# Hydrodynamic limit of a $d$-dimensional exclusion process with conductances ${ }^{1}$ 

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#### Abstract

Fix a polynomial $\Phi$ of the form $\Phi(\alpha)=\alpha+\sum_{2 \leq j \leq m} a_{j} \alpha^{j}$ with $\Phi^{\prime}(1)>0$. We prove that the evolution, on the diffusive scale, of the empirical density of exclusion processes on $\mathbb{T}^{d}$, with conductances given by special class of functions $W$, is described by the unique weak solution of the non-linear parabolic partial differential equation $\partial_{t} \rho=\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}} \Phi(\rho)$. We also derive some properties of the operator $\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}}$.


Résumé. Étant donné un polynôme $\Phi$ de la forme $\Phi(\alpha)=\alpha+\sum_{2 \leq j \leq m} a_{j} \alpha^{j}$ respectant $\Phi^{\prime}(1)>0$, nous démontrons que l'évolution, sur une échelle diffusive, de la densité empirique des processus d'exclusion sur $\mathbb{T}^{d}$, dont les conductances sont données par une classe spéciale de fonctions $W$, est décrite par l'unique solution faible de l'équation aux dérivées partielles parabolique : $\partial_{t} \rho=\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}} \Phi(\rho)$. Nous dérivons également certaines propriétés de l'opérateur $\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}}$.

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## 1. Introduction

The evolution of one-dimensional exclusion processes with random conductances has attracted some attention recently [2,3,6,7]. The purpose of this paper is to extend this analysis to higher dimension.

Let $W: \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a function such that $W\left(x_{1}, \ldots, x_{d}\right)=\sum_{k=1}^{d} W_{k}\left(x_{k}\right)$, where $d \geq 1$ and each function $W_{k}: \mathbb{R} \rightarrow$ $\mathbb{R}$ is strictly increasing, right continuous with left limits (càdlàg), and periodic in the sense that $W_{k}(u+1)-W_{k}(u)=$ $W_{k}(1)-W_{k}(0)$ for all $u \in \mathbb{R}$. Informally, the exclusion process with conductances associated to $W$ is an interacting particle systems on the $d$-dimensional discrete torus $N^{-1} \mathbb{T}_{N}^{d}$, in which at most one particle per site is allowed, and only nearest-neighbor jumps are permitted. Moreover, the jump rate in the direction $e_{j}$ is given by the reciprocal of the increments of $W$ with respect to the $j$ th coordinate.

We show that, on the diffusive scale, the macroscopic evolution of the empirical density of exclusion processes with conductances $W$ is described by the nonlinear differential equation

$$
\begin{equation*}
\partial_{t} \rho=\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}} \Phi(\rho), \tag{1.1}
\end{equation*}
$$

[^0]where $\Phi$ is a polynomial of the form $\Phi(\alpha)=\alpha+\sum_{2 \leq j \leq m} a_{j} \alpha^{j}$ with $\Phi^{\prime}(1)>0$. Furthermore, we denote by $\partial_{W_{k}}$ the generalized derivative with respect to $W_{k}$, see $[1,6]$ and a revision in Section 3. The partial differential equation (1.1) appears naturally as, for instance, scaling limits of interacting particle systems in inhomogeneous media. It may model diffusions in which permeable membranes, at the points of discontinuities of $W$, tend to reflect particles, creating space discontinuities in the density profiles.

The proof of hydrodynamic limit relies strongly on some properties of the differential operator $\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}}$ presented in Theorem 2.1. We prove, among other properties: that the operator $\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}}$, defined on an appropriate domain, is non-positive, self-adjoint and dissipative; that its eigenvalues are countable and have finite multiplicity; and that the associated eigenvectors form a complete orthonormal system.

There is a wide literature on the so-called Feller's generalized diffusion operator ( $\mathrm{d} / \mathrm{d} u)(\mathrm{d} / \mathrm{d} v)$. Where, typically, $u$ and $v$ are strictly increasing functions with $v$ (but not necessarily $u$ ) being continuous. It provides general diffusions operators and an appreciable simplification of the theory of second-order differential operators (see, for instance, $[4,5,9])$. The operator $(\mathrm{d} / \mathrm{d} x)(\mathrm{d} / \mathrm{d} u)$, considered in [6], is the formal adjoint of $(\mathrm{d} / \mathrm{d} u)(\mathrm{d} / \mathrm{d} v)$ in the particular case $v(x)=x$ (as in [5]). The goal of this work is to extend this adjoint operator to higher dimensions and provide some results regarding this extension. The assumption that the function $W$ is a direct sum of one-dimensional functions is technically essential, so that the limit equation (1.1) has a special form. More details, see Sections 4 and 5.

The article is organized as follows: in Section 2 we state the main results of the article; in Section 3 we prove the main properties of the operator $\mathcal{L}_{W}=\sum_{k=1}^{d} \partial_{x_{k}} \partial_{W_{k}}$; in Section 4 we prove the convergence of random walks with random conductances to Markov processes with generator given by $\mathcal{L}_{W}$; in Section 5 we prove the scaling limit of the exclusion process with conductances given by $W$; and, finally, in Section 6 we show that the unique solution of (1.1) has finite energy.

## 2. Notation and results

We examine the hydrodynamic behavior of a $d$-dimensional exclusion process, with $d \geq 1$, with conductances given by a special class of functions $W: \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that:

$$
\begin{equation*}
W\left(x_{1}, \ldots, x_{d}\right)=\sum_{k=1}^{d} W_{k}\left(x_{k}\right) \tag{2.1}
\end{equation*}
$$

where $W_{k}: \mathbb{R} \rightarrow \mathbb{R}$ are strictly increasing right continuous functions with left limits (càdlàg), and periodic in the sense that

$$
W_{k}(u+1)-W_{k}(u)=W_{k}(1)-W_{k}(0)
$$

for all $u \in \mathbb{R}$ and $k=1, \ldots, d$. To keep notation simple, we assume that $W_{k}$ vanishes at the origin, that is, $W_{k}(0)=0$.
Denote by $\mathbb{T}^{d}=[0,1)^{d}$ the $d$-dimensional torus and by $e_{1}, \ldots, e_{d}$ the canonical basis of $\mathbb{R}^{d}$. For this class of functions we have:

- $W(0)=0$;
- $W$ is strictly increasing on each coordinate:

$$
W\left(x+a e_{j}\right)>W(x)
$$

for all $1 \leq j \leq d, a>0, x \in \mathbb{R}^{d}$;

- $W$ is continuous from above:

$$
W(x)=\lim _{y \rightarrow x, y \geq x} W(y)
$$

where we say that $y \geq x$ if $y_{j} \geq x_{j}$ for all $1 \leq j \leq d$;

- $W$ is defined on the torus $\mathbb{T}^{d}$ :

$$
W\left(x_{1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_{d}\right)=W\left(x_{1}, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{d}\right)-W\left(e_{j}\right)
$$

for all $1 \leq j \leq d,\left(x_{1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d}\right) \in \mathbb{T}^{d-1}$.
Unless explicitly stated $W$ belongs to this class. Let $\mathbb{T}_{N}^{d}$ be the $d$-dimensional discrete torus with $N^{d}$ points. Distribute particles throughout $\mathbb{T}_{N}^{d}$ in such a way that each site of $\mathbb{T}_{N}^{d}$ is occupied at most by one particle. Denote by $\eta$ the configurations of the state space $\{0,1\}^{T_{N}^{d}}$, so that $\eta(x)=0$ if site $x$ is vacant and $\eta(x)=1$ if site $x$ is occupied.

Fix $a>-1 / 2$ and $W$. For $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{T}_{N}^{d}$ let

$$
c_{x, x+e_{j}}(\eta)=1+a\left\{\eta\left(x-e_{j}\right)+\eta\left(x+2 e_{j}\right)\right\},
$$

where all sums are modulo $N$, and let

$$
\xi_{x, x+e_{j}}=\frac{1}{N\left[W\left(\left(x+e_{j}\right) / N\right)-W(x / N)\right]}=\frac{1}{N\left[W_{j}\left(\left(x_{j}+1\right) / N\right)-W_{j}\left(x_{j} / N\right)\right]} .
$$

We now describe the stochastic evolution of the process. Let $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{T}_{N}^{d}$. At rate $\xi_{x, x+e_{j}} c_{x, x+e_{j}}(\eta)$ the occupation variables $\eta(x), \eta\left(x+e_{j}\right)$ are exchanged. If $W$ is differentiable at $x / N \in[0,1)^{d}$, the rate at which particles are exchanged is of order 1 for each direction, but if some $W_{j}$ is discontinuous at $x_{j} / N$, it no longer holds. In fact, assume, to fix ideas, that $W_{j}$ is discontinuous at $x_{j} / N$, and smooth on the segments ( $x_{j} / N, x_{j} / N+\varepsilon e_{j}$ ) and $\left(x_{j} / N-\varepsilon e_{j}, x_{j} / N\right)$. Assume, also, that $W_{k}$ is differentiable in a neighborhood of $x_{k} / N$ for $k \neq j$. In this case, the rate at which particles jump over the bonds $\left\{y-e_{j}, y\right\}$, with $y_{j}=x_{j}$, is of order $1 / N$, whereas in a neighborhood of size $N$ of these bonds, particles jump at rate 1. Thus, note that a particle at site $y-e_{j}$ jumps to $y$ at rate $1 / N$ and jumps at rate 1 to each one of the $2 d-1$ other options. Particles, therefore, tend to avoid the bonds $\left\{y-e_{j}, y\right\}$. However, since time will be scaled diffusively, and since on a time interval of length $N^{2}$ a particle spends a time of order $N$ at each site $y$, particles will be able to cross the slower bond $\left\{y-e_{j}, y\right\}$.

Then, this process models membranes that obstruct passages of particles. Note that these membranes are $(d-1)$ dimensional hyperplanes embedded in a $d$-dimensional environment. Moreover, if we consider $W_{j}$ having more than one discontinuity point for more than one $j$, these membranes will be more sophisticated manifolds, for instance, unions of $(d-1)$-dimensional boxes.

The effect of the factor $c_{x, x+e_{j}}(\eta)$ is analogous to the one-dimensional case. If the parameter $a$ is positive, the presence of particles in the neighboring sites of the bond $\left\{x, x+e_{j}\right\}$ speeds up the exchange rate by a factor of order one.

The dynamics informally presented describes a Markov evolution. The generator $L_{N}$ of this Markov process acts on functions $f:\{0,1\}^{\mathbb{T}_{N}^{d}} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
L_{N} f(\eta)=\sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} \xi_{x, x+e_{j}} c_{x, x+e_{j}}(\eta)\left\{f\left(\sigma^{x, x+e_{j}} \eta\right)-f(\eta)\right\}, \tag{2.2}
\end{equation*}
$$

where $\sigma^{x, x+e_{j}} \eta$ is the configuration obtained from $\eta$ by exchanging the variables $\eta(x)$ and $\eta\left(x+e_{j}\right)$ :

$$
\left(\sigma^{x, x+e_{j}} \eta\right)(y)= \begin{cases}\eta\left(x+e_{j}\right) & \text { if } y=x  \tag{2.3}\\ \eta(x) & \text { if } y=x+e_{j}, \\ \eta(y) & \text { otherwise }\end{cases}
$$

A straightforward computation shows that the Bernoulli product measures $\left\{v_{\alpha}^{N}: 0 \leq \alpha \leq 1\right\}$ are invariant, and in fact reversible, for the dynamics. The measure $\nu_{\alpha}^{N}$ is obtained by placing a particle at each site, independently from the other sites, with probability $\alpha$. Thus, $v_{\alpha}^{N}$ is a product measure over $\{0,1\}^{\mathbb{T}_{N}^{d}}$ with marginals given by

$$
v_{\alpha}^{N}\{\eta: \eta(x)=1\}=\alpha
$$

for $x$ in $\mathbb{T}_{N}^{d}$. For more details see [8], Chapter 2. We will often omit the index $N$ on $v_{\alpha}^{N}$.

Denote by $\left\{\eta_{t}: t \geq 0\right\}$ the Markov process on $\{0,1\}^{\mathbb{T}_{N}^{d}}$ associated to the generator $L_{N}$ speeded up by $N^{2}$. Let $D\left(\mathbb{R}_{+},\{0,1\}^{\mathbb{T}_{N}^{d}}\right)$ be the path space of càdlàg trajectories with values in $\{0,1\}^{\mathbb{T}_{N}^{d}}$. For a measure $\mu_{N}$ on $\{0,1\}^{\mathbb{T}_{N}^{d}}$, denote by $\mathbb{P}_{\mu_{N}}$ the probability measure on $D\left(\mathbb{R}_{+},\{0,1\}^{\mathbb{T}_{N}^{d}}\right)$ induced by the initial state $\mu_{N}$, and the Markov process $\left\{\eta_{t}: t \geq 0\right\}$. Expectation with respect to $\mathbb{P}_{\mu_{N}}$ is denoted by $\mathbb{E}_{\mu_{N}}$.

