Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2012, Vol. 48, No. 3, 854–870
DOI: 10.1214/11-AIHP430
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ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Affine Dunkl processes of type $\widetilde{\mathrm{A}}_1$

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Received 27 October 2010; revised 8 March 2011; accepted 26 April 2011

Abstract. We introduce the analogue of Dunkl processes in the case of an affine root system of type $\widetilde{\mathrm{A}}_1$. The construction of the affine Dunkl process is achieved by a skew-product decomposition by means of its radial part and a jump process on the affine Weyl group, where the radial part of the affine Dunkl process is given by a Gaussian process on the ultraspherical hypergroup [0, 1]. We prove that the affine Dunkl process is a càdlàg Markov process as well as a local martingale, study its jumps, and give a martingale decomposition, which are properties similar to those of the classical Dunkl process.

Résumé. Nous introduisons l'analogue des processus de Dunkl dans le cas d'un système de racines affines de type $\widetilde{\mathrm{A}}_1$. La construction du processus de Dunkl affine est obtenue par une décomposition en skew-product de sa partie radiale et d'un processus de sauts sur le groupe de Weyl affine, la partie radiale du processus de Dunkl affine étant définie par un processus gaussien sur l'hypergroupe ultrasphérique [0, 1]. Nous montrons que le processus de Dunkl affine est un processus de Markov càdlàg ainsi qu'une martingale locale, étudions ses sauts, et donnons sa décomposition en martingale, propriétés analogues à celles du processus de Dunkl classique.

MSC: Primary 60J75; secondary 60J60; 60B15; 33C52

Keywords: Dunkl processes; Diffusion processes; Orthogonal polynomials; Skew-product decomposition; Affine root system; Weyl group

1. Introduction

The aim of the following is to study the analogue of Dunkl processes in the case of an affine root system of type $\widetilde{\mathrm{A}}_1$. The affine Dunkl process $Y = (Y_t)_{t\geq 0}$ with parameter k will be defined as the Markov process in $\mathbb{R}$ with infinitesimal generator given by

$$\mathcal{A}u(x) = \frac{1}{2}u''(x) + k\pi\cot(\pi x)u'(x) + \frac{k}{2}\sum_{p\in\mathbb{Z}}\frac{u(s_p(x)) - u(x)}{(x-p)^2},$$

acting on $u \in C_b^2(\mathbb{R})$ and $x \in \mathbb{R}\setminus\mathbb{Z}$, where k is a real number satisfying $k \geq \frac{1}{2}$ and s_p is the affine reflection around $p \in \mathbb{Z}$, i.e. $s_p(x) = -x + 2p$. It is a Markov process with jumps, whose radial part is the continuous Feller process living in the interval $]0, 1[$ and with infinitesimal generator

$$\mathcal{L}u(x) = \frac{1}{2}u''(x) + k\pi\cot(\pi x)u'(x)$$

for $u \in C([0, 1]) \cap C^2(]0, 1[)$, and which can be seen as a Gaussian process on the compact hypergroup [0, 1] associated with ultraspherical polynomials. The main idea to achieve the construction of the process Y with generator $\mathcal{A}$ is to consider a skew-product decomposition, by starting from its radial part and adding the jumps successively at

random times, the jump part of the process being given by some process on the affine Weyl group associated with the affine root system $\widetilde{\mathrm{A}}_1$.

In the rest of this introduction, we briefly present the classical Dunkl processes, and introduce the affine root system $\widetilde{\mathrm{A}}_1$. We also give some heuristics and motivations for the study of the affine case.

1.1. *Dunkl processes*

Recall that the Dunkl processes are a family of càdlàg Feller processes associated with a root system (see [18] for the notion of root systems). From an analytic point of view, the theory was initiated by Dunkl who studied differential-difference operators with some parameter k, associated with a root system [11]. They are at the basis of a rich analytic structure related to them and they are connected to the theory of Riemmanian symmetric space of Euclidean type when k takes only certain values. The probabilistic counterpart, the Dunkl processes, was originated by Rösler and Voit in [26], and then extensively studied by Gallardo and Yor in [14–16], Chybiryakov [7] and Demni [10]. We refer to the book [8] for a good survey of Dunkl operators and Dunkl processes.

In the one-dimensional case, Dunkl processes are a family of càdlàg Feller processes with parameter a nonnegative real number k, and with infinitesimal generator the Dunkl Laplacian given by

$$\mathcal{L}^0 u(x) = \frac{1}{2}u''(x) + k\frac{u'(x)}{x} + k\frac{u(-x) - u(x)}{2x^2}, \tag{1}$$

acting on continuous twice differentiable functions. They correspond to the rank one root system A_1, which is simply given by $\mathcal{R}^0 = \{\pm 1\} \subset \mathbb{R}$. Letting $\mathcal{L}^0$ acting on even function, that is functions invariant by the Weyl group $W^0 = \{Id, \sigma\}$, where σ is the orthogonal reflection with respect to zero, i.e. $\sigma(x) = -x$, we obtain the generator of the Bessel process of index $2k + 1$, given by

$$\mathcal{L}^{0,W^0} u(x) = \frac{1}{2}u''(x) + k\frac{u'(x)}{x}.$$

The radial part of the Dunkl process is thus the Bessel process, and corresponds to the projection of the Dunkl process onto the principal Weyl chamber, which is in the A_1-case the positive real line $\mathbb{R}_+^*$. Note also that the Dunkl process jumps at some random times by orthogonal reflection with respect to zero. Moreover, Dunkl processes, already in the one-dimensional case, satisfy a lot of interesting properties, for instance they are martingales, and we refer to [8] for more details on the theory. We also want to mention that the counterpart of Dunkl processes in the negatively curved setting, which are called Heckman–Opdam processes, is investigated by Schapira in [27].

1.2. *The affine root system* $\widetilde{\mathrm{A}}_1$

We now want to consider not only orthogonal reflections with respect to zero, but also affine reflections relative to integers $p \in \mathbb{Z}$. To this end, we introduce in the following the affine root system of type $\widetilde{\mathrm{A}}_1$. We are just dealing with the rank one case, so we refer to [18] for more general facts on root systems and Weyl groups.

Let $\mathcal{R}^0 = \{\pm 1\} \subset \mathbb{R}$ be the only root system of rank one, denoted A_1. We defined the *affine root system* $\widetilde{\mathrm{A}}_1$ as the product $\mathcal{R} = \mathcal{R}^0 \times \mathbb{Z}$. The *affine refection* associated with $p \in \mathbb{Z}$ is defined by

$$s_p(x) := -x + 2p$$

for $x \in \mathbb{R}$, and the positive affine root system by

$$\mathcal{R}_+ = \{+1\} \cup \big\{(\pm 1, p) | p \le -1\big\}. \tag{2}$$

The *affine Weyl group* W is the infinite group generated by affine reflections s_p, $p \in \mathbb{Z}$, and is isomorphic to the infinite dihedral group. Each connected component of $\mathbb{R} \setminus \mathbb{Z}$ is called an *alcove*, and we single out the particular alcove $]0, 1[$, called the *principal alcove*. Note that, up to some identification of the walls (i.e. the boundary points), the closure of the principal alcove is a fundamental domain for the action of W on $\mathbb{R}$.

