Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2012, Vol. 48, No. 3, 745–773
DOI: 10.1214/11-AIHP434

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
www.imstat.org/aihp

Excited against the tide: A random walk with competing drifts

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Received 20 September 2010; revised 12 April 2011; accepted 9 May 2011

Abstract. We study excited random walks in i.i.d. random cookie environments in high dimensions, where the kth cookie at a site determines the transition probabilities (to the left and right) for the kth departure from that site. We show that in high dimensions, when the expected right drift of the first cookie is sufficiently large, the velocity is strictly positive, regardless of the strengths and signs of subsequent cookies. Under additional conditions on the cookie environment, we show that the limiting velocity of the random walk is continuous in various parameters of the model and is monotone in the expected strength of the first cookie at the origin. We also give non-trivial examples where the first cookie drift is in the opposite direction to all subsequent cookie drifts and the velocity is zero. The proofs are based on a cut-times result of Bolthausen, Sznitman and Zeitouni, the lace expansion for self-interacting random walks of van der Hofstad and Holmes, and a coupling argument.

Résumé. Nous étudions des marches aléatoires excitées dans un environnement de cookies indépendants en grande dimension, où le $k$ième cookie d'un site détermine le taux de transition (vers la droite ou la gauche) pour le $k$ième départ de ce site. Nous montrons qu'en grande dimension, quand le taux de saut moyen vers la droite du premier cookie est suffisamment grand, la vitesse est strictement positive, quelque soit l'amplitude et le signe des cookies suivants. Sous des conditions supplémentaires sur l'environnement des cookies, nous montrons que la vitesse est une fonction continue des divers paramètres du modèle et est monotone en la force moyenne du cookie à l'origine. Nous donnons aussi des examples non-triviaux où la dérive du premier cookie est dans le sens opposé à toutes les autres et où la vitesse est nulle. Les preuves se basent sur un résultat de temps de coupure de Bolthausen, Sznitman et Zeitouni, le développement en lacets de marches aléatoires auto-interagissantes de van der Hofstad et Holmes, et un argument de couplage.

MSC: 60K35; 60K37; 82B41

Keywords: Self-interacting random walk; Cookie environment; Lace expansion; Monotonicity

1. Introduction

The so-called *excited* (or *cookie*) random walks are a class of self-interacting random walks that have received considerable attention in recent years. The *excited random walk* introduced by Benjamini and Wilson [4] is a discrete-time nearest-neighbour random walk $\vec{X} = \{X_0 = o, X_1, X_2, \ldots\}$ in $\mathbb{Z}^d$ that can be described as follows. At each site $x = (x^{[1]}, \ldots, x^{[d]}) \in \mathbb{Z}^d$ (we use a superscript i in square brackets to denote the ith component of a site in $\mathbb{Z}^d$) there is precisely one cookie. When the random walk is at a site at which there is a cookie, it eats that cookie just prior to departure and then has a preference, given by a single parameter $\beta \in [0, 1]$, for departing to the right instead of the left. When the random walk arrives at a site at which there is no cookie (because it has already been eaten on a previous visit), the direction of departure is unbiased. On an appropriate probability space with probability measure Q, we can write this as $Q(X_0 = o) = 1$ and

$$
\begin{aligned}
&Q(X_{n+1} = X_n + u | X_n \notin \vec{X}_{n-1}) = \begin{cases} \frac{1+\beta}{2d} & \text{if } u = +e_1, \\ \frac{1-\beta}{2d} & \text{if } u = -e_1, \\ \frac{1}{2d} & \text{if } u \in \{\pm e_2, \dots, \pm e_d\}, \end{cases} \\
&Q(X_{n+1} = X_n + u | X_n \in \vec{X}_{n-1}) = \frac{1}{2d} \quad \text{for all } u \in \{\pm e_1, \dots, \pm e_d\},
\end{aligned}
\tag{1.1}
$$

where $e_1, \dots, e_d$ are the standard basis vectors in $\mathbb{Z}^d$ and $\vec{X}_n = \{X_0, X_1, \dots, X_n\}$. It is known [5] that there exists a non-random velocity $v = (v^{[1]}, 0, \dots, 0)$, with $v^{[1]}$ strictly positive when $d \geq 2$ and $\beta > 0$, such that $Q(\lim_{n\to\infty} n^{-1} X_n = v) = 1$, while the random walk is recurrent when $d = 1$ and $\beta < 1$ [7]. It is also known [11] that for fixed $d \geq 9$, the velocity $v^{[1]}$ is monotone increasing in β.

The above model has since been generalised considerably (see e.g. [1–3,14,18]) to allow finitely many cookies with different (possibly random) drift parameters, taking values in $[-1, 1]$, at each site. In these generalised settings, almost all of the results obtained are for 1-dimension. For example, in one dimension it is known e.g. [2,3,14,18] that, excluding one degenerate case where the first cookie drift is 1 or -1, if the (i.i.d.) number of cookies per site is bounded, transience criteria and the existence of a non-zero velocity for the random walk depend only on the average total cookie drift per site, δ. The random walk is transient (in the direction of δ) if $|\delta| > 1$ and recurrent otherwise, while the speed is non-zero if and only if $|\delta| > 2$ [14]. In particular in the non-random cookie-environment setting at least two cookies are required per site in order to achieve a transient walk, and at least 3 cookies are required to achieve a walk with non-zero speed. Although it is intuitively clear that the same criteria cannot hold in higher dimensions, almost nothing is known in this case, e.g. see [14], Section 9.

In this paper we make use of the lace expansion for self-interacting random walks [10,11,13] in order to study properties of the speed in high dimensions. Although we are unable to give an explicit criterion for ballisticity such as that appearing in [14] (as remarked in that work it is not even clear what such a criterion should look like), we show in a rather general setting (including possibly infinitely many cookies) that if the expected *first* cookie drift per site, δ_1, is sufficiently large, then the speed of the random walk is non-zero in the direction of δ_1. This provides some rigour to the intuition that the effect of the kth cookie on the velocity $v^{[1]}$ should be decreasing in k (see also [12]). Indeed, it is tempting to think that the velocity can be written in the form

$$
v^{[1]} = \sum_{k=1}^{\infty} a_{k,d} \delta_k,
$$

where $\delta_k \in [-1, 1]$ is the expected drift induced by the kth cookie, and $\{a_{k,d}\}_{k\geq 1}$ is a fixed decreasing sequence that is independent of the distribution of the cookie environment. This is almost certainly not the case, and we do not expect that a quantity of this form is sufficient to characterize positivity of the speed. We show that under certain independence assumptions, in high dimensions the speed $v^{[1]}$ is a continuous function of appropriate parameters of the model, and $v^{[1]}$ is increasing in δ_1. We also give examples of non-trivial random cookie environments for which $v^{[1]} = 0$. These notions and results are stated explicitly in Section 2.

A simple but interesting subclass of the considered models can be defined rather easily, as follows. Suppose that each site in $\mathbb{Z}^d$ is *occupied* with probability $\lambda \in [0, 1]$ and *vacant* with probability $1 - \lambda$, independent of all other sites. The walk has a drift $\frac{\beta}{d}$ in the direction of the first component each time the walker visits a previously unvisited occupied site, and a drift $\frac{\mu}{d}$ in the direction of the first component otherwise, where $(\beta, \mu) \in [-1, 1]^2$. This can be considered as an excited random walk in a cookie environment with infinitely many cookies at each site, or as an excited random walk in a cookie environment with at most one cookie at each site, fighting against a tide (when β and μ have opposite sign). The parameter μ represents the magnitude of the "tide." Part of what makes this subclass easier to deal with is the fact that, under the annealed measure, this is the same as an excited random walk in a *non-random* 1-cookie environment. However this subclass already exhibits many interesting features: the (annealed) velocity is continuous and increasing in various parameters, one can investigate the question of "which drift wins?," and in high dimensions we can find non-trivial values of (β, λ, μ) such that $v^{[1]} = 0$. Moreover, when $\mu = -1$ (resp. 1) we expect that this model has minimum (resp. maximum) velocity (among all cookie-random walks) for any given first cookie drift.

Some of the results in this paper are proved using adaptions of the expansion arguments in [11]. These results are currently out of reach of other methods. However one cannot hope to learn everything about such models using these kinds of arguments alone. For example, they are only applicable in the annealed setting, and only when the models are sufficiently transient. Here we combine such techniques with more traditional renewal and coupling methods.