### 2.1. The operator $\mathcal{L}_{W}$

Fix $W=\sum_{k=1}^{d} W_{k}$ as in (2.1). In [6] it was shown that there exist self-adjoint operators $\mathcal{L}_{W_{k}}: \mathcal{D}_{W_{k}} \subset L^{2}(\mathbb{T}) \rightarrow L^{2}(\mathbb{T})$. Further, the set $\mathcal{A}_{W_{k}}$ of the eigenvectors of $\mathcal{L}_{W_{k}}$ forms a complete orthonormal system in $L^{2}(\mathbb{T})$. Let

$$
\begin{equation*}
\mathcal{A}_{W}=\left\{f: \mathbb{T}^{d} \rightarrow \mathbb{R} ; f\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} f_{k}\left(x_{k}\right), f_{k} \in \mathcal{A}_{W_{k}}, k=1, \ldots, d\right\}, \tag{2.4}
\end{equation*}
$$

and denote by $\operatorname{span}(A)$ the space of finite linear combinations of the set $A$, and let $\mathbb{D}_{W}:=\operatorname{span}\left(\mathcal{A}_{W}\right)$. Define the operator $\mathbb{L}_{W}: \mathbb{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ as follows: for $f=\prod_{k=1}^{d} f_{k} \in \mathcal{A}_{W}$, we have

$$
\begin{equation*}
\mathbb{L}_{W}(f)\left(x_{1}, \ldots, x_{d}\right)=\sum_{k=1}^{d} \prod_{j=1, j \neq k}^{d} f_{j}\left(x_{j}\right) \mathcal{L}_{W_{k}} f_{k}\left(x_{k}\right), \tag{2.5}
\end{equation*}
$$

and then extend to $\mathbb{D}_{W}$ by linearity.
Lemma 3.2, in Section 3, shows that: $\mathbb{L}_{W}$ is symmetric and non-positive; $\mathbb{D}_{W}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$; and the set $\mathcal{A}_{W}$ forms a complete, orthonormal, countable system of eigenvectors for the operator $\mathbb{L}_{W}$. Let $\mathcal{A}_{W}=\left\{h_{k}\right\}_{k \geq 0},\left\{\alpha_{k}\right\}_{k \geq 0}$ be the corresponding eigenvalues of $-\mathbb{L}_{W}$, and consider

$$
\begin{equation*}
\mathcal{D}_{W}=\left\{v=\sum_{k=1}^{\infty} v_{k} h_{k} \in L^{2}\left(\mathbb{T}^{d}\right) ; \sum_{k=1}^{\infty} v_{k}^{2} \alpha_{k}^{2}<+\infty\right\} . \tag{2.6}
\end{equation*}
$$

Define the operator $\mathcal{L}_{W}: \mathcal{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ by

$$
\begin{equation*}
-\mathcal{L}_{W} v=\sum_{k=1}^{+\infty} \alpha_{k} v_{k} h_{k} \tag{2.7}
\end{equation*}
$$

The operator $\mathcal{L}_{W}$ is clearly an extension of the operator $\mathbb{L}_{W}$, and we present in Theorem 2.1 some properties of this operator.

Theorem 2.1. The operator $\mathcal{L}_{W}: \mathcal{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ enjoys the following properties:
(a) the domain $\mathcal{D}_{W}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$. In particular, the set of eigenvectors $\mathcal{A}_{W}=\left\{h_{k}\right\}_{k \geq 0}$ forms a complete orthonormal system;
(b) the eigenvalues of the operator $-\mathcal{L}_{W}$ form a countable set $\left\{\alpha_{k}\right\}_{k \geq 0}$. All eigenvalues have finite multiplicity, and it is possible to obtain a re-enumeration $\left\{\alpha_{k}\right\}_{k \geq 0}$ such that

$$
0=\alpha_{0} \leq \alpha_{1} \leq \cdots \quad \text { and } \quad \lim _{n \rightarrow \infty} \alpha_{n}=\infty ;
$$

(c) the operator $\mathbb{I}-\mathcal{L}_{W}: \mathcal{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ is bijective;
(d) $\mathcal{L}_{W}: \mathcal{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ is self-adjoint and non-positive:

$$
\left\langle-\mathcal{L}_{W} f, f\right\rangle \geq 0
$$

(e) $\mathcal{L}_{W}$ is dissipative.

In view of (a), (b) and (d), we may use Hille-Yosida theorem to conclude that $\mathcal{L}_{W}$ is the generator of a strongly continuous contraction semigroup $\left\{P_{t}: L^{2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right)\right\}_{t \geq 0}$.

Denote by $\left\{G_{\lambda}: L^{2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right)\right\}_{\lambda>0}$ the semigroup of resolvents associated to the operator $\mathcal{L}_{W}: G_{\lambda}=(\lambda-$ $\left.\mathcal{L}_{W}\right)^{-1} . G_{\lambda}$ can also be written in terms of the semigroup $\left\{P_{t} ; t \geq 0\right\}$ :

$$
G_{\lambda}=\int_{0}^{\infty} \mathrm{e}^{-\lambda t} P_{t} \mathrm{~d} t
$$

In Section 4 we derive some properties and obtain some results for these operators.

### 2.2. The hydrodynamic equation

A sequence of probability measures $\left\{\mu_{N}: N \geq 1\right\}$ on $\{0,1\}^{\mathbb{T}_{N}^{d}}$ is said to be associated to a profile $\rho_{0}: \mathbb{T}^{d} \rightarrow[0,1]$ if

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \mu_{N}\left\{\left|\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N) \eta(x)-\int H(u) \rho_{0}(u) \mathrm{d} u\right|>\delta\right\}=0 \tag{2.8}
\end{equation*}
$$

for every $\delta>0$ and every continuous function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$. For details, see [8], Chapter 3 .
Fix a polynomial $\Phi$ of the form

$$
\Phi(\alpha)=\alpha+\sum_{j=2}^{m} a_{j} \alpha^{j}
$$

with $\Phi^{\prime}(1)>0$ and $m$ a positive integer. Let $\gamma: \mathbb{T}^{d} \rightarrow[0,1]$ be a bounded density profile, and consider the parabolic differential equation

$$
\left\{\begin{array}{l}
\partial_{t} \rho=\mathcal{L}_{W} \Phi(\rho),  \tag{2.9}\\
\rho(0, \cdot)=\gamma(\cdot) .
\end{array}\right.
$$

A bounded function $\rho: \mathbb{R}_{+} \times \mathbb{T}^{d} \rightarrow[0,1]$ is said to be a weak solution of the parabolic differential equation (2.9) if

$$
\left\langle\rho_{t}, G_{\lambda} H\right\rangle-\left\langle\gamma, G_{\lambda} H\right\rangle=\int_{0}^{t}\left\langle\Phi\left(\rho_{s}\right), \mathcal{L}_{W} G_{\lambda} H\right\rangle \mathrm{d} s
$$

for every continuous function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$, all $t>0$ and all $\lambda>0$.
Existence of these weak solutions follows from tightness of the sequence of probability measures $\mathbb{Q}_{\mu_{N}}^{W, N}$ introduced in Section 5. The proof of uniquenesses of weak solutions is analogous to [6].

Theorem 2.2. Fix a continuous initial profile $\rho_{0}: \mathbb{T}^{d} \rightarrow[0,1]$, and consider a sequence of probability measures $\mu_{N}$ on $\{0,1\}^{\mathbb{T}_{N}^{d}}$ associated to $\rho_{0}$, in the sense of (2.8). Then, for any $t \geq 0$,

$$
\lim _{N \rightarrow \infty} \mathbb{P}_{\mu_{N}}\left\{\left|\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N) \eta_{t}(x)-\int H(u) \rho(t, u) \mathrm{d} u\right|>\delta\right\}=0
$$

for every $\delta>0$ and every continuous function $H$. Here, $\rho$ is the unique weak solution of the non-linear equation (2.9) with $\gamma=\rho_{0}$, and $\Phi(\alpha)=\alpha+a \alpha^{2}$.

Remark 2.3. As noted in [6], Remark 2.3, the specific form of the rates $c_{x, x+e_{i}}$ is not important, but two conditions must be fulfilled: the rates must be strictly positive, although they may not depend on the occupation variables $\eta(x)$, $\eta\left(x+e_{i}\right)$; but they have to be chosen in such a way that the resulting process is gradient (cf. Chapter 7 in [8] for the definition of gradient processes).

We may define rates $c_{x, x+e_{i}}$ to obtain any polynomial $\Phi$ of the form $\Phi(\alpha)=\alpha+\sum_{2 \leq j \leq m} a_{j} \alpha^{j}, m \geq 1$, with $1+\sum_{2 \leq j \leq m} j a_{j}>0$. Let, for instance, $m=3$. Then the rates

$$
\hat{c}_{x, x+e_{i}}(\eta)=c_{x, x+e_{i}}(\eta)+b\left\{\eta\left(x-2 e_{i}\right) \eta\left(x-e_{i}\right)+\eta\left(x-e_{i}\right) \eta\left(x+2 e_{i}\right)+\eta\left(x+2 e_{i}\right) \eta\left(x+3 e_{i}\right)\right\},
$$

satisfy the above three conditions, where $c_{x, x+e_{i}}$ is the rate defined at the beginning of Section 2 and $a, b$ are such that $1+2 a+3 b>0$. An elementary computation shows that $\Phi(\alpha)=\alpha+a \alpha^{2}+b \alpha^{3}$.

In Section 6 we prove that any limit point $\mathbb{Q}_{W}^{*}$ of the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$ is concentrated on trajectories $\rho(t, u) \mathrm{d} u$, with finite energy in the following sense: for each $1 \leq j \leq d$, there is a Hilbert space $L_{x_{j} \otimes W_{j}}^{2}$, associated to $W_{j}$, such that

$$
\int_{0}^{t} \mathrm{~d} s\left\|\partial_{W_{j}} \Phi(\rho(s, \cdot))\right\|_{x_{j} \otimes W_{j}}^{2}<\infty
$$

where $\|\cdot\|_{x_{j} \otimes W_{j}}$ is the norm in $L_{x_{j} \otimes W_{j}}^{2}$, and $\partial_{W_{j}}$ is the derivative, which must be understood in the generalized sense.

## 3. The operator $\mathcal{L}_{W}$

The operator $\mathcal{L}_{W}: \mathcal{D}_{W} \subset L^{2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ is a natural extension, for the $d$-dimensional case, of the self-adjoint operator obtained for the one-dimensional case in [6]. We begin by presenting one of the main results obtained in [6], and we then present the necessary modifications to conclude similar results for the $d$-dimensional case.

### 3.1. Some remarks on the one-dimensional case

Let $\mathbb{T} \subset \mathbb{R}$ be the one-dimensional torus. Denote by $\langle\cdot, \cdot\rangle$ the inner product of $L^{2}(\mathbb{T})$ :

$$
\langle f, g\rangle=\int_{\mathbb{T}} f(u) g(u) \mathrm{d} u .
$$

Let $W_{1}: \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing right continuous function with left limits (càdlàg), and periodic in the sense that $W_{1}(u+1)-W_{1}(u)=W_{1}(1)-W_{1}(0)$ for all $u$ in $\mathbb{R}$.

Let $\mathcal{D}_{W_{1}}$ be the set of functions $f$ in $L^{2}(\mathbb{T})$ such that

$$
f(x)=a+b W_{1}(x)+\int_{(0, x]} W_{1}(\mathrm{~d} y) \int_{0}^{y} \mathfrak{f}(z) \mathrm{d} z
$$

for some function $\mathfrak{f}$ in $L^{2}(\mathbb{T})$ that satisfies:

$$
\int_{0}^{1} \mathfrak{f}(z) \mathrm{d} z=0, \quad \int_{(0,1]} W_{1}(\mathrm{~d} y)\left(b+\int_{0}^{y} \mathfrak{f}(z) \mathrm{d} z\right)=0 .
$$

Define the operator $\mathcal{L}_{W_{1}}: \mathcal{D}_{W_{1}} \rightarrow L^{2}(\mathbb{T})$ by $\mathcal{L}_{W_{1}} f=\mathfrak{f}$. Formally

$$
\begin{equation*}
\mathcal{L}_{W_{1}} f=\frac{\mathrm{d}}{\mathrm{~d} x} \frac{\mathrm{~d}}{\mathrm{~d} W_{1}} f \tag{3.1}
\end{equation*}
$$

where the generalized derivative $\mathrm{d} / \mathrm{d} W_{1}$ is defined as

$$
\begin{equation*}
\frac{\mathrm{d} f}{\mathrm{~d} W_{1}}(x)=\lim _{\epsilon \rightarrow 0} \frac{f(x+\epsilon)-f(x)}{W_{1}(x+\epsilon)-W_{1}(x)} \tag{3.2}
\end{equation*}
$$

if the above limit exists and is finite.

Theorem 3.1. Denote by $\mathbb{I}$ the identity operator in $L^{2}(\mathbb{T})$. The operator $\mathcal{L}_{W_{1}}: \mathcal{D}_{W_{1}} \rightarrow L^{2}(\mathbb{T})$ enjoys the following properties:
(a) $\mathcal{D}_{W_{1}}$ is dense in $L^{2}(\mathbb{T})$;
(b) the operator $\mathbb{I}-\mathcal{L}_{W_{1}}: \mathcal{D}_{W_{1}} \rightarrow L^{2}(\mathbb{T})$ is bijective;
(c) $\mathcal{L}_{W_{1}}: \mathcal{D}_{W_{1}} \rightarrow L^{2}(\mathbb{T})$ is self-adjoint and non-positive:

$$
\left\langle-\mathcal{L}_{W_{1}} f, f\right\rangle \geq 0 ;
$$

(d) $\mathcal{L}_{W_{1}}$ is dissipative i.e., for all $g \in \mathcal{D}_{W}$ and $\lambda>0$, we have

$$
\|\lambda g\| \leq\left\|\left(\lambda \mathbb{I}-\mathcal{L}_{W_{1}}\right) g\right\| ;
$$

(e) the eigenvalues of the operator $-\mathcal{L}_{W}$ form a countable set $\left\{\lambda_{n}: n \geq 0\right\}$. All eigenvalues have finite multiplicity, $0=\lambda_{0} \leq \lambda_{1} \leq \cdots$, and $\lim _{n \rightarrow \infty} \lambda_{n}=\infty$;
(f) the eigenvectors $\left\{f_{n}\right\}_{n \geq 0}$ of the operator $\mathcal{L}_{W}$ form a complete orthonormal system.

