}(\cdot)$ is defined as in (1.14), with $L(\cdot)$ replaced by $\mathbf{L}(\cdot)$. By exploiting the fact that for every $c > 0$ we have $\sum_{i\le cN} R_{1/2}(i)^2 \overset{N\to\infty}{\sim} \widetilde{\mathbf{L}}(N)$ one sees that

$$\sum_{\underline{i}\in B_1^q} V_k(\underline{i})^2 = \frac{1}{k(\widetilde{\mathbf{L}}(k))^{q-1}} \sum_{0<i_1<\cdots<i_q\le k} \prod_{a=2}^{q} \big(R_{1/2}(i_a - i_{a-1})\big)^2 \overset{k\to\infty}{\sim} 1. \tag{2.16}$$

Therefore,

$$\sum_{\underline{i}\in B_1^q} V_k(\underline{i})^2 \le 2 \tag{2.17}$$

for k sufficiently large.

Let us introduce, for $K > 0$, also

$$\begin{aligned} f_K(x) &:= -K\mathbb{1}_{\{x\ge \exp(K^2)\}}, \\ g_{\mathcal{I}}(\omega) &:= \exp\Big(\sum_{j\in\mathcal{I}} f_K(X_j)\Big), \\ \bar{g}(\omega) &:= \exp\big(f_K(X_1)\big). \end{aligned} \tag{2.18}$$

We are now going to replace, for fixed $\mathcal{I}$, the measure $\mathbb{P}(\mathrm{d}\omega)$ with $g_{\mathcal{I}}(\omega)\mathbb{P}(\mathrm{d}\omega)$. The latter is not a probability measure: we could normalize it, but this is inessential because we are directly exploiting Hölder inequality to get

$$\mathbb{E}\big[\big(\widehat{Z}^{\mathcal{I}}_{\omega}\big)^{\gamma}\big] \le \big(\mathbb{E}\big[g_{\mathcal{I}}(\omega)^{-\gamma/(1-\gamma)}\big]\big)^{1-\gamma} \big(\mathbb{E}\big[g_{\mathcal{I}}(\omega)\widehat{Z}^{\mathcal{I}}_{\omega}\big]\big)^{\gamma}. \tag{2.19}$$

The first factor in the right-hand side is easily controlled, in fact

$$\mathbb{E}\big[g_{\mathcal{I}}(\omega)^{-\gamma/(1-\gamma)}\big]=\mathbb{E}\big[\bar{g}(\omega)^{-\gamma/(1-\gamma)}\big]^{|\mathcal{I}|}=\left[\left(\exp\left(\frac{K\gamma}{1-\gamma}\right)-1\right)\mathbb{P}\big(X_1\geq\exp\big(K^2\big)\big)+1\right]^{|\mathcal{I}|}, \tag{2.20}$$

and since X_1 is centered and its variance coincides with the left-hand side of (2.17), by Chebyshev inequality the term $\exp(K\gamma/(1-\gamma))\mathbb{P}(X_1\geq\exp(K^2))$ can be made arbitrarily small by choosing K large. Therefore for K sufficiently large (depending only on $\gamma(=6/7)$)

$$\mathbb{E}\big[\big(\widehat{Z}^{\mathcal{I}}_{\omega}\big)^{\gamma}\big]\leq 2^{\gamma|\mathcal{I}|}\big(\mathbb{E}\big[g_{\mathcal{I}}(\omega)\widehat{Z}^{\mathcal{I}}_{\omega}\big]\big)^{\gamma}. \tag{2.21}$$

Estimating the remaining factor is a more involved matter. We will actually use the following two statements, that we prove in the next section. Set $P_{\mathcal{I}}:=\mathbf{P}(E_{\mathcal{I}};\delta_N=1)$.

Proposition 2.3. *Assume that* $\alpha=1/2$ *and that* (1.20) *holds for some* $\varepsilon\in(0,1/2]$. *For every* $\eta>0$ *and every* $q>(2\varepsilon)^{-1}$ *we can choose* $A>0$ *such that if* $\beta\leq\beta_0$ *and* $h\leq\Delta(\beta;q,A)$, *for every* $\mathcal{I}\subset\{1,\dots,m\}$ *with* $m\in\mathcal{I}$ *we have*

$$\mathbb{E}\big[g_{\mathcal{I}}(\omega)\widehat{Z}^{\mathcal{I}}_{\omega}\big]\leq\eta^{|\mathcal{I}|}P_{\mathcal{I}}. \tag{2.22}$$

The following technical estimate controls $P_{\mathcal{I}}$ (recall that $\mathcal{I}=\{i_1,\dots,i_{|\mathcal{I}|}\}$).

Lemma 2.4. *Assume* $\alpha=1/2$. *There exist* $C_1=C_1(L(\cdot),k)$, $C_2=C_2(L(\cdot))$ *and* $k_0=k_0(L(\cdot))$ *such that* (*with* $i_0:=0$)

$$P_{\mathcal{I}}\leq C_1C_2^{|\mathcal{I}|}\prod_{j=1}^{|\mathcal{I}|}\frac{1}{(i_j-i_{j-1})^{7/5}} \tag{2.23}$$

for $k\geq k_0$.

Note that in this statement k is just a natural number, but we will apply it with k as in (2.1) so that $k\geq k_0$ is just a requirement on A. Note also that the choice of $7/5$ is arbitrary (any number in $(1,3/2)$ would do: the constants C_1 and C_2 depend on such a number).

Let us now go back to (2.21) and let us plug it into (2.11) and use Proposition 2.3 and Lemma 2.4 to get:

$$\mathbb{E}\big[Z^{\gamma}_{N,\omega}\big]\leq C_1^{\gamma}\sum_{\substack{\mathcal{I}\subset\{1,\dots,m\}\\ m\in\mathcal{I}}}\prod_{j=1}^{|\mathcal{I}|}\left(\frac{(2C_2\eta)^{\gamma}}{(i_j-i_{j-1})^{7\gamma/5}}\right). \tag{2.24}$$

But $7\gamma/5=6/5>1$, so we can choose

$$\eta:=\frac{1}{3C_2(\sum_{i=1}^{\infty}i^{-6/5})^{7/6}}, \tag{2.25}$$

and this implies that $n\mapsto(2C_2\eta)^{\gamma}n^{-7\gamma/5}$ is a sub-probability, which directly entails that

$$\mathbb{E}\big[Z^{\gamma}_{N,\omega}\big]\leq C_1^{\gamma}\quad(\gamma=6/7) \tag{2.26}$$

for every N, which implies, via (2.12), that $\mathrm{F}(\beta,h)=0$ and we are done.

It is important to stress that C_1 may depend on k (we need (2.26) uniform in N, not in k), but C_2 does not (C_2 is just a function of $L(\cdot)$, that is a function of the chosen renewal), so that η may actually be chosen a priori as in (2.25): it is a small but fixed constant that depends only on the underlying renewal τ.

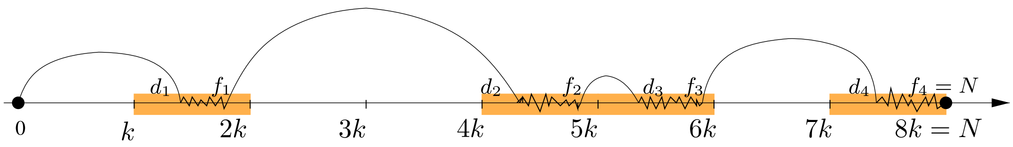

Fig. 2. The figure above explains our coarse graining procedure. Here $N = 8k$, $\mathcal{I} = \{2, 5, 6, 8\}$. The drawn trajectory is a typical trajectory contributing to $\widehat{Z}^{\mathcal{I}}_{N,\omega}$; d_i and f_i, $1 \le i \le 4$, correspond to the indexes of (2.10). The shadowed regions represent the sites on which the change of measure procedure (presented in Section 2.3) acts.

Remark 2.5. *In this section we have actually hidden the role of $\Delta(\beta; q, A)$ in the hypotheses of Proposition 2.3, which are the hypotheses of Theorem 1.7. Let us therefore explain informally why we can prove a critical point shift of $\Delta = 1/k$.*

The coarse graining procedure reduces proving delocalization to Proposition 2.3. As it is quite intuitive from (2.9)–(2.21) and Fig. 2, one has to estimate the expectation, with respect to the $\bar{g}(\omega)$-modified measure, of the partition function Z_{d_j,f_j} (or, equivalently, $Z_{d_j,f_j}/\mathbf{P}(f_j - d_j \in \tau)$) in each visited block (let us assume that $f_j - d_j$ is of the order of k, because if it is much smaller than k one can bound this contribution in a much more elementary way). The Boltzmann factor in Z_{d_j,f_j} is $\exp(\sum_{n=d_j+1}^{f_j}(\beta\omega_n - \log \mathrm{M}(\beta) + h)\delta_n)$ which can be bounded (in an apparently very rough way) by $\exp(\sum_{n=d_j+1}^{f_j}(\beta\omega_n - \log \mathrm{M}(\beta))\delta_n)\exp(hk)$, since $f_j - d_j \le k$. Therefore, if $h \le \Delta(\beta; q, A) \sim 1/k$ we can drop the dependence on h at the expense of the multiplicative factor e that is innocuous because we can show that the expectation (with respect to the $\bar{g}(\omega)$-modified measure) of $Z_{d_j,f_j}/\mathbf{P}(f_j - d_j \in \tau)$ when $h = 0$ can be made arbitrarily small by choosing A sufficiently large.