2. Main results and organisation

We first define our cookie environment, as in [14], but allowing infinitely many cookies at each site. A cookie-environment ω is an element of

$$\Omega = \big\{\big(\omega(x,k)\big)_{x\in\mathbb{Z}^d,k\in\mathbb{N}} : \omega(x,k)\in[0,1], \forall(x,k)\in\mathbb{Z}^d\times\mathbb{N}\big\}. \tag{2.1}$$

For fixed $\omega\in\Omega$ and $x\in\mathbb{Z}^d$, an excited random walk $\vec{X}=\{X_n\}_{n\geq 0}$ starting from x in the cookie environment ω is a stochastic process defined on a probability space with probability measure $Q_{x,\omega}$ satisfying $Q_{x,\omega}(X_0=x)=1$ and

$$Q_{x,\omega}(X_{n+1}=X_n+u|\vec{X}_n)=\begin{cases} d^{-1}\omega\big(X_n,\#\{i\leq n: X_i=X_n\}\big) & \text{if } u=+e_1,\\ d^{-1}\big(1-\omega\big(X_n,\#\{i\leq n: X_i=X_n\}\big)\big) & \text{if } u=-e_1,\\ \frac{1}{2d} & \text{if } u\in\{\pm e_2,\dots,\pm e_d\}.\end{cases} \tag{2.2}$$

The cookie environment ω is chosen according to a measure $\mathbb{Q}$ under which $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ is i.i.d. In other words, at each site in $\mathbb{Z}^d$ there is an infinite stack of cookies chosen according to some probability measure, with stacks at different sites being independent.

Letting $\mathbb{E}$ denote expectation with respect to $\mathbb{Q}$, we define $\delta_i=\mathbb{E}[\omega(o,i)]$. The *annealed*, or *averaged* measure Q_x is defined by

$$Q_x(\cdot,*)=\int_* Q_{x,\omega}(\cdot)\,\mathrm{d}\mathbb{Q}. \tag{2.3}$$

Let $E[\cdot]$ denote expectation with respect to Q_o. In this paper we are interested in the velocity $v=(v^{[1]},0,\dots,0)$, satisfying $Q_o(\lim_{n\to\infty} n^{-1}X_n=v)=1$. It is not even known that such a v exists in general, however a simple extension (see [13]) of [6], Theorem 1.4, from random walks in random environments to excited random walks, yields the following result.

Theorem 2.1. *For each $d\geq 6$ there exists $v\in\mathbb{Z}^d$ such that $Q_o(v=\lim_{n\to\infty} n^{-1}X_n)=1$.*

This result relies on the fact that the projection $\vec{X}^{[2,3,\dots,d]}$ of the excited random walk is a $(d-1)$-dimensional random walk (a simple random walk with geometric$(1-1/d)$ waiting times between steps), independent of the cookie environment. For $d-1\geq 5$, this projection has finite (almost surely) random cut times T_i with the property that the sets of sites visited before and after each cut-time are disjoint. These cut times are independent of the environment, and the cookie environments seen by the random walk in the time intervals $[T_j,T_{j+1}-1]$ and $[T_i,T_{i+1}-1]$ are independent if $i\neq j$. One is able to use these facts to construct a time-shift ergodic sequence from which the law of large numbers can be obtained.

Knowing that a simple random walk in high dimensions has few self-intersections (i.e. sites are typically visited at most once), the following result (proved in Section 4) which indicates that the first cookie has the greatest impact on the speed of the walk, should not be surprising.

Theorem 2.2. *For $d\geq 9$, there exists $\varepsilon_d>0$ such that $v^{[1]}>0$ whenever $\delta_1\geq 1-\varepsilon_d$.*

We may take $\varepsilon_d\nearrow 1$ as $d\nearrow\infty$ in the above theorem. Also by symmetry, we can make the velocity negative by taking δ_1 sufficiently small, irrespective of the distribution of $(\omega(o,i))_{i\geq 2}$. Based on simulations of the subclass of excited against the tide walks (see Fig. 1), we conjecture that Theorem 2.2 is true for all $d\geq 3$ (but not $d=2$).

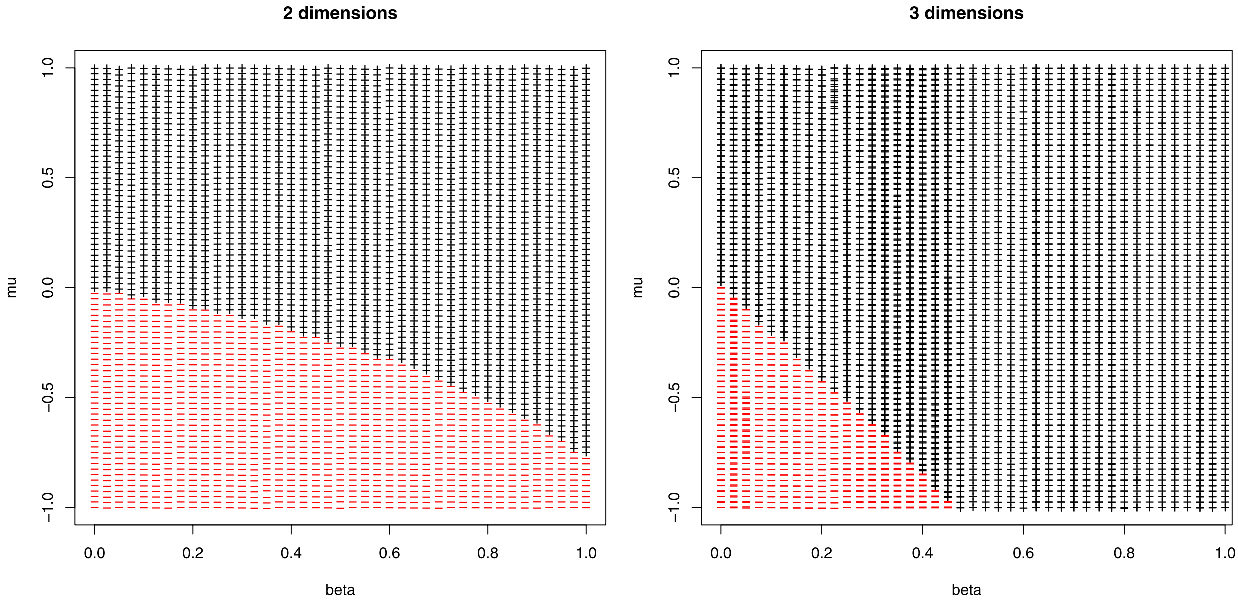

Fig. 1. Estimates of the sign (+, −) of the velocity $v^{[1]}$ of the EAT walk with $\lambda = 1$, in 2 and 3 dimensions. Each point is based on 1000 simulations of 10000-step walks, done in R [15].

Very roughly speaking, $v^{[1]}$ should be monotone in the local drift. Suppose that ω and ω' are two fixed environments, with $\omega(x,i) \le \omega'(x,i)$ for every $(x,i) \in \mathbb{Z}^d \times \mathbb{N}$. Then we expect the velocities to satisfy $v^{[1]}(\omega) \le v^{[1]}(\omega')$ when they exist. One should be more careful when making a statement about monotonicity in terms of the distribution of the random environment. It is not too difficult to think up examples (in any dimension, based on results in this paper for example) of excited random walks $\vec{X}$ and $\vec{X}'$ with $\delta_1 \ll \delta_1'$, but $v^{[1]} > v'^{[1]}$, assuming that the velocities exist, so it is not correct to say that "the velocity is monotone in the average first cookie drift." It is not even clear that we should have $v^{[1]} \le v'^{[1]}$ when $\delta_i \le \delta_i'$ for all i. We think that monotonicity is likely to hold under stochastic domination, that is, if $\mathbb{P}$ and $\mathbb{P}'$ satisfy $\mathbb{P}((\omega(o,i))_{i\in\mathbb{N}} \le \vec{z}) \ge \mathbb{P}'((\omega(o,i))_{i\in\mathbb{N}} \le \vec{z})$ for all $\vec{z} \in [0,1]^{\mathbb{N}}$ then the velocities of the corresponding excited random walks should satisfy $v^{[1]} \le v'^{[1]}$. We prove the following much weaker result, which is an extension of that appearing in [11], and is obtained using similar methods.

Theorem 2.3 (Continuity and monotonicity). *For each finite $A \subset \mathbb{N}$, if $\omega(o,i)$ is independent of $(\omega(o,j))_{j\neq i}$ for each $i \in A$, then for each fixed joint distribution of $\omega(o,A^c) = (\omega(o,i))_{i\notin A}$, the annealed velocity $v^{[1]}$ in dimension d is a continuous function of $(\delta_i)_{i\in A}$ when $d \ge 6$ and is differentiable in δ_i for each $i \in A$ when $d \ge 8$. If $1 \in A$ then $v^{[1]}$ is strictly increasing in δ_1 when $d \ge 12$.*

Results of this kind together with a coupling argument allow one to construct non-trivial examples of excited random walks in high dimensions with $v^{[1]} = 0$. To give explicit examples, we now introduce the "excited against the tide" subclass of models (briefly mentioned in Section 1).

An EAT walk in $\mathbb{Z}^d$ is an excited random walk in an i.i.d. cookie environment in $\mathbb{Z}^d$ such that for some $(\lambda,\beta,\mu) \in [0,1] \times [-1,1]^2$,

$$\begin{aligned} &\mathbb{Q}\big(\omega(o,1) = (1+\beta)/2\big) = \lambda = 1 - \mathbb{Q}\big(\omega(o,1) = (1+\mu)/2\big), \\ &\mathbb{Q}\big(\omega(o,i) = (1+\mu)/2\big) = 1 \quad \text{for } i \ge 2. \end{aligned} \tag{2.4}$$

The original excited random walk model of [4] can be recovered by setting $\lambda = 1$ and $\mu = 0$, while a simple random walk with drift is obtained by setting $\lambda = 0$.