The proof can be found in [6].

### 3.2. The $d$-dimensional case

Consider $W$ as in (2.1). Let $\mathcal{A}_{W_{k}}$ be the countable complete orthonormal system of eigenvectors of the operator $\mathcal{L}_{W_{k}}: \mathcal{D}_{W_{k}} \subset L^{2}(\mathbb{T}) \rightarrow \mathbb{R}$ given in Theorem 3.1.

Let $\mathcal{A}_{W}$ be as in (2.4), and let the operator $\mathbb{L}_{W}: \mathbb{D}_{W}:=\operatorname{span}\left(\mathcal{A}_{W}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ be as in (2.5). By Fubini's theorem, the set $\mathcal{A}_{W}$ is orthonormal in $L^{2}\left(\mathbb{T}^{d}\right)$, and the constant functions are eigenvectors of the operator $\mathcal{L}_{W_{k}}$. Moreover, $\mathcal{A}_{W_{k}} \subset \mathcal{A}_{W}$, in the sense that $f_{k}\left(x_{1}, \ldots, x_{d}\right)=f_{k}\left(x_{k}\right), f_{k} \in \mathcal{A}_{W_{k}}$.

By (3.1), the operators $\mathcal{L}_{W_{k}}$ can be formally extended to functions defined on $\mathbb{T}^{d}$ as follows: given a function $f: \mathbb{T}^{d} \rightarrow \mathbb{R}$, we define $\mathcal{L}_{W_{k}} f$ as

$$
\begin{equation*}
\mathcal{L}_{W_{k}} f=\partial_{x_{k}} \partial_{W_{k}} f \tag{3.3}
\end{equation*}
$$

where the generalized derivative $\partial_{W_{k}}$ is defined by

$$
\begin{equation*}
\partial_{W_{k}} f\left(x_{1}, \ldots, x_{k}, \ldots, x_{d}\right)=\lim _{\epsilon \rightarrow 0} \frac{f\left(x_{1}, \ldots, x_{k}+\epsilon, \ldots, x_{d}\right)-f\left(x_{1}, \ldots, x_{k}, \ldots, x_{d}\right)}{W_{k}\left(x_{k}+\epsilon\right)-W_{k}\left(x_{k}\right)}, \tag{3.4}
\end{equation*}
$$

if the above limit exists and is finite. Hence, by (2.5), if $f \in \mathbb{D}_{W}$

$$
\begin{equation*}
\mathbb{L}_{W} f=\sum_{k=1}^{d} \mathcal{L}_{W_{k}} f \tag{3.5}
\end{equation*}
$$

Note that if $f=\prod_{k=1}^{d} f_{k}$, where $f_{k} \in \mathcal{A}_{W_{k}}$ is an eigenvector of $\mathcal{L}_{W_{k}}$ associated to the eigenvalue $\lambda_{k}$, then $f$ is an eigenvector of $\mathbb{L}_{W}$, with eigenvalue $\sum_{k=1}^{d} \lambda_{k}$.

Lemma 3.2. The following statements hold:
(a) the set $\mathbb{D}_{W}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$;
(b) the operator $\mathbb{L}_{W}: \mathbb{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ is symmetric and non-positive:

$$
\left\langle-\mathbb{L}_{W} f, f\right\rangle \geq 0 .
$$

Proof. The strategy to prove the above lemma is the following. We begin by showing that the set

$$
\mathcal{S}=\operatorname{span}\left(\left\{f \in L^{2}\left(\mathbb{T}^{d}\right) ; f\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} f_{k}\left(x_{k}\right), f_{k} \in \mathcal{D}_{W_{k}}\right\}\right)
$$

is dense in

$$
\mathbb{S}=\operatorname{span}\left(\left\{f \in L^{2}\left(\mathbb{T}^{d}\right) ; f\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} f_{k}\left(x_{k}\right), f_{k} \in L^{2}(\mathbb{T})\right\}\right)
$$

We then show that $\mathbb{D}_{W}$ is dense in $\mathcal{S}$. Since $\mathbb{S}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$, item (a) follows.
We now prove item (a) rigorously. Since $\mathcal{S}$ is a vector space, we only have to show that we can approximate the functions $\prod_{k=1}^{d} f_{k} \in L^{2}\left(\mathbb{T}^{d}\right)$, where $f_{k} \in \mathcal{D}_{W_{k}}$, by functions of $\mathbb{D}_{W}$. By Theorem 3.1 , the set $\mathcal{D}_{W_{k}}$ is dense in $L^{2}(\mathbb{T})$, thus, there exists a sequence $\left(f_{n}^{k}\right)_{n \in \mathbb{N}}$ converging to $f_{k}$ in $L^{2}(\mathbb{T})$. Thus, let

$$
f_{n}\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} f_{n}^{k}\left(x_{k}\right)
$$

By the triangle inequality and Fubini's theorem, the sequence $\left(f_{n}\right)$ converges to $\prod_{k=1}^{d} f_{k}$. Fix $\epsilon>0$, and let

$$
h\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} h_{k}\left(x_{k}\right), \quad h_{k} \in \mathcal{D}_{W_{k}}
$$

Since, for each $k=1, \ldots, d, \mathcal{A}_{W_{k}} \subset \mathcal{D}_{W_{k}}$ is a complete orthonormal set, there exist sequences $g_{j}^{k} \in \mathcal{A}_{W_{k}}$, and $\alpha_{j}^{k} \in \mathbb{R}$, such that

$$
\left\|h_{k}-\sum_{j=1}^{n(k)} \alpha_{j}^{k} g_{j}^{k}\right\|_{L^{2}(\mathbb{T})}<\delta
$$

where $\delta=\epsilon / d M^{d-1}$ and $M:=1+\sup _{k=1: n}\left\|h_{k}\right\|$. Let

$$
g\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} \sum_{j=1}^{n(k)} \alpha_{j}^{k} g_{j}^{k}\left(x_{k}\right) \in \mathbb{D}_{W}
$$

An application of the triangle inequality, and Fubini's theorem, yields $\|h-g\|<\epsilon$. This proves (a).
To prove (b), let

$$
f\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} f_{k}\left(x_{k}\right) \quad \text { and } \quad g\left(x_{1}, \ldots, x_{d}\right)=\prod_{k=1}^{d} g_{k}\left(x_{k}\right)
$$

be functions belonging to $\mathcal{A}_{W}$. We have that

$$
\left\langle f, \mathbb{L}_{W} g\right\rangle=\left\langle\prod_{k=1}^{d} f_{k}, \sum_{k=1}^{d} \prod_{j=1, j \neq k}^{d} g_{j} \mathcal{L}_{W_{k}} g_{k}\right\rangle=\sum_{k=1}^{d}\left\langle\prod_{j=1, j \neq k}^{d} f_{j} g_{j}, f_{k} \mathcal{L}_{W_{k}} g_{k}\right\rangle
$$

where $\langle\cdot, \cdot\rangle$ denotes the inner product in $L^{2}\left(\mathbb{T}^{d}\right)$. Since, by Theorem $3.1, \mathcal{L}_{W_{k}}$ is self-adjoint, we have

$$
\sum_{k=1}^{d}\left\langle\prod_{j=1, j \neq k}^{d} f_{j} g_{j}, g_{k} \mathcal{L}_{W_{k}} f_{k}\right\rangle=\left\langle\mathcal{L}_{W} f, g\right\rangle
$$

In particular, the operator $\mathcal{L}_{W_{k}}$ is non-positive, and, therefore,

$$
\left\langle f, \mathbb{L}_{W} f\right\rangle=\sum_{k=1}^{d}\left\langle\prod_{j=1, j \neq k}^{d} f_{j}^{2}, f_{k} \mathcal{L}_{W_{k}} f_{k}\right\rangle \leq 0
$$

Item (b) follows by linearity.
Lemma 3.2 implies that the set $\mathcal{A}_{W}$ forms a complete, orthonormal, countable, system of eigenvectors for the operator $\mathbb{L}_{W}$.

Let $\mathcal{L}_{W}: \mathcal{D}_{W} \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ be the operator defined in (2.7). The operator $\mathcal{L}_{W}$ is clearly an extension of the operator $\mathbb{L}_{W}$. Formally, by (3.5),

$$
\begin{equation*}
\mathcal{L}_{W} f=\sum_{k=1}^{d} \mathcal{L}_{W_{k}} f, \tag{3.6}
\end{equation*}
$$

where

$$
\mathcal{L}_{W_{k}} f=\frac{\mathrm{d}}{\mathrm{~d} x_{k}} \frac{\mathrm{~d}}{\mathrm{~d} W_{k}} f .
$$

We are now in conditions to prove Theorem 2.1.
Proof of Theorem 2.1. By Lemma 3.2, $\mathbb{D}_{W}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$. Since $\mathbb{D}_{W} \subset \mathcal{D}_{W}$, we conclude that $\mathcal{D}_{W}$ is dense in $L^{2}\left(\mathbb{T}^{d}\right)$.

If $\alpha_{k}$ are eigenvalues of $-\mathcal{L}_{W}$, we may find eigenvalues $\lambda_{j}$, associated to some $f_{j} \in \mathcal{A}_{W_{j}}$, such that $\alpha_{k}=\sum_{j=1}^{d} \lambda_{j}$. By item (e) of Theorem 3.1, (b) follows.

Let $\left\{\alpha_{k}\right\}_{k \geq 0}$ be the set of eigenvalues of $-\mathcal{L}_{W}$. Then, the set of eigenvalues of $\mathbb{I}-\mathcal{L}_{W}$ is $\left\{\gamma_{k}\right\}_{k \geq 0}$, where $\gamma_{k}=\alpha_{k}+1$, and the eigenvectors are the same as the ones of $\mathcal{L}_{W}$. By item (b), we have

$$
1=\gamma_{0} \leq \gamma_{1} \leq \cdots \quad \text { and } \quad \lim _{n \rightarrow \infty} \gamma_{n}=\infty
$$

Thus, $\mathbb{I}-\mathcal{L}_{W}$ is injective. For

$$
v=\sum_{k=1}^{+\infty} v_{k} h_{k} \in L^{2}\left(\mathbb{T}^{d}\right), \quad \text { such that } \quad \sum_{k=1}^{\infty} v_{k}^{2}<+\infty,
$$

let

$$
u=\sum_{k=1}^{+\infty} \frac{v_{k}}{\gamma_{k}} h_{k} .
$$

Then $u \in \mathcal{D}_{W}$ and $\left(\mathbb{I}-\mathcal{L}_{W}\right) u=v$. Hence, item (c) follows.
Let $\mathcal{L}_{W}^{*}: \mathcal{D}_{W^{*}} \subset L^{2}\left(\mathbb{T}^{d}\right) \rightarrow L^{2}\left(\mathbb{T}^{d}\right)$ be the adjoint of $\mathcal{L}_{W}$. Since $\mathcal{L}_{W}$ is symmetric, we have $\mathcal{D}_{W} \subset \mathcal{D}_{W^{*}}$. So, to show the equality of the operators it suffices to show that $\mathcal{D}_{W^{*}} \subset \mathcal{D}_{W}$. Given

$$
\varphi=\sum_{k=1}^{+\infty} \varphi_{k} h_{k} \in \mathcal{D}_{W^{*}},
$$

let $\mathcal{L}_{W *} \varphi=\psi \in L^{2}\left(\mathbb{T}^{d}\right)$. Therefore, for all $v=\sum_{k=1}^{+\infty} v_{k} h_{k} \in \mathcal{D}_{W}$,

$$
\langle v, \psi\rangle=\left\langle v, \mathcal{L}_{W *} \varphi\right\rangle=\left\langle\mathcal{L}_{W} v, \varphi\right\rangle=\sum_{k=1}^{+\infty}-\alpha_{k} v_{k} \varphi_{k} .
$$

Hence

$$
\psi=\sum_{k=1}^{+\infty}-\alpha_{k} \varphi_{k} h_{k}
$$

In particular,

$$
\sum_{k=1}^{+\infty} \alpha_{k}^{2} \varphi_{k}^{2}<+\infty \quad \text { and } \quad \varphi \in \mathcal{D}_{W}
$$

Thus, $\mathcal{L}_{W}$ is self-adjoint. Let $v=\sum_{k=1}^{+\infty} v_{k} h_{k} \in \mathcal{D}_{W}$. From item (b), $\alpha_{k} \geq 0$, and

$$
\left\langle-\mathcal{L}_{W} v, v\right\rangle=\sum_{k=1}^{+\infty} \alpha_{k} v_{k}^{2} \geq 0
$$

Therefore $\mathcal{L}_{W}$ is non-positive, and item (d) follows.
Fix a function $g$ in $\mathcal{D}_{W}, \lambda>0$, and let $f=\left(\lambda \mathbb{I}-\mathcal{L}_{W}\right) g$. Taking inner product, with respect to $g$, on both sides of this equation, we obtain

$$
\lambda\langle g, g\rangle+\left\langle-\mathcal{L}_{W} g, g\right\rangle=\langle g, f\rangle \leq\langle g, g\rangle^{1 / 2}\langle f, f\rangle^{1 / 2} .
$$

Since $g$ belongs to $\mathcal{D}_{W}$, by (d), the second term on the left-hand side is non-negative. Thus, $\|\lambda g\| \leq\|f\|=\|(\lambda \mathbb{I}-$ $\left.\mathcal{L}_{W}\right) g \|$.