Remark 2.6. *How does one guess the change of measure in (2.18)? An argument that suggests how to choose the density $g_{\mathcal{I}}(\omega)$ goes as follows. The reason why $\mathbb{E}[Z_{N,\omega}]$ does not capture the correct typical behavior of $Z_{N,\omega}$ for N large lies in the atypically large values of $Z_{N,\omega}$ that dominate the expectation. In order to get closer to the typical behavior one can introduce a penalization in terms of a density like $\exp(f_K(Z_{N,\omega}))$. However this change of measure is very difficult to handle, because it essentially requires being able to compute quantities involving $Z_{N,\omega}$, that is precisely the object that we are investigating. So what we do is to attempt the substitution of $Z_{N,\omega}$ with a much simpler quantity that is expected to capture some crucial features of it. A naive, but in the end fruitful, way of proceeding is to expand the Boltzmann factor $\exp(H) = 1 + H + H^2/2 + \cdots$. If one considers the term of order q one has to take the q*th *power of $\sum_{n=1}^{N}(\beta\omega_n + h - \log \mathrm{M}(\beta))\delta_n$ thus obtaining in particular $\beta^q \sum_{i_1<i_2<\cdots<i_q} \omega_{i_1}\omega_{i_2}\cdots\omega_{i_q}\delta_{i_1}\delta_{i_2}\cdots\delta_{i_q}$. By replacing $\beta^q\delta_{i_1}\delta_{i_2}\cdots\delta_{i_q}$ by its expectation one gets essentially $V_k(i_1, \dots, i_q)$* (cf. (2.15)) *for $N = k$: this can be seen by using the asymptotic behavior of $\mathbf{E}[\delta_i]$, that is (2.3), and the definition of k that tells us that $L(k)/\widetilde{L}(k)^{(q-1)/2}$ is about β^q (in this rough approximation procedure we have replaced i_1 with k: this is because in most of the terms of the sum i_1 is actually of this order of magnitude).*

Remark 2.7. *A last observation on the proof is about β_0. It can be chosen arbitrarily, but for the sake of simplifying the constants appearing in the proofs we choose $\beta_0 \in (0, \infty)$ such that*

$$\frac{1}{2} \le \frac{\mathrm{d}^2}{\mathrm{d}\beta^2} \log \mathrm{M}(\beta) \le 2 \tag{2.27}$$

for $\beta \in [0, \beta_0]$. Choosing β_0 arbitrarily just boils down to changing the constants in the right-most and left-most terms in (2.27).

3. Coarse graining estimates

We start by proving Lemma 2.4, namely (2.23). The proof is however more clear if instead of working with the exponent $7/5$ we work with $3/2 - \xi$ ($\xi \in (0, 1/2)$, in the end, plug in $\xi = 1/10$).

Proof of Lemma 2.4. First of all, in the product on the right-hand side of (2.23) one can clearly ignore the terms such that $i_j - i_{j-1} = 1$. We then express $\mathcal{I}$ in a more practical way by observing that we can define, in a unique way, an integer $p \le l := |\mathcal{I}|$ and increasing sequences of integers $\{a_j\}_{j=1,\dots,p}$, $\{b_j\}_{j=1,\dots,p}$ with $b_p = m$, $a_j \ge b_{j-1} + 2$ (for $j > 1$) and $b_j \ge a_j$ such that

$$\mathcal{I} = \bigcup_{j=1}^{p} [a_j, b_j] \cap \mathbb{N}. \tag{3.1}$$

For instance, if $\mathcal{I} = \{1, 2, 4, 5, 6, 9\}$ we write $\mathcal{I} = \{1, 2\} \cup \{4, 5, 6\} \cup \{9\}$ and $a_1 = 1, b_1 = 2, a_2 = 4, b_2 = 6, a_3 = b_3 = 9$, so that $p = 3$.

With this definition, it is sufficient to show

$$P_{\mathcal{I}} \le C_1 C_2^l \frac{1}{a_1^{3/2-\xi}} \prod_{j=1}^{p-1} \frac{1}{(a_{j+1} - b_j)^{3/2-\xi}}. \tag{3.2}$$

We start then by writing

$$P_{\mathcal{I}} \le \sum_{\substack{d_1 \in B_{a_1} \\ f_1 \in B_{b_1}}} \cdots \sum_{\substack{d_{p-1} \in B_{a_{p-1}} \\ f_p \in B_{b_{p-1}}}} \sum_{d_p \in B_{a_p}} K(d_1) \mathbf{P}(f_1 - d_1 \in \tau) \cdots K(d_p - f_{p-1}) \mathbf{P}(N - d_p \in \tau), \tag{3.3}$$

where the inequality comes from neglecting the constraint that τ has to intersect $B_{a_j+1}, \dots, B_{b_j-1}$. Note that the meaning of the d and f indexes is somewhat different with respect to (2.10) and that in the above sum we always have

$$\begin{aligned} &d_1 \in B_{a_1}, \\ &(a_j - b_{j-1} - 1)k \le d_j - f_{j-1} \le (a_j - b_{j-1} + 1)k, \\ &(b_j - a_j - 1)k \vee 0 \le f_j - d_j \le (b_j - a_j + 1)k. \end{aligned} \tag{3.4}$$

In particular, $f_j \ge d_j$ is guaranteed by the fact that $\mathbf{P}(f_j - d_j \in \tau) = 0$, otherwise.

Observe now that for k sufficiently large

$$\sum_{x \in B_{a_1}} K(x) \le \begin{cases} 1 & \text{if } a_1 = 1, \\ 3 \frac{L((a_1-1)k)}{k^{1/2}(a_1-1)^{3/2}} & \text{if } a_1 = 2, 3, \dots \end{cases} \le c_1(k) \frac{L(a_1 k)}{k^{1/2} a_1^{3/2}}, \tag{3.5}$$

where $c_1(k) := \max(10, k^{1/2}/L(k))$. Moreover, there exists a constant c_2 depending on $L(\cdot)$ such that for $j > 1$

$$\begin{aligned} &\sum_{x=(a_j-b_{j-1}-1)k}^{(a_j-b_{j-1}+1)k} K(x) \le c_2 \frac{L(k(a_j - b_{j-1}))}{k^{1/2}(a_j - b_{j-1})^{3/2}}, \\ &\sum_{x=(b_j-a_j-1)k \vee 0}^{(b_j-a_j+1)k} \mathbf{P}(x \in \tau) \le c_2 \frac{k^{1/2}}{(b_j - a_j + 1)^{1/2} L(k(b_j - a_j + 1))}. \end{aligned} \tag{3.6}$$

The first inequality is obtained by making use of $a_j \ge b_{j-1} + 2$. Neglecting $\mathbf{P}(N - d_p \in \tau)$ which is smaller than one, we can bound the right-hand side of (3.3) and get

$$P_{\mathcal{I}} \le c_1(k) c_2^{2p} \frac{L(a_1 k)}{k^{1/2} a_1^{3/2}} \prod_{j=1}^{p-1} \left(\frac{L(k(a_{j+1} - b_j))}{(a_{j+1} - b_j)^{3/2}} \right) \left(\frac{1}{(b_j - a_j + 1)^{1/2} L(k(b_j - a_j + 1))} \right). \tag{3.7}$$

Notice now that since $L(\cdot)$ grows slower than any power, $\sup_{a_1} L(a_1k)/(k^{1/2}a_1^{\xi})$ is o(1) for k large. To control the other terms we use the *Potter bound* [4], Theorem 1.5.6: given a slowly varying function $L(\cdot)$ which is locally bounded away from zero and infinity (which we may assume in our set up without loss of generality), for every $a > 0$ there exists $c_a > 0$ such that for every $x, y > 0$

$$\frac{L(x)}{L(y)} \le c_a \max\left(\frac{x}{y}, \frac{y}{x}\right)^a. \tag{3.8}$$

This bound implies that for large enough k

$$\sup_{x\ge 1} \frac{L(k)}{\sqrt{x}L(kx)} \le 2 \quad \text{and} \quad \sup_{x\ge 1} \frac{L(kx)}{L(k)x^{\xi}} \le 2. \tag{3.9}$$

In fact consider the second bound (the argument for the first one is identical): by choosing $a = \xi/2$ we have $L(kx)/(L(k)x^{\xi}) \le c_{\xi/2}x^{-\xi/2} \le 2$ and the second inequality holds for x larger than a suitable constant C_{ξ}. For $x (\ge 1)$ smaller than C_{ξ} instead it suffices to choose k sufficiently large so that $L(kx)/L(k) \le 2$ for every $x \in [1, C_{\xi}]$. Using the two bounds (3.9) in (3.7) we complete the proof. □

The proof of Proposition 2.3 depends on the following lemma that will be proven in the next section.

Lemma 3.1. *Set $h = 0$, fix $q \in \mathbb{N}$, $q > (2\varepsilon)^{-1}$ as in Theorem 1.7, and recall the definition of $k = k(\beta; q, A)$ (2.1). For every ε and $\delta > 0$ there exists $A_0 > 0$ such that for $A \ge A_0$*

$$\mathbb{E}\big[\bar g(\omega) z_d Z_{d,f}\big] \le \delta \mathbf{P}(f - d \in \tau) \tag{3.10}$$

for every d and f such that $0 \le d \le d + \varepsilon k \le f \le k$ and $\beta \le \beta_0$.

Proof of Proposition 2.3. Recalling (2.10) and the notations for the set $\mathcal{I}$ in there, we have

$$\begin{aligned}
&\mathbb{E}\big[g_{\mathcal{I}}(\omega)\widehat{Z}_{\omega}^{\mathcal{I}}\big] \\
&\quad = \sum_{\substack{d_1, f_1 \in B_{i_1} \\ d_1 \le f_1}} \sum_{\substack{d_2, f_2 \in B_{i_2} \\ d_2 \le f_2}} \cdots \sum_{d_l \in B_{i_l}} K(d_1)\mathbb{E}\big[\bar g(\omega) z_{d_1-k(i_1-1)} Z_{d_1-k(i_1-1), f_1-k(i_1-1)}\big] K(d_2 - f_1)\cdots \\
&\qquad \times K(d_l - f_{l-1})\mathbb{E}\big[\bar g(\omega) z_{d_l-k(m-1)} Z_{d_l-k(m-1),k}\big] \\
&\quad \le \mathrm{e}^l \sum_{\substack{d_1, f_1 \in B_{i_1} \\ d_1 \le f_1}} \sum_{\substack{d_2, f_2 \in B_{i_2} \\ d_2 \le f_2}} \cdots \sum_{d_l \in B_{i_l}} K(d_1)(\delta + \mathbb{1}_{\{f_1 - d_1 \le \varepsilon k\}})\mathbf{P}(f_1 - d_1 \in \tau) K(d_2 - f_1)\cdots \\
&\qquad \times K(d_l - f_{l-1})(\delta + \mathbb{1}_{\{N - d_l \le \varepsilon k\}})\mathbf{P}(N - d_l \in \tau),
\end{aligned} \tag{3.11}$$

where the factor e^l in the last expression comes from bounding the contribution due to h (recall that $hk \le 1$). We now consider B_{i_j} as the union of two sub-blocks

$$\begin{aligned}
B_{i_j}^{(1)} &:= \big\{(i_j - 1)k + 1, \dots, (i_j - 1)k + \lfloor k/2 \rfloor\big\}, \\
B_{i_j}^{(2)} &:= \big\{(i_j - 1)k + \lceil k/2 \rceil, \dots, i_j k\big\}.
\end{aligned} \tag{3.12}$$