1.3. *Heuristics and motivations*

Since the link between the operator $\mathcal{A}$ defined in the Introduction and the affine root system $\widetilde{\mathrm{A}}_1$ is not so obvious, we give in the following some little heuristics, without being rigorous. We have seen that in the rank one case, the Dunkl process is the Markov process with infinitesimal generator $\mathcal{L}^0$ given by (1), and with parameter k a nonnegative real number. The idea here is then to replace the positive root system associated with $\mathcal{L}^0$ by the positive affine root system $\mathcal{R}_+$ given by (2), and hence to define the affine Dunkl Laplacian in the $\widetilde{\mathrm{A}}_1$ case by

$$\begin{aligned}\mathcal{A}u(x) &= \frac{1}{2}u''(x) + k\frac{u'(x)}{x} + k\frac{u(-x)-u(x)}{2x^2}\\ &\quad + k\sum_{p\le -1}\left\{\frac{u'(x)}{x-p} + \frac{u'(x)}{x+p} + \frac{u(-x+2p)-u(x)}{2(x-p)^2} + \frac{u(-x-2p)-u(x)}{2(x+p)^2}\right\}.\end{aligned}$$

Recall the series expansion of the cotangent function

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n\ge 1}\left(\frac{1}{x+n} + \frac{1}{x-n}\right)$$

for $x \in \mathbb{R}\setminus\mathbb{Z}$, see [1], which can be written more elegantly

$$\pi\cot(\pi x) = \sum_{n\in\mathbb{Z}}\frac{1}{x-n}.$$

Note that the latter formula is quite dangerous since it is not absolutely convergent, and we have to be cautious with the summation order. Hence, using the cotangent expansion, the affine Dunkl Laplacian writes

$$\mathcal{A}u(x) = \frac{1}{2}u''(x) + k\pi\cot(\pi x)u'(x) + \frac{k}{2}\sum_{p\in\mathbb{Z}}\frac{u(s_p(x))-u(x)}{(x-p)^2},$$

where s_p is the affine reflection associated with $p \in \mathbb{Z}$.

Let us now make some remarks on the operator $\mathcal{A}$. First we note that, using the periodicity of the cotangent function, a direct computation shows that for all $u \in \mathrm{Dom}(\mathcal{A})$,

$$\mathcal{A}(u\circ s_p)(x) = (\mathcal{A}u)(s_p x)$$

for all affine reflexion s_p, $p \in \mathbb{Z}$ and $x \in \mathbb{R}$. This implies that $\mathcal{A}$ becomes invariant with respect to the action of the affine Weyl group W on $\mathbb{R}$, that is $w\mathcal{A}w^{-1} = \mathcal{A}$, for all $w \in W$. Hence, letting $\mathcal{A}$ acting on W-invariant function, that is functions u such that $u(w\cdot x) = u(x)$ for all $w \in W$, where $\cdot$ denotes the action of W on $\mathbb{R}$, the difference part vanishes, and we recover the expression of the operator $\mathcal{L}$ given by

$$\mathcal{L}u(x) = \frac{1}{2}u''(x) + k\pi\cot(\pi x)u'(x).$$

Note that we can identify W-invariant functions with functions on the principal alcove $]0,1[$. As we will see in the next section, the operator $\mathcal{L}$ corresponding to the radial part of $\mathcal{A}$ is the infinitesimal generator of a diffusion process on $[0,1]$ given by the well-known theory of ultraspherical polynomials as studied by Bochner [6], and is moreover a Gaussian process on the ultraspherical hypergroup $[0,1]$. For k a half-integer, it has also an interesting geometrical interpretation in terms of the radial part of the Laplace–Beltrami operator on the sphere and compact homogeneous spaces. There is a rich literature about Gaussian processes on hypergroup structures, and we refer to [5] for standard facts on hypergroups.

The following is divided in two parts. In the first one, corresponding to Section 2, we introduce the radial affine Dunkl process as the diffusion on the hypergroup $[0,1]$ associated with ultraspherical polynomials. Some functional of the radial process is also studied. The second part, which is Section 3, is devoted to the construction of the affine Dunkl process with generator $\mathcal{A}$, using a skew-product decomposition by means of the radial process and a pure jump process on the affine Weyl group, inspired by [7]. We study its jumps and also give a martingale decomposition.

2. The radial affine Dunkl process

2.1. *Definition of the radial process*

In what follows, the parameter k of the affine Dunkl process is a nonnegative number satisfying $k \geq \frac{1}{2}$. We start by studying the radial part of the affine Dunkl process, which is the following diffusion process.

Definition 2.1. *The continuous Feller process* $(X_t)_{t\geq 0}$ *in* $[0,1]$, *with infinitesimal generator* $\mathcal{L}$ *given by*

$$\mathcal{L}u(x) = \frac{1}{2}u''(x) + k\pi\cot(\pi x)u'(x),$$

acting on $u \in C([0,1]) \cap C^2(]0,1[)$, *and* $X_0 \in]0,1[$ *a.s., is called the radial affine Dunkl process with parameter* k.

This process first appears in Bochner [6] (see also [20]), who studied the heat equation

$$\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,x) + k\pi\cot(\pi x)\frac{\partial g}{\partial x}(t,x) = \frac{\partial g}{\partial t}(t,x), \tag{3}$$

more exactly the same equation up to the transformation $x \mapsto \cos(\pi x)$, and shown that the Green kernel associated with this equation can be expressed in terms of ultraspherical polynomials, and is the transition kernel of a diffusion with generator $\mathcal{L}$. The generalization of these results involving Jacobi polynomials are also known, see [17], and the associated process is the so-called Jacobi process.

So first, let us recall some standard facts about ultraspherical polynomials (also known as Gegenbauer polynomials), which can be found in [22] or [29] for example. Ultraspherical polynomials $G_n^{(k)}$ (which can be expressed in terms of Jacobi polynomials $P_n^{(k-1/2,k-1/2)}$) are polynomials orthogonal for the weight $(1-x^2)^{k-1/2}\mathbb{1}_{[-1,1]}(x)\,\mathrm{d}x$, i.e.

$$\int_{[-1,1]} G_n^{(k)}(x)G_m^{(k)}(x)\left(1-x^2\right)^{k-1/2}\mathrm{d}x = \pi\left(\omega_n^{(k)}\right)^{-1}\delta_{n,m}$$

for $k > -\frac{1}{2}$, where

$$\omega_n^{(k)} = \frac{n!(k+n)\Gamma(k)^2}{2^{1-2k}\Gamma(n+2k)}.$$

The first polynomials are (for $k \neq 0$), $G_0^{(k)}(y) = 1$, $G_1^{(k)}(y) = 2ky, \ldots$. They are of the same parity than n, and $G_n^{(k)}(-y) = (-1)^n G_n^{(k)}(y)$. The value at 1 is $G_n^{(k)}(1) = \frac{\Gamma(2k+n)}{n!\Gamma(2k)}$. For $k > 0$, they admit an explicit expression, given by

$$G_n^{(k)}(y) = \frac{1}{\Gamma(k)}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{\Gamma(k+n-m)}{m!(n-2m)!}(2y)^{n-2m}. \tag{4}$$

Furthermore, for all $n \geq 0$, $G_n^{(k)}$ is (up to the normalization $G_n^{(k)}(1) = \frac{\Gamma(2k+n)}{n!\Gamma(2k)}$) the unique polynomial solution of the equation

$$\left(1-x^2\right)f''(x) - (2k+1)xf'(x) + n(n+2k)f(x) = 0$$

and so $x \mapsto G_n^{(k)}(\cos\pi x)$ is solution of

$$\frac{1}{2}g''(x) + k\pi\cot(\pi x)g'(x) = -\lambda_n g(x),$$

with eigenvalue $\lambda_n = \frac{\pi^2}{2} n(n+2k)$.