## 4. Random walk with conductances

Recall the decomposition obtained in (3.6) for the operator $\mathcal{L}_{W}$. In next subsection, we present the discrete version $\mathbb{L}_{N}$ of $\mathcal{L}_{W}$ and we describe, informally, the Markovian dynamics generated by $\mathbb{L}_{N}$.

### 4.1. Discrete approximation of the operator $\mathcal{L}_{W}$

Consider the random walk $\left\{X_{t}^{N}\right\}_{t \geq 0}$ in $\frac{1}{N} \mathbb{T}_{N}^{d}$, which jumps from $x / N$ (resp. $\left.\left(x+e_{j}\right) / N\right)$ to $\left(x+e_{j}\right) / N$ (resp. $x / N$ ) with rate

$$
N^{2} \xi_{x, x+e_{j}}=N /\left\{W_{j}\left(\left(x_{j}+1\right) / N\right)-W_{j}\left(x_{j} / N\right)\right\} .
$$

The generator $\mathbb{L}_{N}$ of this Markov process acts on local functions $f: \frac{1}{N} \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
\mathbb{L}_{N} f(x / N)=\sum_{j=1}^{d} \mathbb{L}_{N}^{j} f(x / N) \tag{4.1}
\end{equation*}
$$

where

$$
\mathbb{L}_{N}^{j} f(x / N)=N^{2}\left\{\xi_{x, x+e_{j}}\left[f\left(\left(x+e_{j}\right) / N\right)-f(x / N)\right]+\xi_{x-e_{j}, x}\left[f\left(\left(x-e_{j}\right) / N\right)-f(x / N)\right]\right\} .
$$

Note that $\mathbb{L}_{N}^{j} f(x / N)$ is, in fact, a discrete version of the operator $\mathcal{L}_{W_{j}}$. The counting measure $m_{N}$ on $\mathbb{T}_{N}^{d}$ is reversible for this process. The following estimate is a key ingredient for proving the results in Section 5:

Lemma 4.1. Let $f$ be a function on $\frac{1}{N} \mathbb{T}_{N}^{d}$. Then, for each $j=1, \ldots, d$ :

$$
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N}^{j} f(x / N)\right)^{2} \leq \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N} f(x / N)\right)^{2}
$$

Proof. Let $X_{N^{d}}$ be the linear space of functions $f$ on $\frac{1}{N^{d}} \mathbb{T}_{N}^{d}$ over the field $\mathbb{R}$. Note that the dimension of $X_{N^{d}}$ is $N^{d}$. Denote by $\langle\cdot, \cdot\rangle_{N^{d}}$ the following inner product in $X_{N^{d}}$ :

$$
\langle f, g\rangle_{N^{d}}=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} f(x / N) g(x / N)
$$

For each $j=1, \ldots, d$, consider the linear operators $\mathcal{L}_{N}^{j}$ on $X_{N}$ (i.e., $d=1$ ) given by

$$
\mathcal{L}_{N}^{j} f=\partial_{x}^{N} \partial_{W_{j}}^{N} f
$$

where $\partial_{x}^{N}$ and $\partial_{W_{j}}^{N}$ are the difference operators:

$$
\begin{aligned}
\partial_{x}^{N} f(x / N) & =N[f((x+1) / N)-f(x / N)] \quad \text { and } \\
\partial_{W_{j}}^{N} f(x / N) & =\frac{f((x+1) / N)-f(x / N)}{W_{j}((x+1) / N)-W_{j}(x / N)}
\end{aligned}
$$

The operators $\mathcal{L}_{N}^{j}$ are symmetric and non-positive. In fact, a simple computation shows that

$$
\left\langle\mathcal{L}_{N}^{j} f, g\right\rangle_{N}=-\sum_{x \in \mathbb{T}_{N}}\left(W_{j}((x+1) / N)-W_{j}(x / N)\right) \partial_{W_{j}}^{N} f(x / N) \partial_{W_{j}}^{N} g(x / N)
$$

Using the spectral theorem, we obtain an orthonormal basis $\mathcal{A}_{N}^{j}=\left\{h_{1}^{j}, \ldots, h_{N}^{j}\right\}$ of $X_{N}$ formed by the eigenvectors of $\mathcal{L}_{N}^{j}$, i.e.,

$$
\mathcal{L}_{N}^{j} h_{i}^{j}=\alpha_{i}^{j} h_{i}^{j} \quad \text { and } \quad\left\langle h_{i}^{j}, h_{k}^{j}\right\rangle_{N}=\delta_{i, k}
$$

where $\delta_{i, k}$ is the Kronecker's delta, which equals 0 if $i \neq k$, and equals 1 if $i=k$. Since $\mathcal{L}_{N}^{j}$ is non-positive, we have that the eigenvalues $\alpha_{i}^{j}$ are non-positive: $\alpha_{i}^{j} \leq 0, j=1, \ldots, d$ and $i=1, \ldots, N$.

Let $\mathcal{A}_{N}=\left\{\phi_{1}, \ldots, \phi_{N^{d}}\right\} \subset X_{N^{d}}$ be set of functions of the form $\phi_{i}\left(x_{1}, \ldots, x_{d}\right)=\prod_{j=1}^{d} h^{j}\left(x_{j}\right)$, with $h^{j} \in \mathcal{A}_{N}^{j}$.
Let $\alpha^{j}$ be the eigenvalue of $h^{j}$, i.e., $\mathcal{L}_{N}^{j} h^{j}=\alpha^{j} h^{j}$. The linear operator $\mathbb{L}_{N}$ on $X_{N^{d}}$, defined in (4.1), is such that $\mathbb{L}_{N}^{j} \phi_{i}=\alpha^{j} \phi_{i}$ and $\mathbb{L}_{N} \phi_{i}=\sum_{j=1}^{d} \alpha^{j} \phi_{i}$. Furthermore, if $\phi_{i}\left(x_{1}, \ldots, x_{d}\right)=\prod_{j=1}^{d} h^{j}\left(x_{j}\right)$ and $\phi_{k}\left(x_{1}, \ldots, x_{d}\right)=$ $\prod_{j=1}^{d} g^{j}\left(x_{j}\right), \phi_{i}, \phi_{k} \in \mathcal{A}_{N}$, we have that

$$
\left\langle\phi_{i}, \phi_{k}\right\rangle_{N^{d}}=\prod_{j=1}^{d}\left\langle h^{j}, g^{j}\right\rangle_{N}=\delta_{i, k}
$$

for $i, k=1, \ldots, N^{d}$. So, the set $\mathcal{A}_{N}$ is an orthonormal basis of $X_{N^{d}}$ formed by the eigenvectors of $\mathbb{L}_{N}$ and $\mathbb{L}_{N}^{j}$. In particular, for each $f \in X_{N^{d}}$, there exist $\beta_{i} \in \mathbb{R}$ such that $f=\sum_{i=1}^{N^{d}} \beta_{i} \phi_{i}$. Thus,

$$
\begin{aligned}
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N}^{j} f(x / N)\right)^{2} & =\left\|\mathbb{L}_{N}^{j} f\right\|_{N^{d}}^{2}=\left\|\mathbb{L}_{N}^{j} \sum_{i=1}^{N^{d}} \beta_{i} \phi_{i}\right\|_{N^{d}}^{2}=\sum_{i=1}^{N^{d}}\left(\alpha_{i}^{j} \beta_{i}\right)^{2} \\
& \leq \sum_{i=1}^{N^{d}}\left(\sum_{j=1}^{d} \alpha_{i}^{j}\right)^{2}\left(\beta_{i}\right)^{2}=\left\|\mathbb{L}_{N} f\right\|_{N^{d}}^{2}=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N} f(x / N)\right)^{2},
\end{aligned}
$$

where $\alpha_{i}^{j} \leq 0$ is the eigenvalue of the operator $\mathbb{L}_{N}^{j}$ associated to the eigenvector $\phi_{i}$. This concludes the proof of the lemma.

### 4.2. Semigroups and resolvents

In this subsection we introduce families of semigroups and resolvents associated to the generators $\mathbb{L}_{N}$ and $\mathcal{L}_{W}$. We present some properties and results regarding the convergence of these operators.

Denote by $\left\{P_{t}^{N}: t \geq 0\right\}$ (resp. $\left\{G_{\lambda}^{N}: \lambda>0\right\}$ ) the semigroup (resp. the resolvent) associated to the generator $\mathbb{L}_{N}$, by $\left\{P_{t}^{N, j}: t \geq 0\right\}$ the semigroup associated to the generator $\mathbb{L}_{N}^{j}$, by $\left\{P_{t}^{j}: t \geq 0\right\}$ the semigroup associated to the generator $\mathcal{L}_{W_{j}}$ and by $\left\{P_{t}: t \geq 0\right\}$ (resp. $\left\{G_{\lambda}: \lambda>0\right\}$ ) the semigroup (resp. the resolvent) associated to the generator $\mathcal{L}_{W}$.

Since the jump rates from $x / N$ (resp. $\left.\left(x+e_{j}\right) / N\right)$ to $\left(x+e_{j}\right) / N$ (resp. $x / N$ ) are equal, $P_{t}^{N}$ is symmetric: $P_{t}^{N}(x, y)=P_{t}^{N}(y, x)$.

Using the decompositions (3.6) and (4.1), we obtain

$$
P_{t}^{N}(x, y)=\prod_{j=1}^{d} P_{t}^{N, j}\left(x_{j}, y_{j}\right) \quad \text { and } \quad P_{t}(x, y)=\prod_{j=1}^{d} P_{t}^{j}\left(x_{j}, y_{j}\right)
$$

By definition, for every $H: N^{-1} \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$,

$$
G_{\lambda} H=\int_{0}^{\infty} \mathrm{d} t \mathrm{e}^{-\lambda t} P_{t} H=\left(\lambda \mathbb{I}-\mathcal{L}_{W}\right)^{-1} H
$$

where $\mathbb{I}$ is the identity operator.
Lemma 4.2. Let $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ be a continuous function. Then

$$
\begin{equation*}
\lim _{N \rightarrow+\infty} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-P_{t} H(x / N)\right|=0 . \tag{4.2}
\end{equation*}
$$

Proof. If $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ has the form $H\left(x_{1}, \ldots, x_{d}\right)=\prod_{j=1}^{d} H_{j}\left(x_{j}\right)$, we have

$$
\begin{equation*}
P_{t}^{N} H(x)=\prod_{j=1}^{d} P_{t}^{N, j} H_{j}\left(x_{j}\right) \quad \text { and } \quad P_{t} H(x)=\prod_{j=1}^{d} P_{t}^{j} H_{j}\left(x_{j}\right) . \tag{4.3}
\end{equation*}
$$

Now, for any continuous function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$, and any $\epsilon>0$, we can find continuous functions $H_{j, k}: \mathbb{T} \rightarrow \mathbb{R}$, such that $H^{\prime}: \mathbb{T}^{d} \rightarrow \mathbb{R}$, which is given by

$$
H^{\prime}(x)=\sum_{j=1}^{m} \prod_{k=1}^{d} H_{j, k}\left(x_{k}\right),
$$

satisfies $\left\|H^{\prime}-H\right\|_{\infty} \leq \epsilon$. Thus,

$$
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-P_{t} H(x / N)\right| \leq 2 \epsilon+\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H^{\prime}(x / N)-P_{t} H^{\prime}(x / N)\right| .
$$

By (4.3) and similar identities for $P_{t} H^{\prime}$ and $P_{t}^{N, j} H^{\prime}$, the sum on the right-hand side in the previous inequality is less than or equal to

$$
\begin{aligned}
& \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} \sum_{j=1}^{m}\left|\prod_{k=1}^{d} P_{t}^{N, k} H_{j, k}\left(x_{k} / N\right)-\prod_{k=1}^{d} P_{t}^{k} H_{j, k}\left(x_{k} / N\right)\right| \\
& \quad \leq \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} \sum_{j=1}^{m} C_{j} \sum_{k=1}^{d}\left|P_{t}^{N, k} H_{j, k}\left(x_{k} / N\right)-P_{t}^{k} H_{j, k}\left(x_{k} / N\right)\right|,
\end{aligned}
$$

where $C_{j}$ is a constant that depends on the product $\prod_{k=1}^{d} H_{j, k}$. The previous expressions can be rewritten as

$$
\begin{gathered}
\sum_{j=1}^{m} C_{j} \sum_{k=1}^{d} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d-1}} \sum_{i=1}^{N}\left|P_{t}^{N, k} H_{j, k}(i / N)-P_{t}^{k} H_{j, k}(i / N)\right| \\
\quad=\sum_{j=1}^{m} C_{j} \sum_{k=1}^{d} \frac{1}{N} \sum_{i=1}^{N}\left|P_{t}^{N, k} H_{j, k}(i / N)-P_{t}^{k} H_{j, k}(i / N)\right| .
\end{gathered}
$$

Moreover, by [3], Lemma 4.5, item iii, when $N \rightarrow \infty$, the last expression converges to 0 .
Corollary 4.3. Let $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ be a continuous function. Then

$$
\begin{equation*}
\lim _{N \rightarrow+\infty} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|G_{\lambda}^{N} H(x / N)-G_{\lambda} H(x / N)\right|=0 . \tag{4.4}
\end{equation*}
$$

Proof. By the definition of resolvent, for each $N$, the previous expression is less than or equal to

$$
\int_{0}^{\infty} \mathrm{d} t \mathrm{e}^{-\lambda t} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-P_{t} H(x / N)\right| .
$$

Corollary now follows from the previous lemma.
Let $f_{N}: \frac{1}{N} \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$ be any function. Then, whenever needed, we consider $f: \mathbb{T}^{d} \rightarrow \mathbb{R}$ as the extension of $f_{N}$ to $\mathbb{T}^{d}$ given by:

$$
f(y)=f_{N}(x), \quad \text { if } x \in \mathbb{T}_{N}^{d}, y \geq x \text { and }\|y-x\|_{\infty}<\frac{1}{N} .
$$

Let $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ be a continuous function. Then the extension of $P_{t}^{N} H: \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$ to $\mathbb{T}^{d}$ belongs to $L^{1}\left(\mathbb{T}^{d}\right)$, and by symmetry of the transition probability $P_{t}^{N}(x, y)$ we have

$$
\begin{equation*}
\int_{\mathbb{T}^{d}} \mathrm{~d} u P_{t}^{N} H(u)=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}^{d}} H(x / N) . \tag{4.5}
\end{equation*}
$$

The next lemma shows that $H$ can be approximated by $P_{t}^{N} H$. As an immediate consequence, we obtain an approximation result involving the resolvent.