If $d_j \in B_{i_j}^{(1)}$ then if ε is sufficiently small ($\varepsilon \le 1/10$ suffices) we have that for k sufficiently large (i.e., $k \ge k_0(L(\cdot), \varepsilon)$)

$$\sum_{f=d_j}^{d_j+\varepsilon k} \mathbf{P}(f - d_j \in \tau) K(d_{j+1} - f) \le 4\left(\sum_{x=1}^{k\varepsilon} \mathbf{P}(x \in \tau)\right) K\big(k(i_{j+1} - i_j)\big). \tag{3.13}$$

This can be compared to

$$\sum_{f=d_j}^{ki_j} \mathbf{P}(f-d_j \in \tau)K(d_{j+1}-f) \geq \frac{1}{3}\left(\sum_{x=1}^{\lfloor k/4 \rfloor} \mathbf{P}(x\in\tau)\right) K\big(k(i_{j+1}-i_j)\big), \tag{3.14}$$

that holds once again for k large. By using that $\sum_{x=1}^{n} \mathbf{P}(x \in \tau)$ behaves for n large like $\sqrt{n}$ times a slowly varying function (cf. (2.3)) we therefore see that given $\delta > 0$ we can find ε such that for any $d_j \in B_{i_j}^{(1)}$ we have

$$\sum_{f=d_j}^{d_j+\varepsilon k} \mathbf{P}(f-d_j\in\tau)K(d_{j+1}-f) \leq \delta \sum_{f=d_j}^{ki_j} \mathbf{P}(f-d_j\in\tau)K(d_{j+1}-f). \tag{3.15}$$

Using the same argument in the opposite way one finds that if $f_j \in B_{i_j}^{(2)}$

$$\sum_{d=f_j-\varepsilon k}^{f_j} K(d-f_{j-1})\mathbf{P}(f_j-d\in\tau) \leq \delta \sum_{d=k(i_j-1)}^{f_j} K(d-f_{j-1})\mathbf{P}(f_j-d\in\tau). \tag{3.16}$$

Since either $d_j \in B_{i_j}^{(1)}$ or $f_j \in B_{i_j}^{(2)}$, we conclude that

$$\begin{aligned}
&\sum_{\substack{d_j, f_j \in B_{i_j}\\ d_j \leq f_j}} \mathbb{1}_{\{f_j - d_j \leq k\varepsilon\}} K(d_j - f_{j-1})\mathbf{P}(f_j-d_j\in\tau)K(d_{j+1}-f_j)\\
&\quad \leq \delta \sum_{\substack{d_j, f_j \in B_{i_k}\\ d_j \leq f_j}} K(d_j - f_{j-1})\mathbf{P}(f_j-d_j\in\tau)K(d_{j+1}-f_j).
\end{aligned} \tag{3.17}$$

The analog estimate can be obtained for the sum over d_l in (3.11) (rather, it is easier). Using this inequality for $j = 1, \ldots, l$ we get our result for $\eta = 2e\delta$. □

4. The q-body potential estimates

In what follows $X = X_1$ and we fix $\delta \in (0,1)$. The positive (small) number ε is fixed too, as well as $q > (2\varepsilon)^{-1}$, where ε is the same which appears in the statement of Theorem 1.7.

Proof of Lemma 3.1. We start by observing that, since $h = 0$,

$$\mathbb{E}\big[\bar{g}(\omega) z_d Z_{d,f}\big] = \mathbf{E}_{d,f}\left[\mathbb{E}\left[\bar{g}(\omega)\exp\left(\sum_{n=d}^{f}\big(\beta\omega_n - \log \mathrm{M}(\beta)\big)\delta_n\right)\right]\right]\mathbf{P}(f-d\in\tau), \tag{4.1}$$

where $\mathbf{P}_{d,f}$ is the law of $\tau \cap [d,f]$, conditioned to $f, d \in \tau$. Given the random set (or renewal trajectory) τ we introduce the probability measure

$$\widehat{\mathbb{P}}_\tau(\mathrm{d}\omega) := \exp\left(\sum_{n=d}^{f}\big(\beta\omega_n - \log \mathrm{M}(\beta)\big)\delta_n\right)\mathbb{P}(\mathrm{d}\omega). \tag{4.2}$$

Note that ω, under $\widehat{\mathbb{P}}_\tau$, is still a sequence of independent random variables, but they are no longer identically distributed. We will use that, for $d \leq n \leq f$,

$$\widehat{\mathbb{E}}_\tau \omega_n = \mathrm{m}_\beta \delta_n \overset{\beta\searrow 0}{\sim} \beta\delta_n \quad (\text{so that } \beta/2 \leq \mathrm{m}_\beta \leq 2\beta) \quad \text{and} \quad \mathrm{var}_{\widehat{\mathbb{P}}_\tau}(\omega_n) \leq 2, \tag{4.3}$$

where the inequalities hold for $\beta \le \beta_0$ (recall (2.27)) and all relations hold uniformly in the renewal trajectory τ. On the other hand, for $n \notin \{d, \dots, f\}$ the ω_n's are IID exactly as under $\mathbb{P}$. We have

$$\begin{aligned}\frac{\mathbb{E}[\bar{g}(\omega) z_d Z_{d,f}]}{\mathbf{P}(f-d \in \tau)} &= \mathbf{E}_{d,f} \widehat{\mathbb{E}}_\tau\big[\bar{g}(\omega)\big] \\ &= \exp(-K)\mathbf{E}_{d,f}\widehat{\mathbb{P}}_\tau\big[X \ge \exp\big(K^2\big)\big] + \mathbf{E}_{d,f}\widehat{\mathbb{P}}_\tau\big[X < \exp\big(K^2\big)\big] \\ &\le \exp(-K) + \mathbf{E}_{d,f}\widehat{\mathbb{P}}_\tau\big[X < \exp\big(K^2\big)\big] \le \frac{\delta}{3} + \mathbf{E}_{d,f}\widehat{\mathbb{P}}_\tau\big[X < \exp\big(K^2\big)\big],\end{aligned} \tag{4.4}$$

where in the last step we have chosen K such that $\exp(-K) \le \delta/3$. We are now going to use the following lemma.

Lemma 4.1. *If d and f are chosen such that $f - d \ge \varepsilon k$ and $X(= X_1)$ is defined as in (2.13), that is $X = \sum_{\underline{i} \in B_1^q} V_k(\underline{i})\omega_{\underline{i}}$, we have that for every $\zeta > 0$ we can find $a > 0$ and A_0 such that*

$$\mathbf{P}_{d,f}\big(\widehat{\mathbb{E}}_\tau X > aA^{(q-1)/2}\big) \ge 1 - \zeta \tag{4.5}$$

for $\beta \le \beta_0$ and $A \ge A_0$.

We apply this lemma by setting $\zeta = \delta/3$ (so a is fixed once δ is chosen) so that, if we choose K such that $2\exp(K^2) = aA^{(q-1)/2}$ (note that, by choosing A large we make K large and we automatically satisfy the previous requirements on K), we have $\mathbf{P}_{d,f}(\widehat{\mathbb{E}}_\tau X < 2\exp(K^2)) \le \delta/3$, so that, in view of (4.4), we obtain

$$\begin{aligned}\frac{\mathbb{E}[\bar{g}(\omega) z_d Z_{d,f}]}{\mathbf{P}(f-d \in \tau)} &\le \frac{2\delta}{3} + \mathbf{E}_{d,f}\widehat{\mathbb{P}}_\tau\big[X - \widehat{\mathbb{E}}_\tau X \le -\exp\big(K^2\big)\big] \\ &\le \frac{2\delta}{3} + \frac{4}{a^2 A^{q-1}}\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X - \widehat{\mathbb{E}}_\tau X)^2\big].\end{aligned} \tag{4.6}$$

The claim of Lemma 3.1 now follows as soon as we can show that the second moment appearing in the last term of (4.6) is $o(A^{q-1})$ for A large. But this is precisely what is granted by the next lemma. □

Lemma 4.2. *There exists $A_0 > 0$ such that*

$$\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X - \widehat{\mathbb{E}}_\tau X)^2\big] \le A^{(q-1)^2/q} \tag{4.7}$$

for every $\beta \le \beta_0$ and every $A \ge A_0$.

Proof. We start by introducing the notation $\widehat{\omega}_n := \omega_n - \mathtt{m}_\beta \delta_n \mathbb{1}_{\{d \le n \le f\}}$ and by observing that

$$\begin{aligned}\widehat{\mathbb{E}}_\tau\big[(X - \widehat{\mathbb{E}}_\tau X)^2\big] &= \widehat{\mathbb{E}}_\tau\left[\left(\sum_{\underline{i} \in B_1^q} V_k(\underline{i}) \prod_{a=1}^q (\widehat{\omega}_{i_a} + \mathtt{m}_\beta \delta_{i_a} \mathbb{1}_{\{d \le i_a \le f\}}) - \mathtt{m}_\beta^q \sum_{\underline{i} \in \{d,\dots,f\}^q} V_k(\underline{i})\delta_{\underline{i}}\right)^2\right] \\ &\le C(q)\widehat{\mathbb{E}}_\tau\left[\left(\sum_{\ell=0}^{q-1} \mathtt{m}_\beta^\ell \sum_{\underline{i} \in B_1^{q-\ell}} \sum_{\underline{j} \in \{d,\dots,f\}^\ell} V_k(\underline{i}\,\underline{j})\widehat{\omega}_{\underline{i}}\delta_{\underline{j}}\right)^2\right] \\ &\le C(q)\sum_{\ell=0}^{q-1} \mathtt{m}_\beta^{2\ell} \sum_{\underline{i} \in B_1^{q-\ell}} \sum_{\underline{j},\underline{m} \in \{d,\dots,f\}^\ell} V_k(\underline{i}\,\underline{j})V_k(\underline{i}\,\underline{m})\delta_{\underline{j}}\delta_{\underline{m}},\end{aligned} \tag{4.8}$$

where $\underline{i}\,\underline{j} \in B_1^q$ is the concatenation of $\underline{i}$ and $\underline{j}$ and in the last step we have first used the Cauchy–Schwarz inequality, then the fact that the $\widehat{\omega}$ variables are independent and centered and (4.3).