Bochner shown in [6] that the Green kernel of the heat equation (3) is given, for $k \geq 0$, by

$$q_t(x, y) = \sum_{n \geq 0} \mathrm{e}^{-\lambda_n t} G_n^{(k)}(\cos \pi x) G_n^{(k)}(\cos \pi y) \omega_n^{(k)} (\sin \pi y)^{2k} \tag{5}$$

for $x, y \in [0, 1] \times [0, 1]$. Furthermore, he proved that q_t is a Markov kernel and gives rise to a diffusion on $[0, 1]$ with infinitesimal generator $\mathcal{L}$, hence the radial affine Dunkl process $(X_t)_{t \geq 0}$ is the Feller process with transition probability given by

$$\mathbb{P}(X_{t+s} \in \mathrm{d}y | X_s = x) = q_t(x, \mathrm{d}y).$$

Another way to introduce the radial process is to say that $(X_t)_{t \geq 0}$ is the unique strong solution of the stochastic differential equation

$$\mathrm{d}X_t = \mathrm{d}B_t + k\pi \cot(\pi X_t)\, \mathrm{d}t,$$

with initial condition $X_0 = x \in]0, 1[$ a.s., and where $(B_t)_{t \geq 0}$ is a standard Brownian motion. One can prove, using for instance the standard scale function technique as in [19] or [21], that if $k \geq \frac{1}{2}$, $(X_t)_{t \geq 0}$ lives almost surely in $]0, 1[$, and the boundaries 0 and 1 are entrance ones, that is X can be started at a boundary point and moves quickly to the interior of the interval and never returns to the boundary. The same arguments show that if $k < \frac{1}{2}$, X can a.s. reach the boundary in finite time.

Remark 2.2. *By the periodicity of the cotangent function, we can define the radial affine Dunkl process in any alcove $I \subset \mathbb{R} \setminus \mathbb{Z}$.*

Let us mention that for $k = 1$, the radial affine Dunkl process is the Brownian motion conditioned to stay in the interval $]0, 1[$, also known as the Legendre process (see [25]). Indeed, in that case, the generator of the process writes

$$\mathcal{L}u(x) = \frac{1}{2} u''(x) + \frac{h'(x)}{h(x)} u'(x),$$

where $h(x) = \sin(\pi x)$ is an eigenfunction for the Laplacian Δ, and hence the corresponding process X is a Doob h-transform at the bottom of the spectrum of Brownian motion killed when it reached the walls of $]0, 1[$. Let us also mention that Brownian motions in alcoves are related to the process of eigenvalues of the Brownian motion with values in the special unitary group $\mathrm{SU}(n)$, see [4]. The last two remarks are analogues of the same kind of properties for the radial Dunkl process in the classical case, see [8]. We also mention that Dunkl processes associated with dihedral root systems as in [9] are related to radial affine Dunkl processes.

2.2. *Gaussian process on compact hypergroup structures*

Before dealing with Gaussian processes on hypergroup, we first note that for half-integers $k = \frac{d-1}{2}$, with $d \geq 2$, the radial affine Dunkl processes are projections onto a diameter of the spherical Brownian motion on the sphere $\mathbb{S}^d$, since the operator $\mathcal{L}$ is the radial part of the Laplace–Beltrami operator on $\mathbb{S}^d$ [3].

Actually the radial affine Dunkl process $(X_t)_{t \geq 0}$ can be seen as a Gaussian process on the compact hypergroup $[0, 1]$ associated with ultraspherical polynomials. We will not go in much details here and we refer to the monograph of Bloom and Heyer [5] for all facts on hypergroups. For all $x, y \in [0, 1]$, there exists a unique probability measure $\delta_x * \delta_y$ on $[0, 1]$ such that

$$G_n^{(k)}(\cos \pi x) G_n^{(k)}(\cos \pi y) = \int_0^1 G_n^{(k)}(\cos \pi z)(\delta_x * \delta_y)(\mathrm{d}z).$$

The convolution $\delta_x * \delta_y$ can be extended uniquely to a bilinear, commutative, associative and weakly continuous convolution $*$ on the Banach space $M_b([0, 1])$ of all bounded Borel measures on $[0, 1]$, and moreover defines a commutative hypergroup structure on $K = [0, 1]$. We refer to [5] for the precise definition of the notion of hypergroups,

and also for the explicit representation of this convolution which is in fact given by Gegenbauer's product formula. The normalized Haar measure ν on K is given by the normalized orthogonality measure of ultraspherical polynomials, i.e.

$$\nu(\mathrm{d}x) = \frac{\Gamma(k+1)\sqrt{\pi}}{\Gamma(k+1/2)}(\sin \pi x)^{2k} \mathbb{1}_{[0,1]}(x)\,\mathrm{d}x.$$

The dual space $\hat{K}$ of characters can be identified with $\mathbb{N}$, the Plancherel measure π on $\hat{K}$ being given by

$$\pi\big(\{n\}\big) = \frac{(n+k)\Gamma(2k+n)}{n!k\Gamma(2k)}, \quad n \geq 0.$$

The hypergroup Fourier transform of a measure $\mu \in M_b(K)$ is given by

$$\hat{\mu}(n) = \int_0^1 G_n^{(k)}(\cos \pi x) \frac{1}{G_n^{(k)}(1)} \mu(\mathrm{d}x), \quad n \geq 0.$$

Now define the convolution semigroup (in the sense of the hypergroup structure) $(\mu_t)_{t\geq 0}$ on K by

$$\mu_t(\mathrm{d}y) := q_t(0, y)\,\mathrm{d}y = \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t} G_n^{(k)}(1) G_n^{(k)}(\cos \pi y) \omega_n^{(k)} (\sin \pi y)^{2k}\,\mathrm{d}y.$$

Then, since $q_t(x, \mathrm{d}y) = (\delta_x * \mu_t)(\mathrm{d}y)$, the radial affine Dunkl process $(X_t)_{t\geq 0}$ is a Markov process on the hypergroup K, that is, for all $s, t \geq 0$, $x \in [0, 1]$, and Borel set A in $[0, 1]$,

$$\mathbb{P}(X_{t+s} \in A | X_s = x) = (\delta_x * \mu_t)(A).$$

Actually, $(\mu_t)_{t\geq 0}$ is known as the Gaussian semigroup on the hypergroup K, see [24] or [31] for instance, and has Lévy–Khinchin representation given by

$$\hat{\mu}_t(n) = \mathrm{e}^{-t\lambda_n},$$

which follows from the definition of the hypergroup Fourier transform and the orthogonal relations of ultraspherical polynomials. Furthermore, using the characterization of recurrence/transience property of Lévy processes on hypergroups, see [5], Chapter 6, it is easily seen that the radial affine Dunkl process $(X_t)_{t\geq 0}$ is recurrent on $]0, 1[$.

