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Limit laws of transient excited random walks on integers

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Abstract. We consider excited random walks (ERWs) on $\mathbb{Z}$ with a bounded number of i.i.d. cookies per site without the non-negativity assumption on the drifts induced by the cookies. Kosygina and Zerner [15] have shown that when the total expected drift per site, δ, is larger than 1 then ERW is transient to the right and, moreover, for $\delta > 4$ under the averaged measure it obeys the Central Limit Theorem. We show that when $\delta \in (2,4]$ the limiting behavior of an appropriately centered and scaled excited random walk under the averaged measure is described by a strictly stable law with parameter $\delta/2$. Our method also extends the results obtained by Basdevant and Singh [2] for $\delta \in (1,2]$ under the non-negativity assumption to the setting which allows both positive and negative cookies.

Résumé. On considère des marches aléatoires excitées sur $\mathbb{Z}$ avec un nombre borné de cookies i.i.d. à chaque site, ceci sans l'hypothèse de positivité. Auparavent, Kosygina et Zerner [15] ont établi que si la dérive totale moyenne par site, δ, est strictement superieur à 1, alors la marche est transiente (vers la droite) et, de plus, pour $\delta > 4$ il y a un théorème central limite pour la position de la marche. Ici, on démontre que pour $\delta \in (2,4]$ cette position, convenablement centrée et réduite, converge vers une loi stable de paramètre $\delta/2$. L'approche permet également d'étendre les résultats de Basdevant et Singh [2] pour $\delta \in (1,2]$ à notre cadre plus général.

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1. Introduction and main results

Excited random walk (ERW) on $\mathbb{Z}^d$ was introduced by Benjamini and Wilson in [3]. They proposed to modify the nearest neighbor simple symmetric random walk by giving it a positive drift ("excitation") in the first coordinate direction upon reaching a previously unvisited site. If the site had been visited before, then the walk made unbiased jumps to one of its nearest neighbor sites. See [4,13,17] and references therein for further results about this particular model.

Zerner [19,20] generalized excited random walks by allowing to modify the transition probabilities at each site not just once but any number of times and, moreover, choosing them according to some probability distribution. He obtained the criteria for recurrence and transience and the law of large numbers for i.i.d. environments on $\mathbb{Z}^d$ and strips and also for general stationary ergodic environments on $\mathbb{Z}$. It turned out that this generalized model had interesting behavior even for $d = 1$, and this case was further studied in [1,2,18].

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Results obtained in all these works rely on the assumption that projections of all possible drifts on some fixed direction are non-negative. In fact, the branching processes framework introduced in [14] for random walks in random environment ($d = 1$) and employed in [1,2] for excited random walks, does not depend on the positivity assumption, and it seems natural to use this approach for extending the analysis to environments which allow both positive and negative drifts. This was done in [15], where the authors discussed recurrence and transience, laws of large numbers, positive speed, and the averaged central limit theorem for multi-excited random walks on $\mathbb{Z}$ in i.i.d. environments with bounded number of "excitations" per site. We postpone further discussion of known results for $d = 1$ and turn to a precise description of the model considered in this paper.

Given an arbitrary positive integer M let

$$\Omega_M := \big\{\big((\omega_z(i))_{i\in\mathbb{N}}\big)_{z\in\mathbb{Z}} \mid \omega_z(i) \in [0,1],\ \text{for } i \in \{1,2,\ldots,M\} \text{ and } \omega_z(i) = 1/2,\ \text{for } i > M, z \in \mathbb{Z}\big\}.$$

An element of Ω_M is called a cookie environment. For each $z \in \mathbb{Z}$, the sequence $\{\omega_z(i)\}_{i\in\mathbb{N}}$ can be thought of as a pile of cookies at site z, and $\omega_z(i)$ is referred to as "the ith cookie at z". The number $\omega_z(i)$ is equal to the transition probability from z to $z+1$ of a nearest-neighbor random walk upon the ith visit to z. If $\omega_z(i) > 1/2$ (resp., $\omega_z(i) < 1/2$) the corresponding cookie will be called positive (resp. negative), $\omega_z(i) = 1/2$ will correspond to a "placebo" cookie or, equivalently, the absence of an effective ith cookie at site z.

Let $\mathbb{P}$ be a probability measure on Ω_M, which satisfies the following two conditions:

(A1) Independence: the sequence $(\omega_z(\cdot))_{z\in\mathbb{Z}}$ is i.i.d. under $\mathbb{P}$.
(A2) Non-degeneracy:

$$\mathbb{E}\left[\prod_{i=1}^{M}\omega_0(i)\right] > 0 \quad \text{and} \quad \mathbb{E}\left[\prod_{i=1}^{M}\big(1-\omega_0(i)\big)\right] > 0.$$

Notice that we do not make any independence assumptions on the cookies at the same site.

It will be convenient to define our ERW model using a coin-toss construction. Let $(\Sigma, \mathcal{F})$ be some measurable space equipped with a family of probability measures $P_{x,\omega}$, $x \in \mathbb{Z}$, $\omega \in \Omega_M$, such that for each choice of $x \in \mathbb{Z}$ and $\omega \in \Omega_M$ we have ± 1-valued random variables $B_i^{(z)}$, $z \in \mathbb{Z}$, $i \ge 1$, which are independent under $P_{x,\omega}$ with distribution given by

$$P_{x,\omega}\big(B_i^{(z)} = 1\big) = \omega_z(i) \quad \text{and} \quad P_{x,\omega}\big(B_i^{(z)} = -1\big) = 1 - \omega_z(i). \tag{1.1}$$

Let X_0 be a random variable on $(\Sigma, \mathcal{F}, P_{x,\omega})$ such that $P_{x,\omega}(X_0 = x) = 1$. Then an ERW starting at $x \in \mathbb{Z}$ in the environment ω, $X := \{X_n\}_{n\ge 0}$, can be defined on the probability space $(\Sigma, \mathcal{F}, P_{x,\omega})$ by the relation

$$X_{n+1} := X_n + B^{(X_n)}_{\#\{r\le n \mid X_r = X_n\}}, \quad n \ge 0. \tag{1.2}$$

Informally speaking, upon each visit to a site the walker eats a cookie and makes one step to the right or to the left with probabilities prescribed by this cookie. Since $\omega_z(i) = 1/2$ for all $i > M$, the walker will make unbiased steps from z starting from the $(M+1)$th visit to z.

Events $\{B_i^{(z)} = 1\}$, $i \in \mathbb{N}$, $z \in \mathbb{Z}$, will be referred to as "successes" and events $\{B_i^{(z)} = -1\}$ will be called "failures".

The consumption of a cookie $\omega_z(i)$ induces a drift of size $2\omega_z(i) - 1$ with respect to $P_{x,\omega}$. Summing up over all cookies at one site and taking the expectation with respect to $\mathbb{P}$ gives the parameter

$$\delta := \mathbb{E}\left[\sum_{i\ge 1}\big(2\omega_0(i) - 1\big)\right] = \mathbb{E}\left[\sum_{i=1}^{M}\big(2\omega_0(i) - 1\big)\right], \tag{1.3}$$

which we call the *average total drift per site*. It plays a key role in the classification of the asymptotic behavior of the walk.

We notice that there is an obvious symmetry between positive and negative cookies: if the environment $(\omega_z)_{z\in\mathbb{Z}}$ is replaced by $(\omega'_z)_{z\in\mathbb{Z}}$ where $\omega'_z(i) = 1-\omega_z(i)$, for all $i\in\mathbb{N}$, $z\in\mathbb{Z}$, then $X' := \{X'_n\}_{n\ge 0}$, the ERW corresponding to the new environment, satisfies

$$X' \overset{\mathcal{D}}{=} -X, \tag{1.4}$$

where $\overset{\mathcal{D}}{=}$ denotes the equality in distribution. Thus, it is sufficient to consider, say, only non-negative δ (this, of course, allows both negative and positive cookies), and we shall always assume this to be the case.

Define the *averaged* measure P_x by setting $P_x(\cdot) = \mathbb{E}(P_{x,\omega}(\cdot))$. Below we summarize known results about this model.

Theorem 1.1 ([15]). *Assume* (A1) *and* (A2).

(i) *If $\delta\in[0,1]$ then X is recurrent, i.e. for $\mathbb{P}$-a.a. ω it returns $P_{0,\omega}$-a.s. infinitely many times to its starting point. If $\delta>1$ then X is transient to the right, i.e. for $\mathbb{P}$-a.a. ω, $X_n\to\infty$ as $n\to\infty$ $P_{0,\omega}$-a.s.*

(ii) *There is a deterministic $v\in[0,1]$ such that X satisfies for $\mathbb{P}$-a.a. ω the strong law of large numbers,*

$$\lim_{n\to\infty}\frac{X_n}{n} = v, \quad P_{0,\omega}\text{-a.s.} \tag{1.5}$$

Moreover, $v=0$ for $\delta\in[0,2]$ and $v>0$ for $\delta>2$.

(iii) *If $\delta>4$ then the sequence*

$$B_t^n := \frac{X_{[tn]}-[tn]v}{\sqrt{n}}, \quad t\ge 0,$$

converges weakly under P_0 to a non-degenerate Brownian motion with respect to the Skorohod topology on the space of cádlág functions.

This theorem does not discuss the rate of growth of the ERW when it is transient but has zero linear speed ($1<\delta\le 2$). It also leaves open the question about fluctuations when $\delta\le 4$.

The rate of growth of the transient cookie walk with zero linear speed was studied in [2] for the case of deterministic spatially homogeneous non-negative cookie environments. For further discussion we need some notation for the limiting stable distributions that appear below. Given $\alpha\in(0,2]$ and $b>0$, denote by $Z_{\alpha,b}$ a random variable (on some probability space) whose characteristic function is determined by the relation

$$\log E e^{\mathrm{i}uZ_{\alpha,b}} = \begin{cases} -b|u|^\alpha\bigl(1-i\frac{u}{|u|}\tan\bigl(\frac{\pi\alpha}{2}\bigr)\bigr), & \text{if } \alpha\ne 1,\\ -b|u|\bigl(1+\frac{2\mathrm{i}}{\pi}\frac{u}{|u|}\log|u|\bigr), & \text{if } \alpha=1.\end{cases} \tag{1.6}$$

Observe that $Z_{2,b}$ is a centered normal random variable with variance $2b$. The weak convergence with respect to P_0 will be denoted by $\Rightarrow$.

Theorem 1.2 ([2]). *Let $\omega_z(i)=p_i\in[1/2,1)$, $i\in\mathbb{N}$ for all $z\in\mathbb{Z}$, where $p_i=1/2$ for $i>M$, and δ be as in* (1.3), *that is $\delta=\sum_{i=1}^M(2p_i-1)$.*

(i) *If $\delta\in(1,2)$ then there is a positive constant b such that as $n\to\infty$*

$$\frac{X_n}{n^{\delta/2}} \Rightarrow (Z_{\delta/2,b})^{-\delta/2}.$$

(ii) *If $\delta=2$ then $(X_n\log n)/n$ converges in probability to some constant $c>0$.*

The above results also hold if X_n is replaced by $\sup_{i\le n}X_i$ or $\inf_{i\ge n}X_i$.

The proof of Theorem 1.2 used the non-negativity of cookies, though this assumption does not seem to be essential for most parts of the proof. It is certainly possible that the approach presented in [2] could yield the same results without the non-negativity assumption.

The functional central limit theorem for ERWs with $\delta \in [0,1)$ in stationary ergodic non-negative cookie environments was obtained in [7]. The limiting process is shown to be Brownian motion perturbed at extrema (see, for example, [5,6]).

The main results of this paper deal with the case when $\delta \in (2,4]$, though they apply also to $\delta \in (1,2]$. Moreover, our approach provides an alternative proof of Theorem 1.2 for general cookie environments that satisfy conditions (A1) and (A2) (see Remark 9.2).

We establish the following theorem.

Theorem 1.3. *Let $T_n = \inf\{j \ge 0 | X_j = n\}$ and v be the speed of the ERW (see (1.5)). The following statements hold under the averaged measure P_0.*

(i) *If $\delta \in (2,4)$ then there is a constant $b > 0$ such that as $n \to \infty$*

$$\frac{T_n - v^{-1}n}{n^{2/\delta}} \Rightarrow Z_{\delta/2,b} \quad \text{and} \tag{1.7}$$

$$\frac{X_n - vn}{n^{2/\delta}} \Rightarrow -v^{1+2/\delta} Z_{\delta/2,b}. \tag{1.8}$$

(ii) *If $\delta = 4$ then there is a constant $b > 0$ such that as $n \to \infty$*

$$\frac{T_n - v^{-1}n}{\sqrt{n \log n}} \Rightarrow Z_{2,b} \quad \text{and} \tag{1.9}$$

$$\frac{X_n - vn}{\sqrt{n \log n}} \Rightarrow -v^{3/2} Z_{2,b}. \tag{1.10}$$

Moreover, (1.8) and (1.10) hold if X_n is replaced by $\sup_{i \le n} X_i$ or $\inf_{i \ge n} X_i$.