Lemma 4.4. Let $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ be a continuous function. Then,

$$
\begin{equation*}
\lim _{t \rightarrow 0} \limsup _{N \rightarrow+\infty} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-H(x / N)\right|=0 \tag{4.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\lim _{\lambda \rightarrow+\infty} \limsup _{N \rightarrow+\infty} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|\lambda G_{\lambda}^{N} H(x / N)-H(x / N)\right|=0 \tag{4.7}
\end{equation*}
$$

Proof. Fix $\epsilon>0$, and consider $H^{\prime}$ as in the proof of Lemma 4.2. Thus,

$$
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-H(x / N)\right| \leq 2 \epsilon+\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H^{\prime}(x / N)-H^{\prime}(x / N)\right|,
$$

where the second term on the right-hand side is less than or equal to

$$
C_{0} \sup _{j, k} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N, k} H_{j, k}\left(x_{k} / N\right)-H_{j, k}\left(x_{k} / N\right)\right|
$$

with $C_{0}$ being a constant that depends on $H^{\prime}$. By [3], Lemma 4.6, the last expression converges to 0 , when $N \rightarrow \infty$, and then $t \rightarrow 0$. This proves the first equality.

To obtain the second limit, note that, by definition of the resolvent, the second expression is less than or equal to

$$
\int_{0}^{\infty} \mathrm{d} t \lambda \mathrm{e}^{-\lambda t} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|P_{t}^{N} H(x / N)-H(x / N)\right|
$$

By (4.5) the sum is uniformly bounded in $t$ and $N$. Furthermore, it vanishes as $N \rightarrow \infty$ and $t \rightarrow 0$. This proves the second part.

Fix a function $H: \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$. For $\lambda>0$, let $H_{\lambda}^{N}=G_{\lambda}^{N} H$ be the solution of the resolvent equation

$$
\begin{equation*}
\lambda H_{\lambda}^{N}-\mathbb{L}_{N} H_{\lambda}^{N}=H . \tag{4.8}
\end{equation*}
$$

Taking inner product on both sides of this equation with respect to $H_{\lambda}^{N}$, we obtain

$$
\begin{aligned}
& \lambda \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(H_{\lambda}^{N}(x / N)\right)^{2}-\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H_{\lambda}^{N}(x / N) \mathbb{L}_{N} H_{\lambda}^{N} \\
& \quad=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H_{\lambda}^{N}(x / N) H(x / N) .
\end{aligned}
$$

A simple computation shows that the second term on the left-hand side is equal to

$$
\frac{1}{N^{d}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} \xi_{x, x+e_{j}}\left[\nabla_{N, j} H_{\lambda}^{N}(x / N)\right]^{2},
$$

where $\nabla_{N, j} H(x / N)=N\left[H\left(\left(x+e_{j}\right) / N\right)-H(x / N)\right]$ is the discrete derivative of the function $H$ in the direction of the vector $e_{j}$. In particular, by Schwarz inequality,

$$
\begin{align*}
& \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H_{\lambda}^{N}(x / N)^{2} \leq \frac{1}{\lambda^{2}} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N)^{2} \quad \text { and } \\
& \frac{1}{N^{d}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} \xi_{x, x+e_{j}}\left[\nabla_{N, j} H_{\lambda}^{N}(x / N)\right]^{2} \leq \frac{1}{\lambda} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N)^{2} . \tag{4.9}
\end{align*}
$$

## 5. Scaling limit

Let $\mathcal{M}$ be the space of positive measures on $\mathbb{T}^{d}$ with total mass bounded by one, and endowed with the weak topology. Recall that $\pi_{t}^{N} \in \mathcal{M}$ stands for the empirical measure at time $t$. This is the measure on $\mathbb{T}^{d}$ obtained by rescaling space by $N$, and by assigning mass $1 / N^{d}$ to each particle:

$$
\begin{equation*}
\pi_{t}^{N}=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} \eta_{t}(x) \delta_{x / N} \tag{5.1}
\end{equation*}
$$

where $\delta_{u}$ is the Dirac measure concentrated in $u$.

For a continuous function $H: \mathbb{T}^{d} \rightarrow \mathbb{R},\left\langle\pi_{t}^{N}, H\right\rangle$ stands for the integral of $H$ with respect to $\pi_{t}^{N}$ :

$$
\left\langle\pi_{t}^{N}, H\right\rangle=\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N) \eta_{t}(x)
$$

This notation is not to be mistaken with the inner product in $L^{2}\left(\mathbb{T}^{d}\right)$ introduced earlier. Also, when $\pi_{t}$ has a density $\rho$, $\pi(t, \mathrm{~d} u)=\rho(t, u) \mathrm{d} u$, we sometimes write $\left\langle\rho_{t}, H\right\rangle$ for $\left\langle\pi_{t}, H\right\rangle$.

For a local function $g:\{0,1\}^{\mathbb{Z}^{d}} \rightarrow \mathbb{R}$, let $\tilde{g}:[0,1] \rightarrow \mathbb{R}$ be the expected value of $g$ under the stationary states:

$$
\tilde{g}(\alpha)=E_{\nu_{\alpha}}[g(\eta)] .
$$

For $\ell \geq 1$ and $d$-dimensional integer $x=\left(x_{1}, \ldots, x_{d}\right)$, denote by $\eta^{\ell}(x)$ the empirical density of particles in the box $\mathbb{B}_{+}^{\ell}(x)=\left\{\left(y_{1}, \ldots, y_{d}\right) \in \mathbb{Z}^{d} ; 0 \leq y_{i}-x_{i}<\ell\right\}:$

$$
\eta^{\ell}(x)=\frac{1}{\ell^{d}} \sum_{y \in \mathbb{B}_{+}^{\ell}(x)} \eta(y) .
$$

Fix $T>0$, and let $D([0, T], \mathcal{M})$ be the space of $\mathcal{M}$-valued càdlàg trajectories $\pi:[0, T] \rightarrow \mathcal{M}$ endowed with the uniform topology. For each probability measure $\mu_{N}$ on $\{0,1\}^{\mathbb{T}_{N}^{d}}$, denote by $\mathbb{Q}_{\mu_{N}}^{W, N}$ the measure on the path space $D([0, T], \mathcal{M})$ induced by the measure $\mu_{N}$ and the process $\pi_{t}^{N}$ introduced in (5.1).

Fix a continuous profile $\rho_{0}: \mathbb{T}^{d} \rightarrow[0,1]$, and consider a sequence $\left\{\mu_{N}: N \geq 1\right\}$ of measures on $\{0,1\}^{\mathbb{T}_{N}^{d}}$ associated to $\rho_{0}$ in the sense (2.8). Further, we denote by $\mathbb{Q}_{W}$ be the probability measure on $D([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, \mathrm{~d} u)=\rho(t, u) \mathrm{d} u$, where $\rho$ is the unique weak solution of (2.9) with $\gamma=\rho_{0}, l_{k}=0, r_{k}=1$, $k=1, \ldots, d$, and $\Phi(\alpha)=\alpha+a \alpha^{2}$.

In Section 5.1 we show that the sequence $\left\{\mathbb{Q}_{\mu_{N}}^{W, N}: N \geq 1\right\}$ is tight, and in Section 5.2 we characterize the limit points of this sequence.

### 5.1. Tightness

The proof of tightness of sequence $\left\{\mathbb{Q}_{\mu_{N}}^{W, N}: N \geq 1\right\}$ is motivated by $[6,7]$. We consider, initially, the auxiliary $\mathcal{M}$ valued Markov process $\left\{\Pi_{t}^{\lambda, N}: t \geq 0\right\}, \lambda>0$, defined by

$$
\Pi_{t}^{\lambda, N}(H)=\left\langle\pi_{t}^{N}, G_{\lambda}^{N} H\right\rangle=\frac{1}{N^{d}} \sum_{x \in \mathbb{Z}^{d}}\left(G_{\lambda}^{N} H\right)(x / N) \eta_{t}(x)
$$

for $H$ in $C\left(\mathbb{T}^{d}\right)$, where $\left\{G_{\lambda}^{N}: \lambda>0\right\}$ is the resolvent associated to the random walk $\left\{X_{t}^{N}: t \geq 0\right\}$ introduced in Section 4.

We first prove tightness of the process $\left\{\Pi_{t}^{\lambda, N}: 0 \leq t \leq T\right\}$ for every $\lambda>0$, and we then show that $\left\{\lambda \Pi_{t}^{\lambda, N}: 0 \leq\right.$ $t \leq T\}$ and $\left\{\pi_{t}^{N}: 0 \leq t \leq T\right\}$ are not far apart if $\lambda$ is large.

It is well-known [8], Proposition 4.1.7, that to prove tightness of $\left\{\Pi_{t}^{\lambda, N}: 0 \leq t \leq T\right\}$ it is enough to show tightness of the real-valued processes $\left\{\Pi_{t}^{\lambda, N}(H): 0 \leq t \leq T\right\}$ for a set of smooth functions $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$ dense in $C\left(\mathbb{T}^{d}\right)$ for the uniform topology.

Fix a smooth function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$. Denote by the same symbol the restriction of $H$ to $N^{-1} \mathbb{T}_{N}^{d}$. Let $H_{\lambda}^{N}=G_{\lambda}^{N} H$, and keep in mind that $\Pi_{t}^{\lambda, N}(H)=\left\langle\pi_{t}^{N}, H_{\lambda}^{N}\right\rangle$. Denote by $M_{t}^{N, \lambda}$ the martingale defined by

$$
\begin{equation*}
M_{t}^{N, \lambda}=\Pi_{t}^{\lambda, N}(H)-\Pi_{0}^{\lambda, N}(H)-\int_{0}^{t} \mathrm{~d} s N^{2} L_{N}\left\langle\pi_{s}^{N}, H_{\lambda}^{N}\right\rangle \tag{5.2}
\end{equation*}
$$

Clearly, tightness of $\Pi_{t}^{\lambda, N}(H)$ follows from tightness of the martingale $M_{t}^{N, \lambda}$ and tightness of the additive functional $\int_{0}^{t} \mathrm{~d} s N^{2} L_{N}\left\langle\pi_{s}^{N}, H_{\lambda}^{N}\right\rangle$.