For half-integers $k = \frac{d-1}{2}$, with $d \geq 2$, the geometrical interpretation stated in the beginning of this subsection is recovered in the following way. The sphere $\mathbb{S}^d$ can be identified with the homogeneous space $\mathrm{SO}(d+1)/\mathrm{SO}(d)$ (endowed with the quotient topology), and the hypergroup K is isomorphic to the double coset space $\mathrm{SO}(d+1)//\mathrm{SO}(d) = \{\mathrm{SO}(d) g\, \mathrm{SO}(d) | g \in \mathrm{SO}(d+1)\}$. The radial affine Dunkl process is then the image of the spherical Brownian motion on $\mathbb{S}^d$ by the canonical projection $\mathrm{SO}(d+1)/\mathrm{SO}(d) \to \mathrm{SO}(d+1)//\mathrm{SO}(d)$, see [30] or [31] for more details.

2.3. *Some properties of the radial process*

From now on, we will denote by $\mathbb{P}_x$ the distribution of the radial affine Dunkl process starting from $x \in \mathbb{R} \setminus \mathbb{Z}$.

As we will see, an important functional of the radial affine Dunkl process, is the continuous process $\frac{1}{\sin^2(\pi X_\cdot)}$. First, note that since X is continuous and never reaches (for $k \geq \frac{1}{2}$) the walls of the alcove where is started from, we have that for all $t \geq 0$, there exists some random $\varepsilon_t > 0$ such that

$$\inf_{s\in[0,t]} \sin^2(\pi X_s) > \varepsilon_t \quad \mathbb{P}_x\text{-a.s.}$$

Hence, we obtain that for all $t \geq 0$,

$$\int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)} < +\infty \quad \mathbb{P}_x\text{-a.s.}$$

Note also that, since $\sin^2(\pi X_s) < 1$ for all $s \geq 0$, we have that $\int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)} \geq t$, so

$$\int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)} \underset{t\to+\infty}{\longrightarrow} +\infty \quad \mathbb{P}_x\text{-a.s.}$$

To study some properties of the radial affine Dunkl process and more particularly of the last functional, we will need a few lemmas.

Lemma 2.3. *Let $k > \frac{1}{2}$. For all $n \geq 0$, we have*

$$\int_{-1}^1 G_n^{(k)}(x)\big(1-x^2\big)^{k-3/2}\,\mathrm{d}x = \frac{\Gamma(k-1/2)\sqrt{\pi}}{\Gamma(k)} \quad \textit{if } n \textit{ is even}$$

and 0 *if n is odd.*

Proof. Since $G_n^{(k)}$ is odd for n odd, we just have to look at the even case. Recall the ultraspherical polynomials expansion of $G_n^{(k)}$, see for instance Askey [2], that is

$$G_n^{(k)}(x) = \sum_{j=0}^{\lfloor n/2\rfloor} g_{n,j}^{k,l} G_{n-2j}^{(l)}(x),$$

with connection coefficients given by

$$g_{n,j}^{k,l} = \frac{\Gamma(l)(n-2j+l)\Gamma(j+k-l)\Gamma(n-j+k)}{\Gamma(k)j!\Gamma(k-l)\Gamma(n-j+l+1)}.$$

Hence, for $l = k-1$ (note that this is well defined since $l > -\frac{1}{2}$), and using the fact that the $G_n^{(k-1)}$'s are orthogonal for the weight $(1-x^2)^{k-3/2}$, we find that, for n even,

$$\begin{aligned}\int_{-1}^1 G_n^{(k)}(x)\big(1-x^2\big)^{k-3/2}\,\mathrm{d}x &= \sum_{j=0}^{n/2} g_{n,j}^{k,k-1} \int_{-1}^1 G_{n-2j}^{(k-1)}(x)\big(1-x^2\big)^{k-3/2}\,\mathrm{d}x \\ &= g_{n,n/2}^{k,k-1}\pi\big(\omega_0^{(k-1)}\big)^{-1}.\end{aligned}$$

Since

$$g_{n,n/2}^{k,k-1} = 1$$

and

$$\big(\omega_0^{(k-1)}\big)^{-1} = \frac{\Gamma(2k-2)2^{3-2k}}{\Gamma(k-1)^2(k-1)} = \frac{\Gamma(k-1/2)}{\sqrt{\pi}\Gamma(k)}$$

by the duplication formula of the Gamma function, i.e.

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

we obtain

$$\int_{-1}^1 G_n^{(k)}(x)\big(1-x^2\big)^{k-3/2}\,\mathrm{d}x = \frac{\Gamma(k-1/2)\sqrt{\pi}}{\Gamma(k)}.$$

□

First, we make the following remark.

Lemma 2.4. *Let* $(X_t)_{t\geq 0}$ *be the radial affine Dunkl process, and* $k > \frac{1}{2}$. *Then, for all* $x \in]0,1[$, *and all* $t \geq 0$, *we have*

$$\mathbb{E}_x\left(\frac{1}{\sin^2(\pi X_t)}\right) < +\infty.$$

Proof. We have,

$$\begin{aligned}\mathbb{E}_x\left(\frac{1}{\sin^2(\pi X_t)}\right) &= \int_{[0,1]} \frac{1}{\sin^2(\pi y)} q_t(x,y)\,\mathrm{d}y \\ &= \int_{[0,1]} \sum_{n\geq 0} \mathrm{e}^{-\lambda_n t} G_n^{(k)}(\cos \pi x) G_n^{(k)}(\cos \pi y) \omega_n^{(k)} (\sin \pi y)^{2k-2}\,\mathrm{d}y,\end{aligned}$$

which is integrable as soon as $2k-2 > -1$, i.e. $k > \frac{1}{2}$, the summability of the series being guaranteed by the term $\mathrm{e}^{-\lambda_n t}$. □

Note that the proof of this lemma gives that for $k = \frac{1}{2}$, $\mathbb{E}_x(\frac{1}{\sin^2(\pi X_t)}) = +\infty$ for all $t > 0$. Actually, Lemma 2.3 leads to the following proposition.

Proposition 2.5. *Let* $k > \frac{1}{2}$. *For all* $x \in]0,1[$, *and all* $t \geq 0$, *we have*

$$\mathbb{E}_x\left(\int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)}\right) < +\infty.$$

Proof. For all $t > 0$, we have,

$$\begin{aligned}&\int_0^t \mathbb{E}_x\left(\frac{\mathrm{d}s}{\sin^2(\pi X_s)}\right) \\ &= \int_0^t \int_0^1 \frac{1}{\sin^2(\pi y)} q_s(x,y)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \int_0^1 \sum_{n\geq 0} \mathrm{e}^{-\lambda_n s} G_n^{(k)}(\cos \pi x) G_n^{(k)}(\cos \pi y) \omega_n^{(k)} (\sin \pi y)^{2k-2}\,\mathrm{d}y\,\mathrm{d}s \\ &= \sum_{n\geq 0} \frac{1}{\lambda_n}\big(1-\mathrm{e}^{-\lambda_n t}\big) G_n^{(k)}(\cos \pi x) \omega_n^{(k)} \int_0^1 G_n^{(k)}(\cos \pi y)(\sin \pi y)^{2k-2}\,\mathrm{d}y.\end{aligned}$$