The paper is organized as follows. In Section 2 we recall the branching processes framework and formulate two statements (Theorems 2.1 and 2.2), from which we later infer Theorem 1.3. Section 3 explains the idea of the proof of Theorem 2.2 and studies properties of the approximating diffusion process. In Section 4 we determine sufficient conditions for the validity of Theorem 2.2. Section 5 contains the main technical lemma (Lemma 5.3). It is followed by three sections, where we use the results of Section 5 to verify the sufficient conditions of Section 4 and prove Theorem 2.1. The proof of Theorem 1.3 is given in Section 9. The Appendix contains proofs of several technical results.

2. Reduction to branching processes

Suppose that the random walk $\{X_n\}_{n \ge 0}$ starts at 0. Since $\delta \ge 0$, Lemma 5 of [15] implies that $P_0(T_n < \infty) = 1$ for all $n \in \mathbb{N}$. At first, we recall the framework used in [1,2,15]. The main ideas go back at least to [16] and [14].

For $n \in \mathbb{N}$ and $k \le n$ define

$$D_{n,k} = \sum_{j=0}^{T_n - 1} \mathbb{1}_{\{X_j = k, X_{j+1} = k-1\}},$$

the number of jumps from k to $k-1$ before time T_n. Then

$$T_n = n + 2\sum_{k \le n} D_{n,k} = n + 2 \sum_{0 \le k \le n} D_{n,k} + 2 \sum_{k < 0} D_{n,k}. \tag{2.1}$$

The last sum is bounded above by the total time spent by X_n below 0. When $\delta > 1$, i.e. X_n is transient to the right, the time spent below 0 is P_0-a.s. finite, and, therefore, for any $\alpha > 0$

$$\lim_{n \to \infty} \frac{\sum_{k<0} D_{n,k}}{n^\alpha} = 0, \quad P_0\text{-a.s.} \tag{2.2}$$

This will allow us to conclude that for transient ERWs fluctuations of T_n are determined by those of $\sum_{0\le k\le n} D_{n,k}$, once we have shown that the latter are of order $n^{2/\delta}$.

We now consider the "reversed" process $(D_{n,n}, D_{n,n-1}, \dots, D_{n,0})$. Obviously, $D_{n,n}=0$ for every $n\in\mathbb{N}$. Moreover, given $D_{n,n}, D_{n,n-1}, \dots, D_{n,k+1}$, we can write

$$D_{n,k} = \sum_{j=1}^{D_{n,k+1}+1} \bigl(\#\text{ of jumps from } k \text{ to } k-1 \text{ between the } (j-1)\text{th}$$
$$\text{and } j\text{th jump from } k \text{ to } k+1 \text{ before time } T_n\bigr), \quad k=0,1,\dots,n-1.$$

Here we used the observation that the number of jumps from k to $k+1$ before time T_n is equal to $D_{n,k+1}+1$ for all $k\le n-1$. The expression "between the 0th and the 1st jump" above should be understood as "prior to the 1st jump".

Fix an $\omega\in\Omega_M$ and denote by $F_m^{(k)}$ the number of "failures" in the sequence $B^{(k)}$ (see (1.1) with z replaced by k) before the mth "success". Then, given $D_{n,k+1}$,

$$D_{n,k} = F^{(k)}_{D_{n,k+1}+1}.$$

Since the sequences $B^{(k)}$, $k\in\mathbb{Z}$, are i.i.d. under P_0, we have that $F_m^{(k)} \stackrel{\mathcal{D}}{=} F_m^{(n-k-1)}$ and can conclude that the distribution of $(D_{n,n}, D_{n,n-1}, \dots, D_{n,0})$ coincides with that of $(V_0, V_1, \dots, V_n)$, where $V=\{V_k\}_{k\ge 0}$ is a Markov chain defined by

$$V_0=0, \qquad V_{k+1}=F^{(k)}_{V_k+1}, \quad k\ge 0.$$

For $x\ge 0$ we shall denote by $[x]$ the integer part of x and by P_x^V the measure associated to the process V, which starts with $[x]$ individuals in the 0th generation. Observe that V is a branching process with the following properties:

(i) V has exactly 1 immigrant in each generation (the immigration occurs before the reproduction) and, therefore, does not get absorbed at 0.

(ii) The number of offspring of the mth individual in generation k is given by the number of failures between the $(m-1)$th and mth success in the sequence $B^{(k)}$. In particular, if $V_k\ge M$ then the offspring distribution of each individual after the Mth one is Geom(1/2) (i.e., geometric on $\{0\}\cup\mathbb{N}$ with parameter 1/2).

Therefore (here and throughout taking any sum from k to ℓ for $k>\ell$ to be zero) we can write

$$V_{k+1} = \sum_{m=1}^{M\wedge(V_k+1)} \zeta_m^{(k)} + \sum_{m=1}^{V_k-M+1} \xi_m^{(k)}, \quad k\ge 0, \tag{2.3}$$

where $\{\xi_m^{(k)}; k\ge 0, m\ge 1\}$ are i.i.d. Geom(1/2) random variables, vectors $(\zeta_1^{(k)}, \zeta_2^{(k)}, \dots, \zeta_M^{(k)})$, $k\ge 0$, are i.i.d. under P_x^V and independent of $\{\xi_m^{(k)}; k\ge 0, m\ge 1\}$. For each $k\ge 0$ the random variables $\{\zeta_m^{(k)}\}_{m=1}^M$ are neither independent nor identically distributed, but, given that for some $j<M$

$$\sum_{m=1}^{j} \zeta_m^{(k)} \ge M,$$

that is all cookies at site k have been eaten before the jth jump from k to $(k+1)$, we are left with $\{\zeta_m^{(k)}\}_{m=j+1}^M$ that are independent Geom(1/2) random variables. Define

$$\sigma_0^V = \inf\{j>0 \mid V_j=0\}, \qquad S^V = \sum_{j=0}^{\sigma_0^V-1} V_j. \tag{2.4}$$

Detailed information about the tails of σ_0^V and S^V will enable us to use the renewal structure and characterize the behavior of $\sum_{0\le k\le n} D_{n,k}$, and, therefore, of T_n as $n\to\infty$ for transient ERWs. We shall show in Section 9 that the following two statements imply Theorem 1.3.

Theorem 2.1. *Let $\delta>0$. Then*

$$\lim_{n\to\infty} n^\delta P_0^V\big(\sigma_0^V>n\big)=C_1\in(0,\infty). \tag{2.5}$$

Theorem 2.2. *Let $\delta>0$. Then*

$$\lim_{n\to\infty} n^{\delta/2} P_0^V\big(S^V>n\big)=C_2\in(0,\infty). \tag{2.6}$$

Remark 2.3. *In fact, a weaker result than (2.5) is sufficient for our purpose: there is a constant B such that $n^\delta P_0^V(\sigma_0^V>n)\le B$ for all $n\in\mathbb{N}$ (see condition (A) in Lemma 4.1). We also would like to point out that the limits in (2.5) and (2.6) exist for every starting point $x\in\mathbb{N}\cup\{0\}$ with C_1 and C_2 depending on x. The proofs simply repeat those for $x=0$.*

For the model described in Theorem 1.2, the convergence (2.5) starting from $x\in\mathbb{N}$ is shown in [2], Proposition 3.1, (for $\delta\in(1,2)$), and (2.6) for $\delta\in(1,2]$ is the content of [2], Proposition 4.1. Theorem 2.1 can also be derived from the construction in [15] (see Lemma 17) and [12]. We use a different approach and obtain both results directly without using the Laplace transform and Tauberian theorems.

We close this section by introducing some additional notation. For $x\ge 0$ we set

$$\tau_x^V=\inf\{j>0\mid V_j\ge x\}, \tag{2.7}$$

$$\sigma_x^V=\inf\{j>0\mid V_j\le x\}. \tag{2.8}$$

We shall drop the superscript whenever there is no possibility of confusion.

When the random walk X is transient to the right, $P_y^V(\sigma_0^V<\infty)=1$ for every $y\ge 0$. This implies that $P_y^V(\sigma_x^V<\infty)=1$ for every $x\in[0,y)$.

Let us remark that when we later deal with a continuous process on $[0,\infty)$ we shall simply use the first hitting time of x to record the entrance time in $[x,\infty)$ (or $[0,x]$), given that the process starts outside of the mentioned interval. We hope that denoting the hitting time of x for such processes also by τ_x will not result in ambiguity.

3. The approximating diffusion process and its properties

The bottom-line of our approach is that the main features of branching process V killed upon reaching 0 are reasonably well described by a simple diffusion process.

The parameters of such diffusion processes can be easily computed at the heuristic level. For $V_k\ge M$, (2.3) implies that

$$V_{k+1}-V_k=\sum_{m=1}^{M}\zeta_m^{(k)}-M+1+\sum_{m=1}^{V_k-M+1}\big(\xi_m^{(k)}-1\big). \tag{3.1}$$

By conditioning on the number of successes in the first M tosses it is easy to compute (see Lemma 3.3 in [1] or Lemma 17 in [15] for details) that for all $x\ge 0$

$$E_x^V\left(\sum_{m=1}^{M}\zeta_m^{(k)}-M+1\right)=1-\delta. \tag{3.2}$$

The term $\sum_{m=1}^M\zeta_m^{(k)}-M+1$ is independent of $\sum_{m=1}^{V_k-M+1}(\xi_m^{(k)}-1)$. When V_k is large, the latter is approximately normal with mean 0 and variance essentially equal to $2V_k$.

Therefore, the relevant diffusion should be given by the following stochastic differential equation:

$$\mathrm{d}Y_t = (1-\delta)\,\mathrm{d}t + \sqrt{2Y_t}\,\mathrm{d}B_t, \qquad Y_0 = y > 0, \quad t \in \big[0, \tau_0^Y\big], \tag{3.3}$$

where for $x \ge 0$ we set

$$\tau_x^Y = \inf\{t \ge 0 \mid Y_t = x\}. \tag{3.4}$$

Throughout the rest of the paper, unless stated otherwise, we shall assume that $\delta > 0$. Observe that $\tau_0^Y < \infty$ a.s., since $2Y_t$ is a squared Bessel process of dimension $2(1-\delta) < 2$ (for a proof, set $a = 0$ and let $b \to \infty$ in part (ii) of Lemma 3.2).

The above heuristics are justified by the next lemma.

Lemma 3.1. *Let $Y = \{Y_t\}_{t\ge 0}$ be the solution of (3.3). Fix an arbitrary $\varepsilon > 0$. For $y \in (\varepsilon, \infty)$ let $V_0 = [ny]$, and define*

$$Y_t^{\varepsilon,n} = \frac{V_{[nt]\wedge\sigma_{\varepsilon n}^V}}{n}, \quad t \in [0, \infty),$$

where σ_x^V is given by (2.8). Then the sequence of processes $Y^{\varepsilon,n} = \{Y_t^{\varepsilon,n}\}_{t\ge 0}$ converges in distribution as $n \to \infty$ with respect to the Skorohod topology on the space of càdlàg functions to the stopped diffusion $Y^\varepsilon = \{Y_{t\wedge\tau_\varepsilon^Y}\}_{t\ge 0}$, $Y_0 = y$.

Proof. We simply apply the (much more general) results of [10]. We first note that our convergence result considers the processes up to the first entry into $(-\infty, \varepsilon]$ for $\varepsilon > 0$ fixed. So we can choose to modify the rules of evolution for V when $V_k \le \varepsilon n$: we consider the process $(V_k^{n,\varepsilon})_{k\ge 0}$ where, with the existing notation,

$$V_0^{n,\varepsilon} = [ny], \qquad V_{k+1}^{n,\varepsilon} = \sum_{m=1}^{M} \zeta_m^{(k)} + \sum_{m=1}^{V_k^{n,\varepsilon}\vee(\varepsilon n)-M+1} \xi_m^{(k)}, \quad k \ge 0. \tag{3.5}$$

Then (given the regularity of points for the limit process) it will suffice to show the convergence of processes

$$\widetilde{Y}_t^{\varepsilon,n} = \frac{V_{[nt]}^{n,\varepsilon}}{n}, \quad t \in [0, \infty),$$

to the solution of the stochastic integral equation

$$\mathrm{d}Y_t = (1-\delta)\,\mathrm{d}t + \sqrt{2(Y_t \vee \varepsilon)}\,\mathrm{d}B_t, \qquad Y_0 = y > 0, \quad t \in [0, \infty). \tag{3.6}$$

We can now apply Theorem 4.1 of Chapter 7 of [10] with $X_n(t) = \widetilde{Y}_t^{\varepsilon,n}$. The needed uniqueness of the martingale problem corresponding to operator

$$Gf = (x \vee \varepsilon) f'' + (1-\delta) f' \tag{3.7}$$

follows from [10], Chapter 5, Section 3 (Theorems 3.6 and 3.7 imply the distributional uniqueness for solutions of the corresponding stochastic integral equation, and Proposition 3.1 shows that this implies the uniqueness for the martingale problem). □

We shall see in a moment that this diffusion has the desired behavior of the extinction time and of the total area under the path before the extinction (see Lemmas 3.3 and 3.5). Unfortunately, these properties in conjunction with Lemma 3.1 do not automatically imply Theorems 2.1 and 2.2, and work needs to be done to "transfer" these results to the corresponding quantities of the process V. Nevertheless, Lemma 3.1 is very helpful when V stays large as we shall see later.