A simple computation shows that the quadratic variation $\left\langle M^{N, \lambda}\right\rangle_{t}$ of the martingale $M_{t}^{N, \lambda}$ is given by:

$$
\frac{1}{N^{2 d}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}^{d}} \xi_{x, x+e_{j}}\left[\nabla_{N, j} H_{\lambda}^{N}(x / N)\right]^{2} \int_{0}^{t} c_{x, x+e_{j}}\left(\eta_{s}\right)\left[\eta_{s}\left(x+e_{j}\right)-\eta_{s}(x)\right]^{2} \mathrm{~d} s
$$

In particular, by (4.9),

$$
\left\langle M^{N, \lambda}\right\rangle_{t} \leq \frac{C_{0} t}{N^{2 d}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} \xi_{x, x+e_{j}}\left[\left(\nabla_{N, j} H_{\lambda}^{N}\right)(x / N)\right]^{2} \leq \frac{C(H) t}{\lambda N^{d}}
$$

for some finite constant $C(H)$ which depends only on $H$. Thus, by Doob inequality, for every $\lambda>0, \delta>0$,

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \mathbb{P}_{\mu_{N}}\left[\sup _{0 \leq t \leq T}\left|M_{t}^{N, \lambda}\right|>\delta\right]=0 \tag{5.3}
\end{equation*}
$$

In particular, the sequence of martingales $\left\{M_{t}^{N, \lambda}: N \geq 1\right\}$ is tight for the uniform topology.
It remains to be examined the additive functional of the decomposition (5.2). The generator of the exclusion process $L_{N}$ can be decomposed in terms of generators of the random walks $\mathbb{L}_{N}^{j}$. By (4.1) and a long but simple computation, we obtain that $N^{2} L_{N}\left\langle\pi^{N}, H_{\lambda}^{N}\right\rangle$ is equal to

$$
\begin{aligned}
& \sum_{j=1}^{d} \\
& \quad+\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N}^{j} H_{\lambda}^{N}\right)(x / N) \eta(x) \\
& \quad-\frac{a}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left[\left(\mathbb{L}_{N}^{j} H_{\lambda}^{N}\right)\left(\left(x+e_{j}^{d}\right) / N\right)+\left(\mathbb{L}_{N}^{j} H_{\lambda}^{N}\right)(x / N)\right]\left(\tau_{x} h_{1, j}^{j}\right)(\eta) \\
&
\end{aligned}
$$

where $\left\{\tau_{x}: x \in \mathbb{Z}^{d}\right\}$ is the group of translations, so that $\left(\tau_{x} \eta\right)(y)=\eta(x+y)$ for $x, y$ in $\mathbb{Z}^{d}$, and the sum is understood modulo $N$. Also, $h_{1, j}, h_{2, j}$ are the cylinder functions

$$
h_{1, j}(\eta)=\eta(0) \eta\left(e_{j}\right), \quad h_{2, j}(\eta)=\eta\left(-e_{j}\right) \eta\left(e_{j}\right)
$$

For all $0 \leq s<t \leq T$, we have

$$
\left|\int_{s}^{t} \mathrm{~d} r N^{2} L_{N}\left\langle\pi_{r}^{N}, H_{\lambda}^{N}\right\rangle\right| \leq \frac{(1+3|a|)(t-s)}{N^{d}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}}\left|\mathbb{L}_{N}^{j} H_{\lambda}^{N}(x / N)\right|
$$

from Schwarz inequality and Lemma 4.1, the right-hand side of the previous expression is bounded above by

$$
(1+3|a|)(t-s) d \sqrt{\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left(\mathbb{L}_{N} H_{\lambda}^{N}(x / N)\right)^{2}}
$$

Since $H_{\lambda}^{N}$ is the solution of the resolvent equation (4.8), we may replace $\mathbb{L}_{N} H_{\lambda}^{N}$ by $U_{\lambda}^{N}=\lambda H_{\lambda}^{N}-H$ in the previous formula. In particular, It follows from the first estimate in (4.9), that the right-hand side of the previous expression is bounded above by $d C(H, a)(t-s)$ uniformly in $N$, where $C(H, a)$ is a finite constant depending only on $a$ and $H$. This proves that the additive part of the decomposition (5.2) is tight for the uniform topology and therefore that the sequence of processes $\left\{\Pi_{t}^{\lambda, N}: N \geq 1\right\}$ is tight.

Lemma 5.1. The sequence of measures $\left\{\mathbb{Q}_{\mu^{N}}^{W, N}: N \geq 1\right\}$ is tight for the uniform topology.
Proof. It is enough to show that for every smooth function $H: \mathbb{T} \rightarrow \mathbb{R}$, and every $\epsilon>0$, there exists $\lambda>0$ such that

$$
\lim _{N \rightarrow \infty} \mathbb{P}_{\mu^{N}}\left[\sup _{0 \leq t \leq T}\left|\Pi_{t}^{\lambda, N}(\lambda H)-\left\langle\pi_{t}^{N}, H\right\rangle\right|>\epsilon\right]=0,
$$

since, in this case, tightness of $\pi_{t}^{N}$ follows from tightness of $\Pi_{t}^{\lambda, N}$. Since there is at most one particle per site, the expression inside the absolute value is less than or equal to

$$
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|\lambda H_{\lambda}^{N}(x / N)-H(x / N)\right| .
$$

By Lemma 4.4, this expression vanishes as $N \uparrow \infty$ and then $\lambda \uparrow \infty$.

### 5.2. Uniqueness of limit points

We prove in this subsection that all limit points $\mathbb{Q}^{*}$ of the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$ are concentrated on absolutely continuous trajectories $\pi(t, \mathrm{~d} u)=\rho(t, u) \mathrm{d} u$, whose density $\rho(t, u)$ is a weak solution of the hydrodynamic equation (2.9) with $l=0<r=1$ and $\Phi(\alpha)=\alpha+a \alpha^{2}$.

Let $\mathbb{Q}^{*}$ be a limit point of the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$ and assume, without loss of generality, that $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges to $\mathbb{Q}^{*}$.
Since there is at most one particle per site, it is clear that $\mathbb{Q}^{*}$ is concentrated on trajectories $\pi_{t}(\mathrm{~d} u)$ which are absolutely continuous with respect to the Lebesgue measure, $\pi_{t}(\mathrm{~d} u)=\rho(t, u) \mathrm{d} u$, and whose density $\rho$ is non-negative and bounded by 1 .

Fix a continuously differentiable function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}$, and $\lambda>0$. Recall the definition of the martingale $M_{t}^{N, \lambda}$ introduced in the previous section. By (5.2) and (5.3), for fixed $0<t \leq T$ and $\delta>0$,

$$
\lim _{N \rightarrow \infty} \mathbb{Q}_{\mu_{N}, N}^{W,}\left[\left|\left\langle\pi_{t}^{N}, G_{\lambda}^{N} H\right\rangle-\left\langle\pi_{0}^{N}, G_{\lambda}^{N} H\right\rangle-\int_{0}^{t} \mathrm{~d} s N^{2} L_{N}\left\langle\pi_{s}^{N}, G_{\lambda}^{N} H\right\rangle\right|>\delta\right]=0 .
$$

Since there is at most one particle per site, we may use Corollary 4.3 to replace $G_{\lambda}^{N} H$ by $G_{\lambda} H$ in the expressions $\left\langle\pi_{t}^{N}, G_{\lambda}^{N} H\right\rangle,\left\langle\pi_{0}^{N}, G_{\lambda}^{N} H\right\rangle$ above. On the other hand, the expression $N^{2} L_{N}\left\langle\pi_{s}^{N}, G_{\lambda}^{N} H\right\rangle$ has been computed in the previous subsection. Since $E_{\nu_{\alpha}}\left[h_{i, j}\right]=\alpha^{2}, i=1,2$ and $j=1, \ldots, d$, Lemma 4.1 and the estimate (4.9), permit us use Corollary 5.4 to obtain, for every $t>0, \lambda>0, \delta>0, i=1,2$,

$$
\lim _{\varepsilon \rightarrow 0} \limsup _{N \rightarrow \infty} \mathbb{P}_{\mu_{N}}\left[\left|\int_{0}^{t} \mathrm{~d} s \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} \mathbb{L}_{N}^{j} H_{\lambda}^{N}(x / N)\left\{\tau_{x} h_{i, j}\left(\eta_{s}\right)-\left[\eta_{s}^{\varepsilon N}(x)\right]^{2}\right\}\right|>\delta\right]=0 .
$$

Recall that $\mathbb{L}_{N} G_{\lambda}^{N} H=\lambda G_{\lambda}^{N} H-H$. As before, we may replace $G_{\lambda}^{N} H$ by $G_{\lambda} H$. Let $U_{\lambda}=\lambda G_{\lambda} H-H$. Since $\eta_{s}^{\varepsilon N}(x)=\varepsilon^{-d} \pi_{s}^{N}\left(\prod_{j=1}^{d}\left[x_{j} / N, x_{j} / N+\varepsilon e_{j}\right]\right)$, we obtain, from the previous considerations, that

$$
\lim _{\varepsilon \rightarrow 0} \limsup _{N \rightarrow \infty} \mathbb{Q}_{\mu_{N}}^{W, N}\left[\left|\left\langle\pi_{t}^{N}, G_{\lambda} H\right\rangle-\left\langle\pi_{0}^{N}, G_{\lambda} H\right\rangle-\int_{0}^{t} \mathrm{~d} s\left\langle\Phi\left(\varepsilon^{-d} \pi_{s}^{N}\left(\prod_{j=1}^{d}\left[\cdot, \cdot+\varepsilon e_{j}\right]\right)\right), U_{\lambda}\right\rangle\right|>\delta\right]=0
$$

Since $H$ is a smooth function, $G_{\lambda} H$ and $U_{\lambda}$ can be approximated, in $L^{1}\left(\mathbb{T}^{d}\right)$, by continuous functions. Since we assumed that $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges in the uniform topology to $\mathbb{Q}^{*}$, we have that

$$
\lim _{\varepsilon \rightarrow 0} \mathbb{Q}^{*}\left[\left|\left\langle\pi_{t}, G_{\lambda} H\right\rangle-\left\langle\pi_{0}, G_{\lambda} H\right\rangle-\int_{0}^{t} \mathrm{~d} s\left\langle\Phi\left(\varepsilon^{-d} \pi_{s}\left(\prod_{j=1}^{d}\left[\cdot, \cdot+\varepsilon e_{j}\right]\right)\right), U_{\lambda}\right\rangle\right|>\delta\right]=0 .
$$

Using the fact that $\mathbb{Q}^{*}$ is concentrated on absolutely continuous paths $\pi_{t}(\mathrm{~d} u)=\rho(t, u) \mathrm{d} u$, with positive density bounded by $1, \varepsilon^{-d} \pi_{s}\left(\prod_{j=1}^{d}\left[\cdot, \cdot+\varepsilon e_{j}\right]\right)$ converges in $L^{1}\left(\mathbb{T}^{d}\right)$ to $\rho(s, \cdot)$ as $\varepsilon \downarrow 0$. Thus,

$$
\mathbb{Q}^{*}\left[\left|\left\langle\pi_{t}, G_{\lambda} H\right\rangle-\left\langle\pi_{0}, G_{\lambda} H\right\rangle-\int_{0}^{t} \mathrm{~d} s\left\langle\Phi\left(\rho_{s}\right), \mathcal{L}_{W} G_{\lambda} H\right\rangle\right|>\delta\right]=0,
$$

because $U_{\lambda}=\mathcal{L}_{W} G_{\lambda} H$. Letting $\delta \downarrow 0$, we see that, $\mathbb{Q}^{*}$ a.s.,

$$
\left\langle\pi_{t}, G_{\lambda} H\right\rangle-\left\langle\pi_{0}, G_{\lambda} H\right\rangle=\int_{0}^{t} \mathrm{~d} s\left\langle\Phi\left(\rho_{s}\right), \mathcal{L}_{W} G_{\lambda} H\right\rangle
$$

This identity can be extended to a countable set of times $t$. Taking this set to be dense, by continuity of the trajectories $\pi_{t}$, we obtain that it holds for all $0 \leq t \leq T$. In the same way, it holds for any countable family of continuous functions $H$. Taking a countable set of continuous functions, dense for the uniform topology, we extend this identity to all continuous functions $H$, because $G_{\lambda} H_{n}$ converges to $G_{\lambda} H$ in $L^{1}\left(\mathbb{T}^{d}\right)$, if $H_{n}$ converges to $H$ in the uniform topology. Similarly, we can show that it holds for all $\lambda>0$, since, for any continuous function $H, G_{\lambda_{n}} H$ converges to $G_{\lambda} H$ in $L^{1}\left(\mathbb{T}^{d}\right)$, as $\lambda_{n} \rightarrow \lambda$.

Proposition 5.2. As $N \uparrow \infty$, the sequence of probability measures $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges in the uniform topology to $\mathbb{Q}_{W}$.
Proof. In the previous subsection we showed that the sequence of probability measures $\mathbb{Q}_{\mu_{N}}^{W, N}$ is tight for the uniform topology. Moreover, we just proved that all limit points of this sequence are concentrated on weak solutions of the parabolic equation (2.9). The proposition now follows from a straightforward adaptation of the uniquenesses of weak solutions proved in [6] for the $d$-dimensional case.

Proof of Theorem 2.2. Since $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges in the uniform topology to $\mathbb{Q}_{W}$, a measure which is concentrated on a deterministic path. For each $0 \leq t \leq T$ and each continuous function $H: \mathbb{T}^{d} \rightarrow \mathbb{R},\left\langle\pi_{t}^{N}, H\right\rangle$ converges in probability to $\int_{\mathbb{T}} \mathrm{d} u \rho(t, u) H(u)$, where $\rho$ is the unique weak solution of (2.9) with $\gamma=\rho_{0}$ and $\Phi(\alpha)=\alpha+a \alpha^{2}$.

