# Universality for certain Hermitian Wigner matrices under weak moment conditions ${ }^{1}$ 

Kurt Johansson<br>Department of Mathematics, KTH Royal Institute of Technology, SE-100 44 Stockholm, Sweden. E-mail: kurtj@kth.se

Received 26 January 2010; revised 7 April 2011; accepted 11 April 2011


#### Abstract

We study the universality of the local eigenvalue statistics of Gaussian divisible Hermitian Wigner matrices. These random matrices are obtained by adding an independent GUE matrix to an Hermitian random matrix with independent elements, a Wigner matrix. We prove that Tracy-Widom universality holds at the edge in this class of random matrices under the optimal moment condition that there is a uniform bound on the fourth moment of the matrix elements. Furthermore, we show that universality holds in the bulk for Gaussian divisible Wigner matrices if we just assume finite second moments.


Résumé. Nous étudions l'universalité des statistiques locales du spectre des matrices de Wigner hermitiennes divisibles par une gaussienne. Ces matrices aléatoires sont obtenues en ajoutant à une matrice de Wigner hermitienne avec des coefficients indépendants une matrice du GUE indépendante. Nous montrons que la classe d'universalité de la loi de Tracy-Widom pour les valeurs propres extrêmes est vérifíé sous la condition optimale d'une borne uniforme sur le quatrième moment des coefficients de la matrice. De plus, nous démontrons l'universalité des fluctuations dans l'intérieur du spectre dès lors que le second moment est fini.

MSC: 60B20; 82B44
Keywords: Wigner matrix; Gaussian divisible; Optimal moment condition; Universality; Tracy-Widom distribution

## 1. Introduction and results

### 1.1. Introduction

An Hermitian Wigner matrix is a random Hermitian matrix with independent elements respecting the Hermitian symmetry. The local eigenvalue statistics of these random matrices is expected to be universal in the sense that it is independent of the distribution of the individual matrix elements, at least under suitable assumptions on the moments of the elements. There are two basic cases. We can either look in the bulk of the spectrum or at the edge around the largest eigenvalue. It is conjectured that, if we assume that the real and imaginary parts of the elements all have mean value zero, variance $\sigma^{2}>0$ and that there is a uniform bound on the fourth moment, then the appropriately scaled eigenvalue point process at the edge should converge to the Airy kernel point process. Furthermore the largest eigenvalue should asymptotically fluctuate according to the Tracy-Widom distribution. This problem is still open, but there are results under stronger moment assumptions. The breakthrough result by Soshnikov, [20], showed that the result is true if the distribution is symmetric and has sub-gaussian tails. Soshnikov's result is based on moment methods. The condition on the moments has been weakened to $18+\varepsilon$ moments (or $36+\varepsilon$ moments, see [1]) in [16].

In the bulk it is expected that the local eigenvalue point process converges to the sine-kernel point process. The exact conditions needed for this to be true are not clear. The result in the bulk was proved for a sub-class of Wigner

[^0]matrices, so called Gaussian divisible Hermitian Wigner matrices in [14]. A Gaussian divisible Hermitian Wigner matrix is an Hermitian Wigner matrix $W$ of the form $W=X+\sqrt{\kappa} V$, where $X$ is an Hermitian Wigner matrix and $V$ an independent GUE matrix. In [14] it was assumed that the elements of $X$ have uniformly bounded $6+\varepsilon$ moments. Spectacular progress has recently been made on this problem by Tao and Vu, [23], with their four-moment theorem, and by Erdös, Ramirez, Schlein and H.-T. Yau using a different approach, [11]. Tao and Vu assume subexponential tails for the distribution of the matrix elements. Erdös, Ramirez, Schlein and H.-T. Yau make rather strong regularity assumptions on the distribution and parts of the argument use methods related to the approach in [14] and this paper. A combined effort, [12], removed some of the assumptions in [23]. Thus, the universality result in the bulk in now established under the assumption of subexponential decay of the tails of the distributions of the matrix elements. ${ }^{2}$

Very recently, Tao and Vu, [22], also generalized Soshnikov's result using an approach analogous to that in their paper on bulk universality. They obtain universality at the edge under the assumption of subexponential deacy and vanishing third moments. The result in this paper can be used to remove this third moment assumption, see Theorem 1.5.

The four-moment theorem indicates that the class of Gaussian divisible Wigner matrices is a good testing ground for what we can expect for general Wigner matrices. In this paper we therefore return to the case of Gaussian divisible Hermitian Wigner matrices with the aim of establishing universality results within this class under weak moment conditions. In particular, we prove universality at the edge under the optimal assumption that the fourth moment is finite. It is known that if we have fewer than four moments then the behaviour around the largest eigenvalue is instead described by a Poisson process, see [1,8,21].

We also show universality in the bulk within the class of Gaussian divisible Hermitian Wigner matrices under the assumption that the second moment is finite. It is not clear that this is the optimal condition. Rather, close to the origin we may still expect sine-kernel universality even if the second moment is infinite, see [10].

The results are obtained using a development of the techniques in [14] which were based on a contour integral formula for a correlation kernel from [9]. In [14] an important tool was a concentration of measure estimate for the empirical eigenvalue distribution for a Wigner matrix from [13]. Here, due to the weak moment assumptions we have to proceed differently and in particular the choice of contours in the contour integral formula becomes more delicate. Hence, the analysis that was done in [14] has to be modified in the technical details and this is somewhat subtle as can be expected since we are at the borderline of the validity of the conclusions of the theorem.

### 1.2. Results

We turn now to precise statements of our results. The $n \times n$ random matrix $X$ is an Hermitian Wigner matrix if $X=\left(x_{i j}\right)$ is Hermitian, $\operatorname{Re} x_{i j}, \operatorname{Im} x_{i j}, 1 \leq i<j \leq n$, and $X_{j j}, 1 \leq j \leq n$, are all independent and satisfy
(i) $\mathbb{E}\left[\operatorname{Re} x_{i j}\right]=\mathbb{E}\left[\operatorname{Im} x_{i j}\right]=0,1 \leq i \leq j \leq n$,
(ii) $\mathbb{E}\left[\left(\operatorname{Re} x_{i j}\right)^{2}\right]=\mathbb{E}\left[\left(\operatorname{Im} x_{i j}\right)^{2}\right]=\sigma^{2} / 2,1 \leq i<j \leq n$,
(iii) $\mathbb{E}\left[x_{j j}^{2}\right]=\sigma^{2}$,
where the variance $\sigma^{2}<\infty$. The distribution of the different real and imaginary parts need not be identical and could depend on $n$. We will also assume that
(iv) $\lim _{n \rightarrow \infty} \frac{1}{n^{2}} \sum_{1 \leq i \leq j \leq n} \mathbb{E}\left[\left|x_{j k}\right|^{2} 1\left(\left|x_{j k}\right|>\eta \sqrt{n}\right)\right]=0$
for any constant $\eta>0$. Here $1(A)$ denotes the inicator function for the event $A$. This last condition is automatic if we have i.i.d. elements. Under assumptions (i)-(iv) we know that the semi-circle law holds, [5].

We will say that $W$ is a Gaussian divisible Hermitian Wigner matrix if it can be written

$$
\begin{equation*}
W=X+\sqrt{\kappa} V \tag{1.1}
\end{equation*}
$$

where $X$ is an Hermitian Wigner matrix, $\kappa$ a positive constant and $V$ an independent GUE-matrix. We take the GUEmeasure to be

$$
\frac{1}{Z_{n}} \mathrm{e}^{-\operatorname{tr} V^{2} / 2} \mathrm{~d} V
$$

[^1]Without loss of generality we can choose the variance $\sigma^{2}=1 / 4$.
Let $\left\{\lambda_{j}\right\}$ be the eigenvalues of $\sqrt{n} W$. The sequence $\left\{\lambda_{j} / n\right\}$ is asymptotically distributed according to the Wigner semi-circle law, [5],

$$
\begin{equation*}
\rho(x)=\frac{2}{\pi(1+4 \kappa)} \sqrt{\left(1+4 \kappa-x^{2}\right)_{+}} . \tag{1.2}
\end{equation*}
$$

Let $C_{c}(\mathbb{R})$ denote the set of all continuous functions with compact support, and $C_{c}^{+}(\mathbb{R})$ the subset of $C_{c}(\mathbb{R})$ of nonnegative functions. For $b>0$ let

$$
\begin{equation*}
K_{\mathrm{sine}}^{b}(u, v)=\frac{\sin b(u-v)}{\pi(u-v)} \tag{1.3}
\end{equation*}
$$

be the sine kernel with density $b / \pi$. The sine-kernel point process on infinite point configurations $\left\{\mu_{j}\right\}$ on the real line is the determinantal point process defined by

$$
\begin{equation*}
\mathbb{E}_{\mathrm{sine}}^{b}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right]=\operatorname{det}\left(I-\phi^{1 / 2} K_{\text {sine }}^{b} \phi^{1 / 2}\right) \tag{1.4}
\end{equation*}
$$

for all $\psi \in C_{c}^{+}(\mathbb{R})$, where $\phi=1-\mathrm{e}^{-\psi}$. Here, the right-hand side is the Fredholm determinant on $L^{2}(\mathbb{R})$ with kernel $\phi^{1 / 2} K_{\text {sine }}^{b} \phi^{1 / 2}$.

Theorem 1.1. Let $W$ be a Gaussian divisible Hermitian Wigner matrix as in (1.1), where $X$ satisfies the conditions (i)-(iv) and let $\left\{\lambda_{j}\right\}$ be the eigenvalues of $\sqrt{n} W$. Assume that $d_{n} / n \rightarrow d$ as $n \rightarrow \infty$, where $|d|<\sqrt{1+4 \kappa}$, and let

$$
\begin{equation*}
\beta=\frac{2}{1+4 \kappa} \sqrt{1+4 \kappa-d^{2}} . \tag{1.5}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\lambda_{j}-d_{n}\right)\right)\right]=\mathbb{E}_{\text {sine }}^{\beta}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right] \tag{1.6}
\end{equation*}
$$

for all $\psi \in C_{c}^{+}(\mathbb{R})$.
The theorem will be proved in Section 2.2. The theorem shows that the appropriately scaled eigenvalue point process converges weakly in the bulk, i.e. in the interior of the support of the semi-circle law, (1.2), to the sine kernel point process with density given by the semi-circle law. This theorem is an extension of the main result theorem in [14], see also [7].

We turn now to the edge behaviour. It is known that if the matrix elements are heavy-tailed with no fourth moment, then the eigenvalue point process at the edge converges to a Poisson point process with a certain density, see $[1,8]$ and [21]. Thus, in order to get the same edge behaviour as for GUE we have to assume at least that the fourth moment is finite. It is known, see [3], that finite fourth moments is necessary and sufficient for the largest eigenvalue to converge to the edge of the support of the semi-circle. We will show that within the class of Gaussian divisible Wigner matrices finite fourth moments suffices for Tracy-Widom asymptotics.

The eigenvalue statistics of a GUE-matrix at the edge is described by the Airy kernel point process. The Airy kernel is defined by

$$
\begin{equation*}
A(x, y)=\int_{0}^{\infty} \operatorname{Ai}(x+t) \operatorname{Ai}(y+t) \mathrm{d} t=\frac{\operatorname{Ai}(x) \operatorname{Ai}^{\prime}(y)-\operatorname{Ai}^{\prime}(x) \operatorname{Ai}(y)}{x-y} . \tag{1.7}
\end{equation*}
$$

The Airy kernel point process on infinite point configurations $\left\{\mu_{j}\right\}$ on the real line is the determinantal point process defined by

$$
\begin{equation*}
\mathbb{E}_{\text {Airy }}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right]=\operatorname{det}\left(I-\phi^{1 / 2} A \phi^{1 / 2}\right) \tag{1.8}
\end{equation*}
$$

for all $\psi \in C_{c}^{+}(\mathbb{R})$, where $\phi=1-\mathrm{e}^{-\psi}$. The Airy kernel point process has almost surely a last particle $\mu_{\max }$ whose distribution is given by the Tracy-Widom distribution,

$$
\begin{equation*}
\mathbb{P}_{\mathrm{Airy}}\left[\mu_{\max } \leq t\right]=F_{\mathrm{TW}}(t)=\operatorname{det}(I-A)_{L^{2}(t, \infty)} . \tag{1.9}
\end{equation*}
$$

Here $\operatorname{det}(I-A)_{L^{2}(t, \infty)}$ is the Fredholm determinant of the trace-class operator on $L^{2}(t, \infty)$ with integral kernel $A(x, y)$.

We can now state our result on the edge statistics.
Theorem 1.2. Let $W$ be a Gaussian divisible Hermitian Wigner matrix, (1.1), wth finite fourth moments, i.e. there is a constant $K<\infty$ independent of $n$ such that

$$
\begin{equation*}
\max _{1 \leq i \leq j \leq n} \mathbb{E}\left[\left|x_{i j}\right|^{4}\right] \leq K \tag{1.10}
\end{equation*}
$$

Let $\left\{\lambda_{j}\right\}$ be the eigenvalues of $\sqrt{n} W$, and let

$$
\gamma=\sqrt{1+4 \kappa}, \quad \delta=\frac{1}{2} \sqrt{1+4 \kappa} .
$$

Then,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(\lambda_{j}-\gamma n\right) / \delta n^{1 / 3}\right)\right)\right]=\mathbb{E}_{\text {Airy }}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right] \tag{1.11}
\end{equation*}
$$

for all $\psi \in C_{c}^{+}(\mathbb{R})$. Furthermore, if $\lambda_{\max }=\max _{1 \leq j \leq n} \lambda_{j}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}\left[\left(\lambda_{\max }-\gamma n\right) / \delta n^{1 / 3} \leq t\right]=F_{\mathrm{TW}}(t) \tag{1.12}
\end{equation*}
$$

for all $t \in \mathbb{R}$.
The theorem will be proved in Section 3.2.
Remark 1.3. When we have two but not four moments we have asymptotically the semi-circle law, the local eigenvalue statistics in the bulk is given by the sine-kernel point process, but the local eigenvalue statistics around the largest eigenvalue, which lies outside the semi-circle, is given by a Poisson process. It would be interesting to investigate the change in statistics as we move towards the edge. In terms of eigenvectors we should move from localized eigenvectors to de-localized eigenvectors. This problem is perhaps even more interesting when we have heavy-tailed distributions with unbounded variance. The global eigenvalue distribution is then no longer given by the semi-circle law and the scaling is different, [6]. See [10] for a discussion. It is possible that the methods of the present paper could be extended to yield e.g. the sine-kernel point process close to the origin in this case also. This would probably require an improvement of the estimate (2.40), which still holds, but is not good enough.

Remark 1.4. When revising the present paper for publication we learnt about the papers [17] and [18]. The results of [17] can be used to give another proof of Theorem 1.1. That paper also uses the contour integral formula but the technical details are different. The very recent paper [18] gives alternative approach to Theorem 1.2, again with the same starting point but different technical details.

As mentioned in the introduction Tao and Vu have recently extended the four-moment theorem to the edge, but since they compared with GUE they had to assume vanishing third moment. By combining with Theorem 1.2 we can see that the third moment condition is not necessary. We formulate this only for the fluctuations of the largest eigenvalue.

Theorem 1.5. Assume that $M=\left(m_{i j}\right)$ is an Hermitian Wigner matrix with subexponential decay, i.e. there are constants $C, C^{\prime}>0$ such that

$$
\mathbb{P}\left[\left|m_{i j}\right| \geq t^{C}\right] \leq \mathrm{e}^{-t}
$$

for all $t \geq C^{\prime}$ and all $1 \leq i \leq j \leq n$. Let $\lambda_{\max }$ be the largest eigenvalue of $\sqrt{n} M$, and assume that the variance $\sigma^{2}=1$. Then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}\left[\left(\lambda_{\max }-2 n\right) / n^{1 / 3} \leq t\right]=F_{\mathrm{TW}}(t) \tag{1.13}
\end{equation*}
$$

for all $t \in \mathbb{R}$.
Proof. We can choose a Gaussian divisible Wigner matrix $M^{\prime}$ so that the moments of $M$ and $M^{\prime}$ match up to order three, see [23]. The result then follows from (1.12) and [22], Theorem 1.13; compare the proof of Theorem 1.16 in [22].

## 2. Bulk universality

### 2.1. Convergence to the sine kernel point process

Consider $n$ Brownian motions $x_{1}(t), \ldots, x_{n}(t)$ on $\mathbb{R}$ starting at $\nu_{1}, \ldots, v_{n}$ and conditioned never to intersect. The random positions at time $S$ then form a determinantal point process with correlation kernel

$$
\begin{equation*}
K_{n, S}^{v}(u, v)=\frac{1}{(2 \pi \mathrm{i})^{2} S} \int_{\gamma_{L}} \mathrm{~d} z \int_{\Gamma_{M}} \mathrm{~d} w \mathrm{e}^{\left(w^{2}-2 v w-z^{2}+2 u z\right) / 2 S} \frac{1}{w-z} \prod_{j=1}^{n} \frac{w-v_{j}}{z-v_{j}}, \tag{2.1}
\end{equation*}
$$

where $\nu=\left\{\nu_{j}\right\}_{j=1}^{n}, \gamma_{L}$ is the contour given by the positively oriented rectangle with corners at $\pm L \pm \mathrm{i}$ and $\Gamma_{M}$ is the contour given by $s \rightarrow M+\mathrm{i} s$, with $M>L$, see [14]. Here $L$ is chosen so large that all the points $v_{j}$ lie inside $\gamma_{L}$. Let $\mathbb{E}^{\nu}$ denote the expectation with respect to the family of non-intersecting Brownian motions, and let $\phi \in C_{c}(\mathbb{R})$ satisfy $0 \leq \phi \leq 1$. Then,

$$
\begin{equation*}
\mathbb{E}^{\nu}\left[\prod_{j=1}^{n}\left(1-\phi\left(x_{j}(S)\right)\right)\right]=\operatorname{det}\left(I-\phi^{1 / 2} K_{n, S}^{v} \phi^{1 / 2}\right) \tag{2.2}
\end{equation*}
$$

where the right-hand side is a Fredholm determinant on $L^{2}(\mathbb{R})$ with respect to the finite rank kernel $\phi^{1 / 2} K_{n, S}^{v} \phi^{1 / 2}$.
This is useful for studying Gaussian divisible Wigner matrices because of the following fact. Let $\mathbb{E}_{X}$ denote the expectation with respect to the Wigner matrix $X$ and let $y(X)=\left\{y_{j}(X)\right\}_{j=1}^{n}$ be the eigenvalues of $\sqrt{n} X$. Furthermore let $\mathbb{E}_{W}$ denote the expectation with respect to the Gaussian divisible Wigner matrix $W$, (1.1). Then, [14], for $\psi \in$ $C_{c}^{+}(\mathbb{R})$,

$$
\begin{equation*}
\mathbb{E}_{W}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\lambda_{j}\right)\right)\right]=\mathbb{E}_{X}\left[\mathbb{E}^{y(X)}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)\right)\right)\right]\right], \tag{2.3}
\end{equation*}
$$

where $\left\{\lambda_{j}\right\}$ are the eigenvalues of $W$ and $S_{n}=\kappa n$. To use this formula we need good control of the kernel $K_{n, S}^{v}$ given by (2.1) for all $\nu=y(X)$ except those in a set of negligible probability.