The following result, proved via a coupling argument and comparison with a walk in an environment that is renewed every 3 steps, states that the random walker drifts with the tide if the opposing excitement is sufficiently weak.

Lemma 2.4. *For any $d \geq 2$ and $\mu \in [-1, 0)$ there exist $\varepsilon > 0$ and $\gamma_*(\mu, d) > 0$ such that for every $\lambda\beta < \gamma_*$, the (λ, β, μ)-EAT walk satisfies $Q_o(\limsup_{n\to\infty} \frac{X_n^{[1]}}{n} < -\varepsilon) = 1$.*

Lemma 2.4 implies transience of $X_n^{[1]}$ for $d \geq 2$ and $\lambda\beta < \gamma_*$, whence regeneration techniques (e.g. [5,17,18]) can be used to prove the existence of the velocity.

Lemma 2.4, Theorem 2.2 and a version of Theorem 2.3 for EAT walks imply the following result, which states that in high dimensions, for any $\mu \leq 0$: if $\lambda > \lambda_*(\mu, d) = \inf\{\lambda\colon v^{[1]}(\lambda, \beta = 1, \mu, d) > 0\}$ we can find a unique β so that the speed is zero (and vice versa).

Corollary 2.5. *For each $d \geq 9$ and $\mu \in [-1, 0]$, there exists $\lambda_* < 1$ (resp. $\beta_* < 1$) such that for each $\lambda > \lambda_*$ (resp. $\beta > \beta_*$) there exists $\beta_0(\mu, d, \lambda) \in [0, 1]$ (resp. $\lambda_0(\mu, d, \beta)$) for which $v = 0$, i.e. $Q_o(\lim_{n\to\infty} \frac{X_n}{n} = 0) = 1$. For each $d \geq 12$ there is a unique such $\beta_0(\mu, d, \lambda)$ (resp. λ_0).*

Apart from Lemma 2.4, all of the above results are proved in this paper in high dimensions only. We expect all of these results to hold for the EAT walk for all $d \geq 2$, with the exception of Theorem 2.2 and Corollary 2.5 which are not expected to hold for μ close to -1, when $d = 2$. In other words, when $d = 2$ and μ is close to -1 we believe that the speed is negative regardless of λ, β. See Fig. 1 in the case $\lambda = 1$.

Conjecture 2.6. *For each $d \geq 2$ and $(\mu, \beta, \lambda) \in [-1, 1]^2 \times [0, 1]$, the velocity $v^{[1]} = \lim_{n\to\infty} n^{-1} X_n^{[1]}$ of the (μ, β, λ)-EAT walk exists and is continuous and monotone increasing in β (resp. λ) for fixed μ and $\lambda > 0$ (resp. fixed μ and $\beta > \mu$) and in μ for fixed β, λ respectively. For each $d \geq 3$ and $\mu \in [-1, 0]$ there exists $\lambda_* < 1$ (resp. $\beta_* < 1$) such that for all $\lambda > \lambda_*$ (resp. $\beta > \beta_*$) there exists a unique $\beta_0(\mu, d, \lambda) \in [0, 1]$ (resp. $\lambda_0 > (\mu, d, \beta)$) such that $v(d, \mu, \beta_0, \lambda) = 0$.*

It would also be interesting to determine whether the first coordinate of the walk is recurrent when the parameters are chosen so that the resulting speed is zero. Perhaps this can be achieved when $d \geq 14$ using [6], Theorem 2.2.

The remainder of this paper is organised as follows. In Section 3 we recall some notation and results from [10], and give an infinite series representation for the velocity v that is valid when the series converges and the velocity is known to exist. In Section 4 we show that this formula converges when $d \geq 6$ and prove Theorem 2.2. In Section 5 we examine this formula further, under the assumptions of Theorem 2.3, and prove Theorem 2.3. In Section 6 we prove Lemma 2.4 using a coupling argument. Finally in Section 7 we discuss possible generalisations of these results and limitations of the methods used in this paper.

3. Results from the lace expansion

In this section we recall notation and results from [10] and [11] and give a formula for the velocity.

A nearest-neighbour random walk path $\vec{x}_n$ is a sequence $\{x_i\}_{i=0}^n$ for which $x_i = (x_i^{[1]}, \ldots, x_i^{[d]}) \in \mathbb{Z}^d$ and $|x_{i+1} - x_i| = 1$ (Euclidean distance) for each i. If $\vec{\eta}$ and $\vec{x}$ are two such paths of length at least j and m respectively and such that $\eta_j = x_0$, then the concatenation $\vec{\eta}_j \circ \vec{x}_m$ is defined by

$$(\vec{\eta}_j \circ \vec{x}_m)_i := \begin{cases} \eta_i & \text{when } 0 \leq i \leq j, \\ x_{i-j} & \text{when } j \leq i \leq m + j. \end{cases} \tag{3.1}$$

In particular, when $m = 0$, $(\vec{\eta}_j \circ \vec{x}_m)_i$ is defined for $0 \leq i \leq j$ and is equal to η_i.

For a general nearest-neighbour path $\vec{x}_i$, we use the notation $p^{\vec{x}_i}(x, y)$ for the conditional probability that the walk steps from x (where $x = x_i$ is implicit in the notation) to y, given the history of the path $\vec{x}_i = \{x_0, \ldots, x_i\}$. In other words, for any finite path $\vec{x}_i$ of non-zero Q_{x_0} measure,

$$p^{\vec{x}_i}(x_i, x_{i+1}) := Q_{x_0}(X_{i+1} = x_{i+1} | \vec{X}_i = \vec{x}_i). \tag{3.2}$$

Given $\vec{\eta}_m$ such that $Q_{\eta_0}(\vec{X}_m=\vec{\eta}_m)>0$, we define a conditional probability measure $Q^{\vec{\eta}_m}$ on walks starting from η_m by

$$Q^{\vec{\eta}_m}(\vec{X}_n=\vec{x}_n):=\prod_{i=0}^{n-1}p^{\vec{\eta}_m\circ\vec{x}_i}(x_i,x_{i+1})=Q(\vec{X}_{m+n}=\vec{\eta}_m\circ\vec{x}_n|\vec{X}_m=\vec{\eta}_m). \tag{3.3}$$

Note that by definition, $Q_{\eta_0}(\vec{X}_m=\vec{x}_m)=Q^{\eta_0}(\vec{X}_m=\vec{x}_m)$.

For excited random walks, the transition probability for the first step of the walk is

$$p^{x_0}(x_0,x_0+u)=Q_{x_0}(X_1=x_0+u)=\begin{cases} d^{-1}\mathbb{E}\big[\omega(o,1)\big] & \text{if } u=+e_1,\\ d^{-1}\mathbb{E}\big[1-\omega(o,1)\big] & \text{if } u=-e_1,\\ (2d)^{-1} & \text{if } u\in\{\pm e_2,\dots,\pm e_d\}.\end{cases}$$

More generally, we have from (3.2) that

$$p^{\vec{x}_i}(x_i,x_{i+1})=\frac{Q_{x_0}(\vec{X}_{i+1}=\vec{x}_{i+1})}{Q_{x_0}(\vec{X}_i=\vec{x}_i)}=\frac{\mathbb{E}[Q_{x_0,\omega}(\vec{X}_{i+1}=\vec{x}_{i+1})]}{\mathbb{E}[Q_{x_0,\omega}(\vec{X}_i=\vec{x}_i)]}. \tag{3.4}$$

For $n\ge 0$, let $L(\vec{x}_n)=\#\{0\le i\le n:\ x_i=x_n\}$ and $L^0(\vec{x}_n)=\#\{1\le i\le n:\ x_i-x_{i-1}\notin\{\pm e_1\}\}$. Then with the convention that an empty product is equal to 1,

$$\begin{aligned}&Q_{x_0,\omega}(\vec{X}_{i+1}=\vec{x}_{i+1})\\&\quad=\left(\frac{1}{2d}\right)^{L^0(\vec{x}_{i+1})}\prod_{z\in\vec{x}_i}\prod_{\substack{r\le i:x_r=z\\x_{r+1}=z+e_1}}d^{-1}\omega\big(z,L(\vec{x}_r)\big)\prod_{\substack{r\le i:x_r=z\\x_{r+1}=z-e_1}}d^{-1}\big(1-\omega\big(z,L(\vec{x}_r)\big)\big).\end{aligned}$$