### 5.3. Replacement lemma

We will use some results from [8], Appendix A1. Denote by $H_{N}\left(\mu_{N} \mid \nu_{\alpha}\right)$ the relative entropy of a probability measure $\mu_{N}$ with respect to a stationary state $v_{\alpha}$, see [8], Section A1.8, for a precise definition. By the explicit formula given in [8], Theorem A1.8.3, we see that there exists a finite constant $K_{0}$, depending only on $\alpha$, such that

$$
\begin{equation*}
H_{N}\left(\mu_{N} \mid v_{\alpha}\right) \leq K_{0} N^{d} \tag{5.4}
\end{equation*}
$$

for all measures $\mu_{N}$.
Denote by $\langle\cdot, \cdot\rangle_{\nu_{\alpha}}$ the inner product of $L^{2}\left(\nu_{\alpha}\right)$ and denote by $I_{N}^{\xi}$ the convex and lower semicontinuous [8], Corollary A1.10.3, functional defined by

$$
I_{N}^{\xi}(f)=\left\langle-L_{N} \sqrt{f}, \sqrt{f}\right\rangle_{\nu_{\alpha}}
$$

for all probability densities $f$ with respect to $v_{\alpha}$ (i.e., $f \geq 0$ and $\int f \mathrm{~d} \nu_{\alpha}=1$ ). By [8], Proposition A1.10.1, an elementary computation shows that

$$
I_{N}^{\xi}(f)=\sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} I_{x, x+e_{j}}^{\xi}(f)
$$

where

$$
I_{x, x+e_{j}}^{\xi}(f)=(1 / 2) \xi_{x, x+e_{j}} \int c_{x, x+e_{j}}(\eta)\left\{\sqrt{f\left(\sigma^{x, x+e_{j}} \eta\right)}-\sqrt{f(\eta)}\right\}^{2} \mathrm{~d} \nu_{\alpha}
$$

By [8], Theorem A1.9.2, if $\left\{S_{t}^{N}: t \geq 0\right\}$ stands for the semigroup associated to the generator $N^{2} L_{N}$,

$$
H_{N}\left(\mu_{N} S_{t}^{N} \mid v_{\alpha}\right)+2 N^{2} \int_{0}^{t} I_{N}^{\xi}\left(f_{s}^{N}\right) \mathrm{d} s \leq H_{N}\left(\mu_{N} \mid v_{\alpha}\right),
$$

where $f_{s}^{N}$ stands for the Radon-Nikodym derivative of $\mu_{N} S_{s}^{N}$ with respect to $v_{\alpha}$.
Recall the definition of $\mathbb{B}_{+}^{\ell}(x)$ in begin of this section. For each $y \in \mathbb{B}_{+}^{\ell}(x)$, such that $y_{1}>x_{1}$, let

$$
\begin{equation*}
\Lambda_{x+e_{1}, y}^{\ell}=\left(z_{k}^{y}\right)_{0 \leq k \leq M(y)} \tag{5.5}
\end{equation*}
$$

be a path from $x+e_{1}$ to $y$ such that:
(1) $\Lambda_{x+e_{1}, y}^{\ell}$ begins at $x+e_{1}$ and ends at $y$, i.e.:

$$
z_{0}^{y}=x+e_{1} \quad \text { and } \quad z_{M(y)}^{y}=y ;
$$

(2) the distance between two consecutive sites of the $\Lambda_{x+e_{1}, y}^{\ell}=\left(z_{k}^{y}\right)_{0 \leq k \leq M(y)}$ is equal to 1, i.e.:

$$
z_{k+1}^{y}=z_{k}^{y}+e_{j} \quad \text { for some } j=1, \ldots, d \text { and for all } k=0, \ldots, M(y)-1
$$

(3) $\Lambda_{x+e_{1}, y}^{\ell}$ is injective:

$$
z_{i}^{y} \neq z_{j}^{y} \quad \text { for all } 0 \leq i<j \leq M(y) ;
$$

(4) the path begins by jumping in the direction of $e_{1}$. Furthermore, the jump in the direction of $e_{j+1}$ is only allowed when it is not possible to jump in the direction of $e_{j}$, for $j=1, \ldots, d-1$.

Lemma 5.3. Fix a function $F: N^{-1} \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$. There exists a finite constant $C_{0}=C_{0}(a, g, W)$, depending only on $a$, $g$ and $W$, such that

$$
\begin{aligned}
& \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F(x / N) \int\left\{\tau_{x} g(\eta)-\tilde{g}\left(\eta^{\varepsilon N}(x)\right)\right\} f(\eta) \nu_{\alpha}(\mathrm{d} \eta) \\
& \quad \leq \frac{C_{0}}{\varepsilon N^{d+1}} \sum_{x \in \mathbb{T}_{N}^{d}}|F(x / N)|+\frac{C_{0} \varepsilon}{\delta N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F(x / N)^{2}+\frac{\delta}{N^{d-2}} I_{N}^{\xi}(f)
\end{aligned}
$$

for all $\delta>0, \varepsilon>0$ and all probability densities $f$ with respect to $\nu_{\alpha}$.
Proof. Any local function can be written as a linear combination of functions in the form $\prod_{x \in A} \eta(x)$, where $A$ is a finite set. It is therefore enough to prove the lemma for such functions. We will only prove the result for $g(\eta)=$ $\eta(0) \eta\left(e_{1}\right)$. The general case can be handled in a similar way.

We begin by estimating

$$
\begin{equation*}
\frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F(x / N) \int \eta(x)\left\{\eta\left(x+e_{1}\right)-\frac{1}{(\varepsilon N)^{d}} \sum_{y \in \mathbb{B}_{+}^{N \varepsilon}(x)} \eta(y)\right\} f(\eta) v_{\alpha}(\mathrm{d} \eta) \tag{5.6}
\end{equation*}
$$

in terms of the functional $I_{N}^{\xi}(f)$. The integral in (5.6) can be rewritten as:

$$
\frac{1}{(N \varepsilon)^{d}} \sum_{y \in \mathbb{B}_{+}^{N \varepsilon}(x)} \int \eta(x)\left[\eta\left(x+e_{1}\right)-\eta(y)\right] f(\eta) \nu_{\alpha}(\mathrm{d} \eta) .
$$

For each $y \in \mathbb{B}_{+}^{N \varepsilon}(x)$, such that $y_{1}>x_{1}$, let $\Lambda_{x+e_{1}, y}^{\ell}=\left(z_{k}^{y}\right)_{0 \leq k \leq M(y)}$ be a path like the one in (5.5). Then, by property (1) of $\Lambda_{x+e_{1}, y}^{\ell}$ and using telescopic sum we have the following:

$$
\eta\left(x+e_{1}\right)-\eta(y)=\sum_{k=0}^{M(y)-1}\left[\eta\left(z_{k}^{y}\right)-\eta\left(z_{k+1}^{y}\right)\right] .
$$

We can, therefore, bound (5.6) above by

$$
\frac{1}{N^{d}} \frac{1}{(N \varepsilon)^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} \sum_{y \in \mathbb{B}_{+}^{N \varepsilon}(x)} \sum_{k=0}^{M(y)-1} \int F(x / N) \eta(x)\left[\eta\left(z_{k}^{y}\right)-\eta\left(z_{k+1}^{y}\right)\right] f(\eta) v_{\alpha}(\mathrm{d} \eta)+\frac{1}{\varepsilon N^{d+1}} \sum_{x \in \mathbb{T}_{N}^{d}}|F(x / N)|,
$$

where the last term in the previous expression comes from the contribution of the points $y \in \mathbb{B}_{+}^{N \varepsilon}(x)$, such that $y_{1}=x_{1}$. Recall that, by property (2) of $\Lambda_{x+e_{1}, y}^{\ell}$, we have that $z_{k+1}^{y}=z_{k}^{y}+e_{j}$, for some $j=1, \ldots, d$.

For each term of the form

$$
\int F(x / N) \eta(x)\left\{\eta(z)-\eta\left(z+e_{j}\right)\right\} f(\eta) v_{\alpha}(\mathrm{d} \eta)
$$

we can use the change of variables $\eta^{\prime}=\sigma^{z, z+e_{j}} \eta$ to write the previous integral as

$$
(1 / 2) \int F(x / N) \eta(x)\left\{\eta(z)-\eta\left(z+e_{j}\right)\right\}\left\{f(\eta)-f\left(\sigma^{z, z+e_{j}} \eta\right)\right\} \nu_{\alpha}(\mathrm{d} \eta) .
$$

Since $a-b=(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b})$ and $\sqrt{a b} \leq a+b$, by Schwarz inequality the previous expression is less than or equal to

$$
\begin{aligned}
& \frac{A}{4\left(1-2 a^{-}\right) \xi_{z, z+e_{j}}} \int F(x / N)^{2} \eta(x)\left\{\eta(z)-\eta\left(z+e_{j}\right)\right\}^{2}\left\{\sqrt{f(\eta)}+\sqrt{f\left(\sigma^{z, z+e_{j}} \eta\right)}\right\}^{2} v_{\alpha}(\mathrm{d} \eta) \\
& \quad+\frac{\xi_{z, z+e_{j}}}{A} \int c_{z, z+e_{j}}(\eta)\left\{\sqrt{f(\eta)}-\sqrt{f\left(\sigma^{z, z+e_{j}} \eta\right)}\right\}^{2} v_{\alpha}(\mathrm{d} \eta)
\end{aligned}
$$

for every $A>0$. In this formula we used the fact that $c_{z, z+e_{j}}(\eta)$ is bounded below by $1-2 a^{-}$, where $a^{-}=$ $\max \{-a, 0\}$. Since $f$ is a density with respect to $\nu_{\alpha}$, the first expression is bounded above by $A /\left(1-2 a^{-}\right) \xi_{z, z+e_{j}}$, whereas the second one is equal to $2 A^{-1} I_{z, z+e_{j}}^{\xi}(f)$.

So, using all the previous calculations together with properties (3) and (4) of the path $\Lambda_{x+e_{1}, y}^{\ell}$, we obtain that (5.6) is less than or equal to

$$
\frac{1}{\varepsilon N^{d+1}} \sum_{x \in \mathbb{T}_{N}^{d}}|F(x / N)|+\frac{A}{\left(1-2 a^{-}\right) N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F(x / N)^{2} \sum_{j=1}^{d} \sum_{k=1}^{\varepsilon N} \xi_{x+(k-1) e_{j}, x+k e_{j}}^{-1} \frac{2 \varepsilon}{A N^{d-1}} \sum_{j=1}^{d} \sum_{x \in \mathbb{T}_{N}^{d}} I_{x, x+e_{j}}^{\xi}(f) .
$$

By definition of the sequence $\left\{\xi_{x, x+e_{j}}\right\}, \sum_{k=1}^{\varepsilon N} \xi_{x+k e_{j}, e_{j}}^{-1} \leq N\left[W_{j}(1)-W_{j}(0)\right]$. Thus, choosing $A=2 \varepsilon N^{-1} \delta^{-1}$, for some $\delta>0$, we obtain that the previous sum is bounded above by

$$
\frac{C_{0}}{\varepsilon N^{d+1}} \sum_{x \in \mathbb{T}_{N}^{d}}|F(x / N)|+\frac{C_{0} \varepsilon}{\delta N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F(x / N)^{2}+\frac{\delta}{N^{d-2}} I_{N}^{\xi}(f) .
$$

Up to this point we have succeeded to replace $\eta(x) \eta\left(x+e_{1}\right)$ by $\eta(x) \eta^{\varepsilon N}(x)$. The same arguments permit to replace this latter expression by $\left[\eta^{\varepsilon N}(x)\right]^{2}$, which concludes the proof of the lemma.

Corollary 5.4. Fix a cylinder function $g$, and a sequence of functions $\left\{F_{N}: N \geq 1\right\}, F_{N}: N^{-1} \mathbb{T}_{N}^{d} \rightarrow \mathbb{R}$ such that

$$
\limsup _{N \rightarrow \infty} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)^{2}<\infty .
$$

Then, for any $t>0$ and any sequence of probability measures $\left\{\mu_{N}: N \geq 1\right\}$ on $\{0,1\}^{\mathbb{T}_{N}^{d}}$,

$$
\underset{\varepsilon \rightarrow 0}{\limsup } \limsup _{N \rightarrow \infty} \mathbb{E}_{\mu_{N}}\left[\left|\int_{0}^{t} \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)\left\{\tau_{x} g\left(\eta_{s}\right)-\tilde{g}\left(\eta_{s}^{\varepsilon N}(x)\right)\right\} \mathrm{d} s\right|\right]=0 .
$$

Proof. Fix $0<\alpha<1$. By the entropy and Jensen inequalities, the expectation appearing in the statement of the Lemma is bounded above by

$$
\frac{1}{\gamma N^{d}} \log \mathbb{E}_{v_{\alpha}}\left[\exp \left\{\gamma\left|\int_{0}^{t} \mathrm{~d} s \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)\left\{\tau_{x} g\left(\eta_{s}\right)-\tilde{g}\left(\eta_{s}^{\varepsilon N}(x)\right)\right\}\right|\right\}\right]+\frac{H_{N}\left(\mu_{N} \mid \nu_{\alpha}\right)}{\gamma N^{d}}
$$

for all $\gamma>0$. In view of (5.4), in order to prove the corollary it is enough to show that the first term vanishes as $N \uparrow$ $\infty$, and then $\varepsilon \downarrow 0$, for every $\gamma>0$. We may remove the absolute value inside the exponential by using the elementary inequalities $\mathrm{e}^{|x|} \leq \mathrm{e}^{x}+\mathrm{e}^{-x}$ and $\limsup { }_{N \rightarrow \infty} N^{-1} \log \left\{a_{N}+b_{N}\right\} \leq \max \left\{\lim \sup _{N \rightarrow \infty} N^{-1} \log a_{N}\right.$, $\left.\lim \sup _{N \rightarrow \infty} N^{-1} \log b_{N}\right\}$. Thus, to prove the corollary, it is enough to show that

$$
\limsup _{\varepsilon \rightarrow 0} \limsup _{N \rightarrow \infty} \frac{1}{N^{d}} \log \mathbb{E}_{v_{\alpha}}\left[\exp \left\{\gamma \int_{0}^{t} \mathrm{~d} s \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)\left\{\tau_{x} g\left(\eta_{s}\right)-\tilde{g}\left(\eta_{s}^{\varepsilon N}(x)\right)\right\}\right\}\right]=0
$$

for every $\gamma>0$.
By Feynman-Kac formula, for each fixed $N$ the previous expression is bounded above by

$$
t \gamma \sup _{f}\left\{\int \frac{1}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)\left\{\tau_{x} g(\eta)-\tilde{g}\left(\eta^{\varepsilon N}(x)\right)\right\} f(\eta) \mathrm{d} \nu_{\alpha}-\frac{1}{N^{d-2}} I_{N}^{\xi}(f)\right\},
$$

where the supremum is carried over all density functions $f$ with respect to $\nu_{\alpha}$. Letting $\delta=1$ in Lemma 5.3 , we obtain that the previous expression is less than or equal to

$$
\frac{C_{0} \gamma t}{\varepsilon N^{d+1}} \sum_{x \in \mathbb{T}_{N}^{d}}\left|F_{N}(x / N)\right|+\frac{C_{0} \gamma \varepsilon t}{N^{d}} \sum_{x \in \mathbb{T}_{N}^{d}} F_{N}(x / N)^{2}
$$

for some finite constant $C_{0}$ which depends on $a, g$ and $W$. By assumption on the sequence $\left\{F_{N}\right\}$, for every $\gamma>0$, this expression vanishes as $N \uparrow \infty$ and then $\varepsilon \downarrow 0$. This concludes the proof of the lemma.