Remark 4.3. *Here and in the following, we adopt the convention that $C(a, b, \dots)$ is a positive constant (which depends on the parameters $a, b, \dots$), whose numerical value may change from line to line.*

Therefore,

$$\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X-\widehat{\mathbb{E}}_\tau X)^2\big] \le C(q)\sum_{\ell=0}^{q-1} \mathtt{m}_\beta^{2\ell} \sum_{\underline{i}\in B_1^{q-\ell}} \sum_{\underline{j},\underline{m}\in\{d,\dots,f\}^\ell} V_k(\underline{i}\,\underline{j})V_k(\underline{i}\underline{m})\mathbf{E}_{d,f}[\delta_{\underline{j}}\delta_{\underline{m}}]. \tag{4.9}$$

Let us point out immediately that we know how to deal with the $\ell = 0$ case: it is simply $C(q)\sum_{\underline{i}\in B_1^q} V_k(\underline{i})^2$ and it is therefore bounded by $2C(q)$ (cf. (2.17)). By using the notation and the bounds in Remarks 2.1 and 2.2, together with the renewal property, we readily see that

$$\mathbf{E}_{d,f}[\delta_{\underline{j}}\delta_{\underline{m}}] \le \frac{\mathtt{c}_L^{-(2\ell+1)}}{\mathbf{P}(f-d\in\tau)}\prod_{a=1}^{2\ell+1} R_{1/2}(r_a-r_{a-1}) \le \frac{\mathtt{c}_L^{-(2\ell+1)}}{R_{1/2}(f-d)}\prod_{a=1}^{2\ell+1} R_{1/2}(r_a-r_{a-1}) \tag{4.10}$$

for $\underline{j},\underline{m}\in\{d,\dots,f\}^\ell$, $r = \mathtt{s}(\underline{j}\underline{m})$, $r_0 := d$ and $r_{2\ell+1} := f$. A notational simplification may be therefore achieved by exploiting further Remark 2.2, namely by using (2.14), so that (4.10) becomes

$$\begin{aligned}\mathbf{E}_{d,f}[\delta_{\underline{j}}\delta_{\underline{m}}] &\le \mathtt{c}_L^{-(2\ell+1)} R_{1/2}(f-d)^{-1}R_{1/2}\big(\min(\underline{j}\underline{m})-d\big)U(\underline{j}\underline{m})R_{1/2}\big(f-\max(\underline{j}\underline{m})\big)\\ &= \mathtt{c}_L^{-(2\ell+1)} R_{1/2}(f-d)^{-1}U(d\underline{j}\underline{m}f).\end{aligned} \tag{4.11}$$

By inserting (4.11) and (2.15) into (4.9) we get to

$$\begin{aligned}&\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X-\widehat{\mathbb{E}}_\tau X)^2\big]\\ &\quad\le C\left(1+\frac{1}{k\widetilde{L}(k)^{q-1}R_{1/2}(f-d)}\sum_{\ell=1}^{q-1}\mathtt{m}_\beta^{2\ell}\sum_{\underline{i}\in B_1^{q-\ell}}\sum_{\underline{j},\underline{m}\in\{d,\dots,f\}^\ell} U(\underline{i}\,\underline{j})U(\underline{i}\underline{m})U(d\underline{j}\underline{m}f)\right)\\ &\quad\le C\left(1+\frac{1}{k\widetilde{L}(k)^{q-1}R_{1/2}(f-d)}\sum_{\ell=1}^{q-1}\mathtt{m}_\beta^{2\ell}\sum_{\underline{i}\in \mathtt{s}(B_1^{q-\ell})}\sum_{\underline{j},\underline{m}\in\mathtt{s}(\{d,\dots,f\}^\ell)} U(\underline{i}\,\underline{j})U(\underline{i}\underline{m})U(d\underline{j}\underline{m}f)\right),\end{aligned} \tag{4.12}$$

where of course $\mathtt{s}(\{1,\dots,a\}^n) = \{\underline{i}\in\{1,\dots,a\}^n\colon i_1\le i_2\le\dots\le i_n\}$ and $C = C(q, L(\cdot))$, with the convention of Remark 4.3.

The rest of the proof is devoted to bounding

$$T_{q,\ell} := \sum_{\underline{i}\in\mathtt{s}(B_1^{q-\ell})}\sum_{\underline{j},\underline{m}\in\mathtt{s}(\{d,\dots,f\}^\ell)} U(\underline{i}\,\underline{j})U(\underline{i}\underline{m})U(d\underline{j}\underline{m}f). \tag{4.13}$$

This is relatively heavy, because, while $\underline{i}$, $\underline{j}$ and $\underline{m}$ are ordered, $\underline{i}\,\underline{j}$, $\underline{i}\underline{m}$ and $\underline{j}\underline{m}$ are not. We have therefore to estimate the contributions given by every mutual arrangement of $\underline{i}$, $\underline{j}$ and $\underline{m}$. This will be done in a systematic way with the help of a *diagram representation* (the diagrams will correspond to groups of configurations $\underline{i}$, $\underline{j}$ and $\underline{m}$ that have the same *mutual order*).

Fix q and ℓ and choose $\underline{i}\in\mathtt{s}(\{1,\dots,k\}^{q-\ell})$ and $\underline{j},\underline{m}\in\mathtt{s}(\in\{d,\dots,f\}^\ell)$. The construction of the diagram of $\underline{i}$, $\underline{j}$ and $\underline{m}$ is done in steps:

(1) Mark with $\square$'s on the horizontal axis (the dotted line in Figs 3 and 4) the positions $i_1\le i_2\le\dots\le i_{q-\ell}$. Do the same for $\underline{j}$ (using $\circ$) and $\underline{m}$ (using $\bullet$). As explained in Remark 4.4 below, we may and do assume that symbols do not sit on the same position (this amounts to assuming strict inequality between all indexes).

(2) Consider the set of $\square$'s and $\circ$'s, and connect all nearest neighbors with a line (the line may be straight or curved for the sake of visual clarity).

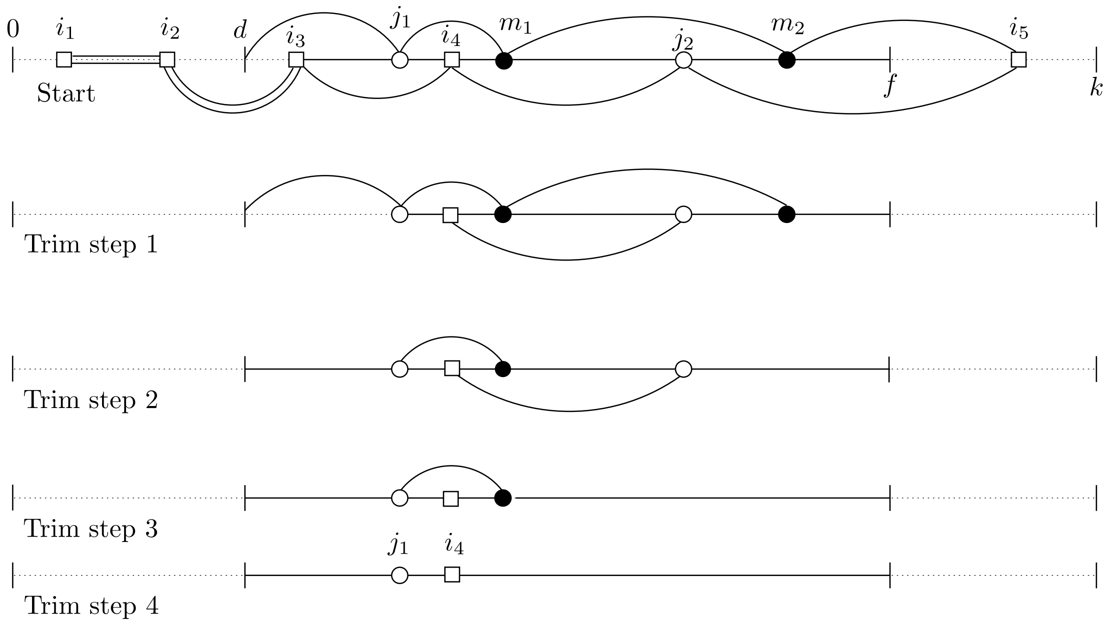

Fig. 3. A diagram arising for $q = 7$ and $\ell = 2$ and the successive trimming procedure explained in the text. In this case twin edges occur between two □'s which have neither ◦'s nor •'s between them.

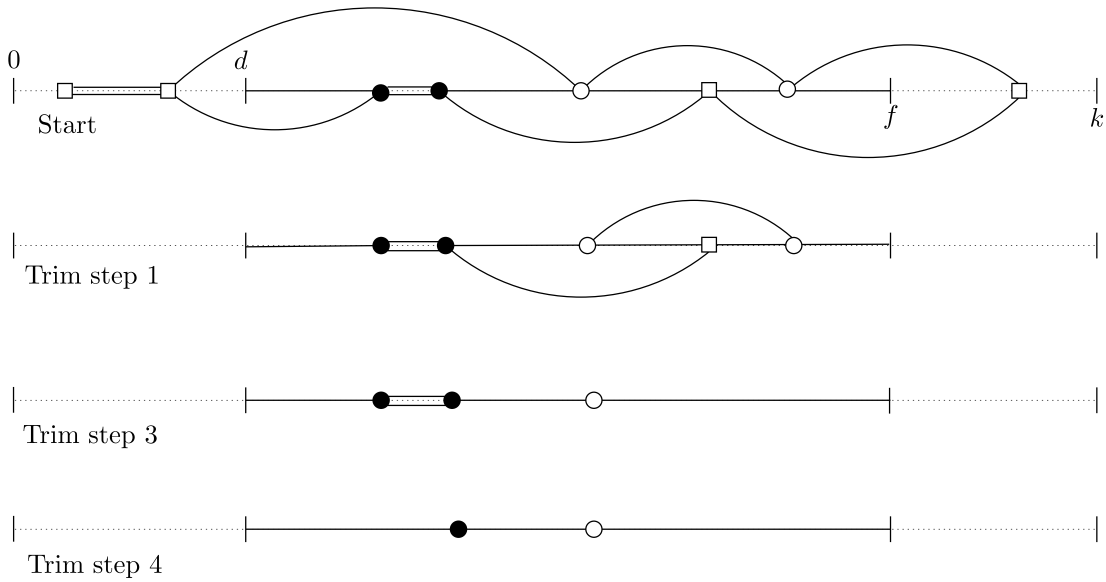

Fig. 4. Another diagram, this time for $q = 6$ and $\ell = 2$ and the successive trimming procedure explained in the text. In this case a twin edge occurs between two •'s with nothing in between. Note that at the third step the right-most (internal) vertex, a ◦, does not have two edges toward the left, so we continue the trimming procedure from the left.