Using the fact that the environment is i.i.d. (over sites $z\in\mathbb{Z}^d$) under $\mathbb{Q}$, the product over z in this expression can be taken outside the expectations in (3.4), along with the factors of $(2d)^{-1}$. The products over $z\neq x_i$ all cancel out in the numerator and denominator in (3.4) and we arrive at

$$\begin{aligned}p^{\vec{x}_i}(x_i,x_{i+1})&=\left(\frac{1}{2d}\right)^{L^0(\vec{x}_{i+1})-L^0(\vec{x}_i)}\\&\quad\times\frac{\mathbb{E}[\prod_{\substack{r\le i:x_r=x_i\\x_{r+1}=x_i+e_1}}d^{-1}\omega(x_i,L(\vec{x}_r))\prod_{\substack{r\le i:x_r=x_i\\x_{r+1}=x_i-e_1}}d^{-1}(1-\omega(x_i,L(\vec{x}_r)))]}{\mathbb{E}[\prod_{\substack{r\le i-1:x_r=x_i\\x_{r+1}=x_i+e_1}}d^{-1}\omega(x_i,L(\vec{x}_r))\prod_{\substack{r\le i-1:x_r=x_i\\x_{r+1}=x_i-e_1}}d^{-1}(1-\omega(z,L(\vec{x}_r)))]}.\end{aligned}$$

The exponent $L^0(\vec{x}_{i+1})-L^0(\vec{x}_i)$ is identical to the indicator that $x_{i+1}-x_i\in\{\pm e_2,\dots,\pm e_d\}$, while the expectations in the numerator and denominator above cancel out if $x_{i+1}-x_i\in\{\pm e_2,\dots,\pm e_d\}$. Thus we obtain

$$p^{\vec{x}_i}(x_i,x_{i+1})=\begin{cases}\frac{\mathbb{E}[B(\vec{x}_i)\omega(x_i,L(\vec{x}_i))]}{d\mathbb{E}[B(\vec{x}_i)]} & \text{if } x_{i+1}-x_i=e_1,\\ \frac{\mathbb{E}[B(\vec{x}_i)(1-\omega(x_i,L(\vec{x}_i)))]}{d\mathbb{E}[B(\vec{x}_i)]} & \text{if } x_{i+1}-x_i=-e_1,\\ \frac{1}{2d} & \text{if } x_{i+1}-x_i\in\{\pm e_2,\dots,\pm e_d\},\end{cases} \tag{3.5}$$

where

$$B(\vec{x}_i)=\prod_{\substack{r\le i-1:x_r=x_i\\x_{r+1}=x_i+e_1}}d^{-1}\omega\big(x_i,L(\vec{x}_r)\big)\prod_{\substack{r\le i-1:x_r=x_i\\x_{r+1}=x_i-e_1}}d^{-1}\big(1-\omega\big(x_i,L(\vec{x}_r)\big)\big). \tag{3.6}$$

Define $j_0 = 0$, and for $n \geq 1$, $j_n \geq 0$ and fixed paths $\vec{x}^{(n-1)}_{j_{n-1}+1}$ and $\vec{x}^{(n)}_{j_n+1}$ (with $x^{(n)}_0 = x^{(n-1)}_{j_{n-1}+1}$) let

$$\Delta_n := \Delta_n\bigl(\vec{x}^{(n-1)}_{j_{n-1}+1}, \vec{x}^{(n)}_{j_n+1}\bigr) = p^{\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n}}\bigl(x^{(n)}_{j_n}, x^{(n)}_{j_n+1}\bigr) - p^{\vec{x}^{(n)}_{j_n}}\bigl(x^{(n)}_{j_n}, x^{(n)}_{j_n+1}\bigr), \tag{3.7}$$

which is the difference in the probabilities of stepping from $x^{(n)}_{j_n}$ to $x^{(n)}_{j_n+1}$ with two different histories (one containing the other).

It is trivially true that Δ_n is zero if $x^{(n)}_{j_n+1} - x^{(n)}_{j_n} \notin \{\pm e_1\}$. We will now show that for (annealed) excited random walk in an i.i.d. random environment, Δ_n is also zero if $x^{(n)}_{j_n} \notin \vec{x}^{(n-1)}_{j_{n-1}}$. This is equivalent to the statement that if $\vec{x}^{(n-1)}_{j_{n-1}+1}$ has not eaten a cookie at site $x^{(n)}_{j_n}$ then both $\vec{x}^{(n-1)}_{j_{n-1}+1} \circ \vec{x}^{(n)}_{j_n}$ and $\vec{x}^{(n)}_{j_n}$ have eaten the same number of cookies at site $x^{(n)}_{j_n}$ and made the same steps away from that site. We have from (3.5) that

$$\begin{aligned}
\Delta_n = &\Biggl(\frac{\mathbb{E}[B(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n})\omega(x^{(n)}_{j_n}, L(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n}))]}{d\mathbb{E}[B(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n})]} \\
&- \frac{\mathbb{E}[B(\vec{x}^{(n)}_{j_n})\omega(x^{(n)}_{j_n}, L(\vec{x}^{(n)}_{j_n}))]}{d\mathbb{E}[B(\vec{x}^{(n)}_{j_n})]}\Biggr) I_{\{x^{(n)}_{j_n+1}-x^{(n)}_{j_n}=e_1\}} \\
&+ \Biggl(\frac{\mathbb{E}[B(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n})(1-\omega(x^{(n)}_{j_n}, L(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n})))]}{d\mathbb{E}[B(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n})]} - \frac{\mathbb{E}[B(\vec{x}^{(n)}_{j_n})(1-\omega(x^{(n)}_{j_n}, L(\vec{x}^{(n)}_{j_n})))]}{d\mathbb{E}[B(\vec{x}^{(n)}_{j_n})]}\Biggr) \\
&\times I_{\{x^{(n)}_{j_n+1}-x^{(n)}_{j_n}=-e_1\}}.
\end{aligned} \tag{3.8}$$

This quantity is zero if both

$$B\bigl(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n}\bigr) = B\bigl(\vec{x}^{(n)}_{j_n}\bigr) \quad \text{and} \quad L\bigl(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n}\bigr) = L\bigl(\vec{x}^{(n)}_{j_n}\bigr).$$

The second equality holds if and only if $x^{(n)}_{j_n} \notin \vec{x}^{(n-1)}_{j_{n-1}}$, by definition of $L(\cdot)$ and (3.1). Similarly if $x^{(n)}_{j_n} \notin \vec{x}^{(n-1)}_{j_{n-1}}$ then from (3.6) we have that $B(\vec{x}^{(n-1)}_{j_{n-1}+1}\circ \vec{x}^{(n)}_{j_n}) = B(\vec{x}^{(n)}_{j_n})$.

Let $\mathbb{Z}_+$ denote the non-negative integers and define $\mathcal{A}_{m,N} := \{(j_1, \dots, j_N) \in \mathbb{Z}^N_+ \colon \sum_{l=1}^N j_l = m - N - 1\}$, and

$$\pi^{(N)}_m(x, y) := \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{x}^{(0)}_1} \sum_{\vec{x}^{(1)}_{j_1+1}} \cdots \sum_{\vec{x}^{(N)}_{j_N+1}} I_{\{x^{(N)}_{j_N}=x, x^{(N)}_{j_N+1}=y\}} p^o\bigl(o, x^{(0)}_1\bigr) \prod_{n=1}^N \Delta_n Q^{\vec{x}^{(n-1)}_{j_{n-1}+1}}\bigl(\vec{X}_{j_n} = \vec{x}^{(n)}_{j_n}\bigr), \tag{3.9}$$

where (here and throughout this paper), each $\sum_{\vec{x}^{(n)}_{j_n+1}}$ is the sum over nearest-neighbour paths $(x^{(n)}_1, \dots, x^{(n)}_{j_n+1}) \in \mathbb{Z}^{d(j_n+1)}$ of length $j_n + 1 \geq 1$, starting at $x^{(n)}_0 = x^{(n-1)}_{j_{n-1}+1}$, and where $x^{(0)}_0 = o$. Recall from (3.1) that when $i_n = 0$, $\vec{x}^{(n-1)}_{j_{n-1}+1} \circ \vec{x}^{(n)}_{i_n} = \vec{x}^{(n-1)}_{j_{n-1}+1}$. Recall also that $j_0 = 0$ and from (3.7) that each Δ_n depends on the paths $\vec{x}^{(n)}_{j_n+1}$ and $\vec{x}^{(n-1)}_{j_{n-1}+1}$. In particular Δ_n may depend on the length of these paths. The summand is zero if the paths are not nearest-neighbour paths, so that we don't need to include this restriction in the summation notation. An empty sum is defined to be zero, while an empty product is defined to be 1. Note that the sum over $\vec{j}$ is empty when $m < N + 1$, so that $\pi^{(N)}_m(x, y)$ is non-zero only when $m \geq N + 1$.

Define in addition the quantities

$$\pi_m(x, y) := \sum_{N=1}^\infty \pi^{(N)}_m(x, y), \qquad \pi^{(N)}(x, y) := \sum_m \pi^{(N)}_m(x, y) \quad \text{and} \quad \pi_m(y) := \sum_{N=1}^\infty \sum_x \pi^{(N)}_m(x, y), \tag{3.10}$$

where (here and throughout this paper) $\sum_x$ denotes summation over $x \in \mathbb{Z}^d$. The summand is usually zero unless x is a nearest-neighbour of another variable, e.g. $\pi^{(N)}_m(x, y)$ is zero if x is not a nearest-neighbour of y. Furthermore,

$\sum_y \pi_m^{(N)}(x,y)=0$ since summing Δ_N over $x_{j_N+1}^{(N)}$ gives $1-1=0$ by (3.7). In view of Theorem 2.1, the following result gives a formula for the velocity, provided the formula converges.