We can make a change of variables $z \rightarrow S z, w \rightarrow S w$ in (2.1) to get

$$
\begin{equation*}
K_{n, S}^{v}(u, v)=\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{L}} \mathrm{~d} z \int_{\Gamma_{M}} \mathrm{~d} w \mathrm{e}^{S\left(w^{2}-z^{2}\right) / 2+u z-v w} \frac{1}{w-z} \prod_{j=1}^{n} \frac{S w-v_{j}}{S z-v_{j}} \tag{2.4}
\end{equation*}
$$

with $\gamma_{L}$ and $\Gamma_{M}$ as above and where all the $\nu_{j} / S$ lie inside $\gamma_{L}$. Let $D$ be a constant that could depend on $\nu$ and $S$. It follows from (2.4) that

$$
\begin{equation*}
K_{n, S}^{v}(u-S D, v-S D)=\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{L}} \mathrm{~d} z \int_{\Gamma_{M}} \mathrm{~d} w \mathrm{e}^{S(f(z)-f(w))+u z-v w} \frac{1}{w-z}, \tag{2.5}
\end{equation*}
$$

where

$$
\begin{equation*}
f(z)=\frac{z^{2}}{2}+D z+\frac{1}{S} \sum_{j=1}^{n} \log \left(S z-v_{j}\right) \tag{2.6}
\end{equation*}
$$

We want to do a saddle point analysis as $S \rightarrow \infty$. The condition $f^{\prime}(a+\mathrm{i} b)=0$ gives the equations

$$
\begin{equation*}
\sum_{j=1}^{n} \frac{S}{\left(S a-v_{j}\right)^{2}+S^{2} b^{2}}=1 \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=1}^{n} \frac{S a-v_{j}}{\left(S a-v_{j}\right)^{2}+S^{2} b^{2}}+a+D=0 . \tag{2.8}
\end{equation*}
$$

Note that if we take

$$
\begin{equation*}
D=D(v)=\sum_{j=1}^{n} \frac{v_{j}}{v_{j}^{2}+b^{2} S^{2}}, \tag{2.9}
\end{equation*}
$$

then we can take $a=0$ and let $b$ be the solution of

$$
\begin{equation*}
\sum_{j=1}^{n} \frac{S}{v_{j}^{2}+b^{2} S^{2}}=1 . \tag{2.10}
\end{equation*}
$$

We can now show an approximation result for $K_{n, S}^{v}(u-S D(v), v-S D(v))$ in terms of the sine-kernel and this will suffice for our investigation in the bulk of Gaussian divisible Wigner matrices. Define, for a given set $v$ and a positive number $S$

$$
\begin{equation*}
B_{n, S}=\left\{v \text {; there is a } b>0 \text { such that } \sum_{j=1}^{n} \frac{S}{v_{j}^{2}+b^{2} S^{2}}=1\right\} . \tag{2.11}
\end{equation*}
$$

Hence, if $v \in B_{n, S}$, there is a unique $b=b(v)$ such that (2.10) holds. Furthermore, define for for $v \in B_{n, S}$,

$$
\begin{equation*}
A(v)=\sum_{j=1}^{n} \frac{S^{3} b^{2}}{\left(v_{j}^{2}+b^{2} S^{2}\right)^{2}} . \tag{2.12}
\end{equation*}
$$

We have the following proposition.
Proposition 2.1. If $v \in B_{n, S}$ there is a numerical constant $C$ such that

$$
\begin{equation*}
\left|K_{n, S}^{v}(u-S D(v), v-S D(v))-\frac{\sin b(v)(u-v)}{\pi(u-v)}\right| \leq \frac{C}{\sqrt{S A(v)}} \mathrm{e}^{3 u^{2} / S A(v)} . \tag{2.13}
\end{equation*}
$$

Proof. We let $f(z)$ be defined by (2.6) with $D=D(\nu)$. Let the contours $\gamma_{ \pm}$be given by $\gamma_{ \pm}: t \rightarrow \mp t \pm i b, t \in \mathbb{R}$, and $\Gamma=\Gamma_{0}: s \rightarrow$ is, $s \in \mathbb{R}$. Set, with $\gamma=\gamma_{+}+\gamma_{-}$,

$$
\begin{equation*}
\tilde{K}_{n, S}^{v}(u, v)=\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma} \mathrm{d} z \int_{\Gamma} \mathrm{d} w \mathrm{e}^{u z-v w} \frac{1}{w-z} \mathrm{e}^{S(f(w)-f(z))} . \tag{2.14}
\end{equation*}
$$

We can deform the contour $\gamma_{L}$ to a rectangular contour $\gamma_{L}^{\prime}$ with corners in $\pm L \pm b \mathrm{i}$, and then move the contour $\Gamma_{M}$ to $\Gamma_{0}$ in the integral (2.4). We then pick up a contribution from the pole at $w=z$ for each $z$ on $\gamma_{L}^{\prime}$ with $\operatorname{Re} z>0$. The part of $\gamma_{L}^{\prime}$ with $\operatorname{Re} z>0$ is a contour from $-b \mathrm{i}$ to $b \mathrm{i}$ and we can deform it to the straight line segment from $-b \mathrm{i}$ to $b \mathrm{i}$. Thus

$$
\begin{align*}
K_{n, S}^{v}(u-S D(v), v-s D(v))= & \frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{L}^{\prime}} \mathrm{d} z \int_{\Gamma_{0}} \mathrm{~d} w \mathrm{e}^{S\left(w^{2}-z^{2}\right)+u z-v w} \frac{1}{w-z} \prod_{j=1}^{n} \frac{S w-v_{j}}{S z-v_{j}} \\
& +\frac{1}{2 \pi \mathrm{i}} \int_{-b \mathrm{i}}^{b \mathrm{i}} \mathrm{e}^{(u-v) z} \mathrm{~d} z . \tag{2.15}
\end{align*}
$$

We can now let $L \rightarrow \infty$ and get, using (2.14),

$$
\begin{equation*}
K_{n, S}^{v}(u-S D(v), v-S D(v))-\frac{\sin b(u-v)}{\pi(u-v)}=\tilde{K}_{n, S}^{v}(u, v) . \tag{2.16}
\end{equation*}
$$

Hence, Theorem 2.1, follows from

$$
\begin{equation*}
\left|\tilde{K}_{n, S}^{v}(u, v)\right| \leq \frac{C}{\sqrt{S A(v)}} \mathrm{e}^{3 u^{2} / S A(v)} \tag{2.17}
\end{equation*}
$$

for all $v \in B_{n, S}$. In order to prove this inequality we have to choose the right contours in (2.14). The following computation motivates the choice of contours.

Let $z(t)=x(t)+\mathrm{i} y(t)$ and set $g(t)=\operatorname{Re} f(z(t))$. Then, using (2.9) and (2.10) we see that

$$
\begin{equation*}
g^{\prime}=\sum_{j=1}^{n}\left[\frac{S\left(x x^{\prime}-y y^{\prime}\right)+x^{\prime} v_{j}}{v_{j}^{2}+b^{2} S^{2}}+\frac{S\left(x x^{\prime}+y y^{\prime}\right)-x^{\prime} v_{j}}{\left(S x-v_{j}\right)^{2}+S^{2} y^{2}}\right] \tag{2.18}
\end{equation*}
$$

If we write the sum of the two fractions in (2.18) as one fraction the numerator becomes

$$
S^{2}\left[-x^{2} x^{\prime}+2 x y y^{\prime}+y^{2} x^{\prime}-b^{2} x^{\prime}\right] v_{j}+S^{3}\left[\left(x x^{\prime}-y y^{\prime}\right)\left(x^{2}+y^{2}\right)+b^{2}\left(x x^{\prime}+y y^{\prime}\right)\right] .
$$

We try to choose $z(t)$ so that the expression in the numerator is independent of $v_{j}$. This gives

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left[-\frac{1}{3} x^{3}+y^{2} x-b^{2} x\right]=0
$$

or

$$
x\left[-\frac{1}{3} x^{2}+y^{2}-b^{2}\right]=C
$$

If $x(0)=0, y(0)= \pm b$ we get $C=0$ and two possibilities $z(t)=\mathrm{i}(t \pm b)$ or $z(t)=t \pm \mathrm{i} \sqrt{t^{2} / 3+b^{2}}$.
If we take $z(t)=\mathrm{i}(t \pm b)$ we get

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Re} f(z(t))=-S t \sum_{j=1}^{n} \frac{S^{2}(t \pm b)(t \pm 2 b)}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(v_{j}^{2}+(t+b)^{2} S^{2}\right)} \tag{2.19}
\end{equation*}
$$

If instead we take $z(t)=t \pm \mathrm{i} \sqrt{t^{2} / 3+b^{2}}$ we obtain

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \operatorname{Re} f(z(t))=S t \sum_{j=1}^{n} \frac{8 S^{2} t^{2} / 9+2 b^{2} S^{2}}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(\left(S t-v_{j}\right)^{2}+\left(t^{2} / 3+b^{2}\right) S^{2}\right)} \tag{2.20}
\end{equation*}
$$

Using this result we can prove
Lemma 2.2. Let $w_{ \pm}(s)=\mathrm{i}(s \pm b)$ and $z_{ \pm}(t)=t \pm \mathrm{i} \sqrt{t^{2} / 3+b^{2}}$. Assume that $v \in B_{n, s}$.
(i) If $\pm s+b \geq 0$, then

$$
\begin{equation*}
\operatorname{Re}\left(f\left(w_{ \pm}(s)\right)-f( \pm b \mathrm{i})\right) \leq-\frac{1}{6} A(v) s^{2} . \tag{2.21}
\end{equation*}
$$

(ii) For each $t \in \mathbb{R}$,

$$
\begin{equation*}
\operatorname{Re}\left(f( \pm b \mathrm{i})-f\left(z_{ \pm}(t)\right)\right) \leq-\frac{1}{6} A(v) t^{2} . \tag{2.22}
\end{equation*}
$$

Proof. We see that, for $-b \leq s \leq 0$,

$$
\begin{aligned}
\operatorname{Re}\left(f\left(w_{+}(s)\right)-f(b \mathrm{i})\right) & =S^{3} \int_{s}^{0} t \sum_{j=1}^{n} \frac{(t+b)(t+2 b)}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(v_{j}^{2}+(b+t)^{2} S^{2}\right)} \mathrm{d} t \\
& \leq S^{3} \int_{s}^{0} t \sum_{j=1}^{n} \frac{(b+t) b}{\left(v_{j}^{2}+b^{2} S^{2}\right)^{2}} \mathrm{~d} t \\
& =\frac{A(\nu)}{b}\left(-\frac{s^{2}}{3}\right)\left(\frac{3}{2} b+s\right) \leq-\frac{A(\nu)}{6} s^{2} .
\end{aligned}
$$

If $s \geq 0$, we get

$$
\begin{aligned}
\operatorname{Re}\left(f\left(w_{+}(s)\right)-f(b \mathrm{i})\right) & =S^{3} \int_{0}^{s} t \sum_{j=1}^{n} \frac{(t+b)(t+2 b)}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(v_{j}^{2}+(b+t)^{2} S^{2}\right)} \mathrm{d} t \\
& \leq-\int_{0}^{s} t \sum_{j=1}^{n} \frac{S^{3}(t+b)^{2}}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(v_{j}^{2}+(b+t)^{2} S^{2}\right)} \mathrm{d} t
\end{aligned}
$$

If we use the fact that $x \rightarrow x^{2}\left(v^{2}+x^{2}\right)^{-1}$ is increasing in $x \geq b$, we see that the last expression is

$$
\leq-A(\nu) \int_{0}^{s} t \mathrm{~d} t=-\frac{1}{2} A(\nu) s^{2} .
$$

The contour $w_{-}(s)$ is treated analogously. This proves (i) in the lemma.
Now, for $t \geq 0$,

$$
\begin{aligned}
\operatorname{Re}\left(f\left(z_{+}(s)\right)-f(b \mathrm{i})\right) & =S \int_{0}^{t} \tau \sum_{j=1}^{n} \frac{8 S^{2} \tau^{2} / 9+2 b^{2} S^{2}}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(\left(S \tau-v_{j}\right)^{2}+S^{2}\left(\tau^{2} / 3+b^{2}\right)\right)} \mathrm{d} \tau \\
& \geq S \int_{0}^{t} \tau \sum_{j=1}^{n} \frac{8 S^{2} \tau^{2} / 9+2 b^{2} S^{2}}{\left(v_{j}^{2}+b^{2} S^{2}\right)\left(2 v_{j}^{2}+7 S^{2} \tau^{2} / 3+b^{2} S^{2}\right)} .
\end{aligned}
$$

It is easy to see that

$$
\frac{8 S^{2} \tau^{2} / 9+2 b^{2} S^{2}}{2 v_{j}^{2}+7 S^{2} \tau^{2} / 3+b^{2} S^{2}} \geq \frac{1}{3} \frac{S^{2} b^{2}}{v_{j}^{2}+b^{2} S^{2}}
$$

and hence we obtain (2.22) for $z_{+}(t)$ and $t \geq 0$. The argument for $t \leq 0$ and the argument for $z_{-}(t)$ are similar.
We can now prove the estimate (2.17). Let $\gamma_{+}$be given by $z_{+}(-t), t \in \mathbb{R}, \gamma_{-}$by $z_{-}(t), t \in \mathbb{R}, \Gamma_{+}$by $w_{+}(s)$, $s \geq-b$, and $\Gamma_{-}$by $w_{-}(s), s \leq b$, where $z_{ \pm}$and $w_{ \pm}$are as in Lemma 2.2. Then,

$$
\tilde{K}_{n, S}^{v}(u, v)=\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{+}+\gamma_{-}} \mathrm{d} z \int_{\Gamma_{+}+\Gamma_{-}} \mathrm{d} w \mathrm{e}^{u z-v w} \frac{1}{w-z} \mathrm{e}^{S(f(w)-f(z))} .
$$

Consider the case when $z$ lies on $\gamma_{+}$and $w$ on $\Gamma_{+}$. The other cases are similar. By Lemma 2.2

$$
\begin{align*}
& \left|\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{+}} \mathrm{d} z \int_{\Gamma_{+}} \mathrm{d} w \mathrm{e}^{u z-v w} \frac{1}{w-z} \mathrm{e}^{S(f(w)-f(z))}\right| \\
& \quad \leq \frac{1}{4 \pi^{2}} \int_{-\infty}^{\infty} \mathrm{d} t \int_{-b}^{\infty} \mathrm{d} s \frac{\mathrm{e}^{u t}}{\sqrt{t^{2}+\left(b+s-\sqrt{t^{2} / 3+b^{2}}\right)^{2}}} \mathrm{e}^{-S A(\nu)\left(s^{2}+t^{2}\right) / 6} . \tag{2.23}
\end{align*}
$$

Since $t^{2}+\left(b+s-\sqrt{t^{2} / 3+b^{2}}\right)^{2} \geq\left(t^{2}+s^{2}\right) / 3$, we see that the expression in the right-hand side of (2.23) is

$$
\leq \frac{\sqrt{2}}{4 \pi^{2}} \int_{\mathbb{R}^{2}} \frac{\mathrm{e}^{u t}}{\sqrt{t^{2}+s^{2}}} \mathrm{e}^{-S A(\nu)\left(s^{2}+t^{2}\right) / 6} \leq \frac{C}{S A(\nu)} \mathrm{e}^{3 u^{2} / A(\nu)}
$$

where $C$ is a numerical constant. This completes the proof of Proposition 2.1.
Assume now that we have a probability measure $\mathbb{P}_{\nu}$ with expectation $\mathbb{E}_{\nu}$ on the point configurations $v=\left\{v_{j}\right\}$. We can then define a point process $\mu=\left\{\mu_{j}\right\}_{j=1}^{n}$ on $\mathbb{R}$ depending on $S$ by

$$
\begin{equation*}
\mathbb{E}_{n, S}\left[\prod_{j=1}^{n}\left(1-\phi\left(\mu_{j}\right)\right)\right]=\mathbb{E}_{\nu}\left[\mathbb{E}^{\nu}\left[\prod_{j=1}^{n}\left(1-\phi\left(x_{j}(S)\right)\right)\right]\right] \tag{2.24}
\end{equation*}
$$

for every $\phi \in C_{c}(\mathbb{R})$ with $0 \leq \phi \leq 1$.
We can now state the following proposition on convergence to the sine kernel point process defined by (1.4).
Proposition 2.3. Let $\alpha_{n}, \beta_{n}, \delta_{n}, \omega_{n}$ and $S_{n}$ be sequences such that $S_{n}>0, \omega_{n} \rightarrow \infty, \omega_{n} / \log \left(S_{n} \alpha_{n}\right) \rightarrow 0$ and $\beta_{n} \rightarrow$ $\beta>0$ as $n \rightarrow \infty$. Define, with $S=S_{n}$ in the above definitions,

$$
\begin{equation*}
C_{n}=\left\{v \in B_{n, S_{n}} ; A(v) \geq \alpha_{n},\left|b(v)-\beta_{n}\right| \leq 1 / \omega_{n},\left|D(v)-\delta_{n}\right| \leq \sqrt{\omega_{n} \alpha_{n} / S_{n}}\right\} . \tag{2.25}
\end{equation*}
$$

Assume that the sequences can be chosen in such a way that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}_{v}\left[C_{n}\right]=1 \tag{2.26}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{n, S_{n}}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\mu_{j}+S_{n} \delta_{n}\right)\right)\right]=\mathbb{E}_{\text {sine }}^{\beta}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right] \tag{2.27}
\end{equation*}
$$

for every $\psi \in C_{c}^{+}(\mathbb{R})$.