(3) Do the same for the set of □'s and •'s.
(4) Do the same for the set of ◦'s and •'s.
(5) Consider the set of ◦'s and •'s and connect the element that is closest to d with d. Do the analogous action with the element which is closest to f. The point d is always to the left of ◦'s and •'s and the point f is always to the right.

We have now a graph with vertex set $\{d, f, \underline{i}, \underline{j}, \underline{m}\}$. Vertices have a type ($\square$, $\circ$ and $\bullet$): d and f have their own type too, graphically this type is $|$. We actually consider the *richer* graph with vertex set given by the points and the type of the point. The edges are the ones built with the above procedure; note that there may be double edges: we keep them and call them *twin* edges (see also the caption of Figs 3 and 4). Two indexes configurations are equivalent if they can be transformed into each other by translating the indexes without allowing them cross (and, of course, keeping their type; the vertices d and f are fixed). This leads to equivalence classes and a class is denoted by $\mathcal{G}$: we split the sum in (4.13) according to these classes, that is $T_{q,\ell} = \sum_{\mathcal{G}} T_{q,\ell,\mathcal{G}}$. The bound we are going to find is rather rough: we are going in fact to bound $\max_{\mathcal{G}} T_{q,\ell,\mathcal{G}}$. This is sufficient, since the number of equivalence classes depends only on q and ℓ.

Remark 4.4. *We have built equivalent classes of non-superposing points only. However in estimating $T_{q,\ell,\mathcal{G}}$ we will allow the index summations to include coinciding indexes so in the end we include (and over-estimate) the contributions of all the configurations of indexes.*

In order to estimate $T_{q,\ell,\mathcal{G}}$ we proceed to a graph trimming procedure that will be then matched to successive estimates on $T_{q,\ell,\mathcal{G}}$.

The trimming procedure is the following:

(1) If there are $\square$ vertices that are left of leftmost element of the set of $\circ$ and $\bullet$ vertices (we may call these $\square$ vertices *external vertices*), we erase them and we trim the edges linking them. Note that if we do this procedure left to right, we erase one vertex and two edges at a time: at each step we trim a couple of twin edges, except at the last step in which the edges are not twin. We do the same with the $\square$ vertices that are right of the rightmost element of the set of $\circ$ and $\bullet$ vertices (if any, of course). The trimming procedure goes this time right to left. We call *internal* the vertices that are left.
(2) Now we start (say) right and we erase the rightmost internal vertex (in this first step is necessarily a $\circ$ or a $\bullet$, later it may be a $\square$; we do not touch d and f) if it has two edges linking to vertices on the left (and it has one edge linking it to f). We trim these three edges and we add an edge linking the rightmost vertex (it can have any type among $\square$, $\circ$ and $\bullet$) that is still present to f with an edge.
(3) We repeat step (2) till it is possible. If it is no longer possible there are two possibilities: either one is left with only four vertices (among them, only one can be a $\square$) and three edges, see *Trim step* 4 in Fig. 3 (this is a fully trimmed configuration and the procedure stops), or we switch to the left and there is a vertex with two edges connecting it to vertexes on the right (and one connecting it to d on the left). In the second case we perform step (2) in a specular fashion, that is we trim the tree edges and we add an edge linking the leftmost vertex with d (this is the case of *Trim step* 4 in Fig. 4). We then repeat step (2) from the left till a fully trimmed configuration (four vertexes and three edges).

Let us now explain the link between the trimming procedure and quantitative estimates on $T_{q,\ell,\mathcal{G}}$. Also this is done by steps corresponding precisely to the three steps of the trimming procedure:

(1) Consider the external $\square$ vertices connected to the rest of the graph by twin edges, if any. We start by the leftmost (if there is at least one on the left: the procedure from the right is absolutely analogous) and notice that we can sum over the index, that is i_1, and use that, thanks to (2.2) (recall (1.14) and (2.5)), there exists C_L such that for $0 < n \le k$

$$\sum_{i=0}^{n} \big(R_{1/2}(n-i)\big)^2 \le C_L \widetilde{L}(k). \tag{4.14}$$

We are of course over-estimating the real sums that are, in most cases, restricted to small portions of B_1. This estimate allows *trimming* $T_{q,\ell,\mathcal{G}}$ in the sense that it gives the bound $T_{q,\ell,\mathcal{G}} \le C_L^r \widetilde{L}(k)^r T_{q-r,\ell,\mathcal{G}'}$, where r is the number of twin edges and $\mathcal{G}'$ is the graph, with $q-r+\ell$ vertices that is left after this procedure. This step can be repeated also for the last external $\Box$'s (there are at most two, one on the left and one on the right). In these cases we simply use that $R_{1/2}(\cdot)$ is decreasing so that if $0 \le n \le n'$

$$\sum_{i=0}^{n} R_{1/2}(n-i) R_{1/2}(n'-i) \le \sum_{i=0}^{n} \big(R_{1/2}(n-i)\big)^2, \tag{4.15}$$

and then (4.14) applies. So this extra trimming yields again $C_L \widetilde{L}(k)$ to the power of half the number of edges trimmed, that is, to the power of the number of the external vertices.

(2) We are left with the internal vertices and we start erasing the vertex (it is necessarily $\circ$ or $\bullet$ at this stage) which is most on the right. So we sum over its index and use the bound: there exists a constant C_L such that for $(0 \le) d \le n' \le n \le f (\le k)$ we have

$$\begin{aligned} \sum_{j=n}^{f} R_{1/2}(j-n) R_{1/2}(j-n') R_{1/2}(f-j) &\le \sum_{j=0}^{f-n} R_{1/2}(j)^2 R_{1/2}\big((f-n)-j\big) \\ &\le C_L \widetilde{L}(f-n) R_{1/2}(f-n) \le C_L \widetilde{L}(k) R_{1/2}(f-n), \end{aligned} \tag{4.16}$$

where in the first inequality we have used the monotonicity of $R_{1/2}(\cdot)$, in the second we have explicitly estimated the sum by using standard results on regularly varying function and (1.15). The last inequality is just the monotonicity of $\widetilde{L}(\cdot)$. This means that this trimming step brings once again a multiplicative factor $C_L \widetilde{L}(k)$: of course this time we have trimmed three edges, but we have also the extra factor $R_{1/2}(f-n)$ which is precisely the contribution of a longer edge that we rebuild (see Fig. 5).

(3) Keep repeating the previous step (the type of the vertices is not really important), trimming each time three edges, but rebuilding one too (so, in total, minus two edges), till the graph with four vertices and three edges. Of course the estimate (4.16) is absolutely analogous when summing over j between d and n.

In order to evaluate the contribution of all the trimming procedure we just need to count the number of vertices that we have erased: $q+\ell-2$. We are now left with the contribution given by the last diagram (four points, three edges: see, for example, *trim step* 4 in Figs 3 and 4), times of course $(C_L \widetilde{L}(k))^{q+\ell-2}$: we bound the last diagram using

$$\sum_{i=d}^{f} \sum_{j=i}^{f} R_{1/2}(i-d) R_{1/2}(j-i) R_{1/2}(f-j) \le C_L \frac{\sqrt{f-d}}{L(f-d)^3} \le C_{L,\varepsilon} \frac{k R_{1/2}(f-d)}{L(k)^2}, \tag{4.17}$$

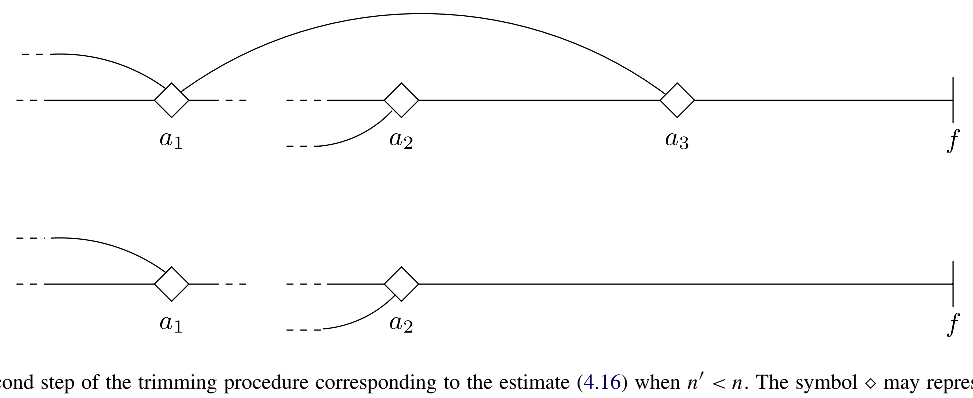

Fig. 5. The second step of the trimming procedure corresponding to the estimate (4.16) when $n' < n$. The symbol $\diamond$ may represent $\Box$, $\circ$ and $\bullet$: the choice is not fully arbitrary, in the sense that, for example, before starting the trimming procedure there is no edge between f (or d) and a $\Box$. However the estimate is independent of the type of symbols.