## 6. Energy estimate

We prove in this section that any limit point $\mathbb{Q}_{W}^{*}$ of the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$ is concentrated on trajectories $\rho(t, u) \mathrm{d} u$ having finite energy. A more comprehensive treatment of energies can be found in [10].

Denote by $\partial_{x_{j}}$ the partial derivative of a function with respect to the $j$ th coordinate, and by $C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right)$ the set of continuous functions with continuous partial derivative in the $j$ th coordinate. Let $L_{x_{j} \otimes W_{j}}^{2}\left([0, T] \times \mathbb{T}^{d}\right)$ be the Hilbert space of measurable functions $H:[0, T] \times \mathbb{T}^{d} \rightarrow \mathbb{R}$ such that

$$
\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d}\left(x_{j} \otimes W_{j}\right) H(s, u)^{2}<\infty
$$

where $\mathrm{d}\left(x_{j} \otimes W_{j}\right)$ represents the product measure in $\mathbb{T}^{d}$ obtained from Lesbegue's measure in $\mathbb{T}^{d-1}$ and the measure induced by $W_{j}$ :

$$
\mathrm{d}\left(x_{j} \otimes W_{j}\right)=\mathrm{d} x_{1} \cdots \mathrm{~d} x_{j-1} \mathrm{~d} W_{j} \mathrm{~d} x_{j+1} \ldots \mathrm{~d} x_{d}
$$

endowed with the inner product $\langle\langle H, G\rangle\rangle_{x_{j} \otimes W_{j}}$ defined by

$$
\langle\langle H, G\rangle\rangle_{x_{j} \otimes W_{j}}=\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d}\left(x_{j} \otimes W_{j}\right) H(s, u) G(s, u)
$$

Let $\mathbb{Q}_{W}^{*}$ be a limit point of the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$, and assume, without loss of generality, that the sequence $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges to $\mathbb{Q}_{W}^{*}$.

Proposition 6.1. The measure $\mathbb{Q}_{W}^{*}$ is concentrated on paths $\rho(t, x) \mathrm{d} x$ with the property that for all $j=1, \ldots, d$ there exists a function in $L_{x_{j} \otimes W_{j}}^{2}\left([0, T] \times \mathbb{T}^{d}\right)$, denoted by $\mathrm{d} \Phi / \mathrm{d} W_{j}$, such that

$$
\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d} x\left(\partial_{x_{j}} H\right)(s, x) \Phi(\rho(s, x))=-\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}} \mathrm{d}\left(x_{j} \otimes W_{j}(x)\right)\left(\mathrm{d} \Phi / \mathrm{d} W_{j}\right)(s, x) H(s, x)
$$

for all functions $H$ in $C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right)$.
The previous proposition follows from the next lemma. Recall the definition of the constant $K_{0}$ given in (5.4).
Lemma 6.2. There exists a finite constant $K_{1}$, depending only on a, such that

$$
E_{\mathbb{Q}_{W}^{*}}\left[\sup _{H}\left\{\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d} x\left(\partial_{x_{j}} H\right)(s, x) \Phi(\rho(s, x))-K_{1} \int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} H(s, x)^{2} \mathrm{~d}\left(x_{j} \otimes W_{j}(x)\right)\right\}\right] \leq K_{0},
$$

where the supremum is carried over all functions $H \in C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right)$.
Proof of Proposition 6.1. Denote by $\ell: C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right) \rightarrow \mathbb{R}$ the linear functional defined by

$$
\ell(H)=\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d} x\left(\partial_{x_{j}} H\right)(s, x) \Phi(\rho(s, x)) .
$$

Since $C^{0,1}\left([0, T] \times \mathbb{T}^{d}\right)$ is dense in $L_{x_{j} \otimes W_{j}}^{2}\left([0, T] \times \mathbb{T}^{d}\right)$, by Lemma 6.2, $\ell$ is $\mathbb{Q}_{W}^{*}$-almost surely finite in $L_{x_{j} \otimes W_{j}}^{2}\left([0, T] \times \mathbb{T}^{d}\right)$. In particular, by Riesz representation theorem, there exists a function $G$ in $L_{x_{j} \otimes W_{j}}^{2}\left([0, T] \times \mathbb{T}^{d}\right)$ such that

$$
\ell(H)=-\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d}\left(x_{j} \otimes W_{j}(x)\right) H(s, x) G(s, x)
$$

This concludes the proof of the proposition.
For a smooth function $H: \mathbb{T}^{d} \rightarrow \mathbb{R}, \delta>0, \varepsilon>0$ and a positive integer $N$, define $W_{N}^{j}(\varepsilon, \delta, H, \eta)$ by

$$
\begin{aligned}
W_{N}^{j}(\varepsilon, \delta, H, \eta)= & \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N) \frac{1}{\varepsilon N}\left\{\Phi\left(\eta^{\delta N}(x)\right)-\Phi\left(\eta^{\delta N}\left(x+\varepsilon N e_{j}\right)\right)\right\} \\
& -\frac{K_{1}}{\varepsilon N} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N)^{2}\left\{W_{j}\left(\left[x_{j}+\varepsilon N+1\right] / N\right)-W_{j}\left(x_{j} / N\right)\right\}
\end{aligned}
$$

The proof of Lemma 6.2 relies on the following result:

Lemma 6.3. Consider a sequence $\left\{H_{\ell}, \ell \geq 1\right\}$ dense in $C^{0,1}\left([0, T] \times \mathbb{T}^{d}\right)$. For every $k \geq 1$, and every $\varepsilon>0$,

$$
\limsup _{\delta \rightarrow 0} \limsup _{N \rightarrow \infty} \mathbb{E}_{\mu^{N}}\left[\max _{1 \leq i \leq k}\left\{\int_{0}^{T} W_{N}^{j}\left(\varepsilon, \delta, H_{i}(s, \cdot), \eta_{s}\right) \mathrm{d} s\right\}\right] \leq K_{0} .
$$

Proof. It follows from the replacement lemma that in order to prove the lemma we just need to show that

$$
\limsup _{N \rightarrow \infty} \mathbb{E}_{\mu^{N}}\left[\max _{1 \leq i \leq k}\left\{\int_{0}^{T} W_{N}^{j}\left(\varepsilon, H_{i}(s, \cdot), \eta_{s}\right) \mathrm{d} s\right\}\right] \leq K_{0},
$$

where

$$
\begin{aligned}
W_{N}^{j}(\varepsilon, H, \eta)= & \frac{1}{\varepsilon N} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N)\left\{\tau_{x} g(\eta)-\tau_{x+\varepsilon N e_{j}} g(\eta)\right\} \\
& -\frac{K_{1}}{\varepsilon N} \sum_{x \in \mathbb{T}_{N}^{d}} H(x / N)^{2}\left\{W_{j}\left(\left[x_{j}+\varepsilon N+1\right] / N\right)-W_{j}\left(x_{j} / N\right)\right\},
\end{aligned}
$$

and $g(\eta)=\eta(0)+a \eta(0) \eta\left(e_{j}\right)$.
By the entropy and Jensen's inequalities, for each fixed $N$, the previous expectation is bounded above by

$$
\frac{H\left(\mu^{N} \mid v_{\alpha}\right)}{N^{d}}+\frac{1}{N^{d}} \log \mathbb{E}_{v_{\alpha}}\left[\exp \left\{\max _{1 \leq i \leq k}\left\{N^{d} \int_{0}^{T} \mathrm{~d} s W_{N}^{j}\left(\varepsilon, H_{i}(s, \cdot), \eta_{s}\right)\right\}\right\}\right] .
$$

By (5.4), the first term is bounded by $K_{0}$. Since $\exp \left\{\max _{1 \leq j \leq k} a_{j}\right\}$ is bounded above by $\sum_{1 \leq j \leq k} \exp \left\{a_{j}\right\}$, and since $\lim \sup _{N} N^{-d} \log \left\{a_{N}+b_{N}\right\}$ is less than or equal to the maximum of $\lim \sup _{N} N^{-d} \log a_{N}$ and $\limsup \sup _{N} N^{-d} \log b_{N}$, the limit, as $N \uparrow \infty$, of the second term in the previous expression is less than or equal to

$$
\max _{1 \leq i \leq k} \limsup _{N \rightarrow \infty} \frac{1}{N^{d}} \log \mathbb{E}_{v_{\alpha}}\left[\exp \left\{N^{d} \int_{0}^{T} \mathrm{~d} s W_{N}^{j}\left(\varepsilon, H_{i}(s, \cdot), \eta_{s}\right)\right\}\right] .
$$

We now prove that, for each fixed $i$, the above limit is non-positive for a convenient choice of the constant $K_{1}$.
Fix $1 \leq i \leq k$. By Feynman-Kac formula and the variational formula for the largest eigenvalue of a symmetric operator, the previous expression is bounded above by

$$
\int_{0}^{T} \mathrm{~d} s \sup _{f}\left\{\int W_{N}^{j}\left(\varepsilon, H_{i}(s, \cdot), \eta\right) f(\eta) v_{\alpha}(\mathrm{d} \eta)-\frac{1}{N^{d-2}} I_{N}^{\xi}(f)\right\}
$$

for each fixed $N$. In this formula the supremum is taken over all probability densities $f$ with respect to $v_{\alpha}$.
To conclude the proof, rewrite

$$
\eta(x) \eta\left(x+e_{j}\right)-\eta\left(x+\varepsilon N e_{j}\right) \eta\left(x+(\varepsilon N+1) e_{j}\right)
$$

as

$$
\eta(x)\left\{\eta\left(x+e_{j}\right)-\eta\left(x+(\varepsilon N+1) e_{j}\right)\right\}+\eta\left(x+(\varepsilon N+1) e_{j}\right)\left\{\eta(x)-\eta\left(x+\varepsilon N e_{j}\right)\right\}
$$

and repeat the arguments presented in the proof of Lemma 5.3.
Proof of Lemma 6.2. Assume without loss of generality that $\mathbb{Q}_{\mu_{N}}^{W, N}$ converges to $\mathbb{Q}_{W}^{*}$. Consider a sequence $\left\{H_{\ell}, \ell \geq\right.$ 1\} dense in $C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right)$. By Lemma 6.3, for every $k \geq 1$

$$
\begin{aligned}
& \limsup _{\delta \rightarrow 0} E_{\mathbb{Q}_{W}^{*}}\left[\operatorname { m a x } _ { 1 \leq i \leq k } \left\{\frac{1}{\varepsilon} \int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d} x H_{i}(s, x)\left\{\Phi\left(\rho_{s}^{\delta}(x)\right)-\Phi\left(\rho_{s}^{\delta}\left(x+\varepsilon e_{j}\right)\right)\right\}\right.\right. \\
& \left.\left.\quad-\frac{K_{1}}{\varepsilon} \int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} \mathrm{~d} x H_{i}(s, x)^{2}\left[W_{j}\left(x_{j}+\varepsilon\right)-W_{j}\left(x_{j}\right)\right]\right\}\right] \leq K_{0},
\end{aligned}
$$

where $\rho_{s}^{\delta}(x)=\left(\rho_{s} * \iota_{\delta}\right)(x)$ and $\iota_{\delta}$ is the approximation of the identity $\iota_{\delta}(\cdot)=(\delta)^{-d} \mathbf{1}\left\{[0, \delta]^{d}\right\}(\cdot)$.
Letting $\delta \downarrow 0$, changing variables, and then letting $\varepsilon \downarrow 0$, we obtain that

$$
E_{\mathbb{Q}_{W}^{*}}\left[\max _{1 \leq i \leq k}\left\{\int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}}\left(\partial_{x_{j}} H_{i}\right)(s, x) \Phi(\rho(s, x)) \mathrm{d} x-K_{1} \int_{0}^{T} \mathrm{~d} s \int_{\mathbb{T}^{d}} H_{i}(s, x)^{2} \mathrm{~d}\left(x_{j} \otimes W_{j}(x)\right)\right\}\right] \leq K_{0} .
$$

To conclude the proof, we apply the monotone convergence theorem, and recall that $\left\{H_{\ell}, \ell \geq 1\right\}$ is a dense sequence in $C^{0,1_{j}}\left([0, T] \times \mathbb{T}^{d}\right)$ for the norm $\|H\|_{\infty}+\left\|\left(\partial_{x_{j}} H\right)\right\|_{\infty}$.

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