Proof. It is clear from (2.24) and (2.26) that it is enough to prove that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{v}\left[1_{C_{n}} \mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)+S_{n} \delta_{n}\right)\right)\right]\right]=\mathbb{E}_{\text {sine }}^{\beta}\left[\exp \left(-\sum_{j} \psi\left(\mu_{j}\right)\right)\right] \tag{2.28}
\end{equation*}
$$

Here $1_{A}$ denotes the indicator function for the event $A$. Write $\phi=1-\mathrm{e}^{-\psi}$. Consider a fixed $v \in C_{n}$ and write

$$
L_{n}^{v}(u, v)=K_{n, S_{n}}^{v}\left(u-S_{n} D(v), v-S_{n} D(v)\right)
$$

and

$$
\phi_{n}(u)=\phi\left(u+S_{n} \delta_{n}-S_{n} D(v)\right) .
$$

It follows from (2.13) that

$$
\begin{align*}
& \left|\phi_{n}^{1 / 2}(u) L_{n}^{v}(u, v) \phi_{n}^{1 / 2}(v)-\phi_{n}^{1 / 2}(u) K_{\text {sine }}^{b(v)} \phi_{n}^{1 / 2}(v)\right| \\
& \quad \leq \frac{C}{\sqrt{S_{n} A(v)}} \mathrm{e}^{C u^{2} / S_{n} A(v)} \phi_{n}^{1 / 2}(u) \phi_{n}^{1 / 2}(v) . \tag{2.29}
\end{align*}
$$

There is a constant $C$ such that $\phi_{n}(u)=0$ if $|u| \geq S_{n}\left|D(\nu)-\delta_{n}\right|+C$. Hence, $\phi_{n}(u)=0$ if $|u| \geq 2 \sqrt{\omega_{n} \alpha_{n} S_{n}}$ for $n$ large since $v \in C_{n}$. If $|u| \leq 2 \sqrt{\omega_{n} \alpha_{n} S_{n}}$, then

$$
\frac{C}{\sqrt{S_{n} A(\nu)}} \mathrm{e}^{C u^{2} / S_{n} A(\nu)} \leq \frac{C}{\sqrt{S_{n} A(\nu)}} \mathrm{e}^{C \omega_{n}} \leq \frac{C}{\left(S_{n} \alpha_{n}\right)^{1 / 4}}
$$

for $n$ large, since $\omega_{n} / \log \left(S_{n} \alpha_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. Thus, by (2.29)

$$
\left|\phi_{n}^{1 / 2}(u) L_{n}^{\nu}(u, v) \phi_{n}^{1 / 2}(v)-\phi_{n}^{1 / 2}(u) K_{\text {sine }}^{b(v)} \phi_{n}^{1 / 2}(v)\right| \leq \frac{C}{\left(S_{n} \alpha_{n}\right)^{1 / 4}} \phi_{n}^{1 / 2}(u) \phi_{n}^{1 / 2}(v)
$$

for all $u, v$. For a given $\varepsilon>0$ we thus have

$$
\begin{equation*}
\left|\phi_{n}^{1 / 2}(u) L_{n}^{v}(u, v) \phi_{n}^{1 / 2}(v)-\phi_{n}^{1 / 2}(u) K_{\text {sine }}^{\beta} \phi_{n}^{1 / 2}(v)\right| \leq \varepsilon \phi_{n}^{1 / 2}(u) \phi_{n}^{1 / 2}(v) \tag{2.30}
\end{equation*}
$$

for all sufficiently large $n$ uniformly in $v \in C_{n}$, since $\left|b(\nu)-\beta_{n}\right| \leq 1 / \omega_{n}$ and $\beta_{n} \rightarrow \beta$ as $n \rightarrow \infty$.
If $A$ is an operator on $L^{2}(\mathbb{R})$ with integral kernel $A(x, y)$ then the Hilbert-Schmidt norm of $A$ is give by $\|A\|_{2}^{2}=$ $\int_{\mathbb{R}^{2}}|A(x, y)|^{2} \mathrm{~d} x \mathrm{~d} y$. We now use the following lemma.

Lemma 2.4. If $A$ and $B$ are trace class operators on $L^{2}(\mathbb{R})$ then

$$
\begin{align*}
& |\operatorname{det}(I-A)-\operatorname{det}(I-B)| \\
& \quad \leq\|A-B\|_{2} \mathrm{e}^{-\operatorname{tr} A+\left(\|A-B\|_{2}+2\|B\|_{2}+1\right)^{2} / 2}+\mathrm{e}^{\left(\|B\|_{2}+1\right)^{2} / 2-\operatorname{tr} B}\left(\mathrm{e}^{-(\operatorname{tr} A-\operatorname{tr} B)}-1\right) . \tag{2.31}
\end{align*}
$$

The lemma is proved in Section 3.4.
It follows from (2.2) and a translation of variables that

$$
\begin{equation*}
\mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)+S_{n} \delta_{n}\right)\right)\right]=\operatorname{det}\left(I-\phi_{n}^{1 / 2} L_{n}^{\nu} \phi_{n}^{1 / 2}\right) . \tag{2.32}
\end{equation*}
$$

Using (2.30), (2.31) and the fact that the sine kernel is translation invariant it is now straightforward to see that

$$
\left|\operatorname{det}\left(I-\phi_{n}^{1 / 2} L_{n}^{v} \phi_{n}^{1 / 2}\right)-\operatorname{det}\left(I-\phi^{1 / 2} K_{\text {sine }}^{\beta} \phi^{1 / 2}\right)\right| \rightarrow 0
$$

uniformly for $v \in C_{n}$ as $n \rightarrow \infty$. This completes the proof by (2.28) and (2.32).

### 2.2. Proof of bulk universality

In this section we will prove Theorem 1.1 on bulk universality for Gaussian divisible Hermitian Wigner matrices with finite second moment using Proposition 2.3. Define

$$
\begin{equation*}
m_{n}(z)=\frac{1}{n} \sum_{j=1}^{n} \frac{1}{y_{j}-z}=\frac{1}{n} \operatorname{tr}(X / \sqrt{n}-z)^{-1} \tag{2.33}
\end{equation*}
$$

for $\operatorname{Im} z \neq 0$. Then

$$
\begin{equation*}
\mathbb{E}_{X}\left[m_{n}(z)\right] \rightarrow m(z)=-2 z+\sqrt{z^{2}-1} \tag{2.34}
\end{equation*}
$$

as $n \rightarrow \infty,[5]$, for each $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. Let $\delta+\beta \mathrm{i}, \beta>0$, be given by

$$
\begin{equation*}
m(d+\kappa(\delta+\beta \mathrm{i}))=\delta+\beta \mathrm{i}, \tag{2.35}
\end{equation*}
$$

which gives

$$
\begin{equation*}
\delta=-\frac{2 d}{1+4 \kappa}, \quad \beta=\frac{2}{1+4 \kappa} \sqrt{1+4 \kappa-d^{2}} . \tag{2.36}
\end{equation*}
$$

Lemma 2.5. There is a sequence $\delta_{n}+\beta_{n} \mathrm{i}, \beta_{n}>0$, such that

$$
\begin{equation*}
\mathbb{E}_{X}\left[m_{n}\left(d_{n} / n+\kappa\left(\delta_{n}+\beta_{n} \mathrm{i}\right)\right)\right]=\delta_{n}+\beta_{n} \mathrm{i} \tag{2.37}
\end{equation*}
$$

and $\delta_{n}+\beta_{n} \mathrm{i} \rightarrow \delta+\beta \mathrm{i}$ as $n \rightarrow \infty$.
Proof. Define $g_{n}(z)=\mathbb{E}_{X}\left[m_{n}\left(d_{n} / n+\kappa z\right)\right]-z$. Then $g_{n}$ is analytic in $\operatorname{Im} z>0$. Since

$$
\left|m_{n}\left(d_{n} / n+\kappa z\right)-m_{n}(d+\kappa z)\right| \leq \frac{\left|d_{n} / n-d\right|}{(\kappa \operatorname{Im} z)^{2}}
$$

and $d_{n} / n \rightarrow d$ as $n \rightarrow \infty$, it follows from (2.34) that $g_{n}(z) \rightarrow g(z)=m(d+\kappa z)-z$ uniformly on compact subsets of $\operatorname{Im} z>0$ as $n \rightarrow \infty$ (by Montel's theorem). Since $g(\delta+\beta \mathrm{i})=0$ by (2.35) it follows by Hurwitz' theorem that there is a sequence $\delta_{n}+\beta_{n} \mathrm{i}$ such that $g_{n}\left(\delta_{n}+\beta_{n} \mathrm{i}\right)=0$ and $\delta_{n}+\beta_{n} \mathrm{i} \rightarrow \delta+\beta \mathrm{i}$.

Set

$$
\begin{equation*}
c_{n}=d_{n} / S_{n}+\delta_{n}, \quad v_{j}=y_{j}-c_{n} S_{n} . \tag{2.38}
\end{equation*}
$$

The probability measure on $X$ induces a probabilty measure on $v=\left\{v_{j}\right\}$ that we denote by $\mathbb{P}_{v}$. Now, using (2.1), we see that

$$
K_{n, S_{n}}^{y}\left(u+c_{n} S_{n}, v+c_{n} S_{n}\right)=\mathrm{e}^{\left(\left(u+c_{n} S_{n}\right)^{2}-\left(v+c_{n} S_{n}\right)^{2}+v^{2}-u^{2}\right) / 2 S_{n}} K_{n, S_{n}}^{v}(u, v)
$$

and from this it follows that

$$
\mathbb{E}^{y}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)-d_{n}\right)\right)\right]=\mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)+\delta_{n} S_{n}\right)\right)\right] .
$$

Hence,

$$
\begin{equation*}
\mathbb{E}_{W}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\lambda_{j}-d_{n}\right)\right)\right]=\mathbb{E}_{v}\left[\mathbb{E}^{v}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(x_{j}\left(S_{n}\right)+S_{n} \delta_{n}\right)\right)\right]\right] \tag{2.39}
\end{equation*}
$$

Choose $\alpha_{n}=\alpha>0$ fixed, to be specified below, $\beta_{n}$ and $\delta_{n}$ as in Lemma 2.5, $S_{n}=\kappa n$ and $\omega_{n}=\sqrt{\log n}$. Then Theorem 1.1 follows if we can show that $\mathbb{P}_{\nu}\left[C_{n}\right] \rightarrow 1$ as $n \rightarrow \infty$ with $C_{n}$ as in (2.25).

To prove this we will use
Lemma 2.6. For each $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ we have the estimate

$$
\begin{equation*}
\mathbb{E}_{X}\left[\left|m_{n}(z)-\mathbb{E}_{X}\left[m_{n}(z)\right]\right|^{2}\right] \leq \frac{2}{n|\operatorname{Im} z|^{2}} . \tag{2.40}
\end{equation*}
$$

This is proved in [2]. For convenience we give the proof in Section 3.4.
Define

$$
M_{n}(\tau)=m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)-\mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)\right] .
$$

Note that, by (2.33) and (2.38)

$$
\begin{equation*}
m_{n}\left(\kappa c_{n}+z\right)=\frac{1}{n} \sum_{j=1}^{n} \frac{1}{v_{j} / n-z} . \tag{2.41}
\end{equation*}
$$

Set

$$
V_{n}=\left\{v ;\left|M_{n}(\tau)\right| \leq \sqrt{\frac{\omega_{n}}{n}} \text { for } \tau=\beta_{n}, \beta / 2,2 \beta \text { and } 3 \beta\right\} .
$$

The result we need now follows from
Lemma 2.7. The following statements hold.
(i) $\mathbb{P}_{\nu}\left[V_{n}\right] \rightarrow 1$ as $n \rightarrow \infty$.
(ii) There is an $\alpha>0$ such that if we choose $\alpha_{n}=\alpha$ and the other sequences as above, then $V_{n} \subseteq B_{n, S_{n}}$ and $V_{n} \cap B_{n, S_{n}} \subseteq C_{n}$, if $n$ is large enough.

Proof. Let $\tau>0$ be fixed. Then by Chebyshev's inequality and Lemma 2.6

$$
\mathbb{P}_{\nu}\left[\left|M_{n}(\tau)\right|>\sqrt{\frac{\omega_{n}}{n}}\right] \leq \frac{n}{\omega_{n}} \mathbb{E}_{X}\left[\left|m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)-\mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)\right]\right|^{2}\right] \leq \frac{2}{\omega_{n} \kappa^{2} \tau^{2}} \rightarrow 0,
$$

as $n \rightarrow \infty$. We can apply this to $\tau=\beta_{n}, \beta / 2,2 \beta$ and $3 \beta$ noting that $\beta_{n} \geq \beta / 2$ if $n$ is large enough. This proves (i).
Note that

$$
\begin{equation*}
\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)=\sum_{j=1}^{n} \frac{v_{j}}{v_{j}^{2}+\tau^{2} S_{n}^{2}} \tag{2.42}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{Im} m_{n}\left(\kappa c_{n}+\kappa \tau \mathrm{i}\right)=\sum_{j=1}^{n} \frac{S_{n} \tau}{v_{j}^{2}+\tau^{2} S_{n}^{2}} . \tag{2.43}
\end{equation*}
$$

Furthermore,

$$
h(\tau)=\frac{1}{\tau} \operatorname{Im} m(x+\mathrm{i} \tau)=\frac{2}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^{2}}}{(t-x)^{2}+\tau^{2}} \mathrm{~d} t
$$

is strictly decreasing in $\tau$ for each fixed $x$.

Define

$$
U_{n}=\left\{v ; \sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+4 \beta^{2} S_{n}^{2}}<1<\sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+\beta^{2} S_{n}^{2} / 4}\right\}
$$

We want to show that $V_{n} \subseteq U_{n}$ if $n$ is large enough. Since $h(\tau)$ is strictly decreasing, (2.35) gives

$$
\frac{1}{2 \beta} \operatorname{Im} m(\kappa c+2 \kappa \beta \mathrm{i})<1-\varepsilon<1=\frac{1}{\beta} \operatorname{Im} m(\kappa c+\kappa \beta \mathrm{i})<1+\varepsilon<\frac{2}{\beta} \operatorname{Im} m(\kappa c+\kappa \beta \mathrm{i} / 2)
$$

if we choose $\varepsilon$ small enough. Here $c=d / \kappa+\delta=\lim _{n \rightarrow \infty} c_{n}$. It follows from this and (2.34) that

$$
\frac{1}{2 \beta} \operatorname{Im} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+2 \kappa \beta \mathrm{i}\right)\right] \leq 1-\varepsilon<1+\varepsilon \leq \frac{2}{\beta} \operatorname{Im} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+\kappa \beta \mathrm{i} / 2\right)\right]
$$

for all $n$ large enough. If $v \in V_{n}$ it follows from this that

$$
\frac{1}{2 \beta} \operatorname{Im} m_{n}\left(\kappa c_{n}+2 \kappa \beta \mathrm{i}\right) \leq 1-\varepsilon+\sqrt{\frac{\omega_{n}}{n}}<1<1+\varepsilon-\sqrt{\frac{\omega_{n}}{n}} \leq \frac{2}{\beta} \operatorname{Im} m_{n}\left(\kappa c_{n}+\kappa \beta \mathrm{i} / 2\right)
$$

and we see from (2.43) that this gives $v \in U_{n}$.
Hence, if $n$ is large enough, then

$$
\begin{equation*}
\beta / 2 \leq b(v) \leq 2 \beta \tag{2.44}
\end{equation*}
$$

for all $v \in V_{n}$. Let $v \in V_{n}$. Then, using (2.44), we see that

$$
\begin{aligned}
A(v) & =\sum_{j=1}^{n} \frac{S_{n}^{3} b^{2}}{\left(v_{j}^{2}+b^{2} S_{n}^{2}\right)^{2}} \geq \frac{1}{4} \sum_{j=1}^{n} \frac{S_{n}^{3} \beta^{2}}{\left(v_{j}^{2}+4 \beta^{2} S_{n}^{2}\right)^{2}} \\
& \geq \frac{S_{n}}{20} \sum_{j=1}^{n} \frac{5 S_{n}^{2} \beta^{2}}{\left(v_{j}^{2}+4 \beta^{2} S_{n}^{2}\right)\left(v_{j}^{2}+9 \beta^{2} S_{n}^{2}\right)}=\frac{1}{20}\left(\sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+4 \beta^{2} S_{n}^{2}}-\sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+9 \beta^{2} S_{n}^{2}}\right)
\end{aligned}
$$

By (2.34), (2.43) and the fact that $v \in V_{n}$ it follows from this that

$$
\begin{aligned}
A(v) \geq & \frac{1}{20}\left(\frac{1}{2 \beta} \operatorname{Im} M_{n}(2 \beta)-\frac{1}{3 \beta} \operatorname{Im} M_{n}(3 \beta)\right) \\
& +\frac{1}{20}\left(\frac{1}{2 \beta} \operatorname{Im} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+2 \kappa \beta \mathrm{i}\right)\right]-\frac{1}{3 \beta} \operatorname{Im} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+3 \kappa \beta \mathrm{i}\right)\right]\right) \\
\geq & \frac{1}{40}\left(\frac{1}{2 \beta} \operatorname{Im} m(\kappa c+2 \kappa \beta \mathrm{i})-\frac{1}{3 \beta} \operatorname{Im} m(\kappa c+3 \kappa \beta \mathrm{i})\right) \doteq \alpha>0
\end{aligned}
$$

for large $n$.
Next, we will show that, if $n$ is large enough,

$$
\begin{equation*}
\left|b(v)-\beta_{n}\right| \leq C \sqrt{\frac{\omega_{n}}{n}} \leq \frac{1}{\omega_{n}} \tag{2.45}
\end{equation*}
$$

for all $v \in V_{n}$. It follows from (2.10), (2.37), (2.43) and $v \in V_{n}$, that

$$
\left|\sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+\beta_{n}^{2} S_{n}^{2}}-\sum_{j=1}^{n} \frac{S_{n}}{v_{j}^{2}+b^{2} S_{n}^{2}}\right| \leq \sqrt{\frac{\omega_{n}}{n}}
$$

which implies

$$
\left|b^{2}-\beta_{n}^{2}\right| \sum_{j=1}^{n} \frac{S_{n}^{3}}{\left(v_{j}^{2}+\beta_{n}^{2} S_{n}^{2}\right)\left(v_{j}^{2}+b^{2} S_{n}^{2}\right)} \leq \sqrt{\frac{\omega_{n}}{n}} .
$$