In the rest of this section we state and prove several facts about Y. When we need to specify that the process Y starts at y at time 0 we shall write Y^y. Again, whenever there is no ambiguity about which process is being considered we shall drop the superscript in τ_x^Y defined in (3.4).

Lemma 3.2. *Fix* $y > 0$.

(i) (*Scaling*) *Let* $\widetilde{Y} = \{\widetilde{Y}_t\}_{t\ge 0}$, *where* $\widetilde{Y}_t = \frac{Y^y_{ty}}{y}$. *Then* $\widetilde{Y} \overset{\mathcal{D}}{=} Y^1$.
(ii) (*Hitting probabilities*) *Let* $0 \le a < y < b$. *Then*

$$P^Y_y(\tau_a < \tau_b) = \frac{b^\delta - y^\delta}{b^\delta - a^\delta}.$$

Proof. Part (i) can be easily checked by Itô's formula applied to $\widetilde{Y}_t$ or seen from scaling properties of the generator. The proof of part (ii) is standard once we notice that the process $(Y^y_t)^\delta$ stopped upon reaching the boundary of $[a, b]$ is a martingale. We omit the details. □

Lemma 3.3. *Let* Y *be the diffusion process defined by* (3.3). *Then*

$$\lim_{x\to\infty} x^\delta P^Y_1(\tau_0 > x) = C_3 \in (0, \infty).$$

Proof. For every $\varepsilon > 0$ and for all $x > 1/\varepsilon$ we have by Lemma 3.2

$$\begin{aligned} x^\delta P^Y_1(\tau_0 > x) &\ge x^\delta P^Y_1(\tau_0 > x \mid \tau_{\varepsilon x} < \tau_0) P^Y_1(\tau_{\varepsilon x} < \tau_0) \\ &\ge x^\delta P^Y_{\varepsilon x}(\tau_0 > x)(\varepsilon x)^{-\delta} = \varepsilon^{-\delta} P^Y_1\left(\tau_0 > \varepsilon^{-1}\right) > 0. \end{aligned}$$

This implies that for each $\varepsilon > 0$

$$\liminf_{x\to\infty} x^\delta P^Y_1(\tau_0 > x) \ge \varepsilon^{-\delta} P^Y_1\left(\tau_0 > \varepsilon^{-1}\right) > 0.$$

Taking the $\limsup_{\varepsilon\to 0}$ in the right-hand side we get

$$\liminf_{x\to\infty} x^\delta P^Y_1(\tau_0 > x) \ge \limsup_{\varepsilon\to 0} \varepsilon^{-\delta} P^Y_1\left(\tau_0 > \varepsilon^{-1}\right) = \limsup_{x\to\infty} x^\delta P^Y_1(\tau_0 > x).$$

This would immediately imply the existence of a finite non-zero limit if we could show that

$$\limsup_{x\to\infty} x^\delta P^Y_1(\tau_0 > x) < \infty.$$

This is the content of the next lemma.

Lemma 3.4. *Let* Y *be the diffusion process defined by* (3.3). *Then*

$$\limsup_{x\to\infty} x^\delta P^Y_1(\tau_0 > x) < \infty.$$

The proof is very similar to the proof of the discrete version (see (A) in Lemma 4.1 and its proof in Section 6) and, thus, is omitted. □

The final result of this section can be viewed as the "continuous counterpart" of Theorem 2.2. It concerns the area under the path of Y.

Lemma 3.5. *Let* Y *be the diffusion process defined by* (3.3). *Then*

$$\lim_{y\to\infty} y^\delta P^Y_1\left(\int_0^{\tau_0} Y_t \, dt > y^2\right) = C_4 \in (0, \infty).$$

Proof. The proof uses scaling and follows the same steps as the proof of Lemma 3.3. For every $\varepsilon > 0$ and $y > 1/\varepsilon$ we have

$$\begin{aligned} y^\delta P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2\right) &\ge y^\delta P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2 | \tau_{\varepsilon y} < \tau_0\right) P_1^Y(\tau_{\varepsilon y} < \tau_0) \\ &\ge y^\delta P_{\varepsilon y}^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2\right)(\varepsilon y)^{-\delta} = \varepsilon^{-\delta} P_{\varepsilon y}^Y\left(\int_0^{\tau_0/(\varepsilon y)} Y_{\varepsilon y s}\,\mathrm{d}s > \frac{y}{\varepsilon}\right) \\ &= \varepsilon^{-\delta} P_{\varepsilon y}^Y\left(\int_0^{\tau_0/(\varepsilon y)} \frac{Y_{\varepsilon y s}}{\varepsilon y}\,\mathrm{d}s > \varepsilon^{-2}\right) = \varepsilon^{-\delta} P_1^Y\left(\int_0^{\tau_0} Y_s\,\mathrm{d}s > \varepsilon^{-2}\right) > 0. \end{aligned}$$

This calculation, in fact, just shows that

$$y^\delta P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2\right)$$

is a non-decreasing positive function of y. Therefore, we only need to prove that it is bounded as $y \to \infty$. But for $y > 1$

$$\begin{aligned} & y^\delta P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2\right) \\ &\quad = P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2 \big| \tau_y < \tau_0\right) + y^\delta P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > y^2, \tau_y > \tau_0\right) \\ &\quad \le 1 + y^\delta P_1^Y(\tau_0 > y, \tau_y > \tau_0) \le 1 + y^\delta P_1^Y(\tau_0 > y). \end{aligned}$$

An application of Lemma 3.4 finishes the proof. □

4. Conditions which imply Theorem 2.2

We have shown that the diffusion process Y has the desired asymptotic behavior of the area under the path up to the exit time τ_0^Y. In this section we give sufficient conditions under which we can "transfer" this result to the process V and obtain Theorem 2.2.

Lemma 4.1. *Suppose that*:

(A) *There is a constant B such that $n^\delta P_0^V(\sigma_0 > n) \le B$ for all $n \in \mathbb{N}$.*
(B) *For every $\varepsilon > 0$*

$$\lim_{n\to\infty} P_{\varepsilon n}^V\left(\sum_{i=0}^{\sigma_0 - 1} V_i > n^2\right) = P_1^Y\left(\int_0^{\tau_0} Y_t\,\mathrm{d}t > \varepsilon^{-2}\right).$$

(C) $\lim_{n\to\infty} n^\delta P_0^V(\tau_n < \sigma_0) = C_5$.

Then

$$\lim_{n\to\infty} n^\delta P_0^V\left(\sum_{i=0}^{\sigma_0 - 1} V_i > n^2\right) = C_4 C_5,$$

where C_4 is the constant from Lemma 3.5.

Proof. Fix an $\varepsilon \in (0,1)$ and split the path-space of V into two parts, the event $H_{n,\varepsilon} := \{\tau_{\varepsilon n}^V < \sigma_0^V\}$ and its complement, $H_{n,\varepsilon}^c$.

First, consider the behavior of the total progeny on the event $H_{n,\varepsilon}^c$. On $H_{n,\varepsilon}^c$, the process V stays below εn until the time σ_0. Estimating each V_i from above by εn and using (A) we get for all $n \in \mathbb{N}$

$$n^\delta P_0^V\left(\sum_{i=0}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon}^c\right) \le n^\delta P_0^V(\sigma_0 > n/\varepsilon) \le (2\varepsilon)^\delta B.$$

Therefore, for all $n \in \mathbb{N}$

$$0 \le n^\delta P_0^V\left(\sum_{i=0}^{\sigma_0-1} V_i > n^2\right) - n^\delta P_0^V\left(\sum_{i=0}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon}\right) \le (2\varepsilon)^\delta B.$$

Hence, we only need to deal with the total progeny on the event $H_{n,\varepsilon}$. The rough idea is that, on $H_{n,\varepsilon}$, it is not unnatural for the total progeny to be of order n^2. This means that the decay of the probability that the total progeny is over n^2 comes from the decay of the probability of $H_{n,\varepsilon}$, which is essentially given by condition (C). This would suffice if we could let $\varepsilon = 1$ but we need ε to be small, thus, some scaling is necessary to proceed with the argument, and this brings into play condition (B) and the result of Lemma 3.5.

To get a lower bound on $F_n := n^\delta P_0^V(\sum_{i=0}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon})$ we use monotonicity of V with respect to the initial number of particles, conditions (B) and (C), and Lemma 3.5:

$$\begin{aligned}\lim_{\varepsilon\to 0}\liminf_{n\to\infty} F_n &\ge \lim_{\varepsilon\to 0}\lim_{n\to\infty} n^\delta P_0^V(H_{n,\varepsilon}) P_{\varepsilon n}^V\left(\sum_{i=0}^{\sigma_0-1} V_i > n^2\right) \\ &= C_5 \lim_{\varepsilon\to 0} \varepsilon^{-\delta} P_1^Y\left(\int_0^{\tau_0} Y_t\, \mathrm{d}t > \varepsilon^{-2}\right) = C_4 C_5.\end{aligned}$$

For an upper bound on F_n we shall need two more parameters, $K \in (1, 1/\varepsilon)$ and $R > 1$. At the end, after taking the limits as $n \to \infty$ and then $\varepsilon \to 0$ we shall let $K \to \infty$ and $R \to 1$.

$$\begin{aligned}n^{-\delta} F_n &= P_0^V\left(\sum_{i=0}^{\tau_{\varepsilon n}-1} V_i + \sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2, H_{n,\varepsilon}\right) \\ &\le P_0^V\left(\sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2(1-K\varepsilon), \sum_{i=0}^{\tau_{\varepsilon n}-1} V_i \le K\varepsilon n^2, H_{n,\varepsilon}\right) \\ &\quad + P_0^V\left(\sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > K\varepsilon n^2, H_{n,\varepsilon}\right).\end{aligned}$$

We bound the first term on the right-hand side by the following sum:

$$P_0^V\left(\sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2(1-K\varepsilon), V_{\tau_{\varepsilon n}} \le R\varepsilon n, H_{n,\varepsilon}\right) + P_0^V\left(\sum_{i=\tau_{\varepsilon n}}^{\sigma_0-1} V_i > n^2(1-K\varepsilon), V_{\tau_{\varepsilon n}} > R\varepsilon n, H_{n,\varepsilon}\right).$$

Estimating these terms in an obvious way and putting everything back together we get

$$\begin{aligned}n^{-\delta} F_n &\le P_{R\varepsilon n}^V\left(\sum_{i=0}^{\sigma_0-1} V_i > n^2(1-K\varepsilon)\right) P_0^V(H_{n,\varepsilon}) \\ &\quad + P_0^V(V_{\tau_{\varepsilon n}} > R\varepsilon n, H_{n,\varepsilon}) + P_0^V\left(\sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > K\varepsilon n^2, H_{n,\varepsilon}\right) \\ &= (I) + (II) + (III).\end{aligned}$$

It only remains to multiply everything by n^δ and consider the upper limits.

Term $n^\delta(I)$ gives the upper bound C_4C_5 in the same way as we got a lower bound by sending $n\to\infty$, $\varepsilon\to 0$, and then $R\to 1$ and using easily verified continuity properties of the relevant distributions. Parameter K disappears when we let $\varepsilon\to 0$.

Term (II) is exponentially small in n for fixed ε and R (see Lemma 5.1), thus $n^\delta(II)$ goes to zero as $n\to\infty$.

Finally, since $V_i\le\varepsilon n$ for all $i<\tau_{\varepsilon n}$, we get

$$\begin{aligned} n^\delta P_0^V\Biggl(\sum_{i=0}^{\tau_{\varepsilon n}-1} V_i > K\varepsilon n^2, H_{n,\varepsilon}\Biggr) &\le n^\delta P_0^V(\tau_{\varepsilon n}>Kn, H_{n,\varepsilon}) \\ &\le n^\delta P_0^V(\sigma_0>Kn, H_{n,\varepsilon}) \le n^\delta P_0^V(\sigma_0>Kn)\le \frac{2^\delta B}{K^\delta}. \end{aligned}$$

□

5. Main tools

The main result of this section is Lemma 5.3, which is a discrete analog of Lemma 3.2(ii).