where C_L is once again a constant that depends only on $L(\cdot)$, while in the last step we have used $k \geq f - d \geq \varepsilon k$ and (2.4). Going back to (4.12) we see that there exists $C = C(\varepsilon, q, L(\cdot))$ such that (with the convention of Remark 4.3)

$$\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X - \widehat{\mathbb{E}}_\tau X)^2\big] \leq C\left(1 + \max_{\ell=1,2,\dots,q-1} \frac{1}{k\widetilde{L}(k)^{q-1}R_{1/2}(f-d)} \frac{\widetilde{L}(k)^{q+\ell-2}kR_{1/2}(f-d)}{L(k)^2}\mathtt{m}_\beta^{2\ell}\right)$$

$$= C\left(1 + \max_{\ell=1,2,\dots,q-1} \frac{\widetilde{L}(k)^{\ell-1}}{L(k)^2}\mathtt{m}_\beta^{2\ell}\right) \leq C\left(1 + \max_{\ell=1,2,\dots,q-1} \frac{\widetilde{L}(k)^{\ell-1}}{L(k)^2}\beta^{2\ell}\right), \tag{4.18}$$

where in the last line we have used $\mathtt{m}_\beta \leq 2\beta$, for $\beta \leq \beta_0$ (cf. (4.3)). We now recall (2.1) that guarantees that

$$\frac{\widetilde{L}(k-1)}{L(k-1)^{2/(q-1)}}\beta^{2q/(q-1)} < A \quad \text{so that} \quad \frac{\widetilde{L}(k)}{L(k)^{2/(q-1)}}\beta^{2q/(q-1)} \leq 2A, \tag{4.19}$$

where the second inequality is a consequence of the slowly varying character of $L(\cdot)$ and $\widetilde{L}(\cdot)$ and it holds for k sufficiently large. But this implies

$$\frac{\widetilde{L}(k)^{\ell-1}}{L(k)^2}\beta^{2\ell} \leq (2A)^{(q-1)\ell/q}\big(\widetilde{L}(k)L(k)^2\big)^{-1+(\ell/q)}, \tag{4.20}$$

so that, by (1.15), by choosing A large we can make the quantity in (4.20) arbitrarily small (recall that $\ell = 1, \dots, q-1$), so that going back to (4.18), we see that

$$\mathbf{E}_{d,f}\widehat{\mathbb{E}}_\tau\big[(X - \widehat{\mathbb{E}}_\tau X)^2\big] \leq C\big(\varepsilon, q, L(\cdot)\big)\Big(1 + A^{(q-1)^2/q} \max_{\ell=1,\dots,q-1}\big(\widetilde{L}(k)L(k)^2\big)^{-1+(\ell/q)}\Big) \leq A^{(q-1)^2/q}, \tag{4.21}$$

where in the last step we have used that, by (1.15), the maximum in the intermediate term can be made arbitrarily small, by choosing k large (that is, A larger than a constant depending on ε, q and $L(\cdot)$). This completes the proof of Lemma 4.2. □

5. Some probability estimates

Proof of Lemma 4.1. The proof is done in four steps.

Step 1: *Reduction to an asymptotic estimate on a constrained renewal.* In this step we show that it is sufficient to establish that for every $\zeta > 0$ there exists $\varrho > 0$ and $N_\zeta \in \mathbb{N}$ such that

$$\mathbf{P}\left(\frac{\mathbf{L}(N)}{\widetilde{\mathbf{L}}(N)^{(q-1)/2}} \sum_{\underline{i}\in\{0,\dots,N\}^q} V_N(\underline{i})\delta_{\underline{i}} \geq \varrho \;\Big|\; N \in \tau\right) \geq 1 - \zeta \tag{5.1}$$

for $N \geq N_\zeta$.

Notice in fact that $\widehat{\mathbb{E}}_\tau X = \mathtt{m}_\beta^q \sum_{\underline{i}} V_k(\underline{i})\delta_{\underline{i}}$, where $\underline{i} \in \{d, \dots, f\}^q$. Since $V_k(\underline{i})$ is invariant under the transformation $\underline{i} = (i_1, \dots, i_q) \mapsto (i_1 + n, \dots, i_q + n)$ (any $n \in \mathbb{Z}$), we may very well work on $\{0, \dots, f - d\}$, that is on an interval $\{0, \dots, N\}$ ($\varepsilon k \leq N \leq k$) and τ is a renewal with $\tau_0 = 0$ and conditioned to $N \in \tau$. With this change of variables, (4.5) reads

$$\mathbf{P}\left(\mathtt{m}_\beta^q \sum_{\underline{i}\in\{0,\dots,N\}^q} V_k(\underline{i})\delta_{\underline{i}} \geq aA^{(q-1)/2} \;\Big|\; N \in \tau\right) \geq 1 - \zeta. \tag{5.2}$$

Now two observations are in order:

- $V_k(\underline{i})/V_N(\underline{i}) = (N/k)^{1/2}[\widetilde{\mathbf{L}}(N)/\widetilde{\mathbf{L}}(k)]^{(q-1)/2}$ so that for k sufficiently large (that is for A larger than a constant depending on ε and $L(\cdot)$) we have

$$\frac{V_k(\underline{i})}{V_N(\underline{i})} \geq \frac{\varepsilon^{1/2}}{2}. \tag{5.3}$$

- By (4.3), (2.1) and (2.2) we see that

$$\mathtt{m}_\beta^q \geq 2^{-q} A^{(q-1)/2} \frac{L(k-1)}{\widetilde{L}(k-1)^{(q-1)/2}} \geq 2^{-q} \mathtt{c}_L A^{(q-1)/2} \frac{\mathbf{L}(N)}{\widetilde{\mathbf{L}}(N)^{(q-1)/2}}. \tag{5.4}$$

These two observations show that for A sufficiently large (5.2) is implied by

$$\mathbf{P}\biggl(\frac{\mathbf{L}(N)}{\widetilde{\mathbf{L}}(N)^{(q-1)/2}} \sum_{\underline{i} \in \{0,\dots,N\}^q} V_N(\underline{i}) \delta_{\underline{i}} \geq \frac{2^q}{\mathtt{c}_L \varepsilon^{1/2}} a \biggm| N \in \tau \biggr) \geq 1 - \zeta. \tag{5.5}$$

Therefore, at least if A is larger than a suitable constant depending on ε and $L(\cdot)$, it is sufficient to prove (5.1).

Step 2: *Removing the constraint.* In this step we claim that there exists a positive constant c, that depends only on $L(\cdot)$, such that if

$$\mathbf{P}\biggl(\frac{\mathbf{L}(N)}{\widetilde{\mathbf{L}}(N)^{(q-1)/2}} \sum_{\underline{i} \in \{1,\dots,\lfloor N/2 \rfloor\}^q} V_N(\underline{i}) \delta_{\underline{i}} \geq \varrho \biggr) \geq 1 - c\zeta, \tag{5.6}$$

then (5.1) holds. Note first of all that the random variable that we are estimating is smaller (since $V_N(\cdot) \geq 0$) than the random variable in (5.1), for every given τ-trajectory. It is therefore sufficient to bound the Radon–Nykodym derivative of the law of $\tau \cap [0, \lfloor N/2 \rfloor]$ without constraint $N \in \tau$ with respect to the law of the same random set with the constraint. Such an estimate can be found, for example, in [14], Lemma A.2.

Step 3: *Reduction to a convergence in law statement.* For $\rho := 1/(2(q-1))$ we define the subset $S_\rho(N)$ of $\mathtt{s}(\{0, 1, \dots, N\}^q)$ (recall that the latter is the set of increasingly rearranged $\underline{i}$ vectors) such that $i_j \leq N((j-1)\rho + (1/2))$ for $j = 1, 2, \dots, q$.

The claim of this step is that (5.6) follows if

$$\eta_N := \frac{\mathbf{L}(N)}{\widetilde{\mathbf{L}}(N)^{(q-1)/2}} \sum_{\underline{i} \in S_\rho(N)} V_N(\underline{i}) \delta_{\underline{i}} \quad \stackrel{N\to\infty}{\Longrightarrow} \quad \eta_\infty \quad \text{with } \eta_\infty > 0 \text{ a.s.}, \tag{5.7}$$

where $\Longrightarrow$ denotes convergence in law.

In order to see why (5.7) implies (5.6) it suffices to observe that replacing N with $\lfloor N/2 \rfloor$ in (5.6) (except when it already appears as $\lfloor N/2 \rfloor$) introduces an error that can be bounded by a multiplicative constant (say, 2) for N sufficiently large, so that it suffices to show that $\mathbf{P}(\eta_N \geq 2\varrho) \geq 1 - c\zeta$. But (5.7) yields $\lim_N \mathbf{P}(\eta_N \geq 2\varrho) \geq \mathbf{P}(\eta_\infty \geq 3\varrho)$. At this point if we choose $\varrho := \varrho(\zeta)$ such that $\mathbf{P}(\eta_\infty \geq 3\varrho) = 1 - (c\zeta/2)$, we are assured that for N sufficiently large (how large depends on ζ) $\mathbf{P}(\eta_N \geq 2\varrho) \geq 1 - c\zeta$ and we are reduced to proving (5.7).

Step 4: *Proof of the convergence in law statement* (5.7). This step depends on the following lemma, that we prove just below.

Lemma 5.1. *For every* $\theta_0 \in (0, 1)$ *we have*

$$\lim_{N\to\infty} \sup_{\theta \in [\theta_0, 1]} \mathbf{E}\Biggl[\Biggl(\frac{1}{\widetilde{\mathbf{L}}(N)} \sum_{j=1}^{\lfloor \theta N \rfloor} R_{1/2}(j) \delta_j - \frac{\mathtt{c}}{2\pi} \Biggr)^2 \Biggr] = 0, \tag{5.8}$$

with $\mathtt{c} := \lim_{x\to\infty} \mathbf{L}(x)/L(x) (\in [1, \mathtt{c}_L^{-1}])$.