Now,

$$
\sum_{j=1}^{n} \frac{S_{n}^{3}}{\left(v_{j}^{2}+\beta_{n}^{2} S_{n}^{2}\right)\left(v_{j}^{2}+b^{2} S_{n}^{2}\right)} \geq \frac{1}{\beta^{2}} \sum_{j=1}^{n} \frac{S_{n}^{3} \beta^{2}}{\left(v_{j}^{2}+4 \beta^{2} S_{n}^{2}\right)\left(v_{j}^{2}+9 \beta^{2} S_{n}^{2}\right)} \geq \frac{20 \alpha}{\beta^{2}}
$$

by the previous argument, since $\beta_{n} \leq 3 \beta$ for large $n$ and $b \leq 2 \beta$ by (3.31). Consequently,

$$
\left|b-\beta_{n}\right| \leq \frac{\beta}{20 \alpha} \sqrt{\frac{\omega_{n}}{n}},
$$

since $b+\beta_{n} \geq \beta / 2+\beta_{n} \geq \beta$ for large $n$. This proves (2.45).
It remains to show that

$$
\begin{equation*}
\left|D(\nu)-\delta_{n}\right| \leq C \sqrt{\frac{\omega_{n}}{n}} \tag{2.46}
\end{equation*}
$$

for all $v \in$ and large $n$. It follows from (2.37) and (2.42) that

$$
\begin{aligned}
D(\nu)-\delta_{n}= & \operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa b \mathrm{i}\right)-\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right) \\
& +\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right)-\operatorname{Re} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right)\right] .
\end{aligned}
$$

We can use (2.44) and (2.45) to show that

$$
\left|\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa b \mathrm{i}\right)-\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right)\right| \leq \frac{\kappa\left|b-\beta_{n}\right|}{\kappa^{2} \beta_{n} b} \leq C \sqrt{\frac{\omega_{n}}{n}} .
$$

Furthermore, the definition of $V_{n}$ gives

$$
\left|\operatorname{Re} m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right)-\operatorname{Re} \mathbb{E}_{X}\left[m_{n}\left(\kappa c_{n}+\kappa \beta_{n} \mathrm{i}\right)\right]\right| \leq \sqrt{\frac{\omega_{n}}{n}}
$$

for all $v \in V_{n}$.
This proves (ii) of Lemma 2.7.

## 3. Edge universality

### 3.1. Convergence to the Airy kernel point process

Let $v=\left\{\nu_{j}\right\}_{j=1}^{n} \subseteq \mathbb{R}$ and $S>0$ be given. We can then choose $b=b(v)$ so that $b S>\max v_{j}$ and

$$
\begin{equation*}
\sum_{j=1}^{n} \frac{S}{\left(b S-v_{j}\right)^{2}}=1 \tag{3.1}
\end{equation*}
$$

Define $a=a(\nu)$ and $d=d(\nu)$ by

$$
\begin{equation*}
a=b+\sum_{j=1}^{n} \frac{1}{b S-v_{j}} \tag{3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
d=\left(\sum_{j=1}^{n} \frac{S^{2}}{\left(b S-v_{j}\right)^{3}}\right)^{1 / 3} . \tag{3.3}
\end{equation*}
$$

Let $0<\alpha_{0}<\beta_{0}$ be given and define

$$
\begin{equation*}
F_{n}=\left\{v ; \alpha_{0} \leq b-v_{j} / S \leq \beta_{0} \text { for } 1 \leq j \leq n\right\} . \tag{3.4}
\end{equation*}
$$

We then have the following estimate and limit result for the correlation kernel given by (2.1).
Proposition 3.1. There are constants $C$ and $S_{0}$ depending only on $\alpha_{0}, \beta_{0}$ so that

$$
\begin{equation*}
d S^{1 / 3} K_{n, S}^{v}\left(a S+\xi d S^{1 / 3}, a S+\xi d S^{1 / 3}\right) \leq C \mathrm{e}^{-\xi} \tag{3.5}
\end{equation*}
$$

for all $v \in F_{n}, \xi \geq 0$ and $S \geq S_{0}$. Furthermore, if $S=\kappa n$, with $\kappa>0$ fixed, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} d S_{n}^{1 / 3} \mathrm{e}^{(\eta-\xi) d S_{n}^{1 / 3}} K_{n, S}^{v}\left(a S_{n}+\xi d S_{n}^{1 / 3}, a S_{n}+\eta d S_{n}^{1 / 3}\right)=A(\xi, \eta) \tag{3.6}
\end{equation*}
$$

uniformly for $v \in F_{n}$ and $\xi, \eta$ in a compact subset of $\mathbb{R}$. Here $A(\xi, \eta)$ is the Airy kernel (1.7).
The proposition will be proved in Section 3.3.
Let $\gamma_{n}$ and $\varepsilon_{n}$ be given sequences of positive numbers, where $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$. Take $S_{n}=\kappa n, \kappa>0$, let $\delta>0$ be given and define

$$
\begin{equation*}
G_{n}=\left\{\nu \in F_{n} ;\left|\frac{a(\nu)-\gamma_{n}}{n^{1 / 3}}\right| \leq \varepsilon_{n},|d(\nu)-\delta| \leq \varepsilon_{n}\right\} . \tag{3.7}
\end{equation*}
$$

Let $\mathbb{P}_{v}$ be a probability measure on point configurations $v=\left\{\nu_{j}\right\}_{j=1}^{n}$ in $\mathbb{R}$, and let $\mathbb{E}_{n, S_{n}}$ be the expectation for the point process $\mu=\left\{\mu_{j}\right\}_{j=1}^{n}$ on $\mathbb{R}$ defined by (2.24).

Proposition 3.2. Assume that there is a choice of $\alpha_{0}, \beta_{0}, \gamma_{n}, \varepsilon_{n}$, $\delta$, where $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$, so that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}_{\nu}\left[G_{n}\right]=1 \tag{3.8}
\end{equation*}
$$

Then, for any $\psi \in C_{c}^{+}(\mathbb{R})$,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{n, S_{n}}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(\mu_{j}-\gamma_{n}\right) / \delta n^{1 / 3}\right)\right)\right]=\mathbb{E}_{\text {Airy }}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\mu_{j}\right)\right)\right] \tag{3.9}
\end{equation*}
$$

## Furthermore,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}_{n, S_{n}}\left[\frac{1}{\delta n^{1 / 3}}\left(\max _{1 \leq j \leq n} \mu_{j}-\gamma_{n}\right) \leq t\right]=F_{\mathrm{TW}}(t) \tag{3.10}
\end{equation*}
$$

for each $t \in \mathbb{R}$.
Proof. We see from (2.24), with $\phi=1-\mathrm{e}^{-\psi}$, and (3.8) that to prove (3.9) it is enough to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{\nu}\left[1_{G_{n}} \mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(\mu_{j}-\gamma_{n}\right) / \delta n^{1 / 3}\right)\right)\right]\right]=\mathbb{E}_{\text {Airy }}\left[\mathrm{e}^{-\sum_{j=1}^{n} \psi\left(\mu_{j}\right)}\right] \tag{3.11}
\end{equation*}
$$

Let

$$
\tilde{K}_{n}^{v}(\xi, \eta)=d S_{n}^{1 / 3} \mathrm{e}^{(\xi-\eta) d S_{n}^{1 / 3}} K_{n, S_{n}}^{v}\left(a S_{n}+\xi d S_{n}^{1 / 3}, a S_{n}+\eta d S_{n}^{1 / 3}\right)
$$

and

$$
\tilde{\phi}_{n}(\xi)=\phi\left(\xi d / \delta+\left(a S_{n}-\gamma_{n}\right) / \delta n^{1 / 3}\right) .
$$

Then,

$$
\begin{equation*}
\mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(\mu_{j}-\gamma_{n}\right) / \delta n^{1 / 3}\right)\right)\right]=\operatorname{det}\left(I-\tilde{\phi}_{n}^{1 / 2} \tilde{K}_{n}^{v} \tilde{\phi}_{n}^{1 / 2}\right) . \tag{3.12}
\end{equation*}
$$

If $v \in G_{n}$ there is a constant $C$, depending on $\phi$, such that

$$
\begin{equation*}
\left|\tilde{\phi}_{n}(\xi)-\phi(\xi)\right| \leq C \varepsilon_{n} . \tag{3.13}
\end{equation*}
$$

If we use (3.6), (3.13) and the fact that $\phi$ has compact support, we can use Lemma 2.4 to show that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \operatorname{det}\left(I-\tilde{\phi}_{n}^{1 / 2} \tilde{K}_{n}^{v} \tilde{\phi}_{n}^{1 / 2}\right)=\operatorname{det}\left(I-\phi^{1 / 2} A \phi^{1 / 2}\right), \tag{3.14}
\end{equation*}
$$

uniformly for $v \in G_{n}$, where $A$ is the Airy kernel, (1.7). The limit (3.9) now follows from (2.24), (3.6), (3.11), (3.12) and (3.13).

It remains to show (3.10). Again, from (3.8), we see that it is enough to show that

$$
\begin{equation*}
\left.\lim _{n \rightarrow \infty} \mathbb{E}_{\nu}\left[1_{G_{n}} \mathbb{E}^{\nu}\left[1_{\#\left(\gamma_{n}+\delta t n^{1 / 3}\right.}, \infty\right)=0\right]\right]=F_{\mathrm{TW}}(t), \tag{3.15}
\end{equation*}
$$

where $\#(x, y)=$ the number of points in $(x, y)$. Take $\tau>t$. Then,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{E}_{\nu}\left[1_{G_{n}} \mathbb{E}^{v}\left[1_{\#\left(\gamma_{n}+\delta t n^{1 / 3}, \gamma_{n}+\tau \delta n^{1 / 3}\right)=0}\right]\right]=\operatorname{det}(I-A)_{L^{2}(t, \tau)} \tag{3.16}
\end{equation*}
$$

follows by an argument analogous to the one above used to prove (3.9). Now,

$$
\begin{align*}
\mathbb{E}^{\nu}\left[1_{\#\left(\gamma_{n}+\delta t n^{1 / 3}, \infty\right)=0}\right]= & \mathbb{E}^{\nu}\left[1_{\#\left(\gamma_{n}+\delta t n^{1 / 3}, \gamma_{n}+\tau \delta n^{1 / 3}\right)=0}\right] \\
& -\mathbb{E}^{v}\left[1_{\#\left(\gamma_{n}+\delta \tau n^{1 / 3}, \infty\right) \geq 1} 1_{\#\left(\gamma_{n}+\delta t n^{1 / 3}, \gamma_{n}+\tau \delta n^{1 / 3}\right)=0}\right] . \tag{3.17}
\end{align*}
$$

The second term in the right-hand side of (3.17) is bounded by

$$
\begin{align*}
& \mathbb{E}^{\nu}\left[1_{\#\left(\gamma_{n}+\delta \tau n^{1 / 3}, \infty\right) \geq 1}\right] \leq \mathbb{E}^{\nu}\left[\#\left(\gamma_{n}+\delta \tau n^{1 / 3}, \infty\right)\right] \\
& \quad=\int_{\gamma_{n}+\delta \tau n^{1 / 3}}^{\infty} K_{n, S_{n}}^{\nu}(x, x) \mathrm{d} x \leq C \int_{\left(\gamma-a S_{n}\right) / d n^{1 / 3}+\delta \tau / d}^{\infty} \mathrm{e}^{-\xi} \mathrm{d} \xi, \tag{3.18}
\end{align*}
$$

where the last inequality follows from (3.5) if $\tau$ is sufficiently large, since then

$$
\begin{equation*}
\left(\gamma-a S_{n}\right) / d n^{1 / 3}+\delta \tau / d \geq-\varepsilon_{n} / \alpha_{0}+\delta \tau / \beta_{0} \geq 0 . \tag{3.19}
\end{equation*}
$$

Hence, by (3.16), (3.17), (3.18) and (3.19),

$$
\begin{align*}
& \limsup _{n \rightarrow \infty}\left|\mathbb{E}_{\nu}\left[1_{G_{n}} \mathbb{E}^{\nu}\left[\#\left(\gamma_{n}+t \delta t n^{1 / 3}, \infty\right)\right]\right]-\operatorname{det}(I-A)_{L^{2}(t, \infty)}\right| \\
& \quad \leq\left|\operatorname{det}(I-A)_{L^{2}(t, \infty)}-\operatorname{det}(I-A)_{L^{2}(t, \tau)}\right|+C \int_{-\varepsilon_{n} / \alpha_{0}+\delta \tau / \beta_{0}}^{\infty} \mathrm{e}^{-\xi} \mathrm{d} \xi . \tag{3.20}
\end{align*}
$$

If we let $\tau \rightarrow \infty$ the right-hand side of (3.20) goes to zero and we have proved (3.10).

### 3.2. Proof of edge universality

In this section we will prove Theorem 1.2 on edge universality for Gaussian divisible Hermitian Wigner matrices with finite fourth moments.

Let $v=y$, where $y=\left\{y_{j}\right\}$ are the eigenvalues of $X$. The expectation $\mathbb{E}_{X}$ on $X$ induces an expectation $\mathbb{E}_{v}$ on $\nu$. By (2.3),

$$
\begin{aligned}
& \mathbb{E}_{W}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(\lambda_{j}-\gamma_{n}\right) / \delta n^{1 / 3}\right)\right)\right] \\
& \quad=\mathbb{E}_{v}\left[\mathbb{E}^{\nu}\left[\exp \left(-\sum_{j=1}^{n} \psi\left(\left(x_{j}\left(S_{n}\right)-\gamma_{n}\right) / \delta n^{1 / 3}\right)\right)\right]\right]
\end{aligned}
$$

By Proposition 3.2 it is enough to show that there is a choice of $\alpha_{0}, \beta_{0}, \gamma_{n}, \varepsilon_{n}$ and $\delta$, where $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$, so that (3.8) holds with $G_{n}$ defined by (3.7) and $F_{n}$ by (3.4).

Let $u(x)=\frac{2}{\pi} \sqrt{1-x^{2}}$ be the Wigner semi-circle law with support in $[-1,1]$. We can choose $b_{0}>1 / \kappa$ so that

$$
\begin{equation*}
\int_{-1}^{1} \frac{\kappa u(x)}{\left(b_{0} \kappa-x\right)^{2}} \mathrm{~d} x=1, \tag{3.21}
\end{equation*}
$$

which gives $b_{0}=(1+2 \kappa)(1+4 \kappa)^{-1 / 2}$ by (2.34). Let

$$
\varepsilon=\frac{1}{3}\left(\frac{1+2 \kappa}{\sqrt{1+4 \kappa}}-1\right),
$$

so that $b_{0} \kappa \geq 1+3 \varepsilon$. We take $\gamma_{n}=n \sqrt{1+4 \kappa}$ and note that

$$
\begin{equation*}
\gamma_{n}=n\left(b_{0} \kappa+\int_{-1}^{1} \frac{\kappa u(x)}{b_{0} \kappa-x} \mathrm{~d} x\right) . \tag{3.22}
\end{equation*}
$$

Also, we choose $\delta=\frac{1}{2} \sqrt{1+4 \kappa}$ and note that

$$
\begin{equation*}
\delta^{3}=\int_{-1}^{1} \frac{\kappa^{3}}{\left(b_{0} \kappa-x\right)^{3}} u(x) \mathrm{d} x . \tag{3.23}
\end{equation*}
$$

Furthermore, we take $\varepsilon_{n}=(\log n)^{-1}, \alpha_{0}=\varepsilon / \kappa$ and $\beta_{0}=b_{0}+(1+2 \varepsilon) / \kappa$.
Define the function $\psi_{\beta}$ by

$$
\psi_{\beta}(x)= \begin{cases}\frac{\kappa}{\kappa \beta-x}, & \text { if }|x| \leq 1+\varepsilon \\ 0, & \text { if }|x| \geq 1+3 \varepsilon\end{cases}
$$

and for $1+\varepsilon \leq|x| \leq 1+3 \varepsilon$ we define $\psi_{\beta}$ so that it becomes a $C^{\infty}$ function.
Set

$$
H_{n}^{\prime}=\left\{\nu ; \max _{1 \leq j \leq n}\left|v_{j} / n\right| \leq 1+\varepsilon\right\}
$$

and

$$
\begin{aligned}
H_{n}= & H_{n}^{\prime} \cap\left\{v ;\left|\sum_{j=1}^{n} \psi_{2 b_{0}}^{\prime}\left(v_{j} / n\right)-n \int_{-1}^{1} \psi_{2 b_{0}}^{\prime}(x) u(x) \mathrm{d} x\right| \leq n^{1 / 6}\right. \text { and } \\
& \left.\left|\sum_{j=1}^{n} \psi_{b_{0}}^{(j)}\left(v_{j} / n\right)-n \int_{-1}^{1} \psi_{b_{0}}^{(j)}(x) u(x) \mathrm{d} x\right| \leq n^{1 / 6} \text { for } j=0,1,2\right\} .
\end{aligned}
$$

We will prove the following lemma below.
Lemma 3.3. Let $H_{n}$ be defined as above. Then,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathbb{P}_{v}\left[H_{n}\right]=1 \tag{3.24}
\end{equation*}
$$

Before we prove the lemma we will use it to show what we want by proving that $H_{n} \subseteq G_{n}$.
Let us first show that there is a constant $C$ so that

$$
\begin{equation*}
\left|b(\nu)-b_{0}\right| \leq C n^{-5 / 6} \tag{3.25}
\end{equation*}
$$

for all $v \in H_{n}$. We see from (3.1), (3.21) and the definition of $\psi_{b_{0}}$ that

$$
\begin{align*}
\left|\int_{-1}^{1} \psi_{b_{0}}^{\prime}(x) u(x) \mathrm{d} x-\frac{1}{n} \sum_{j=1}^{n} \psi_{b_{0}}^{\prime}\left(v_{j} / n\right)\right| & =\left|1-\sum_{j=1}^{n} \frac{S_{n}}{\left(b_{0} S_{n}-v_{j}\right)^{2}}\right| \\
& =S_{n}^{2}\left|b-b_{0}\right| \sum_{j=1}^{n} \frac{b_{0} S_{n}-v_{j}+b S_{n}-v_{j}}{\left(b S_{n}-v_{j}\right)^{2}\left(b_{0} S_{n}-v_{j}\right)^{2}} . \tag{3.26}
\end{align*}
$$