First, remark that the term $n = 0$ is not a problem since $G_0^{(k)}(y) = 1$ and $\omega_0^{(k)} = \frac{k\Gamma(k)^2}{2^{1-2k}\Gamma(2k)}$. Now, by the change of variables $u = \cos(\pi y)$, and Lemma 2.3, we have

$$\int_0^1 G_n^{(k)}(\cos \pi y)(\sin \pi y)^{2k-2}\,\mathrm{d}y = \frac{1}{\pi}\int_{-1}^1 G_n^{(k)}(u)\big(1-u^2\big)^{k-3/2}\,\mathrm{d}u = \frac{1}{\sqrt{\pi}}\frac{\Gamma(k-1/2)}{\Gamma(k)},$$

if n is even, and 0 if n is odd. So,

$$\int_0^t \mathbb{E}_x\left(\frac{\mathrm{d}s}{\sin^2(\pi X_s)}\right) = \sum_{\substack{n\geq 0\\ n \text{ even}}} \frac{1}{\lambda_n}\big(1-\mathrm{e}^{-\lambda_n t}\big) G_n^{(k)}(\cos \pi x) \omega_n^{(k)} \frac{1}{\sqrt{\pi}}\frac{\Gamma(k-1/2)}{\Gamma(k)}.$$

Hence, it suffices to prove that

$$\sum_{\substack{n\geq 0\\ n \text{ even}}} \left|\frac{1}{\lambda_n} G_n^{(k)}(\cos \pi x) \omega_n^{(k)}\right| < +\infty.$$

Using Stirling's formula for the Gamma function, i.e.

$$\Gamma(z) = \frac{\sqrt{2\pi}}{\sqrt{z}} z^z \mathrm{e}^{-z}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right),$$

we find that

$$\omega_n^{(k)} \underset{+\infty}{\sim} n^{2-2k}.$$

Now, using asymptotic expansion of Gegenbauer polynomials (see [22]), that is

$$G_n^{(k)}(\cos \pi x) = 2^{1-k} \frac{\Gamma(n+k)}{n!\Gamma(k)} (\sin \pi x)^{-k} \cos\bigl((n+k)\pi x - k\pi^2/2\bigr) + \mathrm{O}\bigl(n^{k-2}\bigr)$$

for $0 < x < 1$, we have (recall that $\lambda_n = \frac{\pi^2}{2} n(n+2k)$),

$$\left| \frac{1}{\lambda_n} \omega_n^{(k)} G_n^{(k)}(\cos \pi x) \right| \underset{+\infty}{\sim} \frac{1}{n^{k+1}},$$

Hence, since $k > \frac{1}{2}$, the series is convergent, which proves the proposition. □

Remark 2.6. *We obviously obtain the same results if a.s. $X_0 = x$ for some $x \in \mathbb{R} \setminus \mathbb{Z}$ (not only in $]0, 1[$), that is for all $x \in \mathbb{R} \setminus \mathbb{Z}$, $t \geq 0$, and $k > \frac{1}{2}$,*

$$\mathbb{E}_x\left(\int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)}\right) < +\infty.$$

Due to the importance of the process $\frac{1}{\sin^2(\pi X_\cdot)}$ for the construction of the affine Dunkl process as we will see in the next section, we put the following definition.

Definition 2.7. *For all $t \geq 0$, we define*

$$\eta_t = \frac{k}{2}\pi^2 \int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)}.$$

Let us summarize the properties of the process η. For all $x \in \mathbb{R} \setminus \mathbb{Z}$, it is a $\mathbb{P}_x$-almost surely finite continuous increasing process, with $\eta_0 = 0$ and $\eta_t \to +\infty$ a.s. when t goes to infinity. Furthermore, it has finite expectation for $k > \frac{1}{2}$, and for $k = \frac{1}{2}$, $\mathbb{E}_x(\eta_t) = +\infty$ for $t > 0$.

The construction of the affine Dunkl process is now the content of the next section.

3. The affine Dunkl process

We will define in this section the affine Dunkl process with parameter k ($k \geq \frac{1}{2}$) as the Markov process with infinitesimal generator

$$\mathcal{A}f(x) = \frac{1}{2} f''(x) + k\pi \cot(\pi x) f'(x) + \frac{k}{2} \sum_{n \in \mathbb{Z}} \frac{f(s_n(x)) - f(x)}{(x-n)^2}$$

for $f \in C_b^2(\mathbb{R})$ and $x \in \mathbb{R} \setminus \mathbb{Z}$. To achieve this, we will start with the radial affine Dunkl process living in some alcove, and add the jumps at some random times. This is a kind of skew-product decomposition as the one done in [7] (see also [28] for the same decomposition in the Heckman–Opdam setting). More precisely, we construct a pure jump process on the affine Weyl group W, and use the action of W on the radial process.

3.1. *Jump process on the affine Weyl group*

First, since the functional $\eta_t = \frac{k}{2}\pi^2 \int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)}$ is an a.s. finite continuous increasing process, with $\eta_0 = 0$ and $\eta_t \to +\infty$ as $t \to +\infty$, we have that η_t is a well-defined time change. Let us call its inverse $a(t)$, i.e.

$$a(t) = \inf\{s \geq 0 | \eta_s > t\},$$

so a is continuous, increasing, $a(0) = 0$, $a(t) < +\infty$ for all $t \geq 0$, and $a(t) \to +\infty$ as $t \to +\infty$, a.s. It is well known that such time-change transformation preserves the Markovian character of a process (see for example [12]), so the process

$$\tilde{X}_t = X_{a(t)}$$

is a strong Markov process, with infinitesimal generator

$$\tilde{\mathcal{L}} f(y) = \frac{2\sin^2(\pi y)}{k\pi^2} \mathcal{L} f(y)$$

for $f \in C(\bar{I}) \cap C^2(I)$ and $y \in I$, where I is the alcove containing X_0.

Recall that for $x \in \mathbb{R} \setminus \mathbb{Z}$, we have the series expansion

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{n \in \mathbb{Z}} \frac{1}{(x-n)^2}.$$

We denote by σ^x the following probability measure on the affine Weyl group W

$$\sigma^x(\mathrm{d}w) = \sum_{n \in \mathbb{Z}} \frac{\sin^2(\pi x)}{\pi^2} \frac{1}{(x-n)^2} \delta_{s_n}(\mathrm{d}w).$$

Let $(N_t)_{t \geq 0}$ be a Poisson point process with intensity 1, independent of X, i.e.

$$N_t = \sum_{n \geq 1} \mathbb{1}_{\{\tau_n \leq t\}},$$

where $\tau_0 = 0$ and $\tau_n = \sum_{j=1}^n e_j$, where $(e_j)_{j \geq 1}$ is a sequence of independent and identically distributed random variables, with exponential distribution of parameter 1, and independent of X.

Now define recursively the processes $\tilde{X}^j$ and the random variables $(\beta_j)_{j \geq 1}$ on W by

$$\tilde{X}_t^j = \beta_j \cdot \tilde{X}_t^{j-1} \tag{6}$$

for all $j \geq 1$, with $\tilde{X}_t^0 = \tilde{X}_t$, and where conditionally on $\{\tilde{X}_{\tau_j}^{j-1} = x\}$, β_j is distributed according to σ^x. Note that $\tilde{X}^j$ is the Markov process $\tilde{X}$ with initial condition $\tilde{X}_0^j = \beta_j \cdots \beta_1 \cdot \tilde{X}_0$.

Using left multiplication on W, we define the jump process on W

$$w_t = \xi_{N_t} = \beta_n \cdots \beta_1 \quad \text{for } t \in [\tau_n, \tau_{n+1}[, \tag{7}$$

where $\xi_n = \beta_n \cdots \beta_1$, for all $n \geq 1$.