For $p = 1, 2, \dots, q$ we introduce the random variables

$$\eta_{N,p} := \biggl(\frac{2\pi}{\mathtt{c}} \biggr)^{p-q} \frac{\mathbf{L}(N)}{N^{1/2} \widetilde{\mathbf{L}}(N)^{p-1}} \sum_{i_1=0}^{[N/2]} \sum_{i_2=i_1+1}^{[(\rho+(1/2))N]} \cdots \sum_{i_p=i_{p-1}+1}^{[((p-1)\rho+(1/2))N]} \delta_{i_1} \prod_{r=2}^{p} R_{1/2}(i_r - i_{r-1}) \delta_{i_r}, \tag{5.9}$$

where the product in the right-hand side has to be read as 1 if $p = 1$ and, in this case, there is only the sum over i_1. First of all remark that $\eta_{N,q} = \sqrt{q!}\eta_N$ (recall (2.15)) and that $\eta_{N,p-1}$ is obtained from $\eta_{N,p}$ by removing the last term in the product, the corresponding sum and by multiplying by $2\pi\widetilde{\mathbf{L}}(N)/\mathtt{c}$. We now claim that Lemma 5.1 implies that for $p = 2, 3, \dots, q$,

$$\lim_{N\to\infty} \mathbf{E}\big[|\eta_{N,p} - \eta_{N,p-1}|\big] = 0, \tag{5.10}$$

which clearly reduces the problem of proving $\eta_N \Longrightarrow \eta_\infty$ to proving $\eta_{N,1} \Longrightarrow \eta_\infty$, and η_∞ has to be a positive random variable. But in fact we have

$$(2\pi/\mathtt{c})^{q-1}\frac{L(N)}{\mathbf{L}(N)}\eta_{N,1} = \frac{L(N)}{\sqrt{N}}\sum_{i=0}^{\lfloor N/2\rfloor}\delta_i \quad \overset{N\to\infty}{\Longrightarrow} \quad \frac{1}{2\sqrt{\pi}}|Z| \quad \big(Z \sim \mathcal{N}(0,1)\big). \tag{5.11}$$

The convergence in (5.11) is a standard result that we outline briefly. First of all for every choice of $n, m \in \mathbb{N}$ we have

$$\left\{\sum_{i=1}^{n}\delta_i < m\right\} = \{\tau_m > n\}, \tag{5.12}$$

so that the asymptotic law of the *normalized local time* of τ up to n, i.e. $L(n)n^{-1/2}\sum_{i=1}^n \delta_i$, is directly linked to the domain of attraction of the random variable τ_1. Explicitly, one directly verifies that for $\lambda > 0$

$$\mathbf{E}\big[\big(1-\exp(-\lambda\tau_1)\big)\big] \overset{\lambda\searrow 0}{\sim} 2\sqrt{\pi}L(1/\lambda)\sqrt{\lambda}, \tag{5.13}$$

so that, if $a(\cdot)$ is the asymptotic inverse of the regularly varying function $r(\cdot)$, defined by $r(x) := \sqrt{x}/L(x)$ for $x > 0$, that is $a(r(x)) \sim r(a(x)) \sim x$ for $x \to \infty$, we have

$$\lim_{N\to\infty}\mathbf{E}\big[\exp\big(-\lambda\tau_N/a(N)\big)\big] = \exp\big(-2\sqrt{\pi\lambda}\big) = \mathbf{E}\big[\exp(-\lambda Y)\big], \tag{5.14}$$

where Y is a positive random variable with density $f_Y(y)$ equal to $y^{-3/2}\exp(-\pi/y)$ (for $y > 0$). On the other hand, for $t > 0$ by (5.12) we have

$$\mathbf{P}\left(\frac{L(n)}{\sqrt{n}}\sum_{j=1}^{n}\delta_j < t\right) \overset{n\to\infty}{\sim} \mathbf{P}(\tau_{\lfloor t\sqrt{n}/L(n)\rfloor} > n). \tag{5.15}$$

Therefore if we observe that $a(t\sqrt{n}/L(n)) \sim t^2 a(\sqrt{n}/L(n)) \sim t^2 n$, for $n \to \infty$, we directly obtain that

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{L(n)}{\sqrt{n}}\sum_{j=1}^{n}\delta_j < t\right) \overset{n\to\infty}{\sim} \mathbf{P}\left(Y > \frac{1}{t^2}\right). \tag{5.16}$$

By using the (explicit) density of Y, one directly verifies that $\mathbf{P}(Y > 1/t^2)$ coincides with $\mathbf{P}(|Z|/\sqrt{2\pi} < t)$ for every $t > 0$, that is (5.11) is established (recall that in (5.11) the summation is up to $N/2$).

We are therefore left with proving (5.10). This follows by observing that for $p = 3, 4, \dots, q$,

$$\begin{aligned}
\mathbf{E}\big[|\eta_{N,p} - \eta_{N,p-1}|\big] \le{}& \left(\frac{2\pi}{\mathtt{c}}\right)^{p-q}\frac{\mathbf{L}(N)}{N^{1/2}\widetilde{\mathbf{L}}(N)^{p-2}} \\
&\times \sum_{i_1=0}^{\lfloor N/2\rfloor}\sum_{i_2=i_1+1}^{\lfloor(\rho+(1/2))N\rfloor}\cdots\sum_{i_{p-1}=i_{p-2}+1}^{\lfloor((p-2)\rho+(1/2))N\rfloor}\mathbf{E}\left[\delta_{i_1}\prod_{r=2}^{p-1}R_{1/2}(i_r - i_{r-1})\delta_{i_r}\right] \\
&\times \mathbf{E}\left[\left|\frac{1}{\widetilde{\mathbf{L}}(N)}\sum_{i_p=i_{p-1}+1}^{\lfloor((p-1)\rho+(1/2))N\rfloor}R_{1/2}(i_p - i_{p-1})\delta_{i_p} - \frac{\mathtt{c}}{2\pi}\right|\middle|\delta_{i_{p-1}} = 1\right],
\end{aligned} \tag{5.17}$$

and the same expression holds if $p=2$ but in this case the external summation is only over i_1 and $\prod_{r=2}^{p-1} R_{1/2}(i_r - i_{r-1})\delta_{i_r}$ is replaced by 1. The bound (5.17) follows from the triangular inequality and from the renewal property of τ. Next, note that

$$\begin{aligned}
&\mathbf{E}\left[\left|\frac{1}{\widetilde{\mathbf{L}}(N)}\sum_{i_p=i_{p-1}+1}^{\lfloor((p-1)\rho+(1/2))N\rfloor} R_{1/2}(i_p-i_{p-1})\delta_{i_p} - \frac{\mathtt{c}}{2\pi}\right|\Big|\delta_{i_{p-1}}=1\right] \\
&\quad = \mathbf{E}\left[\left|\frac{1}{\widetilde{\mathbf{L}}(N)}\sum_{i=1}^{\lfloor((p-1)\rho+(1/2))N\rfloor - i_{p-1}} R_{1/2}(i)\delta_i - \frac{\mathtt{c}}{2\pi}\right|\right] \stackrel{N\to\infty}{\longrightarrow} 0, \qquad (5.18)
\end{aligned}$$

uniformly in the choice of $i_{p-1}\in\{i_{p-2}+1,\ldots,\lfloor((p-1)\rho+(1/2)N\rfloor\}$. This is because the summation in (5.18) contains at least $[\rho N]$ terms (and no more than N) so that we can apply Lemma 5.1. The fact that $\mathbf{E}[|\eta_{N,p}-\eta_{N,p-1}|] = \mathrm{o}(1)$ as $N\to\infty$ is therefore a consequence of the following explicit estimate:

$$\begin{aligned}
&\frac{\mathbf{L}(N)}{N^{1/2}\widetilde{\mathbf{L}}(N)^{p-2}}\sum_{i_1=0}^{\lfloor N/2\rfloor}\sum_{i_2=i_1+1}^{\lfloor(\rho+(1/2))N\rfloor}\cdots\sum_{i_{p-1}=i_{p-2}+1}^{\lfloor((p-2)\rho+(1/2))N\rfloor}\mathbf{E}\left[\delta_{i_1}\prod_{r=2}^{p-1}R_{1/2}(i_r-i_{r-1})\delta_{i_r}\right] \\
&\quad \le \frac{\mathbf{L}(N)\mathtt{c}_L^{-(p-1)}}{N^{1/2}\widetilde{\mathbf{L}}(N)^{p-2}}\sum_{i_1=0}^{\lfloor N/2\rfloor}\sum_{i_2=i_1+1}^{\lfloor(\rho+(1/2))N\rfloor}\cdots\sum_{i_{p-1}=i_{p-2}+1}^{\lfloor((p-2)\rho+(1/2))N\rfloor} R_{1/2}(i_1)\prod_{r=2}^{p-1}\bigl(R_{1/2}(i_r-i_{r-1})\bigr)^2 \\
&\quad \stackrel{N\to\infty}{\sim} \sqrt{2}\,\mathtt{c}_L^{-(p-1)}, \qquad (5.19)
\end{aligned}$$

where we have used the definition (1.14) of the slowly varying function $\widetilde{\mathbf{L}}(\cdot)$ and the fact that $\int_0^x (y^{1/2}\mathbf{L}(y))^{-1}\,\mathrm{d}y \stackrel{x\to\infty}{\sim} 2x^{1/2}/\mathbf{L}(x)$. This completes the proof of Lemma 4.1. □

Proof of Lemma 5.1. This is very similar to the proof of Lemma 5.4 in [14] (that, in turn generalizes a result of Chung and Erdös [5]). We give it in detail in order to clarify the role of the slowly varying function.

First of all let us remark that

$$\frac{1}{\widetilde{\mathbf{L}}(N)}\sum_{j=1}^{[\theta N]} R_{1/2}(j)\mathbf{E}[\delta_j] \stackrel{N\to\infty}{\sim} \frac{\mathtt{c}\widetilde{\mathbf{L}}(\theta N)}{2\pi\widetilde{\mathbf{L}}(N)} \stackrel{N\to\infty}{\sim} \frac{\mathtt{c}}{2\pi}, \qquad (5.20)$$

where the last asymptotic relation holds uniformly in θ, when θ lies in a compact subinterval of $(0,\infty)$. The statement is therefore reduced to showing that the variance of

$$Y_n := \sum_{j=1}^{n} R_{1/2}(j)\delta_j \qquad (5.21)$$

is $\mathrm{o}(\widetilde{L}(n)^2)$.