We want to show that $b \leq 2 b_{0}$. Since $v \in H_{n}$,

$$
\frac{1}{n} \sum_{j=1}^{n} \psi_{2 b_{0}}^{\prime}\left(v_{j} / n\right) \leq \int_{-1}^{1} \psi_{2 b_{0}}^{\prime}(x) u(x) \mathrm{d} x+n^{-5 / 6}
$$

which gives

$$
\sum_{j=1}^{n} \frac{S_{n}}{\left(2 b_{0} S_{n}-v_{j}\right)^{2}} \leq \int_{-1}^{1} \frac{\kappa u(x)}{\left(2 b_{0} \kappa-x\right)^{2}} \mathrm{~d} x+n^{-5 / 6}<\int_{-1}^{1} \frac{\kappa u(x)}{\left(b_{0} \kappa-x\right)^{2}} \mathrm{~d} x=1
$$

if $n$ is sufficiently large. Hence $b \leq 2 b_{0}$. This gives

$$
\begin{aligned}
\sum_{j=1}^{n} \frac{\left(b_{0} S_{n}-v_{j}+b S_{n}-v_{j}\right) S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{2}\left(b_{0} S_{n}-v_{j}\right)^{2}} & \geq 2 \sum_{j=1}^{n} \frac{\left(b_{0} S_{n}-v_{j}\right)^{1 / 2}\left(b S_{n}-v_{j}\right)^{1 / 2} S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{2}\left(b_{0} S_{n}-v_{j}\right)^{2}} \\
& \geq \sum_{j=1}^{n} \frac{2 S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{3 / 2}\left(b_{0} S_{n}-v_{j}\right)^{3 / 2}} \\
& \geq \frac{2 \kappa^{2}}{\left(2 b_{0} \kappa+1+\varepsilon\right)^{3 / 2}\left(b_{0} \kappa+1+\varepsilon\right)^{3 / 2}} \doteq c_{1}
\end{aligned}
$$

since $v_{j} / n \geq-1-\varepsilon$ if $v \in H_{n}$. Thus (3.26) implies

$$
c_{1}\left|b-b_{0}\right| \leq\left|\int_{-1}^{1} \psi_{b_{0}}^{\prime}(x) u(x) \mathrm{d} x-\frac{1}{n} \sum_{j=1}^{n} \psi_{b_{0}}^{\prime}\left(v_{j} / n\right)\right| \leq n^{-5 / 6} .
$$

This proves (3.25).
Next, we show that for all $v \in H_{n}$,

$$
\begin{equation*}
\left|\frac{a(\nu) S_{n}-\gamma_{n}}{n^{1 / 3}}\right| \leq n^{-1 / 6} . \tag{3.27}
\end{equation*}
$$

Define,

$$
\begin{equation*}
a_{0}=b_{0}+\sum_{j=1}^{n} \frac{1}{b_{0} S_{n}-v_{j}} \tag{3.28}
\end{equation*}
$$

an approximate version of (3.2). Then, by (3.1),

$$
\begin{aligned}
\left(a-a_{0}\right) S_{n} & =\left(b-b_{0}\right) S_{n}\left[1-\sum_{j=1}^{n} \frac{S_{n}}{\left(b S_{n}-v_{j}\right)\left(b_{0} S_{n}-v_{j}\right)}\right] \\
& =-\left(b-b_{0}\right)^{2} S_{n}^{2} \sum_{j=1}^{n} \frac{S_{n}}{\left(b S_{n}-v_{j}\right)^{2}\left(b_{0} S_{n}-v_{j}\right)}
\end{aligned}
$$

Now,

$$
b_{0} S_{n}-v_{j} \geq n\left(b_{0} \kappa-(1+\varepsilon)\right) \geq 2 \varepsilon n,
$$

which gives

$$
\left|a-a_{0}\right| S_{n} \leq \frac{\kappa^{2} n}{2 \varepsilon}\left|b-b_{0}\right|^{2} \sum_{j=1}^{n} \frac{S_{n}}{\left(b S_{n}-v_{j}\right)^{2}} \leq C n^{-2 / 3},
$$

by (3.25) and (3.1). From (3.25) we also obtain

$$
\begin{aligned}
\left|a_{0} S_{n}-\gamma_{n}\right| & =\left|\sum_{j=1}^{n} \frac{\kappa n}{b_{0} \kappa n-v_{j}}-n \int_{-1}^{1} \frac{\kappa u(x)}{b_{0} \kappa-x} \mathrm{~d} x\right| \\
& =\left|\sum_{j=1}^{n} \psi_{b_{0}}\left(v_{j} / n\right)-n \int_{-1}^{1} \psi_{b_{0}}(x) u(x) \mathrm{d} x\right| \leq n^{1 / 6}
\end{aligned}
$$

since $v \in H_{n}$. Hence,

$$
\left|\frac{a(\nu) S_{n}-\gamma_{n}}{n^{1 / 3}}\right| \leq \frac{1}{n^{1 / 3}}\left|a-a_{0}\right| S_{n}+\frac{1}{n^{1 / 3}}\left|a_{0} S_{n}-\gamma_{n}\right| \leq C n^{-1 / 6} .
$$

This proves (3.27).
If $n$ is so large that $\kappa\left|b-b_{0}\right| \leq \varepsilon$ for all $v \in H_{n}$, which we can achieve by (3.25), then using $\left|v_{j} / n\right| \leq 1+\varepsilon$ we get

$$
\begin{aligned}
\kappa\left(b-v_{j} / S_{n}\right) & =\kappa\left(b-b_{0}\right)+\kappa b_{0}-(1+\varepsilon)+1+\varepsilon-v_{j} / n \\
& \geq \kappa b_{0}-(1+\varepsilon)+1+\varepsilon-v_{j} / n-\kappa\left|b-b_{0}\right| \geq \varepsilon,
\end{aligned}
$$

so we have $b-v_{j} / S_{n} \geq \varepsilon / \kappa \doteq \alpha_{0}$. Furthermore,

$$
\kappa\left(b-v_{j} / S_{n}\right)=\kappa\left(b-b_{0}\right)+\kappa b_{0}-v_{j} / n \leq \varepsilon+\kappa b_{0}+1+\varepsilon \doteq \kappa \beta_{0} .
$$

Thus, $H_{n} \subseteq F_{n}$ with these choices of $\alpha_{0}$ and $\beta_{0}$.
Finally, we want to control $|d(\nu)-\delta|$. By (3.3) and (3.23),

$$
\begin{align*}
d^{3}-\delta^{3} & =\sum_{j=1}^{n} \frac{S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{3}}-\int_{-1}^{1} \frac{\kappa^{3} u(x)}{\left(b_{0} \kappa-x\right)^{3}} \mathrm{~d} x \\
& =\sum_{j=1}^{n}\left(\frac{S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{3}}-\frac{S_{n}^{2}}{\left(b_{0} S_{n}-v_{j}\right)^{3}}\right)+\frac{\kappa^{2}}{2 n}\left(\sum_{j=1}^{n} \psi_{b_{0}}^{\prime \prime}\left(v_{j} / n\right)-\int_{-1}^{1} \psi_{b_{0}}^{\prime \prime}(x) u(x) \mathrm{d} x\right) . \tag{3.29}
\end{align*}
$$

Since $v \in H_{n}$,

$$
\begin{equation*}
\frac{\kappa^{2}}{2 n}\left|\sum_{j=1}^{n} \psi_{b_{0}}^{\prime \prime}\left(v_{j} / n\right)-\int_{-1}^{1} \psi_{b_{0}}^{\prime \prime}(x) u(x) \mathrm{d} x\right| \leq \frac{\kappa^{2}}{2} n^{-5 / 6} \tag{3.30}
\end{equation*}
$$

Using $b-v_{j} / n \in\left[\alpha_{0}, \beta_{0}\right]$ and (3.25) we see that

$$
\begin{equation*}
\left|\sum_{j=1}^{n}\left(\frac{S_{n}^{2}}{\left(b S_{n}-v_{j}\right)^{3}}-\frac{S_{n}^{2}}{\left(b_{0} S_{n}-v_{j}\right)^{3}}\right)\right| \leq C n^{-5 / 6} . \tag{3.31}
\end{equation*}
$$

Since $\left|d^{3}-\delta^{3}\right| \geq|d-\delta| \delta^{2}$, (3.29), (3.30) and (3.31) give $|d(v)-\delta| \leq C n^{-5 / 6}$. We see that $H_{n} \subseteq G_{n}$, which is what we wanted to prove.

It remains to prove Lemma 3.3. For this we will use the following estimate.
Lemma 3.4. Let $\left\{v_{j}\right\}$ be the eigenvalues of $\sqrt{n} X$, where $X$ is an Hermitian Wigner matrix with finite fourth moments. Assume that $\phi \in C_{0}^{\infty}(\mathbb{R})$ is real-valued and let $\varepsilon_{0} \in(0,1)$ be given. Then there is a constant $C$, depending on $\phi$ and $\varepsilon_{0}$, so that

$$
\begin{equation*}
\mathbb{E}_{X}\left[\left(\sum_{j=1}^{n} \phi\left(v_{j} / n\right)-n \int_{-1}^{1} \phi(x) u(x) \mathrm{d} x\right)^{2}\right] \leq C n^{\varepsilon_{0}} . \tag{3.32}
\end{equation*}
$$

Before we prove Lemma 3.4 we will use it to prove Lemma 3.3.
Proof of Lemma 3.3. It follows from Theorem 2.12 in [3] that $\mathbb{P}\left[H_{n}^{\prime}\right] \rightarrow 1$ as $n \rightarrow \infty$. If $\phi=\psi_{2 b_{0}}^{\prime}, \psi_{b_{0}}, \psi_{b_{0}}^{\prime}$ or $\psi_{b_{0}}^{\prime \prime}$ then $\phi \in C_{0}^{\infty}(\mathbb{R})$ and Lemma 3.4 with $\varepsilon_{0}=1 / 6$ gives

$$
\mathbb{P}_{v}\left[\left|\sum_{j=1}^{n} \phi\left(v_{j} / n\right)-n \int_{-1}^{1} \phi(x) u(x) \mathrm{d} x\right| \geq n^{1 / 6}\right] \leq \frac{C n^{1 / 6}}{n^{1 / 3}},
$$

by Chebyshev's inequality. This proves Lemma 3.3.
Proof of Lemma 3.4. Pick $A>0$. There is a function $\psi_{A} \in C_{0}^{\infty}(\mathbb{R})$ such that $0 \leq \psi_{A} \leq 1, \psi_{A}(x)=1$ if $|x| \leq A$, $\operatorname{supp} \psi_{A} \subseteq[-(A+1), A+1]$ and $\left|\psi_{A}^{(r)}(x)\right| \leq c_{m}$ for all $x \in \mathbb{R}, 0 \leq r \leq m$, where the constant $c_{m}$ is independent of $A$. For $z \in \mathbb{C}$ we define

$$
\begin{equation*}
\phi_{A}(z)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \psi_{A}(\xi) \hat{\phi}(\xi) \mathrm{e}^{\mathrm{i} \xi z} \mathrm{~d} \xi, \tag{3.33}
\end{equation*}
$$

which is an entire function of $z$. Here,

$$
\begin{equation*}
\hat{\phi}(\xi)=\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i} \xi x} \phi(x) \mathrm{d} x \tag{3.34}
\end{equation*}
$$

is the Fourier transform of $\phi$. The function $\phi_{A}$ has the following properties. There is a constant $C$ independent of $A$ so that

$$
\begin{equation*}
\left|\phi_{A}(z)\right| \leq \frac{C \mathrm{e}^{(A+1)|\operatorname{Im} z|}}{|z|^{2}} \tag{3.35}
\end{equation*}
$$

if $z \neq 0$, and

$$
\begin{equation*}
\left|\phi_{A}(z)\right| \leq C \mathrm{e}^{(A+1)|\operatorname{Im} z|} \tag{3.36}
\end{equation*}
$$

for all $z$. Furthermore, given $m \geq 1$, there is a constant $C_{m}$ so that

$$
\begin{equation*}
\left|\phi(x)-\phi_{A}(x)\right| \leq \frac{C_{m}}{A^{m}} \tag{3.37}
\end{equation*}
$$

for all $x \in \mathbb{R}$. The inequality (3.36) follows immediately from (3.33) and $|\hat{\phi}| \leq\|\phi\|_{\infty}$. Integration by parts gives

$$
\phi_{A}(z)=\frac{1}{2 \pi(\mathrm{i} z)^{2}} \int_{-\infty}^{\infty} \frac{\mathrm{d}^{2}}{\mathrm{~d} \xi^{2}}\left(\psi_{A}(\xi) \hat{\phi}(\xi)\right) \mathrm{e}^{\mathrm{i} \xi x} \mathrm{~d} x
$$

The properties of $\psi_{A}$ and suitable estimates of $\hat{\phi}$ and its derivatives obtained from (3.34) using integration by parts, now gives (3.35). The estimate (3.37) is also easy to prove using integration by parts.

Let $\gamma_{ \pm}$be given by $t \rightarrow \mp t \pm \mathrm{i} v, t \in \mathbb{R}$, where $v>0$ is fixed, and let $\gamma=\gamma_{+}+\gamma_{-}$. Cauchy's integral formula and the estimate (3.35) show that we can represent $\phi_{A}$ by

$$
\begin{equation*}
\phi_{A}(x)=\frac{1}{2 \pi \mathrm{i}} \int_{\gamma} \frac{\phi_{A}(z)}{z-x} \mathrm{~d} z . \tag{3.38}
\end{equation*}
$$

We now turn to the proof of (3.32). Write $r_{A}=\phi-\phi_{A}$. Then,

$$
\begin{align*}
& \mathbb{E}_{X}\left[\left(\sum_{j=1}^{n} \phi\left(v_{j} / n\right)-n \int_{-1}^{1} \phi(x) u(x) \mathrm{d} x\right)^{2}\right]^{1 / 2} \\
& \leq \mathbb{E}_{X}\left[\left(\sum_{j=1}^{n} \phi_{A}\left(v_{j} / n\right)-n \int_{-1}^{1} \phi_{A}(x) u(x) \mathrm{d} x\right)^{2}\right]^{1 / 2} \\
& \quad+\mathbb{E}_{X}\left[\left(\sum_{j=1}^{n} r_{A}\left(v_{j} / n\right)-n \int_{-1}^{1} r_{A}(x) u(x) \mathrm{d} x\right)^{2}\right]^{1 / 2} \tag{3.39}
\end{align*}
$$

The second term in the right-hand side of (3.39) is $\leq n C_{m} / A^{m}$ by (3.37), which is $\leq n^{\varepsilon_{0} / 2}$ if $A=\left(C_{m} n^{1-\varepsilon_{0} / 2}\right)^{1 / m}$. In order to estimate the first term we need the following two lemmas.

Lemma 3.5. Assume that $X$ is an Hermitian Wigner matrix with finite fourth moments and $m_{n}$ is given by (2.33). Then there is a constant $C$ so that

$$
\begin{equation*}
\mathbb{E}_{X}\left[\left|m_{n}(z)-\mathbb{E}_{X}\left[m_{n}(z)\right]\right|^{2}\right] \leq \frac{C}{n^{2}|\operatorname{Im} z|^{4}} \tag{3.40}
\end{equation*}
$$

for all $z$ with $\operatorname{Im} z \neq 0$.
Lemma 3.6. Let $r_{n}$ be a sequence of positive numbers tending to zero. Then there is a constant $C$ such that if $n$ is sufficiently large,

$$
\begin{equation*}
\left|\mathbb{E}_{X}\left[m_{n}(z)\right]-m(z)\right| \leq \frac{C}{n|\operatorname{Im} z|^{5}} \tag{3.41}
\end{equation*}
$$

for all $z$ such that $\left(n r_{n}\right)^{-1 / 5} \leq|\operatorname{Im} z| \leq 1$. Here $m(z)$ is given by (2.34).
These two lemmas can be extracted from [2] and [4], Lemma 2.5, but for completeness and convenience we give somewhat streamlined proofs in Section 3.4.

Combining (3.40) and (3.41) we get

$$
\begin{equation*}
\mathbb{E}_{X}\left[\left|m_{n}(z)-m(z)\right|^{2}\right]^{1 / 2} \leq \frac{C}{n|\operatorname{Im} z|^{5}} \tag{3.42}
\end{equation*}
$$

if $\left(n r_{n}\right)^{-1 / 5} \leq|\operatorname{Im} z| \leq 1$. Now, by (3.38),

$$
\begin{aligned}
& \mathbb{E}_{X}\left[\left(\sum_{j=1}^{n} \phi_{A}\left(\nu_{j} / n\right)-n \int_{-1}^{1} \phi_{A}(x) u(x) \mathrm{d} x\right)^{2}\right] \\
& \quad \leq \frac{n^{2}}{4 \pi^{2}} \int_{\gamma}|\mathrm{d} z| \int_{\gamma}|\mathrm{d} w|\left|\phi_{A}(z)\right|\left|\phi_{A}(w)\right| \mathbb{E}_{X}\left[\left|m_{n}(z)-m(z)\right|^{2}\right]^{1 / 2} \mathbb{E}_{X}\left[\left|m_{n}(w)-m(w)\right|^{2}\right]^{1 / 2} \\
& \\
& \quad \leq \frac{C}{v^{10}} \mathrm{e}^{2 A}
\end{aligned}
$$

by (3.35), (3.36) and (3.42) provided $\left(n r_{n}\right)^{-1 / 5} \leq v \leq 1$. Hence, the first term in the right-hand side of (3.39) is $\leq C v^{-5} \mathrm{e}^{A v}$. We need $A v \leq 1$, which gives $v \leq 1 / A=\left(C_{m} n^{1-\varepsilon_{0} / 2}\right)^{-1 / m}$. Also, we need $v^{-5} \leq n^{\varepsilon_{0} / 2}$, i.e. $v \geq n^{-5 \varepsilon_{0} / 2}$. Take $v=n^{-\delta_{0}}$, where $\delta_{0}=\min \left(1 / 10, \varepsilon_{0}\right), r_{n}=n^{-1 / 2}$ and $m$ so large that $m \delta_{0} \geq 1$. Then all the required inequalities are satified and we have proved Lemma 3.4.