3.2. *Skew-product decomposition*

Given an operator $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$, we say that a càdlàg stochastic process $(Y_t)_{t \geq 0}$ is a solution of the martingale problem for $\mathcal{A}$ if for all $u \in \mathcal{D}(\mathcal{A})$,

$$u(Y_t) - u(Y_0) - \int_0^t \mathcal{A}u(Y_s)\,\mathrm{d}s$$

is a $(\mathcal{F}_t^Y)$-martingale, where $(\mathcal{F}_t^Y)_{t\geq 0}$ is the natural filtration of Y (see [13] for a detailed exposition of the theory of martingale problems).

Now we can state the main result of this paper.

Theorem 3.1. *Let $(X_t)_{t\geq 0}$ be the radial affine Dunkl process with $X_0 = x \in \mathbb{R}\setminus\mathbb{Z}$ a.s., and parameter $k \geq \frac{1}{2}$, and $(w_t)_{t\geq 0}$ be the pure jump process defined by (7). Then the process $(Y_t)_{t\geq 0}$ defined by*

$$Y_t = w_{\eta_t} \cdot X_t,$$

with $\eta_t = \frac{k}{2}\pi^2 \int_0^t \frac{\mathrm{d}s}{\sin^2(\pi X_s)}$, is a Markov process on $\mathbb{R}$, with infinitesimal generator $\mathcal{A}$ given by

$$\mathcal{A}f(y) = \frac{1}{2}f''(y) + k\pi\cot(\pi y)f'(y) + \frac{k}{2}\sum_{p\in\mathbb{Z}} \frac{f(s_p(y)) - f(y)}{(y-p)^2}$$

for $f \in C_b^2(\mathbb{R})$ and $y \in \mathbb{R}\setminus\mathbb{Z}$, and such that $Y_t \in \mathbb{R}\setminus\mathbb{Z}$ for all $t \geq 0$ a.s.

Definition 3.2. *The process $(Y_t)_{t\geq 0}$ defined in the above theorem is called the affine Dunkl process with parameter k.*

Proof of Theorem 3.1. We suppose that X starts in $]0, 1[$ without lost of generality. As notice in Section 2, we have that X lives in $]0, 1[$ almost surely. Consider the process $\tilde{X}$, with generator $\tilde{\mathcal{L}}$, defined previously by $\tilde{X}_t = X_{a(t)}$, where $a(t)$ is the inverse of η_t. Define, for all $t \geq 0$,

$$\tilde{Y}_t = w_t \cdot \tilde{X}_t.$$

Hence, for $t \in [\tau_n, \tau_{n+1}[$, we have

$$\begin{aligned}\tilde{Y}_t &= \beta_n \cdots \beta_1 \cdot \tilde{X}_t \\ &= \tilde{X}_t^n,\end{aligned}$$

where the processes $\tilde{X}^n$ are defined by (6).

By construction $(\tilde{Y}_t)_{t\geq 0}$ is a càdlàg process which jumps at the random times τ_n's. We denote by $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ the natural filtration of $\tilde{Y}$, and by $\tilde{\mathcal{F}}_t^n = \sigma(\tilde{X}_s^n, s \leq t) \vee \sigma(N_s, s \leq t)$. As we shall see, $(\tilde{Y}_t)_{t\geq 0}$ is a solution of the martingale problem for the generator $\tilde{\mathcal{A}}$ given by

$$\tilde{\mathcal{A}}f(y) = \tilde{\mathcal{L}}f(y) + \int_W \big(f(w\cdot y) - f(y)\big)\sigma^y(\mathrm{d}w),$$

acting on $f \in C_b^2(\mathbb{R})$ for $y \in \mathbb{R}\setminus\mathbb{Z}$, where $\sigma^x(\mathrm{d}w) = \sum_{n\in\mathbb{Z}} \frac{\sin^2(\pi x)}{\pi^2}\frac{1}{(x-n)^2}\delta_{s_n}(\mathrm{d}w)$, and $\tilde{\mathcal{L}}$ is the generator of $\tilde{X}$. The proof follows exactly the lines of Proposition 10.2, Chapter 4 of [13], see also Lemma 17 in [7]. First, since $(\tilde{X}_t^n)_{t\geq\tau_n}$ is a Markov process with generator $\tilde{\mathcal{L}}$, and using independence of $\tilde{X}$ and $(\tau_n)_{n\geq 0}$, we have that for $u \in C_b^2(\mathbb{R})$

$$u\big(\tilde{X}^n_{(t\vee\tau_n)\wedge\tau_{n+1}}\big) - u\big(\tilde{X}^n_{\tau_n}\big) - \int_{\tau_n}^{(t\vee\tau_n)\wedge\tau_{n+1}} \tilde{\mathcal{L}}u\big(\tilde{X}^n_s\big)\,\mathrm{d}s$$

is a $(\tilde{\mathcal{F}}_t)$-martingale. Hence, summing over $n \geq 0$, and using

$$u\big(\tilde{X}^n_{(t\vee\tau_n)\wedge\tau_{n+1}}\big) = u\big(\tilde{X}^n_{\tau_n}\big)\mathbb{1}_{\{t<\tau_n\}} + u\big(\tilde{X}^n_t\big)\mathbb{1}_{\{\tau_n\leq t<\tau_{n+1}\}} + u\big(\tilde{X}^n_{\tau_{n+1}}\big)\mathbb{1}_{\{t\geq\tau_{n+1}\}},$$

we get that

$$u(\tilde{Y}_t) - u(\tilde{Y}_0) - \int_0^t \tilde{\mathcal{L}}u(\tilde{Y}_s)\,\mathrm{d}s - \sum_{n=1}^{N_t}\big(u\big(\tilde{X}^n_{\tau_n}\big) - u\big(\tilde{X}^{n-1}_{\tau_n}\big)\big) \tag{8}$$

is a $(\tilde{\mathcal{F}}_t)$-martingale. Note that, since N is a Poisson process, we have that

$$\int_0^t \left(\int_W u(w \cdot \tilde{Y}_{s^-}) \sigma^{\tilde{Y}_{s^-}}(\mathrm{d}w) - u(\tilde{Y}_{s^-}) \right) \mathrm{d}(N_s - s)$$

is a $(\tilde{\mathcal{F}}_t)$-martingale, and is equal to

$$\sum_{n=1}^{N_t} \left(\int_W u\big(w \cdot \tilde{X}^{n-1}_{\tau_n}\big) \sigma^{\tilde{X}^{n-1}_{\tau_n}}(\mathrm{d}w) - u\big(\tilde{X}^{n-1}_{\tau_n}\big) \right) - \int_0^t \left(\int_W u(w \cdot \tilde{Y}_{s^-}) \sigma^{\tilde{Y}_{s^-}}(\mathrm{d}w) - u(\tilde{Y}_{s^-}) \right) \mathrm{d}s. \tag{9}$$