We start with two technical lemmas. The first one will be used many times throughout the paper.

Lemma 5.1. *There are constants $c_1,c_2>0$ and $N\in\mathbb{N}$ such that for every $x\ge N$ and $y\ge 0$,*

$$\sup_{0\le z<x} P_z^V(V_{\tau_x}>x+y\mid \tau_x<\sigma_0)\le c_1\bigl(\mathrm{e}^{-c_2y^2/x}+\mathrm{e}^{-c_2y}\bigr), \tag{5.1}$$

$$\sup_{x<z<4x} P_z^V(V_{\sigma_x\wedge\tau_{4x}}<x-y)\le c_1\mathrm{e}^{-c_2y^2/x}. \tag{5.2}$$

This statement is a consequence of the fact that the offspring distribution of V is essentially geometric. The proof is given in the Appendix.

Lemma 5.2. *Fix $a\in(1,2]$. Consider the process V with $|V_0-a^n|\le a^{2n/3}$ and let $\gamma=\inf\{k\ge 0|V_k\notin(a^{n-1},a^{n+1})\}$. Then for all sufficiently large n*

(i) $P^V\bigl(\operatorname{dist}\bigl(V_\gamma,\bigl(a^{n-1},a^{n+1}\bigr)\bigr)\ge a^{2(n-1)/3}\bigr)\le\exp\bigl(-a^{n/4}\bigr)$;

(ii) $\left|P^V\bigl(V_\gamma\le a^{n-1}\bigr)-\dfrac{a^\delta}{a^\delta+1}\right|\le a^{-n/4}$.

Part (i) is an immediate consequence of Lemma 5.1. The proof of part (ii) is basic but technical and is given in the Appendix.

Lemma 5.3 (Main lemma). *For each $a\in(1,2]$ there is an $\ell_0\in\mathbb{N}$ such that if $\ell,m,u,x\in\mathbb{N}$ satisfy $\ell_0\le\ell<m<u$ and $|x-a^m|\le a^{2m/3}$ then*

$$\frac{h_a^-(m)-1}{h_a^-(u)-1}\le P_x^V(\sigma_{a^\ell}>\tau_{a^u})\le\frac{h_a^+(m)-1}{h_a^+(u)-1},$$

where

$$h_a^\pm(i)=\prod_{r=\ell+1}^{i}\bigl(a^\delta\mp a^{-\lambda r}\bigr),\quad i>\ell,$$

and λ is some small positive number not depending on ℓ.

Remark 5.4. *It is to be noted that for fixed ℓ there are $K_1(\ell)$ and $K_2(\ell)$ such that*

$$K_1(\ell) \le \frac{h_a^{\pm}(i)}{a^{(i-\ell)\delta}} \le K_2(\ell) \quad \text{for all } i > \ell$$

and $K_j(\ell) \to 1$ as $\ell \to \infty$, $j = 1, 2$.

Proof of Lemma 5.3. We will show the upper bound by comparing the process V with another process $\widetilde{V}$, whose exit probabilities can be estimated by further reduction to an exit problem for a birth-and-death-like Markov chain.

For $i \in \mathbb{N}$ set $x_i = [a^i + a^{2i/3}]$. By monotonicity, it is enough to prove the upper bound when the starting point x is equal to x_m. Thus, we set $V_0 = x_m$. The comparison will be done in two steps.

Step 1. We shall construct a sequence of stopping times γ_i, $i \ge 0$, and a comparison process $\widetilde{V} = (\widetilde{V}_k)_{k\ge 0}$ with x_ℓ as an absorbing point so that $\widetilde{V}_k \ge V_k$ for all k before the absorption. Let $\gamma_0 = 0$,

$$\gamma_1 = \inf\{k > 0 \mid V_k \notin (a^{m-1}, a^{m+1})\}, \quad \widetilde{V}_k = V_k \text{ for } k = 0, 1, \ldots, \gamma_1 - 1,$$

and at time γ_1 add to V_{γ_1} the necessary number of particles to get

$$\widetilde{V}_{\gamma_1} = \begin{cases} x_{m-1}, & \text{if } V_{\gamma_1} \le a^{m-1}, \\ x_{m+j}, & \text{if } x_{m+j-1} < V_{\gamma_1} \le x_{m+j},\ j \in \mathbb{N}. \end{cases}$$

Clearly, $\widetilde{V}_{\gamma_1} \ge V_{\gamma_1}$. By construction, $\widetilde{V}_{\gamma_1} = x_n$ for some $n \ge m-1$, $n \ne m$. If $\widetilde{V}_{\gamma_1} = x_\ell$, then we stop the process.

Assume that we have already defined stopping times γ_r, $r = 0, 1, \ldots, i$, and the process $\widetilde{V}_k$ for all $k \le \gamma_i$ so that $\widetilde{V}_{\gamma_i} = x_n$ for some $n > \ell$. We define $\widetilde{V}_k$ for $k > \gamma_i$ by applying to it the same branching mechanism as for V, namely, (2.3) with V replaced by $\widetilde{V}$, $k \ge \gamma_i$. Denote by γ_{i+1} the first time after γ_i when $\widetilde{V}$ exits the interval (a^{n-1}, a^{n+1}). At time γ_{i+1}, if the process exited through the lower end of the interval then we set $\widetilde{V}_{\gamma_{i+1}} = x_{n-1}$, if the process exited the through the upper end we add to $\widetilde{V}$ the minimal number of particles needed to get $\widetilde{V}_{\gamma_{i+1}} = x_s$ for some $s > n$. If $\widetilde{V}_{\gamma_{i+1}} = x_\ell$, then we stop the process. Thus, we obtain a sequence of stopping times γ_i, $i \ge 0$, and the desired dominating process $\widetilde{V}$ absorbed at x_ℓ such that $\widetilde{V}_{\gamma_i} \in \{x_\ell, x_{\ell+1}, \ldots\}$, $i \ge 0$.

Step 2. Define a Markov chain $R = (R_j)_{j\ge 0}$ on $\{\ell, \ell+1, \ldots\}$ by setting

$$R_j = n \quad \text{if } \widetilde{V}_{\gamma_j} = x_n,\ j \ge 0.$$

The state ℓ is absorbing. Let $\sigma_\ell^R = \inf\{j \ge 0 | R_j = \ell\}$ and $\tau_u^R = \inf\{j \ge 0 | R_j \ge u\}$. By construction,

$$P_{x_m}^V(\sigma_{a^\ell}^V > \tau_{a^u}^V) \le P_{x_m}^{\widetilde{V}}(\sigma_{x_\ell}^{\widetilde{V}} > \tau_{x_u}^{\widetilde{V}}) = P_m^R(\sigma_\ell^R > \tau_u^R).$$

We shall show that $(h_a^+(R_j))_{j\ge 0}$ is a supermartingale with respect to the natural filtration. (We set $h_a^+(\ell) = 1$.) The optional stopping theorem and monotonicity of function h_a^+ will immediately imply the upper bound in the statement of the lemma.

For $i > \ell$ we have

$$E_i^R(h_a^+(R_1)) = h_a^+(i-1)P_i^R(R_1 = i-1) + h_a^+(i+1)P_i^R(R_1 = i+1) + \sum_{n=i+2}^{\infty} h_a^+(n)P_i^R(R_1 = n).$$

By the definition of h_a^+ this is less or equal than

$$h_a^+(i)\Bigg[(a^\delta - a^{-\lambda i})^{-1}P_i^R(R_1 = i-1) + (a^\delta - a^{-\lambda(i+1)})P_i^R(R_1 = i+1) + \sum_{n=i+2}^{\infty} a^{\delta(n-i)}P_i^R(R_1 = n)\Bigg]. \quad (5.3)$$

By Lemma 5.2 and Lemma 5.1 we have that for all $i > \ell$, where ℓ is chosen sufficiently large,

$$P_i^R(R_1 = i-1) = \frac{a^\delta}{a^\delta + 1} + O(a^{-i/4}),$$
$$P_i^R(R_1 = i+1) = \frac{1}{a^\delta + 1} + O(a^{-i/4}),$$

and

$$P_i^R(R_1 \ge n) \le P_{n-2}^R(R_1 \ge n) = \mathrm{O}\big(\exp(-a^{n/4})\big) \quad \text{for all } n \ge i+2.$$

Substituting this into (5.3) and performing elementary computations we obtain

$$E_i^R\big(h_a^+(R_1)\big) \le h_a^+(i)\left[1 - \frac{a^{-\lambda i}}{a^\delta + 1}\big(a^{-\lambda} - a^{-\delta}\big) + \mathrm{O}\big(a^{-2\lambda i}\big)\right] \le h_a^+(i),$$

provided that $\lambda < \min\{1/8, \delta\}$ and ℓ (therefore i) is sufficiently large.

For the lower bound we argue in a similar manner, except that now we choose $x_m = [a^m - a^{2m/3}] + 1$, assume that $V_0 = x_m$, and construct a comparison process $\widetilde{V} = (\widetilde{V}_k)_{k\ge 0}$ absorbed at x_ℓ so that $\widetilde{V}_0 = V_0$ and $\widetilde{V}_k \le V_k$ for all k before the absorption.

More precisely, we let $\gamma_0 = 0$ and assume that we have already defined stopping times γ_r, $r = 0, 1, \ldots, i$, and the process $\widetilde{V}_k$ for all $k \le \gamma_i$ so that $\widetilde{V}_{\gamma_i} = x_n$ for some $n \ne \ell$. We define $\widetilde{V}_k$ for $k > \gamma_i$ by (2.3) with V replaced by $\widetilde{V}$, $k \ge \gamma_i$. Denote by γ_{i+1} the first time after γ_i when $\widetilde{V}$ exits the interval (a^{n-1}, a^{n+1}). At time γ_{i+1}, if the process exited through the upper end of the interval we set $\widetilde{V}_{\gamma_{i+1}} = x_{n+1}$, if the process exited through the lower end we reduce the number of particles by removing the minimal number of particles to ensure that $\widetilde{V}_{\gamma_{i+1}} = x_s$ for some $s < n$. If $\widetilde{V}_{\gamma_{i+1}} \le x_\ell$, then we stop the process and redefine $\widetilde{V}_{\gamma_{i+1}}$ to be x_ℓ. This procedure allows us to obtain a sequence of stopping times γ_i, $i \ge 0$, and the desired comparison process $\widetilde{V}$ absorbed at x_ℓ such that $V_{\gamma_i} \in \{x_\ell, x_{\ell+1}, \ldots\}$, $i \ge 0$.

Next, just as in the proof of the upper bound, we construct a Markov chain $R = (R_j)_{j\ge 0}$ and show that $(h_a^-(R_j))_{j\ge 0}$ is a submartingale (with $h_a^-(\ell)$ defined to be 1). The optional stopping theorem and monotonicity of function h_a^- imply the lower bound. □

Corollary 5.5. *For each non-negative integer x there exists a constant $C_6 = C_6(x)$ such that for every $n \in \mathbb{N}$*

$$n^\delta P_x^V(\tau_n < \sigma_0) \le C_6. \tag{5.4}$$

Moreover, for each $\varepsilon > 0$ there is a constant $c_3 = c_3(\varepsilon)$ such that for all $n \in \mathbb{N}$

$$P_n^V(\sigma_0 > \tau_{c_3 n}) < \varepsilon. \tag{5.5}$$

Remark 5.6. *In fact*, (5.4) *will be substantially improved by Lemma* 8.1.