Let us compute and start by observing that

$$\begin{aligned}
\mathrm{var}_{\mathbf{P}}(Y_n) &= \sum_{i,j=1}^{n} R_{1/2}(i)R_{1/2}(j)\bigl[\mathbf{E}[\delta_i\delta_j]-\mathbf{E}[\delta_i]\mathbf{E}[\delta_j]\bigr] \\
&= 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} R_{1/2}(i)R_{1/2}(j)\bigl[\mathbf{E}[\delta_i\delta_j]-\mathbf{E}[\delta_i]\mathbf{E}[\delta_j]\bigr] + \mathrm{O}\bigl(\widetilde{L}(n)\bigr) \\
&=: 2T_n + \mathrm{O}\bigl(\widetilde{L}(n)\bigr), \qquad (5.22)
\end{aligned}$$

and

$$
\begin{aligned}
T_n &= \sum_{i=1}^{n-1} R_{1/2}(i)\mathbf{E}[\delta_i]\Bigg[\sum_{j=1}^{n-i} R_{1/2}(i+j)\mathbf{E}[\delta_j] - \sum_{j=i+1}^{n} R_{1/2}(j)\mathbf{E}[\delta_j]\Bigg] \\
&\le \sum_{i=1}^{n-1} R_{1/2}(i)\mathbf{E}[\delta_i]\Bigg[\sum_{j=1}^{n-i} R_{1/2}(i+j)\mathbf{E}[\delta_j] - \sum_{j=i+1}^{n} R_{1/2}(i+j)\mathbf{E}[\delta_j]\Bigg] \\
&\le \sum_{i=1}^{n-1} R_{1/2}(i)\mathbf{E}[\delta_i]\sum_{j=1}^{i} R_{1/2}(i+j)\mathbf{E}[\delta_j] \le \sum_{i=1}^{n-1}\big(R_{1/2}(i)\big)^2\mathbf{E}[\delta_i]\sum_{j=1}^{i}\mathbf{E}[\delta_j] \\
&\le \mathrm{c}_L^{-2}\sum_{i=1}^{n-1}\big(R_{1/2}(i)\big)^3\sum_{j=1}^{i} R_{1/2}(j) \overset{n\to\infty}{\sim} 2\mathrm{c}_L^{-2}\int_0^n \frac{1}{(1+x)(\mathbf{L}(x))^4}\,\mathrm{d}x,
\end{aligned}
\tag{5.23}
$$

where the first three inequalities follow since $R_{1/2}(\cdot)$ is non-increasing and the fourth follows from (2.4). The conclusion of the proof follows now from Remark 5.2. □

Remark 5.2. *For $x \to \infty$*

$$
\int_0^x \frac{1}{(1+y)(\mathbf{L}(y))^4}\,\mathrm{d}y \ll \big(\widetilde{\mathbf{L}}(x)\big)^2, \tag{5.24}
$$

with $\widetilde{\mathbf{L}}(x)$ defined as in (1.14) *with $L(\cdot)$ replaced by $\mathbf{L}(\cdot)$. This is a consequence of* (1.15) (*which of course holds also for* $\mathbf{L}(\cdot)$):

$$
\int_0^x \frac{1}{(1+y)(\mathbf{L}(y))^4}\,\mathrm{d}y \ll \int_0^x \frac{1}{(1+y)(\mathbf{L}(y))^2}\widetilde{\mathbf{L}}(y)\,\mathrm{d}y \le \widetilde{\mathbf{L}}(x)\int_0^x \frac{1}{(1+y)(\mathbf{L}(y))^2}\,\mathrm{d}y, \tag{5.25}
$$

and the right-most term is $(\widetilde{\mathbf{L}}(x))^2$.

6. A general monotonicity result

We present now a very general result: we give it in our context but a look at the proof suffices to see that it holds also under substantially milder assumptions on the process τ.

Proposition 6.1. *The free energy $\mathrm{F}(\beta,h)$ is a non-increasing function of β on $[0,\infty)$. Therefore*:

(i) *$\beta \mapsto h_c(\beta)$ is a non-decreasing function of β.*
(ii) *There exists a critical value $\beta_c \in [0,\infty]$ such that $h_c(0) = h_c(\beta)$ if and only if $\beta \le \beta_c$.*

This result is of particular relevance when $\sum_n 1/(nL(n)^2) < \infty$, that is when for small β we have $h_c(\beta) = h_c(0)$ (cf. Section 1.4): in this case β_c is the transition point from the irrelevant disorder regime to the relevant one. But also in our set-up, in which $\sum_n 1/(nL(n)^2) = \infty$, it is of some use since it implies that it is sufficient to prove Theorem 1.7 for one value of $\beta_0 > 0$ and the statement holds also for any other value of β_0 (by accepting, of course, a worse estimate on the shift of the critical point if one follows the estimates quantitatively, see Remark 2.7).

Proof of Proposition 6.1. We just need to prove that $\beta \mapsto \mathrm{F}(\beta,h)$ is a non-increasing function on $[0,\infty)$ as the other points are a trivial consequence of this result. To do so, we prove that $\beta \mapsto \mathbb{E}[\log Z_{N,\omega}]$ is a non-increasing function of β, and pass to the limit. The proof is the adaptation of an argument used in [6] for directed polymers with bulk disorder to prove a similar result.

What we will show is

$$\frac{\partial}{\partial\beta}\mathbb{E}[\log Z_{N,\omega}] = \mathbb{E}\left[\frac{\partial}{\partial\beta}\log Z_{N,\omega}\right] \le 0. \tag{6.1}$$

The proof of the equality in (6.1) is standard and can be easily adapted from [6], Lemma 3.3. Recall now that $\mathtt{m}_\beta := \mathrm{M}'(\beta)/\mathrm{M}(\beta)$. We have

$$\begin{aligned}\mathbb{E}\left[\frac{\partial}{\partial\beta}\log Z_{N,\omega}\right] &= \mathbf{E}\left[\mathbb{E}\left[\frac{1}{Z_{N,\omega}}\sum_{n=1}^{N}(\omega_n - \mathtt{m}_\beta)\delta_n \exp\left(\sum_{n=1}^{N}\left[\beta\omega_n + h - \log \mathrm{M}(\beta)\right]\delta_n\right)\delta_N\right]\right]\\ &= \mathbf{E}\left[\exp\left(\sum_{n=1}^{N} h\delta_n\right)\delta_N \widehat{\mathbb{E}}_\tau\left[Z_{N,\omega}^{-1}\sum_{n=1}^{N}(\omega_n - \mathtt{m}_\beta)\delta_n\right]\right].\end{aligned} \tag{6.2}$$

For a fixed trajectory of the renewal, the probability measure $\widehat{\mathbb{P}}_\tau$ (recall definition (4.2)), is a product measure, so that, since $Z_{N,\omega}^{-1}$ is a decreasing function of ω and $\sum_{n=1}^{N}(\omega_n - \mathtt{m}_\beta)\delta_n$ is a non-decreasing function of ω, by the Harris–FKG inequality we have

$$\widehat{\mathbb{E}}_\tau\left[Z_{N,\omega}^{-1}\sum_{n=1}^{N}(\omega_n - \mathtt{m}_\beta)\delta_n\right] \le \widehat{\mathbb{E}}_\tau\left[Z_{N,\omega}^{-1}\right]\widehat{\mathbb{E}}_\tau\left[\sum_{n=1}^{N}(\omega_n - \mathtt{m}_\beta)\delta_n\right] = 0. \tag{6.3}$$

□

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References

[1] K. S. Alexander. The effect of disorder on polymer depinning transitions. *Comm. Math. Phys.* **279** (2008) 117–146. MR2377630
[2] K. S. Alexander and N. Zygouras. Quenched and annealed critical points in polymer pinning models. *Comm. Math. Phys.* **291** (2009) 659–689. MR2534789
[3] K. S. Alexander and N. Zygouras. Equality of critical points for polymer depinning transitions with loop exponent one. *Ann. Appl. Probab.* **20** (2010) 356–366. MR2582651
[4] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge Univ. Press, Cambridge, 1987. MR0898871
[5] K. L. Chung and P. Erdös. Probability limit theorems assuming only the first moment I. *Mem. Amer. Math. Soc.* **6** (1951) 1–19. MR0040612
[6] F. Comets and N. Yoshida. Directed polymers in random environment are diffusive at weak disorder. *Ann. Probab.* **34** (2006) 1746–1770. MR2271480
[7] B. Derrida, G. Giacomin, H. Lacoin and F. L. Toninelli. Fractional moment bounds and disorder relevance for pinning models. *Comm. Math. Phys.* **287** (2009) 867–887. MR2486665
[8] B. Derrida, V. Hakim and J. Vannimenus. Effect of disorder on two-dimensional wetting. *J. Stat. Phys.* **66** (1992) 1189–1213. MR1156401
[9] R. A. Doney. One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107** (1997) 451–465. MR1440141
[10] M. E. Fisher. Walks, walls, wetting, and melting. *J. Stat. Phys.* **34** (1984) 667–729. MR0751710
[11] G. Forgacs, J. M. Luck, T. M. Nieuwenhuizen and H. Orland. Wetting of a disordered substrate: Exact critical behavior in two dimensions. *Phys. Rev. Lett.* **57** (1986) 2184–2187.
[12] G. Giacomin. *Random Polymer Models*. Imperial College Press, London, 2007. MR2380992
[13] G. Giacomin. Renewal sequences, disordered potentials, and pinning phenomena. In *Spin Glasses: Statics and Dynamics* 235–270. *Progress in Probability* **62**. Birkhäuser, Basel, 2009.
[14] G. Giacomin, H. Lacoin and F. L. Toninelli. Marginal relevance of disorder for pinning models. *Comm. Pure Appl. Math.* **63** (2010) 233–265. MR2588461
[15] G. Giacomin and F. L. Toninelli. Smoothing effect of quenched disorder on polymer depinning transitions. *Comm. Math. Phys.* **266** (2006) 1–16; Smoothing of depinning transitions for directed polymers with quenched disorder. *Phys. Rev. Lett.* **96** (2006) 070602. MR2231963
[16] G. Giacomin and F. L. Toninelli. On the irrelevant disorder regime of pinning models. *Ann. Probab.* **37** (2009) 1841–1873. MR2561435
[17] A. B. Harris. Effect of random defects on the critical behaviour of ising models. *J. Phys. C* **7** (1974) 1671–1692.

[18] H. Lacoin. Hierarchical pinning model with site disorder: Disorder is marginally relevant. *Probab. Theory Related Fields*. To appear. Available at arXiv:0807.4864 [math.PR].
[19] F. L. Toninelli. A replica-coupling approach to disordered pinning models. *Comm. Math. Phys.* **280** (2008) 389–401. MR2395475
[20] F. L. Toninelli. Coarse graining, fractional moments and the critical slope of random copolymers. *Electron. J. Probab.* **14** (2009) 531–547. MR2480552