But

$$\sum_{n=1}^{N_t} \left(u\big(\tilde{X}^n_{\tau_n}\big) - \int_W u\big(w \cdot \tilde{X}^{n-1}_{\tau_n}\big) \sigma^{\tilde{X}^{n-1}_{\tau_n}}(\mathrm{d}w) \right) \tag{10}$$

is also a $(\tilde{\mathcal{F}}_t)$-martingale. This follows from the fact that for all $t_1 < \cdots < t_m \le s < t$, and all $h_1, \dots, h_m$ measurable bounded functions, we have

$$\begin{aligned}
&\mathbb{E}\left(\prod_{i=1}^m h_i(Y_{t_i}) \sum_{n \ge 1} \mathbb{1}_{\{s < \tau_n \le t\}} \left(u\big(\tilde{X}^n_{\tau_n}\big) - \int_W u\big(w \cdot \tilde{X}^{n-1}_{\tau_n}\big) \sigma^{\tilde{X}^{n-1}_{\tau_n}}(\mathrm{d}w) \right) \right) \\
&\quad = \mathbb{E}\left(\prod_{i=1}^m h_i(Y_{t_i}) \sum_{n \ge 1} \mathbb{1}_{\{s < \tau_n \le t\}} \mathbb{E}\left(u\big(\tilde{X}^n_{\tau_n}\big) - \int_W u\big(w \cdot \tilde{X}^{n-1}_{\tau_n}\big) \sigma^{\tilde{X}^{n-1}_{\tau_n}}(\mathrm{d}w) \Big| \tilde{\mathcal{F}}^{n-1}_{\tau_n} \right) \right) \\
&\quad = 0,
\end{aligned}$$

since $\tilde{X}^n_{\tau_n} = \beta_n \cdot \tilde{X}^{n-1}_{\tau_n}$, and β_n is distributed according to $\sigma^{\tilde{X}^{n-1}_{\tau_n}}$ conditionally to $\tilde{X}^{n-1}_{\tau_n}$.

Adding (10) and (9) to (8), we get that

$$u(\tilde{Y}_t) - u(\tilde{Y}_0) - \int_0^t \tilde{\mathcal{L}} u(\tilde{Y}_s)\, \mathrm{d}s - \int_0^t \int_W \big(u(w \cdot \tilde{Y}_s) - u(\tilde{Y}_s) \big) \sigma^{\tilde{Y}_s}(\mathrm{d}w)\, \mathrm{d}s$$

is a $(\tilde{\mathcal{F}}_t)$-martingale (since $\tilde{Y}$ is càdlàg and $\{s | \tilde{Y}_{s^-} \neq \tilde{Y}_s\}$ is Lebesgue negligible, we can replace $\tilde{Y}_{s^-}$ by $\tilde{Y}_s$ in the last integral). Hence, we have obtained that for $u \in C_b^2(\mathbb{R})$

$$u(\tilde{Y}_t) - u(\tilde{Y}_0) - \int_0^t \tilde{\mathcal{A}} u(\tilde{Y}_s)\, \mathrm{d}s$$

is a $(\tilde{\mathcal{F}}_t)$-martingale, which proves that $(\tilde{Y}_t)_{t \ge 0}$ is a solution of the martingale problem for the generator $\tilde{\mathcal{A}}$.

We are now interested in solution of

$$Y_t = \tilde{Y}\left(\int_0^t \beta(Y_s)\, \mathrm{d}s \right), \tag{11}$$

with $\beta(y) = \frac{k\pi^2}{2\sin^2(\pi y)}$ for $y \in \mathbb{R} \setminus \mathbb{Z}$, and $\beta(y) = 0$ for $y \in \mathbb{Z}$. First remark that since $\sin^2(\pi \cdot)$ is a W-invariant function, we have $\beta(\tilde{Y}_t) = \beta(\tilde{X}_t)$ almost surely, since $\tilde{Y}_t = w_t \cdot \tilde{X}_t$. Since X never touches 0 and 1 a.s., so is $\tilde{X}$, and $\beta \circ \tilde{X}$ is a.s. bounded on bounded intervals. Define

$$\zeta_1 = \inf\left\{ t \ge 0 \,\Big|\, \int_0^t \frac{\mathrm{d}s}{\beta(\tilde{Y}_s)} = +\infty \right\}$$

and

$$\zeta_0 = \inf\big\{ t \ge 0 \mid \beta(\tilde{Y}_t) = 0 \big\}.$$

Since $0 < \sin^2(\pi \tilde{X}_t) < 1$ for all $t \geq 0$, we easily see that $\zeta_1 = \zeta_0 = +\infty$, so by applying Theorem 1.3, Chapter 6 of [13], we have that equation (11) admits a solution $(Y_t)_{t\geq 0}$, which is a solution of the martingale problem for $\mathcal{A} = \beta\tilde{\mathcal{A}}$, i.e. for all $u \in C_b^2(\mathbb{R})$,

$$u(Y_t) - u(Y_0) - \int_0^t \mathcal{A}u(Y_s)\,\mathrm{d}s$$

is a $(\tilde{\mathcal{F}}_{\tau(t)})$-martingale, where $\tau(t) = \int_0^t \beta(Y_s)\,\mathrm{d}s$. Since we have

$$\begin{aligned}\tilde{\mathcal{A}}u(y) &= \tilde{\mathcal{L}}u(y) + \int_W \big(u(w\cdot y) - u(y)\big)\sigma^y(\mathrm{d}w)\\ &= \frac{2\sin^2(\pi y)}{k\pi^2}\mathcal{L}u(y) + \frac{\sin^2(\pi y)}{\pi^2}\sum_{n\in\mathbb{Z}}\frac{u(s_n(y)) - u(y)}{(y-n)^2},\end{aligned}$$

we obtain

$$\mathcal{A}u(y) = \mathcal{L}u(y) + \frac{k}{2}\sum_{n\in\mathbb{Z}}\frac{u(s_n(y)) - u(y)}{(y-n)^2},$$

acting on $u \in C_b^2(\mathbb{R})$, for $y \in \mathbb{R}\setminus\mathbb{Z}$. Since $\beta(Y_s) = \beta(X_s)$ for all $s \geq 0$ by the W-invariance of $\sin^2(\pi\cdot)$, we have that

$$\int_0^t \beta(Y_s)\,\mathrm{d}s = \eta_t$$

for all $t \geq 0$. Hence, we obtain the skew-product decomposition of $(Y_t)_{t\geq 0}$ given by

$$Y_t = w_{\eta_t}\cdot X_t$$

for all $t \geq 0$. Let π be the projection onto the principal alcove $\mathcal{A}_0 =]0, 1[$. Then, by the invariance of π under the action of the Weyl group W and the skew-product representation of the affine Dunkl process, we see that

$$\pi(Y_t) = X_t \quad \text{a.s.},$$

i.e. $(\pi(Y_t))_{t\geq 0}$ is the radial affine Dunkl process. Hence, if two process Y and Y' are solutions to the martingale problem for $\mathcal{A}$, then $\pi(Y_t) = \pi(Y'_t) = X_t$, so Y and Y' have the same one-dimensional distributions, and by the same arguments as in Theorem 4.2, Chapter 4 in [13], we have that the affine Dunkl process Y is a Markov process with infinitesimal generator $\mathcal{A}$. By construction, Y is càdlàg and lives a.s. in $\mathbb{R}\setminus\mathbb{Z}$, else the radial process X would touch the walls of the principal alcove, which is impossible as already noticed. □

3.3. *Jumps of the affine Dunkl process*

By the construction of the affine Dunkl process, we have the skew-product decomposition

$$Y_t = w_{\eta_t}\cdot X_t$$

for $t \geq 0$. This shows that there is a jump of the process at time t when the functional η_t is equal to one of the τ_n's. Hence, the number of jumps V_t of $(Y_t)_{t\geq 0}$ before time t, i.e.

$$V_t = \sum_{s\leq t}\mathbb{1}_{\{\Delta Y_s \neq 0\}},$$

where $\Delta Y_s = Y_s - Y_{s^-}$, is exactly given by the point process

$$V_t = \sum_{n\geq 1}\mathbb{1}_{\{\eta_t \geq \tau_n\}}.$$

Since η is a well-defined time change, we get the following representation of V in term of a time-change Poisson process

$$V_t = N_{\eta_t}$$

for all $t \geq 0$, where $(N_t)_{t\geq 0}$ is the Poisson process considered previously. Using this representation and the fact that for all $t \geq 0$, $\eta_t < +\infty$ a.s., we get immediately the following proposition.