Proof of Corollary 5.5. We choose arbitrarily $a \in (1, 2]$ and an $\ell \ge \ell_0$ as in Lemma 5.3 but also such that $a^\ell > x$. We note that it is sufficient to prove the statement for n of the form $[a^u]$. We define stopping times β_i, $i \in \mathbb{N}$, by

$$\beta_1 = \inf\big\{k > 0 | V_k \ge a^{\ell+1}\big\},$$
$$\beta_{i+1} = \inf\big\{k > \beta_i:\ V_k \ge a^{\ell+1} \text{ and } \exists s \in (\beta_i, k) | V_s \le a^\ell\big\}.$$

Lemma 5.1 and the monotonicity of V with respect to its starting point imply that

$$\begin{aligned}
&P_x^V\big(\exists r \in [\beta_i, \beta_{i+1})\big| V_r \ge a^u | \beta_i < \sigma_0\big) \\
&\quad \le \frac{h_a^+(\ell+1) - 1}{h_a^+(u) - 1} + \sum_{k=\ell+1}^{\infty} P_x^V\big(V_{\beta_i} \ge a^k + a^{2k/3} | \beta_i < \sigma_0\big) \frac{h_a^+(k+1) - 1}{h_a^+(u) - 1} \\
&\quad = \frac{h_a^+(\ell+1) - 1}{h_a^+(u) - 1}\left(1 + \sum_{k=\ell+1}^{\infty} P_x^V\big(V_{\beta_i} \ge a^k + a^{2k/3} | \beta_i < \sigma_0\big) \frac{h_a^+(k+1) - 1}{h_a^+(\ell+1) - 1}\right) \\
&\quad \le \frac{2(h_a^+(\ell+1) - 1)}{h_a^+(u) - 1}
\end{aligned}$$

supposing, as we may, that ℓ was fixed sufficiently large. Thus,

$$a^{u\delta}P_x^V(\sigma_0>\tau_{a^u})\le 2a^{u\delta}\frac{(h_a^+(\ell+1)-1)}{h_a^+(u)-1}\sum_{i=1}^\infty P_x^V(\beta_i<\sigma_0).$$

The bound (5.4) now follows from noting that $P_x^V(\beta_i<\sigma_0)$ decays geometrically fast to zero (with a rate which may depend on ℓ but does not depend on u) and that $a^{u\delta}(h_a^+(u)-1)^{-1}$ is bounded in u (see Remark 5.4).

To prove (5.5) we notice that by Lemma 5.3 and (5.4) for all $n>a^\ell$

$$\begin{aligned}P_n^V(\sigma_0>\tau_{c_3n})&\le P_n^V(\sigma_0>\tau_{c_3n},\sigma_{a^\ell}>\tau_{c_3n})+P_n^V(\sigma_0>\tau_{c_3n},\sigma_{a^\ell}<\tau_{c_3n})\\&\le P_n^V(\sigma_{a^\ell}>\tau_{c_3n})+\frac{C_6(a^\ell)}{(c_3n)^\delta}=\mathrm{O}\big(c_3^{-\delta}\big).\end{aligned}$$

The constant c_3 can be chosen large enough to get (5.5) for all $n\in\mathbb{N}$. □

6. Proof of (A)

Proposition 6.1. *There is a constant $c_4>0$ such that for all $k,x\in\mathbb{N}$ and $y\ge 0$*

$$P_y^V\left(\sum_{r=1}^{\sigma_0}\mathbb{1}_{\{V_r\in[x,2x)\}}>2xk\right)\le P_y^V(\rho_0<\sigma_0)(1-c_4)^k, \tag{6.1}$$

where $\rho_0=\inf\{j\ge 0|V_j\in[x,2x)\}$.

Proof. First, observe that there is a constant $c>0$ such that for all $x\in\mathbb{N}$

$$\text{(i)}\quad P_{2x}^V(\sigma_{x/2}<x)>c;\qquad \text{(ii)}\quad P_{x/2}^V(\sigma_0<\tau_x)>c.$$

The inequality (i) is an immediate consequence of Lemma 3.1. To prove the second inequality, we fix $x_0\in\mathbb{N}$ and let $x>2x_0+1$. Then by Corollary 5.5

$$P_{x/2}^V(\sigma_0<\tau_x)=P_{x/2}^V(\sigma_0<\tau_x|\sigma_{x_0}<\tau_x)P_{x/2}^V(\sigma_{x_0}<\tau_x)\ge\big(1-C_6(x_0)x^{-\delta}\big)P_{x/2}^V(\sigma_{x_0}<\tau_x).$$

Choosing x_0 large enough and applying Lemma 5.3 to the last term in the right-hand side we obtain (ii) for all sufficiently large x. Adjusting the constant c if necessary we can extend (ii) to all $x\in\mathbb{N}$.

Next, we show that (i) and (ii) imply (6.1) with $c_4=c^2$. Denote by $\rho_0\ge 0$ the first entrance time of V in $[x,2x)$ and set

$$\rho_j=\inf\big\{r\ge\rho_{j-1}+2x\mid V_r\in[x,2x)\big\},\quad j\ge 1.$$

Notice that for each $j\ge 1$, the time spent by V in $[x,2x)$ during the time interval $[\rho_{j-1},\rho_j)$ is at most $2x$. If V spends more than $2xk$ units of time in $[x,2x)$ before time σ_0 then $\rho_k<\sigma_0^V$. Thus,

$$P_y^V\left(\sum_{r=1}^{\sigma_0}\mathbb{1}_{\{V_r\in[x,2x)\}}>2xk\right)\le P_y^V(\rho_k<\sigma_0)=P_y^V(\rho_k<\sigma_0|\rho_{k-1}<\sigma_0)P_y^V(\rho_{k-1}<\sigma_0).$$

Using the strong Markov property, monotonicity with respect to the starting point, and inequalities (i) and (ii) we get

$$\begin{aligned}P_y^V(\rho_k<\sigma_0|\rho_{k-1}<\sigma_0)&\le\max_{x\le z<2x}P_z^V(\rho_1<\sigma_0)\\&\le\max_{x\le z<2x}\big(P_z^V(\rho_1<\sigma_0,\sigma_{x/2}<x)+P_z^V(\rho_1<\sigma_0,\sigma_{x/2}\ge x)\big)\end{aligned}$$

$$\leq \max_{x\leq z<2x}\big(P_z^V(\rho_1<\sigma_0|\sigma_{x/2}<x)P_z^V(\sigma_{x/2}<x)+1-P_z^V(\sigma_{x/2}<x)\big)$$
$$\leq \max_{x\leq z<2x}\big(1-P_z^V(\sigma_{x/2}<x)\big(1-P_z^V(\rho_1<\sigma_0|\sigma_{x/2}<x)\big)\big)$$
$$\leq 1-P_{2x}^V(\sigma_{x/2}<x)\big(1-P_{x/2}^V(\rho_0<\sigma_0)\big)$$
$$\leq 1-P_{2x}^V(\sigma_{x/2}<x)P_{x/2}^V(\tau_x>\sigma_0)\leq 1-c^2.$$

Substituting this in (6.2) and iterating in k gives (6.1). □

Proposition 6.2. *For every $h>0$*

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} n^\delta P_0^V\left(\sum_{i=1}^{\sigma_0}\mathbb{1}_{\{V_i<\varepsilon n\}}>nh\right)=0.$$

Proof. Fix $\varepsilon\in(0,1)$ and let $k=k(n,\varepsilon)$ be the smallest integer such that $2^k\geq\varepsilon n$. Define intervals $I_i=[2^{k-i},2^{k-i+1})$, $i\in\mathbb{N}$, and events

$$A_i=\left\{\sum_{r=1}^{\sigma_0}\mathbb{1}_{\{V_r\in I_i\}}>\frac{h2^{i-1}}{\varepsilon i(i+1)}|I_i|\right\}.$$

Intervals I_i and events A_i depend on n but this is not reflected in our notation. Since $2^{k-1}<\varepsilon n$ and $\sum_{i=1}^k(i(i+1))^{-1}<1$, we have

$$\left\{\sum_{i=1}^{\sigma_0}\mathbb{1}_{\{V_i<\varepsilon n\}}>nh\right\}\subset\bigcup_{i=1}^k A_i,$$

and, therefore,

$$P_0^V\left(\sum_{i=1}^{\sigma_0}\mathbb{1}_{\{V_i<\varepsilon n\}}>nh\right)\leq\sum_{i=1}^k P(A_i).$$

Using (5.4) and (6.1) we get

$$\begin{aligned}n^\delta P_0^V\left(\sum_{i=1}^{\sigma_0}\mathbb{1}_{\{V_i<\varepsilon n\}}>nh\right)&\leq n^\delta\sum_{i=1}^k(1-c_4)^{[h2^{i-2}/(\varepsilon i(i+1))]}P_0^V(\tau_{2^{k-i}}<\sigma_0)\\&\leq C_6(0)\varepsilon^{-\delta}\sum_{i\geq 1}(1-c_4)^{[h2^{i-2}/(\varepsilon i(i+1))]}2^{i\delta},\end{aligned}$$

and this quantity vanishes as $\varepsilon\to 0$. □

Proof of (A). To obtain (A) we apply (5.4) and Proposition 6.2 to the right-hand side of the following inequality:

$$\begin{aligned}n^\delta P_0^V(\sigma_0>n)&\leq n^\delta P_0^V(\tau_{\varepsilon n}<\sigma_0)+n^\delta P_0^V(\sigma_0>n,\tau_{\varepsilon n}>\sigma_0)\\&\leq\varepsilon^{-\delta}(\varepsilon n)^\delta P_0^V(\tau_{\varepsilon n}<\sigma_0)+n^\delta P_0^V\left(\sum_{i=1}^{\sigma_0}\mathbb{1}_{\{V_i<\varepsilon n\}}>n\right).\end{aligned}$$

□

7. Proof of (B)

We shall need the following fact.

Proposition 7.1. *For each $\varepsilon > 0$, there is a constant $C_7 = C_7(\varepsilon) > 0$ such that*

$$P_n^V(\sigma_0 > C_7 n) < \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Proof. We have

$$P_n^V(\sigma_0 > C_7 n) \le P_n^V(\sigma_0 > \tau_{c_3 n}) + P_n^V(\sigma_0 > C_7 n, \sigma_0 < \tau_{c_3 n}).$$

Using (5.5) we can choose $c_3 > 1$ so that $P_n^V(\sigma_0 > \tau_{c_3 n}) < \varepsilon/2$ for all $n \in \mathbb{N}$. Thus, we only need to estimate the last term. Notice that it is bounded above by the probability that the occupation time of the interval $(0, c_3 n)$ up to the moment σ_0 exceeds $C_7 n$. The latter can be estimated by Markov inequality:

$$P_n^V\left(\sum_{r=1}^{\sigma_0} \mathbb{1}_{\{V_r < c_3 n\}} > C_7 n\right) \le (C_7 n)^{-1} E_n^V\left(\sum_{r=1}^{\sigma_0} \mathbb{1}_{\{V_r < c_3 n\}}\right).$$

We claim that the last expectation does not exceed $4nc_3/c_4$ and so we can take $C_7 > 8c_3/(\varepsilon c_4)$. Indeed, let m be the smallest positive integer such that $2^m \ge c_3 n$. Then writing the expectation of our non-negative integer-valued random variable as the sum of the probabilities of its tails and using (6.1) to estimate the tails we get

$$E_n^V\left(\sum_{r=1}^{\sigma_0^V} \mathbb{1}_{\{V_r < c_3 n\}}\right) \le \sum_{j=1}^m E_n^V\left(\sum_{r=1}^{\sigma_0^V} \mathbb{1}_{\{V_r \in [2^{j-1}, 2^j)\}}\right) \le \sum_{j=1}^m \frac{2^j}{c_4} \le \frac{2^{m+1}}{c_4} \le \frac{4nc_3}{c_4}.$$

□

Proof of (B). For every $\alpha \in (0, \varepsilon)$ and $\beta \in (0, 1)$ we have

$$P_{\varepsilon n}^V\left(\sum_{j=0}^{\sigma_{\alpha n}-1} V_j > n^2\right) \le P_{\varepsilon n}^V\left(\sum_{j=0}^{\sigma_0 - 1} V_j > n^2\right) \le P_{\varepsilon n}^V\left(\sum_{j=0}^{\sigma_{\alpha n}-1} V_j > (1-\beta)n^2\right) + P_{\varepsilon n}^V\left(\sum_{j=\sigma_{\alpha n}}^{\sigma_0 - 1} V_j > \beta n^2\right). \tag{7.1}$$

By Lemma 3.1 for every $R > 0$

$$\lim_{n\to\infty} P_{\varepsilon n}^V\left(\sum_{j=0}^{\sigma_{\alpha n}-1} V_j > Rn^2\right) = P_\varepsilon^Y\left(\int_0^{\sigma_\alpha} Y_s \, \mathrm{d}s > R\right), \tag{7.2}$$

since, as is easily verified, under law P_ε^Y the law of $\int_0^{\sigma_\alpha} Y_s \, \mathrm{d}s$ has no atoms. Next, we notice that for all $x, \beta > 0$

$$P_\varepsilon^Y\left(\int_0^{\sigma_0} Y_s \, \mathrm{d}s > (1+\beta)x\right) - P_\varepsilon^Y\left(\int_{\sigma_\alpha}^{\sigma_0} Y_s \, \mathrm{d}s > \beta x\right) \le P_\varepsilon^Y\left(\int_0^{\sigma_\alpha} Y_s \, \mathrm{d}s > x\right) \le P_\varepsilon^Y\left(\int_0^{\sigma_0} Y_s \, \mathrm{d}s > x\right). \tag{7.3}$$