Theorem 3.1 (Theorem 3.1 of [10]). *For $d \geq 6$, the speed of the excited random walk in i.i.d. cookie environment is given by*

$$v = E[X_1] + \sum_{m=2}^{\infty} \sum_y y\pi_m(y), \tag{3.11}$$

whenever this series converges.

Since $\sum_y \pi_m^{(N)}(x,y)=0$, this formula can also be written in the following more useful form

$$v = E[X_1] + \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x,y} (y-x)\pi_m^{(N)}(x,y). \tag{3.12}$$

4. The formula for the speed

In this section we analyse the formula (3.12) by using the fact that the $d-1$ coordinates $(X_n^{[2]},\dots,X_n^{[d]})$ of the excited random walk behave as a simple random walk (with geometric waiting times). Green's functions upper bounds are then used to prove that the formula (3.12) converges in high dimensions and to prove Theorem 2.2.

Let $\mathbb{P}_d$ denote the law of simple symmetric random walk in d dimensions, beginning at the origin. Let $D_d^{*0}(x) = I_{\{x=o\}}$ and let $D_d(x) = (2d)^{-1} I_{\{|x|=1\}}$ be the simple random walk step distribution. Let $G_d(x) = \sum_{k=0}^{\infty} D_d^{*k}(x)$ denote the Green's function for this random walk, where for absolutely summable functions f, g on $\mathbb{Z}^d$, the convolution of f and g is defined by $(f * g)(x) := \sum_y f(y)g(x-y)$, and for $k \geq 1$, $f^{*k}(x)$ denotes the k-fold convolution of f with itself (with $f^{*1}(x) = f(x)$). Then $G_d^{*k}(x) < \infty$ when $d > 2k$. By [9], Lemma B.3, $G_d^{*k} := \sup_x G_d^{*k}(x) = G_d^{*k}(o)$.

For $i \in \mathbb{Z}_+$ define

$$\mathcal{E}_i(d) := \sup_{v\in\mathbb{Z}^{d-1}} \left(\left(\frac{d}{d-1}\right)^{i+1} G_{d-1}^{*(i+1)}(v) - \delta_{o,v} \right) = \left(\frac{d}{d-1}\right)^{i+1} G_{d-1}^{*(i+1)} - 1, \tag{4.1}$$

where the second equality is [11], Eq. (5.1). This quantity is finite if $d-1 > 2(i+1)$.

For $d > 1$ define the following quantity (which is finite only when $d-1 > 4$)

$$a_d := \frac{d}{(d-1)^2} G_{d-1}^{*2}. \tag{4.2}$$

We will now prove the following result which is essentially the content of [11], Lemmas 3.1 and 4.3. In this result $Q^{\leftrightarrow_k,\vec{x}_t}$ denotes the law of a walk that evolves as an excited random walk with history $\vec{x}_t$, except that its $(k+1)$st increment is chosen from $\pm e_1$ with equal probability ($\frac{1}{2}$), independent of both the history of the walk and $\mathbb{Q}$.

Lemma 4.1. *For all $i \in \mathbb{Z}_+$, all $u \in \mathbb{Z}^d$ and all nearest-neighbour paths $\vec{x}_t$, $t \in \mathbb{Z}_+$,*

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{x}_t}(X_j = u) \leq i! \left(\frac{d}{d-1}\right)^{i+1} G_{d-1}^{*(i+1)}, \tag{4.3}$$

$$\sum_{j=1}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{x}_t}(X_j = u) \leq i! \mathcal{E}_i(d), \tag{4.4}$$

$$\sup_{\mathbb{Q}} \sup_{\vec{x}_t,u} \sum_{j=M_0}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{x}_t}(X_j = u) \to 0 \quad \textit{as } M_0 \to \infty, \textit{ if } d > 2(i+1)+1, \tag{4.5}$$

$$\sum_{j=1}^{\infty} \frac{(j+i)!}{j!} \sum_{k=0}^{j-1} Q^{\leftrightarrow_k, \vec{x}_t}(X_j = u) \le (i+1)! \left(\frac{d}{d-1}\right)^{i+2} G_{d-1}^{*(i+2)}, \tag{4.6}$$

$$\sup_{\mathbb{Q}} \sup_{\vec{x}_t, u} \sum_{j=M_0}^{\infty} \frac{(j+i)!}{j!} \sum_{k=0}^{j-1} Q^{\leftrightarrow_k, \vec{x}_t}(X_j = u) \to 0 \quad \textit{as } M_0 \to \infty, \textit{ if } d > 2(i+2)+1, \tag{4.7}$$

where the supremum over $\mathbb{Q}$ is a supremum over all i.i.d. cookie environment measures.

Proof. Independently of both the history $\vec{x}_t$ and the law of the environment $\mathbb{Q}$, under $Q^{\vec{x}_t}$ the projection $\vec{X}_j^{[-1]}$ (defined by $X_j^{[-1]} = (X_j^{[2]}, \dots, X_j^{[d]} \in \mathbb{Z}^{d-1})$) of the excited random walk behaves as a simple random walk in $d-1$ dimensions that does not move with probability $\frac{1}{d}$ and moves (to a uniformly chosen nearest neighbour) with probability $q_d := \frac{d-1}{d}$. Let the number of moves made by $\vec{X}_j^{[-1]}$ be $\mathcal{N}_j$. Then $\mathcal{N}_j \sim$Bin(j, q_d), and $\mathcal{N} = \{\mathcal{N}_j\}_{j \ge 0}$ is an increasing random walk on $\mathbb{Z}_+$ with increments $+1$ (with probability q_d) or 0 (with probability $1-q_d$). Conditioning on $\mathcal{N}_j$ we have

$$\begin{aligned} Q^{\vec{x}_t}(X_j = u) &= \sum_{l=0}^{j} Q^{\vec{x}_t}(X_j = u | \mathcal{N}_j = l) Q^{\vec{x}_t}(\mathcal{N}_j = l) \\ &\le \sum_{l=0}^{j} Q^{\vec{x}_t}\big(X_j^{[-1]} = u^{[-1]} | \mathcal{N}_j = l\big) \mathcal{P}(\mathcal{N}_j = l) \le \sum_{l=0}^{j} \mathbb{P}_{d-1}\big(X_l = (u - x_t)^{[-1]}\big) \mathcal{P}(\mathcal{N}_j = l), \end{aligned} \tag{4.8}$$

where $\mathcal{P}$ is the law of $\mathcal{N}$ (which does not depend on $\mathbb{Q}$, u, $\vec{x}_t$). Therefore for all $r \in \mathbb{Z}_+$,

$$\sum_{j=r}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{x}_t}(X_j = u) \le \sup_{v \in \mathbb{Z}^{d-1}} \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) \sum_{j=r \vee l}^{\infty} \frac{(j+i)!}{j!} \mathcal{P}(\mathcal{N}_j = l) \tag{4.9}$$

$$= \sup_{v \in \mathbb{Z}^{d-1}} \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) E_{\mathcal{P}}\left[\sum_{j=\tau_l \vee r}^{\tau_{l+1}-1} \frac{(j+i)!}{j!} I_{\{\mathcal{N}_j = l\}}\right], \tag{4.10}$$

where $E_{\mathcal{P}}$ denotes expectation with respect to $\mathcal{P}$, and τ_l is the hitting time of level l by $\mathcal{N}$. The expectation is bounded by the corresponding term with $r = 0$ which is

$$\begin{aligned} \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} \mathcal{P}(\mathcal{N}_j = l) &= \sum_{j=l}^{\infty} \frac{(j+i)!}{j!} \frac{j!}{(j-l)! l!} q_d^l (1-q_d)^{j-l} \\ &= q_d^{-i} \frac{(l+i)!}{l!} \sum_{j=l}^{\infty} \frac{(j+i)!}{(j+i-(l+i))!(l+i)!} q_d^{l+i} (1-q_d)^{j+i-(l+i)} \\ &= q_d^{-i} \frac{(l+i)!}{l!} \sum_{j=l}^{\infty} \mathcal{P}(\mathcal{N}_{j+i} = l+i) = q_d^{-(i+1)} \frac{(l+i)!}{l!}. \end{aligned} \tag{4.11}$$

It follows that the summand in (4.10) is bounded by $\mathbb{P}_{d-1}(X_l = v) q_d^{-(i+1)} \frac{(l+i)!}{l!}$, where (see e.g. [11], Eq. (3.2), using the fact that the number of ways of partitioning l into a sum of $i+1$ non-negative integers is $\binom{l+i}{i}$)

$$\sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) q_d^{-(i+1)} \frac{(l+i)!}{l!} = i! q_d^{-(i+1)} G_{d-1}^{*(i+1)}(v) \le i! q_d^{-(i+1)} G_{d-1}^{*(i+1)} < \infty \quad \text{if } G_{d-1}^{*(i+1)} < \infty. \tag{4.12}$$

This establishes the first claim.