### 3.3. The correlation kernel at the edge

In this section we will prove Proposition 3.1. Let

$$
\begin{equation*}
f(z)=\frac{z^{2}}{2}-a z+\frac{1}{S} \sum_{j=1}^{n} \log \left(S z-v_{j}\right) . \tag{3.43}
\end{equation*}
$$

Then, by (2.4),

$$
\begin{equation*}
K_{n, S}^{v}(a S+u, a S+v)=\frac{1}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{L}} \mathrm{~d} z \int_{\Gamma_{M}} \mathrm{~d} w \frac{\mathrm{e}^{-v w+u z}}{w-z} \mathrm{e}^{S(f(w)-f(z))} . \tag{3.44}
\end{equation*}
$$

Note that $a$ and $b$ are chosen so that $f^{\prime}(b)=f^{\prime \prime}(b)=0$. We can now argue as in section the proof of Proposition 2.1 in order to find good contours. Define $g(t)=\operatorname{Re} f(x(t)+\mathrm{i} y(t))$, where $x(0)=b, y(0)=0$. Then

$$
\begin{align*}
g^{\prime} & =x x^{\prime}-y y^{\prime}-a x^{\prime}+\sum_{j=1}^{n} \frac{S\left(x x^{\prime}+y y^{\prime}\right)-v_{j} x^{\prime}}{\left(S x-v_{j}\right)^{2}+S^{2} y^{2}} \\
& =\sum_{j=1}^{n}\left[\frac{S\left(x x^{\prime}-y y^{\prime}\right)-2 b S x^{\prime}+x^{\prime} v_{j}}{\left(b S-v_{j}\right)^{2}}+\frac{S\left(x x^{\prime \prime}+y y^{\prime}\right)-v_{j} x^{\prime}}{\left(S x-v_{j}\right)^{2}+S^{2} y^{2}}\right], \tag{3.45}
\end{align*}
$$

where we have used (3.1) and (3.2) in the second equality. If we write the expression in the last sum in (3.45) as on fraction the numerator becomes

$$
S^{2}\left(-x^{2} x^{\prime}+2 y y^{\prime} x+y^{2} x^{\prime}+2 b x x^{\prime}-2 b y y^{\prime}-b^{2} x^{\prime}\right) v_{j}+S^{3}\left(\left(x x^{\prime}-y y^{\prime}-2 b x^{\prime}\right)\left(x^{2}+y^{2}\right)+b^{2}\left(x x^{\prime}+y y^{\prime}\right)\right) .
$$

We want to choose $x(t)+\mathrm{i} y(t)$ so that the expression in the numerator is independent of $v_{j}$ which gives the equation

$$
-\frac{1}{3} x^{3}+y^{2} x+b x^{2}-b y^{2}-b x=C .
$$

Since $x(0)=b, y(0)=0$ we see that $C=-b^{3} / 3$, and we obtain

$$
y^{2}(x-b)=\frac{1}{3}(x-b)^{3} .
$$

We see that $x(t)=b$ is one possibility and $y(t)= \pm \frac{1}{\sqrt{3}}(x(t)-b)$ another. The choice $x(t)=b, y(t)=t$ gives

$$
\begin{equation*}
g^{\prime}(t)=-\sum_{j=1}^{n} \frac{S^{3} t^{3}}{\left(b S-v_{j}\right)^{2}\left(\left(b S-v_{j}\right)^{2}+S^{2} t^{2}\right)} \tag{3.46}
\end{equation*}
$$

and the choice $x(t)=t, y(t)= \pm \frac{1}{\sqrt{3}}(t-b)$ gives

$$
\begin{equation*}
g^{\prime}(t)=-\sum_{j=1}^{n} \frac{S^{3} t^{3}}{\left(b S-v_{j}\right)^{2}\left(\left(b S-v_{j}+S t\right)^{2}+S^{2} t^{2} / 3\right)} \tag{3.47}
\end{equation*}
$$

This leads us to the following choice of contours. Let $\gamma$ be given by $z(t)$, where

$$
z(t)= \begin{cases}b+t \mathrm{e}^{\pi \mathrm{i} / 6}, & t \leq 0,  \tag{3.48}\\ b+t \mathrm{e}^{5 \pi \mathrm{i} / 6}, & t \geq 0,\end{cases}
$$

and let $\Gamma$ be given by $w(s)=b+$ is, $s \in \mathbb{R}$. We can deform the contour $\gamma_{L}$ in (3.44) to $\gamma$ and $\Gamma_{M}$ to $\Gamma$.
From (3.1), (3.46) and $v \in F_{n}$ we see that for $t \geq 0$

$$
g^{\prime}(t) \leq-\sum_{j=1}^{n} \frac{S t^{3}}{\left(b S-v_{j}\right)^{2}\left(\beta_{0}^{2}+t^{2}\right)}=-\frac{t^{3}}{\beta_{0}^{2}+t^{2}}
$$

and similarly for $t \leq 0$,

$$
g^{\prime}(t) \geq-\frac{t^{3}}{\beta_{0}^{2}+t^{2}}
$$

From this it follows that

$$
\operatorname{Re} f(w(s))-f(b) \leq \begin{cases}-s^{4} / 8 \beta_{0}^{2} & \text { for } 0 \leq|s| \leq \beta_{0}  \tag{3.49}\\ \left(\beta_{0}^{2}-2 s^{2}\right) / 8 & \text { for }|s| \geq \beta_{0}\end{cases}
$$

Using the fact that $\left(S b-v_{j}+S t\right)^{2} \leq 2 \beta_{0}^{2} S^{2}+2 S^{2} t^{2}$ we get in a similar way from (3.47), that

$$
f(b)-\operatorname{Re} f(z(t)) \leq \begin{cases}-t^{4} / 24 \beta_{0}^{2} & \text { for } 0 \leq|s| \leq \beta_{0}  \tag{3.50}\\ \left(\beta_{0}^{2}-2 t^{2}\right) / 24 & \text { for }|s| \geq \beta_{0}\end{cases}
$$

Set $\varepsilon=S^{-5 / 24}$ and let $I_{1}=(-\infty,-\varepsilon], I_{2}=[-\varepsilon, \varepsilon], I_{3}=[\varepsilon, \infty)$. Define $\Gamma_{k}$ by $w(s), s \in I_{k}$, and $\gamma_{k}$ by $z(t), t \in I_{k}$. Let

$$
I_{j k}=\frac{\mathrm{e}^{(v-u) b} d S^{1 / 3}}{(2 \pi \mathrm{i})^{2}} \int_{\gamma_{j}} \mathrm{~d} z \int_{\Gamma_{k}} \mathrm{~d} w \frac{\mathrm{e}^{-v w+u z}}{w-z} \mathrm{e}^{S(f(w)-f(z))}
$$

where $u=d S^{1 / 3} \xi, v=d S^{1 / 3} \eta$. Then

$$
d S^{1 / 3} \mathrm{e}^{(\eta-\xi) d S^{1 / 3}} K_{n, S}^{v}\left(a S+d S^{1 / 3} \xi, a S+d S^{1 / 3} \eta\right)=\sum_{j, k=1}^{3} I_{j k} .
$$

We first show that

$$
\begin{equation*}
\left|I_{1, k}\right|,\left|I_{3, k}\right| \leq C \mathrm{e}^{-c S^{1 / 8} \xi-c S^{1 / 6}} \tag{3.51}
\end{equation*}
$$

for $S \geq 1, k=1,2,3$. Consider $I_{3, k}$, the estimation of $I_{1, k}$ is analogous. If $z \in \gamma_{3}$ and $w \in \Gamma$, then

$$
\begin{equation*}
\left|\frac{\mathrm{e}^{-\eta d S^{1 / 3}(w-b)+\xi d S^{1 / 3}(z-b)}}{w-z}\right| \leq \frac{C}{\varepsilon} \mathrm{e}^{-C \varepsilon d S^{1 / 3} \xi} \leq \frac{C}{\varepsilon} \mathrm{e}^{-C S^{1 / 8} \xi} . \tag{3.52}
\end{equation*}
$$

Here we have used the fact that

$$
\sum_{j=1}^{n} \frac{S^{2}}{\left(b S-v_{j}\right)^{3}}=\sum_{j=1}^{n} \frac{S}{\left(b S-v_{j}\right)^{2}} \frac{1}{b-v_{j} / S} \in\left[1 / \beta_{0}, 1 / \alpha_{0}\right]
$$

by (3.1) and (3.4), which gives $d \in\left[1 / \beta_{0}^{1 / 3}, 1 / \alpha_{0}^{1 / 3}\right]$. It follows from (3.49) that

$$
\begin{equation*}
\int_{-\infty}^{\infty} \mathrm{e}^{S(\operatorname{Re} f(w(s))-f(b))} \mathrm{d} s \leq \frac{C}{S^{1 / 4}} . \tag{3.53}
\end{equation*}
$$

Furthermore, (3.50) gives

$$
\begin{equation*}
\int_{\varepsilon}^{\infty} \mathrm{e}^{S(f(b)-\operatorname{Re} f(z(t)))} \mathrm{d} t \leq \frac{C}{S^{1 / 4}} \mathrm{e}^{-C S \varepsilon^{4}} \leq \frac{C}{S^{1 / 4}} \mathrm{e}^{-C S^{1 / 6}} \tag{3.54}
\end{equation*}
$$

If we combine (3.51), (3.53) and (3.54) we get (3.51).
Next, we show that there are positive constants $C, c, S_{0}$ such that for $S \geq S_{0}$,

$$
\begin{equation*}
\left|I_{2, k}\right| \leq C \mathrm{e}^{-\xi} \mathrm{e}^{-c S^{1 / 6}} . \tag{3.55}
\end{equation*}
$$

We treat $I_{2,3}$, the proof for $I_{2,1}$ is analogous. We have that

$$
\begin{align*}
I_{2,3}= & \frac{d S^{1 / 3}}{(2 \pi \mathrm{i})^{2}} \int_{-\varepsilon}^{\varepsilon} \mathrm{d} t \int_{\varepsilon}^{\infty} \mathrm{d} s \frac{z^{\prime}(t)}{w(s)-z(t)} \\
& \times \mathrm{e}^{-\eta d S^{1 / 3}(w(s)-b)+\xi d S^{1 / 3}(z(t)-b)+S(f(w(s))-f(b))+S(f(b)-f(z(t)))} . \tag{3.56}
\end{align*}
$$

Claim 3.7. If $|z-b| \leq \alpha_{0} / 2$, then

$$
\begin{equation*}
f(z)=f(b)+\frac{1}{3} d^{3}(z-b)^{3}-\lambda d^{4}(z-b)^{4}+R(z-b), \tag{3.57}
\end{equation*}
$$

where

$$
\begin{equation*}
|R(z)| \leq 20 \alpha_{0}^{-5}|z|^{5} \tag{3.58}
\end{equation*}
$$

and $\lambda \in\left[\left(\alpha_{0}^{2 / 3} / \beta_{0}\right)^{2} / 4,\left(\beta_{0}^{2 / 3} / \alpha_{0}\right)^{2} / 4\right]$.
Proof. Let $h(t)=f(b+t(z-b))$. Then Taylor's formula yields (3.57) with $\lambda=-f^{(4)}(b) / 24 d^{4}$ and

$$
R(z)=\frac{z^{5}}{120} f^{(5)}(b)+\frac{z^{5}}{120} \int_{0}^{1}(1-t)^{5} f^{(5)}(b+t(z-b)) \mathrm{d} t .
$$

Now, by (3.1) and (3.4),

$$
-\frac{1}{24} f^{(4)}(b)=\frac{1}{4} \sum_{j=1}^{n} \frac{S^{3}}{\left(b S-v_{j}\right)^{4}} \in\left[\frac{1}{4 \beta_{0}^{2}}, \frac{1}{4 \alpha_{0}^{2}}\right],
$$

and similarly $d^{4} \in\left[1 / \beta_{0}^{4 / 3}, 1 / \alpha_{0}^{4 / 3}\right]$. Hence, the result for $\lambda$ follows. If $|z-b| \leq \alpha_{0} / 2$, then $\left|S(b+t(z-b))-v_{j}\right| \geq$ $\left|b S-v_{j}\right| / 2$ and thus, by (3.1) and (3.4),

$$
\left|f^{(5)}(b+t(z-b))\right| \leq 24 \cdot 2^{5} \sum_{j=1}^{n} \frac{S^{4}}{\left(b S-v_{j}\right)^{5}} \leq \frac{24 \cdot 2^{5}}{\alpha_{0}^{3}}
$$

This gives (3.58).
Using (3.48), (3.57) and making the change of variables $\tau=d S^{1 / 3} t$, the $t$-integral in (3.56) becomes

$$
\begin{align*}
& \mathrm{e}^{5 \pi \mathrm{i} / 6} \int_{0}^{\varepsilon^{\prime}} \frac{1}{\mathrm{i} d S^{1 / 3} S-\tau \mathrm{e}^{5 \pi \mathrm{i} / 6}} \mathrm{e}^{\mathrm{i} \xi \tau \mathrm{e}^{\pi \mathrm{i} / 3}-\mathrm{i} \tau^{3} / 3+\lambda \tau^{4} \mathrm{e}^{4 \pi \mathrm{i} / 3} / S^{1 / 3}-R_{S}\left(\tau \mathrm{e}^{5 \pi \mathrm{i} / 6}\right)} \mathrm{d} \tau \\
& \quad+\mathrm{e}^{\pi \mathrm{i} / 6} \int_{-\varepsilon^{\prime}}^{0} \frac{1}{\mathrm{i} d S^{1 / 3} S-\tau \mathrm{e}^{\pi \mathrm{i} / 6}} \mathrm{e}^{\mathrm{i} \xi \tau \mathrm{e}^{-\pi \mathrm{i} / 3}-\mathrm{i} \tau^{3} / 3+\lambda \tau^{4} \mathrm{e}^{2 \pi \mathrm{i} / 3} / S^{1 / 3}-R_{S}\left(\tau \mathrm{e}^{\pi \mathrm{i} / 6}\right)} \mathrm{d} \tau \tag{3.59}
\end{align*}
$$

where $\varepsilon^{\prime}=d S^{1 / 3} \varepsilon=d S^{1 / 8}$ and $R_{S}(\tau)=S R\left(\tau / d S^{1 / 3}\right)$. Let $C_{+}^{\prime}$ be the curve from 0 to $\varepsilon^{\prime}$ consisting of the line segments from 0 to -i , from -i to $\varepsilon^{\prime}-\mathrm{i}$ and from $\varepsilon^{\prime}-\mathrm{i}$ to $\varepsilon^{\prime}$, and $C_{-}^{\prime}$ the curve from $-\varepsilon^{\prime}$ to 0 consisting of the line segments from $-\varepsilon^{\prime}$ to $-\varepsilon^{\prime}-\mathrm{i}$, from $-\varepsilon^{\prime}-\mathrm{i}$ to -i and from -i to 0 . Now, let $C_{-}$be the curve obtained from $C_{-}^{\prime}$ by rotating it around the origin by an angle $-\pi / 3$, and let $C_{+}$be the curve obtained from $C_{+}^{\prime}$ by rotating it around the origin an angle $\pi / 3$. The sum of the two integrals in (3.59) can then be written

$$
\begin{equation*}
\mathrm{i} \int_{C_{-}+C_{+}} \frac{1}{\mathrm{i} d S^{1 / 3}-\mathrm{i} z} \mathrm{e}^{\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{i} z)} \mathrm{d} z \tag{3.60}
\end{equation*}
$$

The contour $C_{-}+C_{+}$can be deformed into $C_{1}+C_{2}+C_{3}+C_{4}+C_{5}$, where

- $C_{1}$ is the line segment from $\varepsilon^{\prime} \mathrm{e}^{2 \pi \mathrm{i} / 3}$ to $\varepsilon^{\prime} \mathrm{e}^{2 \pi \mathrm{i} / 3}+\mathrm{e}^{-5 \pi \mathrm{i} / 6}$,
- $C_{2}$ is the line segment from $\varepsilon^{\prime} \mathrm{e}^{2 \pi \mathrm{i} / 3}+\mathrm{e}^{-5 \pi \mathrm{i} / 6}$ to $-\sqrt{3}+\mathrm{i}$,
- $C_{3}$ is the line segment from $-\sqrt{3}+\mathrm{i}$ to $\sqrt{3}+\mathrm{i}$,
- $C_{4}$ is the line segment from $\sqrt{3}+\mathrm{i}$ to $\varepsilon^{\prime} \mathrm{e}^{\pi \mathrm{i} / 3}+\mathrm{e}^{-\pi \mathrm{i} / 6}$ and
- $C_{5}$ is the line segment from $\varepsilon^{\prime} \mathrm{e}^{\pi \mathrm{i} / 3}+\mathrm{e}^{-\pi \mathrm{i} / 6}$ to $\varepsilon^{\prime} \mathrm{e}^{\pi \mathrm{i} / 3}$.