By the strong Markov property and scaling,

$$P_\varepsilon^Y\left(\int_{\sigma_\alpha}^{\sigma_0} Y_s \, \mathrm{d}s > \beta x\right) = P_\alpha^Y\left(\int_0^{\sigma_0} Y_s \, \mathrm{d}s > \beta x\right) = P_1^Y\left(\int_0^{\sigma_0} Y_s \, \mathrm{d}s > \beta x \alpha^{-2}\right) \to 0 \quad \text{as } \alpha \to 0. \tag{7.4}$$

Letting $n \to \infty$, then $\alpha \to 0$, and finally $\beta \to 0$ we obtain from (7.1)–(7.4) that

$$\liminf_{n\to\infty} P_{\varepsilon n}^V\left(\sum_{j=0}^{\sigma_0-1} V_j > n^2\right) \ge P_1^Y\left(\int_0^{\sigma_0} Y_s \, \mathrm{d}s > \varepsilon^{-2}\right).$$

To get the matching upper bound it is enough to show that for every $\nu > 0$ and $\beta \in (0, 1)$ there is an $\alpha \in (0, \varepsilon)$ such that for all sufficiently large n

$$P_{\varepsilon n}^V\left(\sum_{j=\sigma_{\alpha n}}^{\sigma_0-1} V_j > \beta n^2\right) < 2\nu. \tag{7.5}$$

The left-hand side of (7.5) does not exceed

$$P^V_{\alpha n}(\tau_{c_3 n\alpha} < \sigma_0) + P^V_{\alpha n}\left(\sum_{j=0}^{\sigma_0-1} V_j > \beta n^2, \tau_{c_3 n\alpha} > \sigma_0\right).$$

Given ν, define $c_3(\nu)$ and $C_7(\nu)$ as in (5.5) and Proposition 7.1 respectively. Let $\alpha < \sqrt{\beta/(c_3C_7)}$. By (5.5) the first term above is less than ν for all $n > 1/\alpha$. On the set $\{\tau_{c_3n\alpha} > \sigma_0\}$ the process V is below $c_3n\alpha$ and, thus, the second term is bounded above by $P^V_{\alpha n}(\sigma_0 > (\beta n)/(c_3\alpha))$. By Proposition 7.1 and our choice of α the latter probability does not exceed ν. Using relations (7.1)–(7.4) and, again, the absence of atoms for the distribution of $\int_0^{\sigma_x} Y_s\,\mathrm{d}s$ under P^Y_1 for each $x \ge 0$, we get the desired upper bound. □

8. Proofs of (C) and Theorem 2.1

First we prove (C) of Lemma 4.1. Then using the approach of Lemma 4.1 we show the convergence claimed in Theorem 2.1.

The next lemma includes (C) as a special case ($k = 0$, $C_5 = f(0)$).

Lemma 8.1. *There is a function* $f : \mathbb{N} \cup \{0\} \to (0,\infty)$, *such that*

$$\lim_{n\to\infty} n^\delta P^V_k(\sigma_0 > \tau_n) = f(k) \quad \textit{for each integer } k \ge 0.$$

We shall need the following proposition.

Proposition 8.2. *For each* $a \in (1,2]$ *and* $k \ge 0$

$$\sum_{j=1}^{\infty} \left| a^\delta P^V_k(\sigma_0 > \tau_{a^j} | \sigma_0 > \tau_{a^{j-1}}) - 1 \right| < \infty. \tag{8.1}$$

Proof. By the monotonicity in the initial number of particles we get a lower bound: for all sufficiently large j

$$P^V_k(\sigma_0 > \tau_{a^j} | \sigma_0 > \tau_{a^{j-1}}) \ge P^V_{a^{j-1}}(\sigma_0 > \tau_{a^j}).$$

For an upper bound we need to take into account the possibility of a large "overshoot". Let $x = a^{j-1} + a^{2(j-1)/3}$, then

$$\begin{aligned} P^V_k(\sigma_0 > \tau_{a^j} | \sigma_0 > \tau_{a^{j-1}}) &\le P^V_x(\sigma_0 > \tau_{a^j}) \\ &\quad + P^V_k(\sigma_0 > \tau_{a^j} | \sigma_0 > \tau_{a^{j-1}}, V_{\tau_{a^{j-1}}} > x) P^V_k(V_{\tau_{a^{j-1}}} > x | \sigma_0 > \tau_{a^{j-1}}) \\ &\le P^V_x(\sigma_0 > \tau_{a^j}) + P^V_k(V_{\tau_{a^{j-1}}} > x | \sigma_0 > \tau_{a^{j-1}}). \end{aligned}$$

The last probability decays faster than any power of a^{-j} as $j \to \infty$ by Lemma 5.1. Therefore, it is enough to show the convergence of the series

$$\sum_{j=1}^{\infty} \left| a^\delta P^V_{x_{j-1}}(\sigma_0 > \tau_{a^j}) - 1 \right|,$$

where $x_j = a^j + \varepsilon_j$ and $0 \le \varepsilon_j \le a^{2j/3}$. We have for all sufficiently large j (with ℓ chosen appropriately for a as in Lemma 5.3),

$$\begin{aligned} &\left| P^V_{x_{j-1}}(\sigma_0 > \tau_{a^j}) - a^{-\delta} \right| \\ &\quad \le \left| P^V_{x_{j-1}}(\sigma_{a^\ell} > \tau_{a^j}) - a^{-\delta} \right| + P^V_{x_{j-1}}(\sigma_0 > \tau_{a^j} > \sigma_{a^\ell}) \end{aligned}$$

$$\le \left|P^V_{x_{j-1}}(\sigma_{a^\ell} > \tau_{a^j}) - a^{-\delta}\right| + P^V_{x_{j-1}}(\sigma_0 > \tau_{a^j} | \tau_{a^j} > \sigma_{a^\ell})$$
$$\le \left|P^V_{x_{j-1}}(\sigma_{a^\ell} > \tau_{a^j}) - a^{-\delta}\right| + P^V_{a^\ell}(\sigma_0 > \tau_{a^j}).$$

By Lemma 5.3

$$\sum_{j=1}^{\infty}\left|P^V_{x_{j-1}}(\sigma_{a^\ell} > \tau_{a^j}) - a^{-\delta}\right| < \infty,$$

and to complete the proof of (8.1) we invoke the bound provided by (5.4) for $x = [a^\ell]$. □

Proof of Lemma 8.1. Fix an arbitrary non-negative integer k and $a \in (1,2]$. For each $n > a$ there is an $m \in \mathbb{N}$ such that $a^m \le n < a^{m+1}$. We have

$$a^{m\delta} P^V_k(\sigma_0 > \tau_{a^{m+1}}) \le n^\delta P^V_k(\sigma_0 > \tau_n) \le a^{(m+1)\delta} P^V_k(\sigma_0 > \tau_{a^m}).$$

If we can show that

$$\lim_{m\to\infty} a^{m\delta} P^V_k(\sigma_0 > \tau_{a^m}) = g(a,k) > 0 \tag{8.2}$$

for some $g(a,k)$, then

$$0 < a^{-\delta} g(a,k) \le \liminf_{n\to\infty} n^\delta P^V_k(\sigma_0 > \tau_n) \le \limsup_{n\to\infty} n^\delta P^V_k(\sigma_0 > \tau_n) \le a^\delta g(a,k).$$

This implies

$$1 \le \frac{\limsup_{n\to\infty} n^\delta P^V_k(\sigma_0 > \tau_n)}{\liminf_{n\to\infty} n^\delta P^V_k(\sigma_0 > \tau_n)} \le a^{2\delta},$$

and we obtain the claimed result by letting a go to 1.

To show (8.2) we set $\ell = \min\{j \in \mathbb{N} | a^j > k\}$ and notice that for $m > \ell$

$$a^{m\delta} P^V_k(\sigma_0 > \tau_{a^m}) = a^\delta P^V_k(\sigma_0 > \tau_{a^m} | \sigma_0 > \tau_{a^{m-1}}) \times a^{(m-1)\delta} P^V_k(\sigma_0 > \tau_{a^{m-1}})$$
$$= \cdots = a^{\ell\delta} P^V_k(\sigma_0 > \tau_{a^\ell}) \prod_{j=\ell+1}^{m} a^\delta P^V_k(\sigma_0 > \tau_{a^j} | \sigma_0 > \tau_{a^{j-1}}).$$

Since all terms in the last product are strictly positive and $a^{\ell\delta} P^V_k(\sigma_0 > \tau_{a^\ell})$ does not depend on m, the convergence (8.2) follows from (8.1). □

Proof of Theorem 2.1. We will show that $\lim_{n\to\infty} n^\delta P^V_0(\sigma_0 > n) = C_3 C_5$, where C_3 and C_5 are the same as in Lemma 3.3 and condition (C).

We begin with a lower bound. Fix positive ε and $\beta \ll \varepsilon$. We have

$$P^V_0(\sigma_0 > n) \ge P^V_0(\tau_{n\varepsilon} < \sigma_0) P^V_{\varepsilon n}(\sigma_0 > n) \ge P^V_0(\tau_{\varepsilon n} < \sigma_0) P^V_{\varepsilon n}(\sigma_{\beta n} > n).$$

By (C), Lemma 3.1, and scaling (Lemma 3.2 (i))

$$\liminf_{n\to\infty} n^\delta P^V_0(\sigma_0 > n) \ge C_5 \varepsilon^{-\delta} P^Y_1\left(\tau^Y_{\beta/\varepsilon} > \varepsilon^{-1}\right).$$

Letting $\beta \to 0$ and then $\varepsilon \to 0$ we obtain via Lemma 3.3 that

$$\liminf_{n\to\infty} n^\delta P^V_0(\sigma_0 > n) \ge C_3 C_5.$$

The upper bound is slightly more complicated. First, notice that

$$n^\delta P_0^V(\sigma_0 > n, \tau_{\varepsilon n} > \sigma_0) \le n^\delta P_0^V\left(\sum_{i=1}^{\sigma_0} \mathbb{1}_{\{V_i < \varepsilon n\}} > n\right).$$

By Proposition 6.2 the right-hand side becomes negligible as $n \to \infty$ and then $\varepsilon \to 0$. Thus, it is enough to estimate $n^\delta P_0^V(\sigma_0 > n, \tau_{\varepsilon n} < \sigma_0)$. Let $R \in (1, 3/2)$. Then

$$\begin{aligned} n^\delta P_0^V(\sigma_0 > n, \tau_{\varepsilon n} < \sigma_0) \le{} & n^\delta P_0^V\bigl(\sigma_0 > \tau_{\varepsilon n} > (R-1)n\bigr) \\ & + n^\delta P_0^V(V_{\tau_{\varepsilon n}} > R\varepsilon n, \tau_{\varepsilon n} < \sigma_0) \\ & + n^\delta P_0^V\bigl(\sigma_0 - \tau_{\varepsilon n} > (2-R)n, V_{\tau_{\varepsilon n}} \le R\varepsilon n\bigr). \end{aligned}$$

By Proposition 6.2 the first term on the right-hand side vanishes for every fixed $R > 1$ when we let $n \to \infty$ and then $\varepsilon \to 0$. By Lemma 5.1 the $\limsup_{n\to\infty}$ of the second term is zero. Thus it will be sufficient to bound the last term. For $\beta \ll \varepsilon$ let us define $\sigma_{\beta n}^{\varepsilon n}$ to be the first time after $\tau_{\varepsilon n}$ that V falls below βn. Then the last term is bounded above by

$$\begin{aligned} & n^\delta\bigl(P_0^V\bigl(\sigma_0 - \sigma_{\beta n}^{\varepsilon n} > (R-1)n\bigr) + P_0^V\bigl(\sigma_{\beta n}^{\varepsilon n} - \tau_{\varepsilon n} > (3-2R)n, V_{\tau_{\varepsilon n}} \le R\varepsilon n\bigr)\bigr) \\ & \quad \le \bigl(P_{\beta n}^V\bigl(\sigma_0 > (R-1)n\bigr) + P_{R\varepsilon n}^V\bigl(\sigma_{\beta n} > (3-2R)n\bigr)\bigr) n^\delta P_0^V(\tau_{\varepsilon n} < \sigma_0). \end{aligned}$$

Taking $\limsup_{n\to\infty}$ and then letting $\beta \to 0$ we obtain (by Proposition 7.1, Lemma 3.1 and (C)) the following upper bound for $\limsup_{n\to\infty} n^\delta P_0^V(\sigma_0 - \tau_{\varepsilon n} > (2-R)n, V_{\tau_{\varepsilon n}} \le R\varepsilon n)$,

$$C_5 \varepsilon^{-\delta} P_{R\varepsilon}^Y\bigl(\tau_0 > (3-2R)\bigr) = C_5 \varepsilon^{-\delta} P_1^Y\bigl(\tau_0 > \varepsilon^{-1}(3-2R)/R\bigr).$$