For the second claim proceeding as above we have

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{(j+i)!}{j!}Q^{\vec{x}_t}(X_j=u) &\le \sum_{l=0}^{\infty}\mathbb{P}_{d-1}\big(X_l=u^{[-1]}\big)\sum_{j=1\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)\\ &\le \sup_{v\in\mathbb{Z}^{d-1}}\sum_{l=0}^{\infty}\mathbb{P}_{d-1}(X_l=v)\Bigg[\sum_{j=l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)-\delta_{0,l}i!\Bigg]\\ &= \sup_{v\in\mathbb{Z}^{d-1}}\sum_{l=0}^{\infty}\mathbb{P}_{d-1}(X_l=v)\Bigg[\sum_{j=l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)-\delta_{0,l}i!\Bigg]\\ &\le i!\sup_{v\in\mathbb{Z}^{d-1}}\big[q_d^{-i}G^{*(i+1)}(v)-\delta_{v,o}\big],\end{aligned}\tag{4.13}$$

as required.

To prove the third claim, for each $K\in\mathbb{Z}_+$ the right hand side of (4.9) is bounded by the supremum over v of

$$\sum_{l=0}^{K}\mathbb{P}_{d-1}(X_l=v)\sum_{j=r\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)+\sum_{l=K+1}^{\infty}\mathbb{P}_{d-1}(X_l=v)\sum_{j=0\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l).\tag{4.14}$$

By first choosing K_0 sufficiently large (not depending on r) and then choosing r sufficiently large depending on K_0, to prove the third claim it is enough to show that

$$\sup_{\mathbb{Q},v}\sum_{l=K_0+1}^{\infty}\mathbb{P}_{d-1}(X_l=v)\sum_{j=l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)\to 0\quad\text{as } K_0\to\infty,\quad\text{and}\tag{4.15}$$

$$\text{for every } l\in\mathbb{Z}_+,\qquad\sum_{j=r\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)\to 0\quad\text{as } r\to\infty.\tag{4.16}$$

By (4.11) we have

$$\begin{aligned}\sum_{l=K_0+1}^{\infty}\mathbb{P}_{d-1}(X_l=v)\sum_{j=0\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l) &\le \sum_{l=K_0+1}^{\infty}\mathbb{P}_{d-1}(X_l=v)q_d^{-(i+1)}\frac{(l+i)!}{l!}\\ &\le \sum_{l=K_0+1}^{\infty}\mathbb{P}_{d-1}(X_l=v)C_{d,i}l^i,\end{aligned}\tag{4.17}$$

for some constant $C_{d,i}$ depending only on d and i. By the local central limit theorem, there exists a constant C_d depending only on d such that for every n, $\sup_v\mathbb{P}_{d-1}(X_n=v)\le C_dn^{-(d-1)/2}$. Thus (4.17) is bounded by

$$\sum_{l=K_0+1}^{\infty}\frac{C'_{d,i}}{l^{(d-1)/2-i}}\to 0\quad\text{as } K_0\to\infty,\tag{4.18}$$

when $\frac{d-1}{2}-i>1$. This establishes (4.15).

For (4.16) recall that

$$\sum_{j=r\vee l}^{\infty}\frac{(j+i)!}{j!}\mathcal{P}(\mathcal{N}_j=l)=E_{\mathcal{P}}\Bigg[\sum_{j=\tau_l\vee r}^{\tau_{l+1}-1}\frac{(j+i)!}{j!}I_{\{\mathcal{N}_j=l\}}\Bigg]\le E_{\mathcal{P}}\big[C_i\tau_{l+1}^{i+1}I_{\{\tau_l>r\}}\big].\tag{4.19}$$

Since $E_{\mathcal{P}}[\tau_{l+1}^{i+1}]<\infty$ and $I_{\{\tau_l>r\}}\to 0$ almost surely as $r\to 0$ the result follows by dominated convergence.

It remains to establish the claims (4.6) and (4.7). Note that the claimed bound (4.6) is the same as that of (4.3) with $i+1$ replacing i. Similarly the condition for convergence in (4.7) is that of (4.5) with $i+1$ instead of i. Conditioning on the number of moves $\mathcal{N}_{j-1} \sim \mathrm{Bin}(j-1, q_d)$ made by $X_j^{[-1]}$ (note that $X_{k+1} - X_k = \pm e_1$)

$$Q^{\leftrightarrow_k, \vec{x}_t}(X_j = u) = \sum_{l=0}^{j-1} Q^{\leftrightarrow_k, \vec{x}_t}(X_j = u | \mathcal{N}_{j-1} = l) Q^{\leftrightarrow_k, \vec{x}_t}(\mathcal{N}_{j-1} = l)$$
$$\le \sum_{l=0}^{j-1} \mathbb{P}_{d-1}\big(X_l = (u - x_t)^{[-1]}\big) \mathcal{P}(\mathcal{N}_{j-1} = l). \tag{4.20}$$

Therefore the left-hand side of (4.6) is bounded by (with $r = 1$)

$$\sup_{v \in \mathbb{Z}^{d-1}} \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) \sum_{j=(l+1)\vee r}^{\infty} \frac{(j+i)!}{j!} j \mathcal{P}(\mathcal{N}_{j-1} = l)$$
$$\le \sup_{v \in \mathbb{Z}^{d-1}} \sum_{l=0}^{\infty} \mathbb{P}_{d-1}(X_l = v) \sum_{j'=l\vee(r-1)}^{\infty} \frac{(j'+i+1)!}{j'!} \mathcal{P}(\mathcal{N}_{j'} = l). \tag{4.21}$$

This is the same as (4.9) with $i+1$ instead of i. The claimed result (4.6) now follows from the bound (4.12) on (4.9). Similarly the claim (4.7) follows from (4.14)–(4.19) with $i+1$ instead of i. □

The following proposition is essentially [11], Proposition 3.2, with β replaced by 2 in each of the bounds. In this proposition, the bounds do not depend on the cookie-environment measure $\mathbb{Q}$.

Proposition 4.2 (Bounds on the expansion coefficients). *For excited random walk with i.i.d. cookie environment*,

$$\sum_{m=2}^{\infty} \sum_{x,y} \big|(y-x)\pi_m^{(N)}(x,y)\big| \le \begin{cases} 2d^{-1}\mathcal{E}_0(d), & N = 1, \\ \big[\frac{2}{d}\mathcal{E}_1(d)\big]\big[(2a_d)^{N-2}\big]\big[\frac{2G_{d-1}}{d-1}\big], & N > 1, \end{cases} \tag{4.22}$$

$$\text{for each } N \ge 1, \quad \sup_{\mathbb{Q}} \sum_{m=M}^{\infty} \sum_{x,y} \big|(y-x)\pi_m^{(N)}(x,y)\big| \to 0 \quad \text{as } M \to \infty \text{ if } d \ge 6 \tag{4.23}$$

where the supremum is over all i.i.d. cookie environment measures $\mathbb{Q}$.

Proof. The given bounds are verified via an inductive argument (in N) and multiple applications of Lemma 4.1. From (3.9), since the sum over x and y of the indicator function becomes 1, and $|y-x| = 1$ if $\pi_m^{(N)}(x,y) \neq 0$ we have

$$\sum_{m=2}^{\infty} \sum_{x,y} \big|(y-x)\pi_m^{(N)}(x,y)\big|$$
$$\le \sum_{j_1=0}^{\infty} \cdots \sum_{j_N=0}^{\infty} \sum_{\vec{x}_1^{(0)}} \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} p^o\big(o, x_1^{(0)}\big) \prod_{n=1}^{N} |\Delta_n| Q^{\vec{x}_{j_{n-1}+1}^{(n-1)}}\big(\vec{X}_{j_n} = \vec{x}_{j_n}^{(n)}\big)$$
$$= \sum_{j_1=0}^{\infty} \sum_{\vec{x}_1^{(0)}} p^o\big(o, x_1^{(0)}\big) \sum_{\vec{x}_{j_1+1}^{(1)}} |\Delta_1| Q^{\vec{x}_{j_0+1}^{(0)}}\big(\vec{X}_{j_1} = \vec{x}_{j_1}^{(1)}\big) \cdots \sum_{j_{N-1}=0}^{\infty} \sum_{\vec{x}_{j_{N-1}+1}^{(N-1)}} |\Delta_{N-1}| Q^{\vec{x}_{j_{N-2}+1}^{(N-2)}}\big(\vec{X}_{j_{N-1}} = \vec{x}_{j_{N-1}}^{(N-1)}\big)$$
$$\times \sum_{j_N=0}^{\infty} \sum_{\vec{x}_{j_N+1}^{(N)}} |\Delta_N| Q^{\vec{x}_{j_{N-1}+1}^{(N-1)}}\big(\vec{X}_{j_N} = \vec{x}_{j_N}^{(N)}\big). \tag{4.24}$$