The integral in (3.60) can then be written

$$
\begin{equation*}
\mathrm{i} \sum_{j=1}^{5} \int_{C_{j}} \frac{1}{\mathrm{i} d S^{1 / 3}-\mathrm{i} z} \mathrm{e}^{\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{i} z)} \mathrm{d} z \tag{3.61}
\end{equation*}
$$

Combining this with (3.56) now leads us to the estimate

$$
\left|I_{2,3}\right| \leq \frac{d S^{1 / 3}}{4 \pi^{2}} \sum_{j=1}^{5} \int_{\varepsilon}^{\infty} \mathrm{d} s \int_{C_{j}}|\mathrm{~d} z| \frac{\mathrm{e}^{\operatorname{Re} S(f(b+\mathrm{i} s)-f(b))}}{\left|d S^{1 / 3} s-z\right|} \mathrm{e}^{\operatorname{Re}\left(\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{i} z)\right)}
$$

Note that $\left|d S^{1 / 3} s-z\right| \geq \varepsilon^{\prime} / 2$ when $s \geq \varepsilon$ and $z \in C_{j}$. Also, by (3.49),

$$
\begin{equation*}
\int_{\varepsilon}^{\infty} \mathrm{e}^{\operatorname{Re} S(f(b+\mathrm{i} s)-f(b))} \mathrm{d} s \leq \frac{C}{S^{1 / 4}} \mathrm{e}^{-c S \varepsilon^{4}}=\frac{C}{S^{1 / 4}} \mathrm{e}^{-c S^{1 / 6}} \tag{3.62}
\end{equation*}
$$

The contour $C_{3}$ is given by $z(t)=t+\mathrm{i},|t| \leq \sqrt{3}$. This gives, using (3.58),

$$
\begin{equation*}
\int_{C_{3}} \mathrm{e}^{\operatorname{Re}\left(\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{i} z)\right)}|\mathrm{d} z| \leq C \mathrm{e}^{-\xi} \tag{3.63}
\end{equation*}
$$

for $S \geq 1$. The curve $C_{4}$ is given by $z(t)=\mathrm{e}^{-\pi \mathrm{i} / 6}+t \mathrm{e}^{\pi \mathrm{i} / 3}, \sqrt{3} \leq t \leq \varepsilon^{\prime}$. Then $\operatorname{Re} z(t)^{4} \leq 0$ if $t \geq \sqrt{3}, \operatorname{Re}(\mathrm{i} \xi z(t)+$ $\left.\mathrm{i} z(t)^{3} / 3\right) \leq-\xi+1 / 3-t^{2}$ and $\left|R_{S}(\mathrm{i} z(t))\right| \leq C S^{-1 / 24}$. This gives

$$
\begin{equation*}
\int_{C_{4}} \mathrm{e}^{\operatorname{Re}\left(\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{i} z)\right.}|\mathrm{d} z| \leq C \mathrm{e}^{-\xi} . \tag{3.64}
\end{equation*}
$$

The curve $-C_{5}$ is given by $z(t)=\varepsilon^{\prime} \mathrm{e}^{\pi \mathrm{i} / 3}+t \mathrm{e}^{-\pi \mathrm{i} / 6}, 0 \leq t \leq 1$, and inserting the parametrization and estimating we see that we get an estimate

$$
\int_{C_{5}} \mathrm{e}^{\operatorname{Re}\left(\mathrm{i} \xi z+\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{iz})\right)}|\mathrm{d} z| \leq C \mathrm{e}^{-\xi}
$$

if $\varepsilon^{\prime} \geq c_{0}$, where $c_{0}$ is a numerical constant. This holds if $S \geq S_{0}=\left(c_{0} \alpha_{0}^{1 / 3}\right)^{8} \geq\left(C_{0} / d\right)^{8}$. The estimates for the integrals along $C_{1}$ and $C_{2}$ are analogous to the estimates for $C_{5}$ and $C_{4}$ respectively. Collecting all the estimates we have proved (3.55).

It remains to estimate and compute the asymptotics of $I_{22}$. Let $C^{\prime}$ be the contour $t \rightarrow t+\mathrm{i},|t| \leq \varepsilon^{\prime}$ and let $C=C_{1}+C_{2}+C_{3}+C_{4}+C_{5}$. The same type of computations that led to the expression (3.60) now gives

$$
\begin{equation*}
I_{22}=-\frac{\mathrm{i}}{(2 \pi \mathrm{i})^{2}} \int_{C} \mathrm{~d} z \int_{C^{\prime}} \mathrm{d} w \frac{\mathrm{e}^{\mathrm{i} \eta w+\mathrm{i} \xi z}}{z+w} \mathrm{e}^{\mathrm{i} w^{3} / 3-\lambda w^{4} / S^{1 / 3}+R_{S}(-\mathrm{i} w)} \mathrm{e}^{\mathrm{i} z^{3} / 3+\lambda z^{4} / S^{1 / 3}-R_{S}(\mathrm{iz})} . \tag{3.65}
\end{equation*}
$$

By introducing the parametrizations of $C$ and $C^{\prime}$ we can now again prove that

$$
\begin{equation*}
\left|I_{22}\right| \leq C \mathrm{e}^{-(\xi+\eta)} \tag{3.66}
\end{equation*}
$$

for $S \geq S_{0}$ with a suitable $S_{0}$ that only depends on $\alpha_{0}$. Combining the estimates (3.51), (3.55) and (3.66) we obtain (3.5).

We now take $S=S_{n}=\kappa n$. It is clear from (3.51) and (3.55) that all contributions except $I_{22}$ go to zero uniformly for $\xi, \eta$ in a compact set and all $v \in F_{n}$ as $n \rightarrow \infty$. Let $\tilde{C}$ be the "limit" of $C$ as $n \rightarrow \infty$, i.e. $\tilde{C}=\tilde{C}_{1}+\tilde{C}_{2}+\tilde{C}_{3}$, where $-\tilde{C}_{1}:-\sqrt{3}+\mathrm{i}+t \mathrm{e}^{2 \pi \mathrm{i} / 3}, t \geq 0, \tilde{C}_{2}: t+\mathrm{i},|t| \leq \sqrt{3}, \tilde{C}_{3}: \sqrt{3}+\mathrm{i}+t \mathrm{e}^{\pi \mathrm{i} / 3}, t \geq 0$. Introducing the parametrizations into the integral in (3.65) we see that we can let $n \rightarrow \infty$ in (3.65) with $S=S_{n}$, to obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} I_{22}=-\frac{\mathrm{i}}{(2 \pi \mathrm{i})^{2}} \int_{\tilde{C}} \mathrm{~d} z \int_{\operatorname{Im} w=1} \mathrm{~d} w \frac{\mathrm{e}^{\mathrm{i} \eta w+\mathrm{i} \xi z}}{z+w} \mathrm{e}^{\mathrm{i}\left(w^{3}+z^{3}\right) / 3} \tag{3.67}
\end{equation*}
$$

uniformly for $\xi, \eta$ in a compact set and all $v \in F_{n}$. A deformation argument now shows that we can deform $\tilde{C}$ to $\operatorname{Im} z=1$, and in this way we see that the right-hand side of (3.67) equals the Airy kernel (1.7), see e.g. Proposition 2, 3 in [15]. This proves (3.6) and finishes the proof of Proposition 3.1.

### 3.4. Proofs of some lemmas

The proofs of Lemmas 3.6 and 3.6 can be extracted from [2] and [4]. The presentation below is somewhat streamlined for our purposes. We use notation similar to that in [2] and [4]. See also [5].

Recall that $X=\left(x_{i j}\right)$ is an Hermitian Wigner matrix, such that $\mathbb{E}\left[\left|x_{i j}\right|^{2}\right]=\sigma^{2}$ and $\mathbb{E}\left[\left|x_{i j}\right|^{4}\right] \leq K$ for all $1 \leq i \leq$ $j \leq n$, where $K<\infty$ is a constant. Let $X_{k}$ be the matrix obtained from $X$ by removing row $k$ and column $k$, and let $\alpha_{k}$ be column $k$ of $X$ with element number $k$ removed. Set

$$
D=\left(\frac{1}{\sqrt{n}} X-z I\right)^{-1}, \quad D_{k}=\left(\frac{1}{\sqrt{n}} X_{k}-z I\right)^{-1}
$$

Write $v=\operatorname{Im} z$. We can assume that $v>0$. We need some identities from matrix theory.

Lemma 3.8. The following identities hold,

$$
\begin{align*}
& \operatorname{tr} D=\sum_{k=1}^{n} \frac{1}{x_{k k} / \sqrt{n}-z-\alpha_{k}^{*} D_{k} \alpha_{k}},  \tag{3.68}\\
& \operatorname{tr} D-\operatorname{tr} D_{k}=\sum_{k=1}^{n} \frac{1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}}{x_{k k} / \sqrt{n}-z-\alpha_{k}^{*} D_{k} \alpha_{k}} . \tag{3.69}
\end{align*}
$$

Proof. The identity (3.68) follows from Cramer's rule and the formula

$$
\operatorname{det}\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)=\operatorname{det}(A) \operatorname{det}\left(D-C A^{-1} B\right),
$$

which holds whenever $A$ is invertible. The formula (3.69) follows from the formula

$$
\left(\begin{array}{ll}
A & B \\
C & D
\end{array}\right)^{-1}=\left(\begin{array}{cc}
A^{-1}+A^{-1} B\left(D-C A^{-1} B\right)^{-1} C A^{-1} & -A^{-1} B\left(D-C A^{-1} B\right)^{-1} \\
-\left(D-C A^{-1} B\right)^{-1} C A^{-1} & \left(D-C A^{-1} B\right)^{-1}
\end{array}\right)
$$

for the inverse of a block matrix.
Let

$$
\begin{aligned}
& \beta_{k}=-x_{k k} / \sqrt{n}+z+\alpha_{k}^{*} D_{k} \alpha_{k}, \\
& \beta_{k}^{*}=z+\frac{\sigma^{2}}{n} \operatorname{tr} D_{k}, \\
& \beta=z+\frac{\sigma^{2}}{n} \operatorname{tr} D, \\
& \varepsilon_{k}^{*}=\beta_{k}-\beta_{k}^{*}=-x_{k k} / \sqrt{n}+\frac{1}{n}\left(\alpha_{k}^{*} D_{k} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}\right) .
\end{aligned}
$$

Let $\mathbb{E}_{k}$ denote expectation with respect to the elemts in row/colum $k$ in $X$. We need the following basic estimates.

## Lemma 3.9.

$$
\begin{align*}
& \operatorname{Im} \beta_{k}=v\left(1+\frac{1}{n} \alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k}\right) \geq v,  \tag{3.70}\\
& \operatorname{Im} \beta_{k}^{*}=v\left(1+\frac{1}{n} \operatorname{tr} D_{k} D_{k}^{*}\right) \geq v, \tag{3.71}
\end{align*}
$$

$$
\begin{equation*}
\left|1+\frac{1}{n} \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right| \leq 1+\frac{1}{n} \alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k}, \tag{3.72}
\end{equation*}
$$

$$
\begin{equation*}
\left|\operatorname{tr} D-\operatorname{tr} D_{k}\right| \leq \frac{1}{v} \tag{3.73}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{E}_{k}\left[\left|\alpha_{k}^{*} D_{k} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}\right|^{2}\right] \leq K \operatorname{tr} D_{k} D_{k}^{*} \tag{3.74}
\end{equation*}
$$

$$
\begin{equation*}
\mathbb{E}_{k}\left[\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}\right|^{2}\right] \leq K \operatorname{tr} D_{k}^{2} D_{k}^{* 2} \tag{3.75}
\end{equation*}
$$

Proof. We see that

$$
\operatorname{Im} \beta_{k}=v+\frac{1}{2 \mathrm{i} n}\left(\alpha_{k}^{*} D_{k} \alpha_{k}-\alpha_{k}^{*} D_{k}^{*} \alpha_{k}\right)=v\left(1+\frac{1}{n} \alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k}\right) \geq v,
$$

which gives (3.70) and a similar argument proves (3.71). To prove (3.72) we write $D_{k}=U^{*} \operatorname{diag}\left(\left(\lambda_{j} / \sqrt{n}-z\right)^{-1}\right) U$, where $\lambda_{1}, \ldots, \lambda_{n-1}$ are the eigenvalues of $X_{k}$ and $U$ is unitary. Then,

$$
\begin{aligned}
\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right| & \leq \sum_{j=1}^{n}\left|\lambda_{j} / \sqrt{n}-z\right|^{-2}\left|\left(U \alpha_{k}\right)_{j}\right|^{2} \\
& =\left(U \alpha_{k}\right)^{*} \operatorname{diag}\left(\left(\lambda_{j} / \sqrt{n}-z\right)^{-1}\right) \operatorname{diag}\left(\left(\lambda_{j} / \sqrt{n}-\bar{z}\right)^{-1}\right) U \alpha_{k}=\alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k} .
\end{aligned}
$$

We see from (3.69), (3.70) and (3.72) that

$$
\left|\operatorname{tr} D-\operatorname{tr} D_{k}\right|=\frac{\left|1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right|}{\left|\beta_{k}\right|} \leq \frac{1+(1 / n) \alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k}}{v\left(1+(1 / n) \alpha_{k}^{*} D_{k} D_{k}^{*} \alpha_{k}\right)}=\frac{1}{v},
$$

which proves (3.73). Let $A=\left(a_{i j}\right)$ be an $(n-1) \times(n-1)$ matrix that does not depend on the elements in row/column $k$. Note that

$$
\begin{equation*}
\mathbb{E}_{k}\left[\alpha_{k}^{*} A \alpha_{k}\right]=\sum_{j=1}^{n-1} \sigma^{2} a_{j j}=\sigma^{2} \operatorname{tr} A \tag{3.76}
\end{equation*}
$$

Hence,

$$
\mathbb{E}_{k}\left[\left|\alpha_{k}^{*} D_{k} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}\right|^{2}\right]=\mathbb{E}_{k}\left[\alpha_{k}^{*} A^{*} \alpha_{k} \alpha_{k}^{*} A \alpha_{k}\right]-\sigma^{4}\left(\operatorname{tr} A^{*}\right)(\operatorname{tr} A) .
$$

Now,

$$
\begin{aligned}
\mathbb{E}_{k}\left[\alpha_{k}^{*} A^{*} \alpha_{k} \alpha_{k}^{*} A \alpha_{k}\right] & =\mathbb{E}_{k}\left[\sum_{i, j, r, s}\left(\alpha_{k}^{*}\right)_{i} \bar{a}_{j i}\left(\alpha_{k}\right)_{j}\left(\alpha_{k}^{*}\right)_{r} a_{r s}\left(\alpha_{k}\right)_{s}\right] \\
& \leq K \sum_{i}\left|a_{i i}\right|^{2}+\sum_{i \neq j} \sigma^{4} \bar{a}_{i i} a_{j j}+\sum_{i \neq j} \sigma^{4} \bar{a}_{j i} a_{j i} \\
& =\left(K-2 \sigma^{4}\right) \sum_{i}\left|a_{i i}\right|^{2}+\sigma^{4}\left(\operatorname{tr} A^{*}\right)(\operatorname{tr} A)+\sigma^{4} \operatorname{tr} A^{*} A \\
& \leq K \operatorname{tr} A^{*} A+\sigma^{4}\left(\operatorname{tr} A^{*}\right)(\operatorname{tr} A) .
\end{aligned}
$$

This proves (3.74) and (3.75).
Let $\mathcal{F}_{k}$ be the $\sigma$-algebra generated by $\operatorname{Im} x_{j k}, \operatorname{Re} x_{j k}, k+1 \leq i \leq j \leq n, \mathcal{F}_{n}=\varnothing$. Define

$$
z_{k}=\mathbb{E}\left[\operatorname{tr} D \mid \mathcal{F}_{k-1}\right]-\mathbb{E}\left[\operatorname{tr} D \mid \mathcal{F}_{k}\right] .
$$

Then,

$$
\begin{equation*}
\mathbb{E}\left[|\operatorname{tr} D-\mathbb{E}[\operatorname{tr} D]|^{2}\right]=\mathbb{E}\left[\sum_{j, k=1}^{n} \bar{z}_{j} z_{k}\right]=\sum_{k=1}^{n} \mathbb{E}\left[\left|z_{k}\right|^{2}\right] \tag{3.77}
\end{equation*}
$$

by orthogonality. Since $\operatorname{tr} D_{k}$ is independent of the elements in row/column $k$,

$$
\mathbb{E}\left[\operatorname{tr} D_{k} \mid \mathcal{F}_{k-1}\right]=\mathbb{E}\left[\operatorname{tr} D_{k} \mid \mathcal{F}_{k}\right]
$$

and hence

$$
\begin{equation*}
z_{k}=\mathbb{E}\left[\operatorname{tr} D-\operatorname{tr} D_{k} \mid \mathcal{F}_{k-1}\right]-\mathbb{E}\left[\operatorname{tr} D-\operatorname{tr} D_{k} \mid \mathcal{F}_{k}\right] . \tag{3.78}
\end{equation*}
$$

We can now give the
Proof of Lemma 2.6. Note that $m_{n}(z)=\frac{1}{n} \operatorname{tr} D$ and that (3.73), (3.77) and (3.78) give

$$
\mathbb{E}\left[|\operatorname{tr} D-\mathbb{E}[\operatorname{tr} D]|^{2}\right] \leq \sum_{k=1}^{n} \frac{2}{v^{2}}=\frac{2 n}{v^{2}}
$$

We turn next to the
Proof of Lemma 3.5. From (3.69) we obtain

$$
\begin{aligned}
\operatorname{tr} D-\operatorname{tr} D_{k} & =\left(1+\frac{1}{n} \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right)\left(\frac{1}{\beta_{k}^{*}}-\frac{1}{\beta_{k}}\right)-\left(1+\frac{1}{n} \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right) \frac{1}{\beta_{k}^{*}} \\
& =\frac{\varepsilon_{k}^{*}\left(1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right)}{\beta_{k}^{*} \beta_{k}}-\frac{1+\left(\sigma^{2} / n\right) \operatorname{tr} D_{k}^{2}}{\beta_{k}^{*}}-\frac{\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}}{n \beta_{k}^{*}} .
\end{aligned}
$$

Since neither $\operatorname{tr} D_{k}^{2}$ or $\beta_{k}^{*}$ depends on row/column $k$, we see that

$$
\mathbb{E}\left[\left.\frac{1+\left(\sigma^{2} / n\right) \operatorname{tr} D_{k}^{2}}{\beta_{k}^{*}} \right\rvert\, \mathcal{F}_{k-1}\right]=\mathbb{E}\left[\left.\frac{1+\left(\sigma^{2} / n\right) \operatorname{tr} D_{k}^{2}}{\beta_{k}^{*}} \right\rvert\, \mathcal{F}_{k}\right] .
$$

Hence, from (3.78), we see that

$$
\begin{aligned}
z_{k}= & \mathbb{E}\left[\left.\frac{\varepsilon_{k}^{*}\left(1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right)}{\beta_{k}^{*} \beta_{k}} \right\rvert\, \mathcal{F}_{k-1}\right]-\mathbb{E}\left[\left.\frac{\varepsilon_{k}^{*}\left(1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right)}{\beta_{k}^{*} \beta_{k}} \right\rvert\, \mathcal{F}_{k}\right] \\
& +\mathbb{E}\left[\left.\frac{\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}}{n \beta_{k}^{*}} \right\rvert\, \mathcal{F}_{k-1}\right]-\mathbb{E}\left[\left.\frac{\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}}{n \beta_{k}^{*}} \right\rvert\, \mathcal{F}_{k}\right] .
\end{aligned}
$$