Proposition 3.3. *For all $x \in \mathbb{R} \setminus \mathbb{Z}$, and all $t \geq 0$*

$$V_t < +\infty \quad \mathbb{P}_x\text{-almost surely},$$

i.e. the number of jumps of the affine Dunkl process in a finite time interval is almost surely finite.

Now define for all $n \geq 1$,

$$T_n = a(\tau_n),$$

where a is the inverse of η, i.e. $a(t) = \inf\{s \geq 0 \mid \eta_s > t\}$, and $T_0 = 0$. The sequence $(T_n)_{n\geq 1}$ corresponds to the jump times of the affine Dunkl process. Since a is increasing and $a(t) \to +\infty$ as $t \to +\infty$ a.s., we have that for all $n \geq 0$,

$$T_n > T_{n-1} \quad \mathbb{P}_x\text{-almost surely}$$

and $T_n \to +\infty$ when $n \to +\infty$ a.s. Note that we can also define the jump times recursively by

$$T_n = \inf\left\{t \geq T_{n-1} \;\middle|\; \frac{k\pi^2}{2}\int_{T_{n-1}}^{t} \frac{\mathrm{d}s}{\sin^2(\pi X_s)} > e_n\right\}, \tag{12}$$

where $(e_n)_{n\geq 1}$ is a sequence of independent and identically distributed random variables with exponential distribution of parameter 1. Indeed, define for all $n \geq 1$ and all $t \geq 0$,

$$\eta_n(t) = \frac{k\pi^2}{2}\int_{T_{n-1}}^{T_{n-1}+t} \frac{\mathrm{d}s}{\sin^2(\pi X_s)},$$

so η_n is a well-defined time-change with inverse given by

$$\eta_n^{-1}(t) = \inf\bigl\{s \geq 0 \mid \eta_n(s) > t\bigr\},$$

which is finite for all $t \geq 0$, and goes to infinity as t goes to infinity. Then, we can rewrite T_n as

$$\begin{aligned} T_n &= T_{n-1} + \inf\bigl\{t \geq 0 \mid \eta_n(t) > e_n\bigr\} \\ &= T_{n-1} + \eta_n^{-1}(e_n). \end{aligned}$$

So,

$$T_n - T_{n-1} = \eta_n^{-1}(e_n),$$

which gives $\eta_n(T_n - T_{n-1}) = e_n$, and

$$\eta_{T_n} = \sum_{j=1}^{n} \eta_j(T_j - T_{j-1}) = \sum_{j=1}^{n} e_j = \tau_n,$$

or equivalently $T_n = a(\tau_n)$. Note that the fact that the sequence $(T_n)_{n\geq 0}$ is well defined can be proved directly using expression (12) and the strong Markov property of $(X_t)_{t\geq 0}$.

Since V_t is a time-change Poisson process, it is not difficult to exhibit its compensator.

Lemma 3.4. *Let $V_t = N_{\eta_t}$. The compensator of V is η, that is*

$$V_t - \eta_t$$

is a martingale with respect to the filtration $(\mathcal{F}^N_{\eta_t})_{t\geq 0}$, where $\mathcal{F}^N_t = \sigma(N_s, s \leq t)$.

Proof. Let $0 \leq s < t$. Define $T = \eta_t$ and $S = \eta_s$. We have $S < T$, since $t \mapsto \eta_t$ is increasing. Note also that S and T are stopping times with respect to $\mathcal{F}_t$. Hence, since N is a Poisson process, $N_t - t$ is a martingale and by the optional sampling theorem, we have

$$\mathbb{E}_x\big(V_t - \eta_t | \mathcal{F}^N_{\eta_s}\big) = \mathbb{E}_x\big(N_T - T | \mathcal{F}^N_S\big) = N_S - S = V_s - \eta_s. \qquad \square$$

Since η is the compensator of V, we have by Proposition 2.5 that

$$\mathbb{E}_x(V_t) < +\infty \quad \text{for } k > \frac{1}{2}$$

and $\mathbb{E}_x(V_t) = +\infty$ for $k = \frac{1}{2}$, that is, the number of jumps of the affine Dunkl process has finite expectation when $k > \frac{1}{2}$, and infinite expectation for $k = \frac{1}{2}$.

3.4. *Martingale decomposition*

First, we remark that Y is a local martingale.

Proposition 3.5. *The affine Dunkl process $(Y_t)_{t\geq 0}$ is a local martingale.*

Proof. Using the formula

$$\pi \cot(\pi x) = \sum_{n\in\mathbb{Z}} \frac{1}{x-n},$$

one can see that the function $f(x) = x$ is killed by the generator $\mathcal{A}$ of $(Y_t)_{t\geq 0}$, which proves that Y is a local martingale. $\square$

We give now the martingale decomposition of Y into its continuous and purely discontinuous parts. First, recall that the Lévy kernel N of a Markov process describes the distribution of its jumps, see [23]. For all $x \in \mathbb{R}$, and for a function f in the domain of the infinitesimal generator which vanishes in a neigbourhood of x, the Lévy kernel N of $(Y_t)_{t\geq 0}$ is given by

$$\mathcal{A}f(x) = \int_{\mathbb{R}} N(x, \mathrm{d}y) f(y).$$

Hence, by the explicit form of the infinitesimal generator $\mathcal{A}$, we get immediately that

$$N(x, \mathrm{d}y) = \frac{k\pi^2}{2} \sum_{n\in\mathbb{Z}} \frac{\delta_{s_n(x)}(\mathrm{d}y)}{(x-n)^2}$$

for all $x \in \mathbb{R} \setminus \mathbb{Z}$. By [23], for all nonnegative measurable function f on $\mathbb{R}^2$, the nonnegative discontinuous functional

$$\sum_{s\geq t} f(Y_{s^-}, Y_s) \mathbb{1}_{\{\Delta Y_s \neq 0\}},$$

where $\Delta Y_s = Y_s - Y_{s^-}$, can be compensated by the process

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}} N(Y_{s^-}, \mathrm{d}y) f(Y_{s^-}, y).$$

Proposition 3.6. *We have the following martingale decomposition*,

$$Y_t = Y_0 + B_t + M_t,$$

where $(B_t)_{t\geq 0}$ *is a standard Brownian motion*, *and* $(M_t)_{t\geq 0}$ *is a purely discontinuous local martingale which can written as the compensated sum of its jumps*:

$$M_t = -\sum_{s\leq t} \Delta Y_s \mathbb{1}_{\{\Delta Y_s \neq 0\}} + \int_0^t k\pi \cot(\pi Y_s)\,\mathrm{d}s.$$

The proof uses Itô's formula and the theory of Lévy kernel and is exactly the same as in the classical case of Dunkl processes, so we refer to [15] for more details.

Acknowledgments

This work is part of my Ph.D. thesis, and I would like to thank my advisor Philippe Biane for his suggestions and his reading of this paper. I also would like to thank Nathanaël Enriquez for some fruitful discussions, and I am indebted to an anonymous referee for comments and remarks which improved the quality of the presented paper.

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