If $N=1$ this quantity is

$$\sum_{j_1=0}^{\infty}\sum_{\vec{x}_1^{(0)}} p^o\big(o,x_1^{(0)}\big)\sum_{\vec{x}_{j_1+1}^{(1)}}|\Delta_1|Q^{\vec{x}_{j_0+1}^{(0)}}\big(\vec{X}_{j_1}=\vec{x}_{j_1}^{(1)}\big). \tag{4.25}$$

From the discussion following (3.7) and the fact that (almost surely) $|\omega(x,k)-\omega(x,k')|\le 1$ for all $k,k'\in\mathbb{N}$, we have that for each $n\in\mathbb{N}$,

$$|\Delta_n|\le \frac{1}{d}I_{\{x_{j_n+1}^{(n)}-x_{j_n}^{(n)}=\pm e_1\}}I_{\{x_{j_n}^{(n)}\in\vec{x}_{j_{n-1}}^{(n-1)}\}}\le \frac{1}{d}I_{\{x_{j_n+1}^{(n)}-x_{j_n}^{(n)}=\pm e_1\}}\sum_{r_{n-1}=0}^{j_{n-1}}I_{\{x_{j_n}^{(n)}=x_{r_{n-1}}^{(n-1)}\}}. \tag{4.26}$$

Therefore since $j_0=0$ and $x_0^{(0)}=o$, (4.25) is bounded by

$$\begin{aligned}&\frac{1}{d}\sum_{j_1}\sum_{\vec{x}_1^{(0)}}p^o\big(o,x_1^{(0)}\big)\sum_{\vec{x}_{j_1}^{(1)}}I_{\{x_{j_1}^{(1)}=x_0^{(0)}\}}Q^{\vec{x}_1^{(0)}}\big(\vec{X}_{j_1}=\vec{x}_{j_1}^{(1)}\big)\sum_{x_{j_1+1}^{(1)}}I_{\{x_{j_1+1}^{(1)}-x_{j_1}^{(1)}=\pm e_1\}}\\ &\quad\le\frac{2}{d}\sum_{j_1}\sum_{\vec{x}_1^{(0)}}p^o\big(o,x_1^{(0)}\big)\sum_{x_{j_1}^{(1)}}I_{\{x_{j_1}^{(1)}=o\}}Q^{\vec{x}_1^{(0)}}\big(X_{j_1}=x_{j_1}^{(1)}\big).\end{aligned} \tag{4.27}$$

The summand is zero if $j_1=0$ since $x_0^{(1)}=x_1^{(0)}\neq x_0^{(0)}$. Hence this is equal to

$$\frac{2}{d}\sum_{j_1=1}^{\infty}\sum_{\vec{x}_1^{(0)}}p^o\big(o,x_1^{(0)}\big)Q^{\vec{x}_1^{(0)}}(X_{j_1}=o)=\frac{2}{d}\sum_{j=2}^{\infty}Q^o(X_j=o), \tag{4.28}$$

and the first bound follows for $N=1$ by (4.4) with $i=0$. Similarly, when $N=1$, the left hand side of (4.23) is bounded by

$$\sup_{\mathbb{Q}}\sum_{j_1=M-2}^{\infty}\sum_{\vec{x}_1^{(0)}}p^o\big(o,x_1^{(0)}\big)\sum_{\vec{x}_{j_1+1}^{(1)}}|\Delta_1|Q^{\vec{x}_{j_0+1}^{(0)}}\big(\vec{X}_{j_1}=\vec{x}_{j_1}^{(1)}\big).$$

Comparing this with (4.25) and proceeding as above, the term inside the supremum is bounded by $\frac{2}{d}\sum_{j=M-1}^{\infty}Q^o(X_j=o)$, which does not depend on $\mathbb{Q}$ and which converges to 0 as $M\to\infty$, as the tail of a convergent series.

For $N>1$, by (4.26) the last line of (4.24) is bounded by

$$\begin{aligned}&\frac{1}{d}\sum_{r_{N-1}=0}^{j_{N-1}}\sum_{j_N=0}^{\infty}\sum_{\vec{x}_{j_N}^{(N)}}I_{\{x_{j_N}^{(N)}=x_{r_{N-1}}^{(N-1)}\}}Q^{\vec{x}_{j_{N-1}+1}^{(N-1)}}\big(\vec{X}_{j_N}=\vec{x}_{j_N}^{(N)}\big)\sum_{x_{j_N+1}^{(N)}}I_{\{x_{j_N+1}^{(N)}-x_{j_N}^{(N)}=\pm e_1\}}\\ &\quad\le\frac{2}{d}\sum_{r_{N-1}=0}^{j_{N-1}}\sup_u\sum_{j_N=0}^{\infty}Q^{\vec{x}_{j_{N-1}+1}^{(N-1)}}(X_{j_N}=u)\le\frac{2(j_{N-1}+1)}{d-1}G_{d-1}\end{aligned} \tag{4.29}$$

by (4.3) with $i=0$.

We can repeat the above procedure for the $n=N-1$ term, however there is now an additional factor of $(j_{N-1}+1)$ in the sum over j_{N-1} that was not present in the computation above. To be precise, from (4.24) and (4.29), we need to bound

$$\sum_{j_{N-1}=0}^{\infty}(j_{N-1}+1)\sum_{\vec{x}_{j_{N-1}+1}^{(N-1)}}|\Delta_{N-1}|Q^{\vec{x}_{j_{N-2}+1}^{(N-2)}}\big(\vec{X}_{j_{N-1}}=\vec{x}_{j_{N-1}}^{(N-1)}\big).$$

Up to a change of indices, this term is the same as the $n = N$ term that we have just bounded (i.e. the last line of (4.24)) except for an extra factor $(j_{N-1} + 1)$. We therefore apply (4.3) with $i = 1$ instead of $i = 0$ at the appropriate point in the argument, giving a bound

$$(j_{N-2}+1)\frac{2}{d}\left(\frac{d}{d-1}\right)^2 G_{d-1}^{*(2)} = 2(j_{N-2}+1)a_d.$$

Repeating this procedure, we eventually encounter the term $n = 1$, which is (4.25) except for an extra factor of (j_1+1), i.e.

$$\sum_{j_1=0}^{\infty}(j_1+1)\sum_{\vec{x}_1^{(0)}} p^o\big(o, x_1^{(0)}\big)\sum_{\vec{x}_{j_1+1}^{(1)}} |\varDelta_1| Q^{\vec{x}_{j_0+1}^{(0)}}\big(\vec{X}_{j_1} = \vec{x}_{j_1}^{(1)}\big). \tag{4.30}$$

Proceeding as for (4.25) but using (4.4) with $i = 1$ instead of $i = 0$ gives a bound on this term of $\frac{2}{d}\mathcal{E}_1(d)$. Collecting the bounds: $\frac{2}{d-1}G_{d-1}$ from the $n = N$ term, $2a_d$ for each of the $N-2$ terms $n = N-1, \dots, n = 2$, and $\frac{2}{d}\mathcal{E}_1(d)$ for the $n = 1$ term, we obtain (4.22) when $N > 1$.

To prove (4.23) for $N > 1$, note that $\sum_{i=1}^N j_i = m - N - 1$ for $\vec{j} \in \mathcal{A}_{m,N}$, so we have that for fixed N, and $m \geq M + N + 1$, some j_i must be at least $\lceil M/N \rceil$. Therefore

$$\begin{aligned}\sum_{m=M+N+1}^{\infty}\sum_{j\in\mathcal{A}_{m,N}} &\leq \sum_{m=M+N+1}^{\infty}\sum_{t=1}^{N}\sum_{j\in\mathcal{A}_{m,N}} I_{\{j_t \geq \lceil M/N\rceil\}} \\ &\leq \sum_{t=1}^{N}\sum_{j_1=0}^{\infty}\cdots\sum_{j_{t-1}=0}^{\infty}\sum_{j_t=\lceil M/N\rceil}^{\infty}\sum_{j_{t+1}=0}^{\infty}\cdots\sum_{j_N=0}^{\infty}.\end{aligned} \tag{4.31}$$

Proceed as above, starting with the term $n = N$, until reaching the term $n = t$. The usual bounds lead us to a quantity

$$\frac{2}{d}\sum_{r_{t-1}=0}^{j_{t-1}}\sup_u \sum_{j_t=\lceil M/N\rceil}^{\infty}\frac{(j+i)!}{j!}Q^{\vec{x}_{j_{t-1}+1}^{(t-1)}}(X_{j_t} = u), \tag{4.32}$$

with $i = 1$ (or $i = 0$ when $t = N$). On this term, instead of using (4.3) (or (4.4) if $t = 1$) we use (4.5) with $M_0 = M/N$ to make this term at most ε for large M. This replaces the factor in (4.22) corresponding to this term by a factor of ε. Then proceed to bound the remaining terms as before. □

Armed with Proposition 4.2, we now require only bounds on the quantities appearing therein (i.e. simple random walk Green's function upper bounds) to prove Theorem 2.2. One such result is the following.