As $\varepsilon \to 0$ and then $R \to 1$, the latter expression converges by Lemma 3.3 to $C_5 C_3$. This completes the proof. □

9. Proof of Theorem 1.3

Let $\delta > 2$. By (2.1) and (2.2), it is enough to show that as $n \to \infty$

$$\frac{2\sum_{k=0}^n D_{n,k} - (v^{-1} - 1)n}{n^{2/\delta}} \stackrel{\mathcal{D}}{=} \frac{2\sum_{j=0}^n V_j - (v^{-1} - 1)n}{n^{2/\delta}} \tag{9.1}$$

converges in distribution to $Z_{\delta/2,b}$ for some $b > 0$. Define the consecutive times when $V_j = 0$,

$$\sigma_{0,0} = 0, \qquad \sigma_{0,i} = \inf\{j > \sigma_{0,i-1} | V_j = 0\}, \quad i \in \mathbb{N},$$

the total progeny of V over each lifetime, $S_i = \sum_{j=\sigma_{0,i-1}}^{\sigma_{0,i}-1} V_j$, $i \in \mathbb{N}$, and the number of renewals up to time n, $N_n = \max\{i \ge 0 | \sigma_{0,i} \le n\}$. Then $(\sigma_{0,i} - \sigma_{0,i-1}, S_i)_{i\ge1}$ are i.i.d. under P_0^V. Moreover, $\sigma_{0,i} - \sigma_{0,i-1} \stackrel{\mathcal{D}}{=} \sigma_0^V$ and $S_i \stackrel{\mathcal{D}}{=} S^V$, $i \in \mathbb{N}$. By Theorem 2.2 the distribution of S^V is in the domain of attraction of the law of $Z_{\delta/2,\widetilde{b}}$ for some $\widetilde{b} > 0$ (see, for example, [8], Chapter 2, Theorem 7.7). Since by Theorem 2.1 (in fact, the upper bound (A) is sufficient) the second moment of σ_0^V is finite, it follows from standard renewal theory (see, for example, [9], Theorems II.5.1 and II.5.2) that

$$\frac{N_n}{n} \xrightarrow{\text{a.s.}} \lambda := (E_0^V \sigma_0)^{-1},$$

and for each $\varepsilon > 0$ there is $c_5 > 0$ such that $P_0^V(|N_n - \lambda n| > c_5\sqrt{n}) < \varepsilon$ for all sufficiently large n. Using the fact that $T_n/n \to v^{-1}$ a.s. as $n \to \infty$ and relations (2.1) and (9.1) we get that $E_0^V S_i = (v^{-1} - 1)/(2\lambda)$.

Proof of part (i). Let $\delta \in (2,4)$. We have

$$\frac{\sum_{j=0}^{n} V_j - (v^{-1}-1)n/2}{n^{2/\delta}} = \frac{\sum_{i=1}^{N_n}(S_i - E_0^V S_i)}{n^{2/\delta}} + E_0^V S_1 \frac{N_n - \lambda n}{n^{2/\delta}} + \frac{\sum_{j=\sigma_{0,N_n}}^{n} V_j}{n^{2/\delta}}.$$

By Theorem I.3.2 [9], the first term converges in distribution to $Z_{\delta/2,\widetilde{b}}$. The second term converges to zero in probability by the above mentioned facts from renewal theory. The last term is bounded above by $S_{N_{n+1}}/n^{2/\delta}$, which converges to zero in probability. This finishes the proof of (1.7), which immediately gives (1.8) with X_n replaced by $\sup_{i\le n} X_i$, since $\{\sup_{i\le n} X_i < m\} = \{T_m > n\}$.

Next we show (1.8) with X_n replaced by $\inf_{i\ge n} X_i$. The proof is the same as in, for example, [2], p. 849. We observe that for all $m, n, p \in \mathbb{N}$

$$\Big\{\sup_{i\le n} X_i < m\Big\} \subset \Big\{\inf_{i\ge n} X_i < m\Big\} \subset \Big\{\sup_{i\le n} X_i < m+p\Big\} \cup \Big\{\inf_{i\ge T_{m+p}} X_i < m\Big\}.$$

The following lemma completes the proof of part (i).

Lemma 9.1.

$$\lim_{k\to\infty} \sup_{n\ge 1} P_0\Big(\inf_{i\ge T_n} X_i < n-k\Big) = 0.$$

We postpone the proof of this lemma until the end of the section.

Proof of part (ii). Let $\delta = 4$. Theorem 2.2 implies that the distribution of S^V is in the domain of attraction of the normal distribution ([11], Chapter XVII.5). Norming constants are easily computed to be (see [11], Chapter XVII.5, formula (5.23) with $C = 1$) $\sqrt{C_2 n \log n}$. The constant b, which appears in the statement is equal to $C_2/2$. Relations (1.9) and (1.10) follow in the same way as for part (i). □

Proof of Lemma 9.1. Let $P_{n,k} := P_0(\inf_{i\ge T_n} X_i < n-k)$.

Step 1. The supremum over $n \ge 1$ can be reduced to the maximum over $n \in \{1,2,\dots,k\}$: $\sup_{n\ge 1} P_{n,k} = \max_{1\le n\le k} P_{n,k}$.

Indeed, consider $P_{k,k}$ and $P_{m+k,k}$ for $m \ge 1$. The corresponding events $\{\inf_{i\ge T_k} X_i < 0\}$ and $\{\inf_{i\ge T_{k+m}} X_i < m\}$ depend on the behavior of the process only at times when X_i is in $[0,\infty)$ and $[m,\infty)$, respectively. But at times T_0 and T_m the walk is at 0 and m, respectively, and the distributions of the environments starting from the current point to the right of it are the same under P_0. We conclude that $P_{k,k} = P_{m+k,k}$. This is essentially the content of Lemma 10 from [19]. The proof does not use the positivity of cookies so it can be applied here.

Step 2. We list four elementary properties of $\{P_{n,k}\}$, $n, k \ge 1$:

(a) $P_{n,k} \ge P_{n,k+m}$ for all $1 \le n \le k$ and $m \ge 0$;
(b) $P_{n,k+m}$ converges to 0 as $m \to \infty$ for each $k \ge n \ge 1$;
(c) $P_{n,k} \ge P_{n+m,k+m}$ for all $n \le k$ and $m \ge 0$;
(d) $P_{n+m,k+m}$ converges to 0 as $m \to \infty$ for each $k \ge n \ge 1$.

Inequality (a) is obvious. Part (b) follows from the transience of X. Namely, $\inf_{i\ge T_n} X_i > -\infty$ a.s. but $n-(k+m) \to -\infty$ as $m \to \infty$. Inequality (c) is also obvious: since $T_n < T_{m+n}$ we have

$$\Big\{\inf_{i\ge T_n} X_i < n-k\Big\} \supset \Big\{\inf_{i\ge T_{n+m}} X_i < n-k\Big\} = \Big\{\inf_{i\ge T_{n+m}} X_i < (n+m)-(k+m)\Big\}.$$

The convergence in (d) again follows from the transience: $X_i \to \infty$ as $i \to \infty$ a.s. implies that $\inf_{i\ge T_{m+n}} X_i \to \infty$ as $m \to \infty$ a.s. but $(k+m)-(n+m)$ stays constant.

Step 3. Take any $\varepsilon > 0$ and using (d) choose an m so that $P_{m,m} < \varepsilon$. Properties (a) and (c) imply that $P_{n,n+i} < \varepsilon$ for all $i \ge 0$ and $n \ge m$. Using (b), for $n = 1,2,\dots,m-1$ choose k_n so that $P_{n,k_n} < \varepsilon$. Let $K = \max_{1\le n\le m} k_n$ (naturally, we set $k_m = m$). Then $P_{n,k} < \varepsilon$ for all $n \le k$ and $k \ge K$ that is $\max_{1\le n\le k} P_{n,k} < \varepsilon$ for all $k \ge K$. □

Remark 9.2. *Theorems* 2.1 *and* 2.2 *imply Theorem* 1.2 *for general cookie environments satisfying conditions* (A1) *and* (A2). *The proof is the same as in Section* 6 *of* [2] *and uses Lemma* 9.1.

Appendix: Proofs of technical results

We shall need the following simple lemma.

Lemma A.1. *Let* $(\xi_i)_{i\in\mathbb{N}}$ *be i.i.d.* Geom(1/2) *random variables. There exists a constant* $c_6 > 0$ *such that for all* $x, y \in \mathbb{N}$

$$P\left(\sum_{i=1}^{x}(\xi_i - 1) \geq y\right) \leq \mathrm{e}^{-c_6 y^2/x} \vee \mathrm{e}^{-c_6 y}.$$

Proof. Let $\varphi(t) = \log E e^{t(\xi_1 - 1)} = -t - \log(2 - e^t)$, $t \in [0, \log 2)$. Then $\varphi'(0) = 0$ and there is a constant $C > 0$ such that $\varphi(t) \leq Ct^2$ for all $t \in [0, (\log 2)/2]$. By Chebyshev's inequality, for each $t \in [0, (\log 2)/2]$

$$\log P\left(\sum_{i=1}^{x}(\xi_i - 1) \geq y\right) \leq x\varphi(t) - yt \leq Cxt^2 - yt,$$

and, therefore,

$$\log P\left(\sum_{i=1}^{x}(\xi_i - 1) \geq y\right) \leq -\max_{t\in[0,(\log 2)/2]}\left(yt - Cxt^2\right) \leq -\frac{1}{4}\min\left\{\frac{y^2}{Cx}, y\log 2\right\}.$$

This gives the desired inequality with $c_6 = \frac{1}{4}\min\{C^{-1}, \log 2\}$. □

Proof of Lemma 5.1. We shall prove part (i). The proof of part (ii) is very similar and is omitted.

To prove part (i) it is enough to show the existence of $c_1, c_2 > 0$ such that (5.1) holds for all $x \geq 2M + 1$ and $y \geq 6M$. The extension to all $y \geq 0$ is done simply by replacing the constant c_1 with

$$\max\left\{c_1, \left(\mathrm{e}^{-c_2(6M)^2/(2M+1)} + \mathrm{e}^{-6Mc_2}\right)^{-1}\right\},$$

since the left-hand side of (5.1) is at most 1 and the right-hand side of (5.1) is increasing in x and decreasing in y. We have

$$\begin{aligned}
P_z(V_{\tau_x} > x + y, \tau_x < \sigma_0) &= \sum_{n=1}^{\infty} P_z(V_{\tau_x} > x + y, \tau_x = n, \tau_x < \sigma_0) \\
&= \sum_{n=1}^{\infty} P_z\left(V_n > x + y, V_n \geq x, \tau_x = n, \tau_x < \sigma_0\right) \\
&= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z\left(V_n > x + y, V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, \dots, n-2\}\right) \\
&= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z(V_n > x + y | V_n \geq x, V_{n-1} = r) \\
&\qquad \times P_z\left(V_n \geq x, V_{n-1} = r, 0 < V_j < x, j \in \{1, \dots, n-2\}\right)
\end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} \frac{P_z(V_n > x+y|V_{n-1}=r)}{P_z(V_n \geq x|V_{n-1}=r)} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)$$

$$= \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)$$

$$\leq \max_{0\leq r<x} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} \sum_{n=1}^{\infty}\sum_{r=1}^{x-1} P_z(\tau_x < \sigma_0, \tau_x = n, V_{n-1} = r)$$

$$= \max_{0\leq r<x} \frac{P_r(V_1 > x+y)}{P_r(V_1 \geq x)} P_z(\tau_x < \sigma_0).$$

Thus,

$$\max_{0\leq z<x} P_z(V_{\tau_x} > x+y|\tau_x < \sigma_0) \leq \max_{0\leq z<x} \frac{P_z(V_1 > x+y)}{P_z(V_1 \geq x)}.$$

To estimate the last ratio we recall that V_1 is the sum of the number of offspring produced by each of z particles and by the immigrant particle. The offspring distribution of at most M particles can be affected by the cookies. For notational convenience we shall use ξ_j, $j = 1, 2, \ldots, z+1$, to denote the number of offspring of the jth particle. All that was said in Section 2 about $\zeta_m^{(k)}$ applies now to ξ_m, $m = 1, 2, \ldots, M$, and ξ_m, $m > M$, are just i.i.d. Geom(1/2) random variables. Abbreviate $\xi_j - 1$ by ξ'_j, $j \in \mathbb{N}$. With this notation, $V_1 = z + 1 + \sum_{m=1}^{z+1} \xi'_m$. Let $\mathcal{G}_0 = \{\varnothing, \Omega\}$, $\mathcal{G}_n$, $n \in \mathbb{N}$, be the σ-algebra generated by $(\xi_m)_{m\leq n}$, and

$$N_x = \inf\left\{n \geq 1 \,\middle|\, z + 1 + \sum_{m=1}^{n} \xi'_m \geq x\right\}.$$