Thus,

$$
\mathbb{E}\left[\left|z_{k}\right|^{2}\right] \leq 2 \mathbb{E}\left[\frac{\left|\varepsilon_{k}^{*}\right|^{2}\left|1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right|^{2}}{\left|\beta_{k}^{*}\right|^{2}\left|\beta_{k}\right|^{2}}\right]+2 \mathbb{E}\left[\frac{\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}\right|^{2}}{n^{2}\left|\beta_{k}^{*}\right|^{2}}\right]
$$

We see from (3.70) and (3.72) that

$$
\begin{equation*}
\frac{\left|1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right|^{2}}{\left|\beta_{k}\right|^{2}} \leq \frac{1}{v^{2}} \tag{3.79}
\end{equation*}
$$

and from (3.70) and (3.74) we obtain

$$
\begin{align*}
\mathbb{E}_{k}\left[\left|\varepsilon_{k}^{*}\right|^{2}\right] & =\frac{\sigma^{2}}{n}+\frac{1}{n^{2}} \mathbb{E}_{k}\left[\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}\right|^{2}\right] \\
& \leq \frac{\sigma^{2}}{n}+\frac{K}{n^{2}} \operatorname{tr} D_{k} D_{k}^{*} \leq \frac{K+\sigma^{2}}{v n} \operatorname{Im} \beta_{k}^{*} \tag{3.80}
\end{align*}
$$

Consequently, by (3.71), (3.79) and (3.80)

$$
\mathbb{E}\left[\frac{\left|\varepsilon_{k}^{*}\right|^{2}\left|1+(1 / n) \alpha_{k}^{*} D_{k}^{2} \alpha_{k}\right|^{2}}{\left|\beta_{k}^{*}\right|^{2}\left|\beta_{k}\right|^{2}}\right] \leq \frac{1}{v^{3}} \mathbb{E}\left[\frac{1}{\left|\beta_{k}^{*}\right|^{2}} \mathbb{E}_{k}\left[\left|\varepsilon_{k}^{*}\right|^{2}\right]\right] \leq \frac{K+\sigma^{2}}{n v^{4}}
$$

Note that,

$$
\operatorname{tr} D_{k}^{2} D_{k}^{* 2}=\sum_{j=1}^{n-1} \frac{1}{\left|\lambda_{j}-z\right|^{4}} \leq \frac{1}{v^{2}} \sum_{j=1}^{n-1} \frac{1}{\left|\lambda_{j}-z\right|^{2}}=\frac{1}{v^{2}} \operatorname{tr} D_{k} D_{k}^{*}
$$

and hence, using also (3.71) and (3.75),

$$
\begin{aligned}
\mathbb{E}\left[\frac{\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}\right|^{2}}{n^{2}\left|\beta_{k}^{*}\right|^{2}}\right] & \leq \frac{1}{n v} \mathbb{E}\left[\frac{1}{n\left|\beta_{k}^{*}\right|} \mathbb{E}_{k}\left[\left|\alpha_{k}^{*} D_{k}^{2} \alpha_{k}-\sigma^{2} \operatorname{tr} D_{k}^{2}\right|^{2}\right]\right] \\
& \leq \frac{K}{n v} \mathbb{E}\left[\frac{1}{\left|\beta_{k}^{*}\right|} \frac{1}{n} \operatorname{tr} D_{k}^{2} D_{k}^{* 2}\right] \leq \frac{K}{n v^{3}} \mathbb{E}\left[\frac{1+(1 / n) \operatorname{tr} D_{k} D_{k}^{*}}{\operatorname{Im} \beta_{k}^{*}}\right]=\frac{K}{n v^{4}} .
\end{aligned}
$$

We see now from (3.77) that

$$
\begin{equation*}
\mathbb{E}\left[|\operatorname{tr} D-\mathbb{E}[\operatorname{tr} D]|^{2}\right] \leq \frac{2 K+\sigma^{2}}{v^{4}} \tag{3.81}
\end{equation*}
$$

We still have to give the
Proof of Lemma 3.6. Set

$$
\begin{equation*}
\delta=\mathbb{E}\left[m_{n}(z)\right]+\frac{1}{z+\sigma^{2} \mathbb{E}\left[m_{n}(z)\right]} . \tag{3.82}
\end{equation*}
$$

We see that

$$
\begin{align*}
\delta & =\mathbb{E}\left[m_{n}(z)+\frac{1}{z+\sigma^{2} m_{n}(z)}-\left(\frac{1}{z+\sigma^{2} m_{n}(z)}-\frac{1}{z+\sigma^{2} \mathbb{E}\left[m_{n}(z)\right]}\right)\right] \\
& =\mathbb{E}\left[m_{n}(z)+\frac{1}{\beta}\right]+\sigma^{2} \mathbb{E}\left[\frac{m_{n}(z)-\mathbb{E}\left[m_{n}(z)\right]}{\beta \mathbb{E}[\beta]}\right] . \tag{3.83}
\end{align*}
$$

By (3.82), $\operatorname{Im} \beta \geq v$, we obtain

$$
\begin{equation*}
\left|\sigma^{2} \mathbb{E}\left[\frac{m_{n}(z)-\mathbb{E}\left[m_{n}(z)\right]}{\beta \mathbb{E}[\beta]}\right]\right| \leq \frac{\sigma^{2}}{v^{2}} \frac{\sqrt{2 K+\sigma^{2}}}{n v^{2}}=\frac{\sigma^{2} \sqrt{2 K+\sigma^{2}}}{n v^{4}} . \tag{3.84}
\end{equation*}
$$

Using (3.68) we find

$$
\begin{align*}
\mathbb{E}\left[m_{n}(z)+\frac{1}{\beta}\right] & =\mathbb{E}\left[\frac{1}{n} \sum_{k=1}^{n}\left(\frac{1}{\beta}-\frac{1}{\beta_{k}^{*}}+\frac{1}{\beta_{k}^{*}} \frac{1}{\beta_{k}}\right)\right] \\
& =\mathbb{E}\left[\frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon_{k}^{*}}{\beta_{k} \beta_{k}^{*}}\right]-\mathbb{E}\left[\frac{\sigma^{2}}{n} \sum_{k=1}^{n} \frac{\operatorname{tr} D-\operatorname{tr} D_{k}}{\beta \beta_{k}^{*}}\right] . \tag{3.85}
\end{align*}
$$

It follows from (3.70), (3.73) and $\operatorname{Im} \beta \geq v$, that

$$
\begin{equation*}
\left|\mathbb{E}\left[\frac{\sigma^{2}}{n} \sum_{k=1}^{n} \frac{\operatorname{tr} D-\operatorname{tr} D_{k}}{\beta \beta_{k}^{*}}\right]\right| \leq \frac{\sigma^{2}}{n v^{3}} . \tag{3.86}
\end{equation*}
$$

Also,

$$
\mathbb{E}\left[\frac{1}{n} \sum_{k=1}^{n} \frac{\varepsilon_{k}^{*}}{\beta_{k} \beta_{k}^{*}}\right]=\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left[\frac{\varepsilon_{k}^{*}}{\beta_{k}^{* 2}}\right]-\frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left[\frac{\varepsilon_{k}^{* 2}}{\beta_{k} \beta_{k}^{* 2}}\right] .
$$

We see that

$$
\mathbb{E}\left[\frac{\varepsilon_{k}^{*}}{\beta_{k}^{* 2}}\right]=\mathbb{E}\left[\frac{1}{\beta_{k}^{* 2}} \mathbb{E}_{k}\left[\varepsilon_{k}^{*}\right]\right]=0
$$

by (3.76). Furthermore, by (3.70), (3.71) and (3.80),

$$
\left|\mathbb{E}\left[\frac{\varepsilon_{k}^{* 2}}{\beta_{k} \beta_{k}^{* 2}}\right]\right| \leq \frac{K+\sigma^{2}}{n v^{3}}
$$

Combining this with (3.86), we see from (3.83), (3.84) and (3.85) that

$$
\begin{equation*}
|\delta| \leq \frac{K+2 \sigma^{2}}{n v^{3}}+\frac{\sigma^{2} \sqrt{2 K+\sigma^{2}}}{n v^{4}} \leq \frac{C_{0}}{n v^{4}} \tag{3.87}
\end{equation*}
$$

if $v \leq 1$.
Solving the equation (3.82) for $\mathbb{E}\left[m_{n}(z)\right]$ we obtain

$$
\begin{equation*}
\mathbb{E}\left[m_{n}(z)\right]=\frac{1}{2 \sigma^{2}}\left(-z+\sigma^{2} \delta+\sqrt{\left(z+\sigma^{2} \delta\right)^{2}-4 \sigma^{2}}\right) \tag{3.88}
\end{equation*}
$$

Since $\operatorname{Im} \mathbb{E}\left[m_{n}(z)\right] \geq 0$ we have to choose the square root with positive imaginary part. Let

$$
m(z)=\frac{1}{2 \sigma^{2}}\left(-z+\sqrt{z^{2}-4 \sigma^{2}}\right)=\frac{1}{2 \sigma^{2} \pi} \int_{-2 \sigma}^{2 \sigma} \frac{\sqrt{4 \sigma^{2}-x^{2}}}{x-z} \mathrm{~d} x .
$$

Then,

$$
\begin{equation*}
\left|\mathbb{E}\left[m_{n}(z)\right]-m(z)\right| \leq \frac{|\delta|}{2}+\frac{1}{2 \sigma^{2}}\left|\sqrt{z^{2}-4 \sigma^{2}}-\sqrt{z^{2}-4 \sigma^{2}}\right| \tag{3.89}
\end{equation*}
$$

Note that, by (3.87) and the assumption that $v \geq\left(n r_{n}\right)^{-1 / 5}$, it follows that $|\delta / v| \leq C_{0} r_{n}$. For $n$ sufficiently large, $C_{0} r_{n} \leq 1 / 3$ and hence $|\delta / v| \leq 1 / 3$. We now take $\sigma^{2}=1 / 4$. It follows from (3.88) and $\operatorname{Im} \mathbb{E}\left[m_{n}(z)\right] \geq 0$ that $\operatorname{Im} \sqrt{(z+\delta / 4)^{2}-1} \geq v$, and similarly we find $\operatorname{Im} \sqrt{z^{2}-1} \geq v$. Thus,

$$
\begin{align*}
\left|\sqrt{(z+\delta / 4)^{2}-1}-\sqrt{z^{2}-1}\right| & =\frac{\left|2 \delta z+\delta^{2}\right|}{\left|\sqrt{(z+\delta / 4)^{2}-1}+\sqrt{z^{2}-1}\right|} \\
& \leq \frac{\delta}{v}(|z|+\delta v) \leq \frac{\delta}{v}(|z|+1) . \tag{3.90}
\end{align*}
$$

If $|z| \leq 4$, (3.87), (3.89) and (3.90) gives (3.41). If $|z| \geq 4$, then (3.90) gives

$$
\left|\sqrt{(z+\delta / 4)^{2}-1}-\sqrt{z^{2}-1}\right| \leq \frac{1}{2}|z|
$$

and since $\left|\sqrt{z^{2}-1}\right| \geq \frac{1}{2}|z|$, we obtain

$$
\left|\sqrt{(z+\delta / 4)^{2}-1}+\sqrt{z^{2}-1}\right| \geq \frac{1}{2}|z|
$$

Thus, (3.90) gives

$$
\left|\sqrt{(z+\delta / 4)^{2}-1}-\sqrt{z^{2}-1}\right| \leq \frac{2|\delta||z|+\delta^{2}}{|z| / 2} \leq 5 \frac{\delta}{v}
$$

and again we get (3.41).

Finally, we prove Lemma 2.4.
Proof of Lemma 2.4. Let $\operatorname{det}_{2}(I-A)$ be the regularized determinant defined for Hilbert-Schmidt operators, see e.g. [19]. If $A$ is a trace-class operator, then

$$
\begin{equation*}
\operatorname{det}(I-A)=\operatorname{det}_{2}(I-A) \mathrm{e}^{-\operatorname{tr} A}, \tag{3.91}
\end{equation*}
$$

where the left-hand side is the Fredholm determinant. Now, see e.g. [19], Chapter 9, for two Hilbert-Schmidt operators $A$ and $B$,

$$
\begin{align*}
\left|\operatorname{det}_{2}(I-A)-\operatorname{det}_{2}(I-B)\right| & \leq\|A-B\|_{2} \mathrm{e}^{(1 / 2)\left(\|A\|_{2}+\|B\|_{2}+1\right)^{2}} \\
& \leq\|A-B\|_{2} \mathrm{e}^{(1 / 2)\left(\|A-B\|_{2}+2\|B\|_{2}+1\right)^{2}} \tag{3.92}
\end{align*}
$$

and

$$
\begin{equation*}
\left|\operatorname{det}_{2}(I-A)\right| \leq \mathrm{e}^{(1 / 2)\|A\|_{2}+1} . \tag{3.93}
\end{equation*}
$$

Using (3.91) we can write

$$
\begin{aligned}
\operatorname{det}_{2}(I-A)-\operatorname{det}_{2}(I-B)= & \left(\operatorname{det}_{2}(I-A)-\operatorname{det}_{2}(I-B)\right) \mathrm{e}^{-\operatorname{tr} A} \\
& +\operatorname{det}_{2}(I-B) e^{-\operatorname{tr} B}\left(\mathrm{e}^{-(\operatorname{tr} A-\operatorname{tr} B)}-1\right)
\end{aligned}
$$

and the inequality (2.31) follows from (3.92) and (3.93).

## Acknowledgement

I would like to thank the referee for careful reading and useful suggestions.

## References

[1] A. Auffinger, G. Ben Arous and S. Péché. Poisson convergence for the largest eigenvalues of heavy-taled matrices. Ann. Inst. Henri Poincaré Probab. Stat. 45 (2009) 589-610. MR2548495
[2] Z. D. Bai. Convergence rate of expected spectral distributions of large random matrices. Part I. Wigner matrices. Ann. Probab. 21 (1993) 625-648. MR1217559
[3] Z. D. Bai. Methodologies in spectral analysis of large dimensional random matrices, a review. Statist. Sinica 9 (1999) 611-677. MR1711663
[4] Z. D. Bai, B. Miao and J. Tsay. Convergence rates of the spectral distribution of large Wigner matrices. Int. Math. J. 1 (2002) 65-90. MR1825933
[5] Z. D. Bai and J. W. Silverstein. Spectral Analysis of Large Dimensional Random Matrices, 2nd edition. Springer, New York, 2010. MR2567175
[6] G. Ben Arous and A. Guionnet. The spectrum of heavy-tailed random matrices. Comm. Math. Phys. 278 (2008) 715-751. MR2373441
[7] G. Ben Arous and S. Péché. Universality of local eigenvalue statistics for some sample covariance matrices. Comm. Pure Appl. Math. 58 (2005) 1316-1357. MR2162782
[8] G. Biroli, J.-P. Bouchaud and M. Potters. On the top eigenvalue of heavy-tailed random matrices. Europhys. Lett. 78 (2007) 10001. MR2371333
[9] E. Brézin and S. Hikami. Spectral form factor in a random matrix theory. Phys. Rev. E 55 (1997) 4067-4083. MR1449379
[10] P. Cizeau and J.-P. Bouchaud. Theory of Lévy matrices. Phys. Rev. E 50 (1994) 1810-1822.
[11] L. Erdös, S. Péché, J. Ramirez, B. Schlein and H.-T. Yau. Bulk universality for Wigner matrices. Comm. Pure Appl. Math. 63 (2010) $895-925$. MR2662426
[12] L. Erdös, J. Ramirez, B. Schlein, T. Tao, V. Vu and H.-T. Yau. Bulk universality for Wigner Hermitian matrices with subexponential decay. Math. Res. Lett. 17 (2010) 667-674. MR2661171
[13] A. Guionnet and O. Zeitouni. Concentration of the spectral measure for large matrices. Electron. Commun. Probab. 5 (2000) 119-136 (electronic). MR1781846
[14] K. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. Comm. Math. Phys. 215 (2001) 683-705. MR1810949
[15] K. Johansson. Discrete polynuclear growth and determinantal processes. Comm. Math. Phys. 242 (2003) 277-329. MR2018275
[16] A. Ruzmaikina. Universality of the edge distribution of the eigenvalues of Wigner random matrices with polynomially decaying distributions of entries. Comm. Math. Phys. 261 (2006) 277-296. MR2191882
[17] T. Shcherbina. On universality of bulk local regime of the deformed Gaussian Unitary Ensemble. J. Math. Phys. Anal. Geom. 5 (2009) 396-433. MR2590774
[18] T. Shcherbina. On universality of local edge regime for the deformed Gaussian Uniraty Ensemble. J. Stat. Phys. 143 (2011) $455-481$.
[19] B. Simon. Trace Ideals and Their Applications, 2nd edition. Math. Surveys Monogr. 120. Amer. Math. Soc., Providence, RI, 2005. MR2154153
[20] A. Soshnikov. Universality at the edge of the spectrum in Wigner random matrices. Comm. Math. Phys. 207 (1999) 697-733. MR1727234
[21] A. Soshnikov. Poisson statistics for the largest eigenvaluet of Wigner matrices with heavy tails. Electron. Commun. Probab. 9 (2004) 82-91. MR2081462
[22] T. Tao and V. Vu. Random matrices: Universality of local eigenvalue statistics up to the edge. Comm. Math. Phys. 298 (2010) 549-572. MR2669449
[23] T. Tao and V. Vu. Random matrices: Universality of local eigenvalue statistics. Acta Math. 206 (2011) 127-204. MR2784665
[24] T. Tao and V. Vu. Random covariance matrices: Universality of local statistics of eigenvalues. Available at arXiv:0912.0966.


[^0]:    ${ }^{1}$ Suported by the Swedish Research Council (VR) and the Knut and Alice Wallenberg Foundation KAW2010.0063.

[^1]:    ${ }^{2}$ Very recently [24] the assumption on the distribution has been reduced to a finite but large number of moments.