Then $\{V_1 > x+y\} \subset \{V_1 \geq x\} \subset \{N_x \leq z+1\}$ and

$$\frac{P_z(V_1 > x+y)}{P_z(V_1 \geq x)} = \frac{P_z(V_1 > x+y)}{E_z(\mathbb{1}_{\{N_x\leq z+1\}} P_z(V_1 \geq x|\mathcal{G}_{N_x}))}.$$

The proof of part (i) will be complete as soon as we show that:

(a) the is a constant $c_7 > 0$ such that $P_z(V_1 \geq x|\mathcal{G}_{N_x}) \geq c_7$ for all $0 \leq z < x$ and $x \geq 2M+1$;
(b) there is a constant $c_8 > 0$ such that for all $x \geq 2M+1$ and $y \geq 6M$

$$P_z(V_1 > x+y|N_x \leq z+1) \leq \left(e^{-c_8 y} \vee e^{-c_8 y^2/x}\right). \tag{A.1}$$

Proof of (a). Consider two cases: (i) $z \geq M$ and (ii) $x - z \geq M+1$. In case (i) at least one of ξ_m, $1 \leq m \leq z+1$, has a Geom(1/2) distribution. The distribution of each ξ_m is supported on the non-negative integers, and, trivially, $\sum_{m=1}^{M} \xi'_m \geq -M$. Therefore, on the event $\{N_x \leq z+1\}$

$$P_z(V_1 \geq x|\mathcal{G}_{N_x}) \geq P_z\left(\sum_{m=N_x+1}^{z+1} \xi'_m \geq 0 \,\middle|\, \mathcal{G}_{N_x}\right) \geq \min_{n\geq M+1} P\left(\sum_{m=M+1}^{n} \xi'_m \geq M\right) \geq c_7 > 0, \tag{A.2}$$

since for all $n \geq M+1$ the probabilities under the minimum sign are strictly positive and by the Central Limit Theorem

$$\lim_{n\to\infty} P\left(\sum_{m=M+1}^{n} \xi'_m \geq M\right) = \frac{1}{2}.$$

We recall that when $N_x = z+1$ the first sum in (A.2) is empty and the probability above is equal to 1, so the lower bond holds for this case as well.

Case (ii) is even simpler. We have $x - z \geq M + 1$. This implies that all cookies at site 0 have already been used for offspring of the first N_x particles. Therefore, the sum $\sum_{m=N_x+1}^{z+1} \xi'_m$ (when it is not zero) is equal to the non-trivial sum of centered Geom(1/2) random variables. Therefore, in case (ii) we again get (A.2) where the rightmost M is replaced by 0.

Proof of (b). Observe that

$$\{V_1 > x + y\} \subset \left\{ \sum_{m=N_x+1}^{z+1} \xi'_m \geq y/2 \right\} \cup \left\{ z + 1 + \sum_{m=1}^{N_x} \xi'_m - x \geq y/2 \right\}.$$

Using the assumption $y \geq 6M$ and Lemma A.1 we get

$$\begin{aligned}
P_z\left(\sum_{m=N_x+1}^{z+1} \xi'_m \geq y/2, N_x \leq z+1 \right) &= E_z\left(\mathbb{1}_{\{N_x \leq z+1\}} P_z\left(\sum_{m=N_x+1}^{z+1} \xi'_m \geq y/2 \,\Big|\, \mathcal{G}_{N_x} \right)\right) \\
&\leq \max_{1\leq n\leq x} P_z\left(\sum_{m=M+1}^{M+n} \xi'_m \geq y/2 - M \right) P_z(N_x \leq z+1) \\
&\leq \max_{1\leq n\leq x} P_z\left(\sum_{m=M+1}^{M+n} \xi'_m \geq y/3 \right) P_z(N_x \leq z+1) \\
&\leq \left(\mathrm{e}^{-c_6 y^2/(9x)} \vee \mathrm{e}^{-c_6 y/3} \right) P_z(N_x \leq z+1).
\end{aligned}$$

Finally, we estimate the probability of the second set:

$$\begin{aligned}
&P_z\left(z + 1 + \sum_{m=1}^{N_x} \xi'_m - x \geq y/2, N_x \leq z+1 \right) \\
&= \sum_{n=1}^{z+1} P_z\left(\sum_{m=1}^{N_x} \xi'_m \geq y/2 + x - z - 1, N_x = n \right) \\
&= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z\left(\sum_{m=1}^{n} \xi'_m \geq y/2 + x - z - 1, N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) \\
&= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z\left(\xi'_n \geq y/2 + x - z - 1 - \ell \,\Big|\, N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) P_z\left(N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) \\
&= \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z\left(\xi_n \geq y/2 + x - z - \ell \,\Big|\, \xi_n \geq x - z - \ell, N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) P_z\left(N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) \\
&\leq P_z(\xi_{M+1} \geq y/2 - M) \sum_{n=1}^{z+1} \sum_{\ell=1-n}^{x-z-2} P_z\left(N_x = n, \sum_{m=1}^{n-1} \xi'_m = \ell \right) \\
&\leq 2^{-y/3} P_z(N_x \leq z+1).
\end{aligned}$$

This finishes the proof. □

Proof of part (ii) of Lemma 5.2. Let $s \in C_0^\infty([0,\infty))$ be a non-negative function such that $s(x) = x^\delta$ on $(2/(3a), 3a/2)$. Fix an n such that $a^{n-1} > M$ and define the process $U^n := (U_k^n)_{k\ge 0}$ by

$$U_k^n = s\left(\frac{V_{k\wedge\gamma}}{a^n}\right).$$

We shall show that when n is large U^n is close to being a martingale (with respect to its natural filtration $(\mathcal{F}_k)_{k\ge 0}$). U^n is just a discrete version of the martingale used in the proof of Lemma 3.2.

On the event $\{\gamma > k\}$ we have

$$E\big(U_{k+1}^n|\mathcal{F}_k\big) = E\left(s\left(\frac{V_{k+1}}{a^n}\right)\Big|\mathcal{F}_k\right) = E\left(s\left(\frac{V_k}{a^n} + \frac{V_{k+1}-V_k}{a^n}\right)\Big|\mathcal{F}_k\right),$$

and

$$\begin{aligned} E\big(U_{k+1}^n|\mathcal{F}_k\big) - U_k^n &= E\left[s'\left(\frac{V_k}{a^n}\right)\frac{V_{k+1}-V_k}{a^n}\Big|\mathcal{F}_k\right] + \frac{1}{2}E\left[s''\left(\frac{V_k}{a^n}\right)\frac{(V_{k+1}-V_k)^2}{a^{2n}}\Big|\mathcal{F}_k\right] + r_k^n \\ &= -\frac{\delta-1}{a^n}s'\left(\frac{V_k}{a^n}\right) + \frac{1}{2}E\left[s''\left(\frac{V_k}{a^n}\right)\frac{(V_{k+1}-V_k)^2}{a^{2n}}\Big|\mathcal{F}_k\right] + r_k^n, \end{aligned}$$

where r_k^n is the error, which we shall estimate later. By (3.1), the second term on the right-hand side of the above equality is equal to

$$\frac{1}{2a^{2n}}s''\left(\frac{V_k}{a^n}\right)E\left[\left(1 + \sum_{m=1}^{M}\zeta_m^{(k)\prime} + \sum_{m=1}^{V_k-M+1}\xi_m^{(k)\prime}\right)^2\Big|\mathcal{F}_k\right],$$

where $\zeta_m^{(k)\prime} = \zeta_m^{(k)} - 1$ and $\xi_m^{(k)\prime} = \xi_m^{(k)} - 1$. Since $\sum_{m=1}^M \zeta_m^{(k)\prime}$ is independent from all $\xi_m^{(k)\prime}$, $m \ge 1$, and V_k, the last formula reduces to

$$\frac{1}{2a^{2n}}s''\left(\frac{V_k}{a^n}\right)\left(2(V_k - M + 1) + E\left[\left(1 + \sum_{m=1}^{M}\zeta_m^{(k)\prime}\right)^2\right]\right).$$

Using the fact that $xs''(x) + (1-\delta)s'(x) = 0$ for $x \in (2/(3a), 3a/2)$ we get that on the event $\{\gamma > k\}$

$$E\big(U_{k+1}^n|\mathcal{F}_k\big) - U_k^n = \frac{1}{a^{2n}}s''\left(\frac{V_k}{a^n}\right)\left(1 - M + \frac{1}{2}E\left[\left(1 + \sum_{m=1}^{M}\zeta_m^{(k)\prime}\right)^2\right]\right) + r_k^n.$$

The first term on the right-hand side is bounded in absolute value by K_1/a^{2n} for some constant K_1. Thus it remains to estimate r_k^n. By Taylor's expansion r_k^n is bounded by

$$\frac{1}{6}\|s'''\|_\infty E\left[\left(\frac{|V_{k+1}-V_k|}{a^n}\right)^3\Big|\mathcal{F}_k\right] \le \frac{1}{6}\|s'''\|_\infty \left(E\left[\left(\frac{V_{k+1}-V_k}{a^n}\right)^4\Big|\mathcal{F}_k\right]\right)^{3/4}.$$

Writing again the difference $V_{k+1} - V_k$ in terms of geometric random variables, using independence of $\sum_{m=1}^M \zeta_m^{(k)\prime}$ from all $\xi_m^{(k)\prime}$, $m \ge 1$, and the fact that $V_k < a^{n+1}$ on $\{\gamma > k\}$ we find that

$$E\left[\left(\frac{V_{k+1}-V_k}{a^n}\right)^4\Big|\mathcal{F}_k\right] \le \frac{K_2}{a^{2n}},$$

and, therefore, $|r_k^n| \le K_3/a^{3n/2}$. Let $R_0^n = 0$ and for $k \ge 1$ set

$$R_k^n = \sum_{j=1}^{k\wedge\gamma}\left[\frac{1}{a^{2n}}s''\left(\frac{V_j}{a^n}\right)\left(1 - M + \frac{1}{2}E\left[\left(1+\sum_{m=1}^{M}\zeta_m^{(j)\prime}\right)^2\right]\right) + r_j^n\right].$$

Then $U_k^n - R_k^n$ is a martingale with the initial value U_0^n. Our bounds on the increments of the process $(R_k^n)_{k\ge 0}$ and Proposition A.2 below imply that

$$E\left|R_\gamma^n\right| \le \frac{K_4}{a^{3n/2}}E\gamma \le \frac{K_5}{a^{n/2}}.$$

This allows us to pass to the limit as $k \to \infty$ and conclude that $U_0^n = EU_\gamma^n - ER_\gamma^n$. Thus,

$$U_0^n - \frac{K_5}{a^{n/2}} \le EU_\gamma^n \le P\left(V_\gamma \in \left[a^{n+1}, a^{n+1} + a^{2(n-1)/3}\right)\right)s\left(a + a^{-(n+2)/3}\right)$$
$$+ P\left(V_\gamma \in \left(a^{n-1} - a^{2(n-1)/3}, a^{n-1}\right]\right)s\left(a^{-1}\right) + E\left(U_\gamma^n \mathbb{1}_{\{d(V_\gamma,(a^{n-1},a^{n+1}))\ge a^{2(n-1)/3}\}}\right).$$

By part (i), we obtain that

$$P\left(V_\gamma \ge a^{n+1}\right)a^\delta + P\left(V_\gamma \le a^{n-1}\right)a^{-\delta} \ge U_0^n - K_6/a^{n/3}.$$

Similarly we get

$$P\left(V_\gamma \ge a^{n+1}\right)a^\delta + P\left(V_\gamma \le a^{n-1}\right)a^{-\delta} \le U_0^n + K_7/a^{n/3}.$$

This completes the proof. □

Proposition A.2. *There exists $C_8 \in (0,\infty)$ so that for all $x > 0$,*

$$\sup_{x\le y\le 2x} E_y^V\left(\sum_{r=0}^{\sigma_{x/2}^V}\mathbb{1}_{V_r\in[x,2x]}\right) < C_8 x.$$

Proof. By the usual compactness considerations and Lemma 3.1, there exists $c > 0$ such that $P_y^V(\sigma_{x/2}^V < x) > c$ for all $x > 0$ and $y \in [x, 2x]$. From this and the Markov property applied to successive re-entries to the interval $[x, 2x]$ (see the proof of Proposition 6.1 for details), we obtain

$$P_y^V\left(\sum_{r=0}^{\sigma_{x/2}^V}\mathbb{1}_{V_r\in[x,2x]} > nx\right) \le (1-c)^n,$$

and the result follows. □

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