Corollary 4.3. *For $d \geq 6$, the annealed velocity of the excited random walk in i.i.d. cookie environment is given by*

$$v = E[X_1] + \sum_{m=2}^{\infty}\sum_{N=1}^{\infty}\sum_{x,y}(y-x)\pi_m^{(N)}(x,y), \tag{4.33}$$

where $E[X_1^{[1]}] = \frac{2\delta_1 - 1}{d}$.

Proof. From [9], G_d^{*i} is finite for $d > 2i$ and for fixed i, G_d^{*i} is decreasing in d. In fact [9] gives the rigorous upper bound $G_5^{*2} < 25/12$, which implies that $2a_6 < 1$. It follows immediately from summing the bound of Proposition 4.2 over $N \geq 1$ that $\sum_{x,y}\sum_{m\geq 2}|(y-x)\pi_m(x,y)|$ converges. It then follows from Theorems 3.1 and 2.1 that we indeed have (3.12). An elementary calculation shows that $\mathbb{E}[X_1^{[1]}] = \frac{2\delta_1 - 1}{d}$ as required. □

Proof of Theorem 2.2. For $d \geq 6$, by taking δ_1 sufficiently close to 1 the first term in (4.33) can be made as close to $\frac{1}{d}$ as we like. Thus to prove Theorem 2.2 it is sufficient to show that there exists $\varepsilon > 0$ such that (regardless of the cookie environment) $d|\sum_{m=2}^{\infty}\sum_{N=1}^{\infty}\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)| < 1-\varepsilon$.

From Proposition 4.2 we have (for any i.i.d. cookie environment) that

$$d\left|\sum_{m=2}^{\infty}\sum_{N=1}^{\infty}\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\right| \leq 2\mathcal{E}_0(d) + 4(d-1)^{-1}G_{d-1}\mathcal{E}_1(d)\sum_{N=2}^{\infty}(2a_d)^{N-2}$$
$$= 2\mathcal{E}_0(d) + \frac{4G_{d-1}\mathcal{E}_1(d)}{(1-2a_d)(d-1)}.$$

The right-hand side is decreasing in d since the G_d^{*i} are. This quantity is indeed smaller than 1 when $d \geq 9$, as can easily be checked using the rigorous upper bounds $G_8 \leq 1.078648$ and $G_8^{(*2)} \leq 1.289003$ [8,9]. □

5. Continuity and differentiability

In this section we prove Theorem 2.3. Recall that in this theorem we are concerned with environments for which there exists some finite $A \subset \mathbb{N}$ such that for each $k \in A$, $\omega(o,k)$ is independent of $(\omega(o,j))_{j\neq k}$. With this assumption, under the annealed measure, for every $k \in A$ we have that the drift induced by the kth cookie at any site is given by $\mathbb{E}[\omega(o,k)]$. This is because for all $j \neq k$, the jth departure from o gives no information about $\omega(o,k)$ at all. To be precise, if $\omega(o,k)$ is independent of $\omega(o,j)$ for all $j \neq k$ under $\mathbb{Q}$, then from (3.5), when $L(\vec{x}_i) = k$ we have

$$p^{\vec{x}_i}(x_i,x_{i+1}) \begin{cases} \frac{\mathbb{E}[B(\vec{x}_i)\omega(x_i,k)]}{d\mathbb{E}[B(\vec{x}_i)]} = \frac{\mathbb{E}[\omega(x_i,k)]}{d} = \frac{\delta_k}{d} & \text{if } x_{i+1}-x_i = e_1, \\ \frac{\mathbb{E}[B(\vec{x}_i)(1-\omega(x_i,k))]}{d\mathbb{E}[B(\vec{x}_i)]} = \frac{\mathbb{E}[(1-\omega(x_i,k))]}{d} = \frac{1-\delta_k}{d} & \text{if } x_{i+1}-x_i = -e_1, \\ \frac{1}{2d}, & \text{if } x_{i+1}-x_i \in \{\pm e_2,\dots,\pm e_d\}, \end{cases} \tag{5.1}$$

since

$$B(\vec{x}_i) = \prod_{\substack{r\leq i-1:x_r=x_i\\x_{r+1}=x_i+e_1}} d^{-1}\omega\big(x_i,L(\vec{x}_r)\big)\prod_{\substack{r\leq i-1:x_r=x_i\\x_{r+1}=x_i-e_1}} d^{-1}\big(1-\omega\big(x_i,L(\vec{x}_r)\big)\big)$$

depends only on $(\omega(x_i,j))_{j<k}$ and hence is independent of $\omega(x_i,k)$. When $L(\vec{x}_i) = j \neq k$, $p^{\vec{x}_i}(x_i,x_{i+1})$ does not depend on δ_k (or $\omega(x_i,k)$) at all. This is because $\omega(x_i,k)$ is independent of all quantities appearing in the expectations in (5.1) if $j<k$, while if $j>k$ the $\omega(x_i,k)$ terms appearing in $B(\vec{x}_i)$ can be factored out (by independence) and cancelled out in the expectations of the numerator and denominator.

Let $M \in \mathbb{N}$ and $A \subset \mathbb{N}$ be finite. Let $\mathbb{Q}$ be an i.i.d. cookie environment measure satisfying the conditions of Theorem 2.3, i.e. $\omega(o,i)$ is independent of $(\omega(o,j))_{j\neq i}$ for each $i \in A$ and let $\vec{\delta} = (\delta_i)_{i\in A}$. For each fixed joint distribution of $\omega(o,A^c) = (\omega(o,i))_{i\notin A}$ and each $k \in A$ define

$$V_M(\vec{\delta}) := \sum_{m=2}^{M}\sum_{N=1}^{m-1}\sum_{x,y}(y-x)\pi_m^{(N)}(x,y), \quad \text{and} \tag{5.2}$$

$$V_M^{\langle k\rangle}(\vec{\delta}) := \frac{\partial}{\partial\delta_k}V_M(\vec{\delta}) = \sum_{m=2}^{M}\sum_{N=1}^{m-1}\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y). \tag{5.3}$$

Each V_M is continuous in $\vec{\delta}$ and is differentiable with respect to δ_k for any $k \in A$, as it consists of finite sums and products of continuous and differentiable functions, namely the transition probabilities $p^{\vec{x}_i}(x_i,x_{i+1})$. In particular, by (5.1) and the discussion following it,

$$\frac{\partial}{\partial\delta_k}p^{\vec{x}_i}(x_i,x_{i+1}) = \frac{1}{d}I_{\{L(\vec{x}_i)=k\}}[I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]. \tag{5.4}$$

Similarly,

$$\frac{\partial}{\partial \delta_k}\big(p^{\vec{x}_i}(x_i,x_{i+1}) - p^{\vec{\eta}_j\circ\vec{x}_i}(x_i,x_{i+1})\big) = \frac{1}{d}[I_{\{L(\vec{x}_i)=k\}} - I_{\{L(\vec{\eta}_j\circ\vec{x}_i)=k\}}][I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]$$
$$= \frac{1}{d}[I_{\{L(\vec{x}_i)=k\}} I_{\{L(\vec{\eta}_j\circ\vec{x}_i)>k\}}][I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}], \tag{5.5}$$

where we have used the fact that $L(\vec{\eta}_j \circ \vec{x}_i) \geq L(\vec{x}_i)$.

We wish to study the limits of the functions V_M and $V_M^{\langle k\rangle}$ as $M\to\infty$. In particular we want that $V(\vec{\delta}) = \lim_{M\to\infty} V_M(\vec{\delta})$ is continuous and that $\frac{\partial}{\partial\delta_k}V(\vec{\delta})$ exists and is equal to $\lim_{M\to\infty} V_M^{\langle k\rangle}(\vec{\delta})$. For the differentiability result, our treatment here includes steps missing from [11], where taking the derivative through the infinite series was not handled properly.

For continuity of $V(\vec{\delta})$ it is sufficient (e.g. [16], Theorem 7.12) to show that V_M converges uniformly to V, i.e. that $\sup_{\vec{\delta}\in[0,1]^{|A|}} |V_M(\vec{\delta}) - V(\vec{\delta})| \to 0$ as $M\to\infty$. Now

$$\begin{aligned}
\sup_{\vec{\delta}}\big|V_M(\vec{\delta}) - V(\vec{\delta})\big| &= \sup_{\vec{\delta}}\Bigg|\sum_{m=M+1}^{\infty}\sum_{N=1}^{m-1}\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \\
&\leq \sup_{\vec{\delta}}\sum_{m=M+1}^{\infty}\sum_{N=1}^{m-1}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \\
&\leq \sup_{\vec{\delta}}\sum_{m=M+1}^{\infty}\sum_{N=1}^{N_0}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| + \sup_{\vec{\delta}}\sum_{m=M+1}^{\infty}\sum_{N=N_0+1}^{m-1}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \\
&\leq \sum_{N=1}^{N_0}\sup_{\vec{\delta}}\sum_{m=M+1}^{\infty}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| + \sup_{\vec{\delta}}\sum_{N=N_0+1}^{\infty}\sum_{m=3}^{\infty}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg|.
\end{aligned} \tag{5.6}$$

Therefore to prove that the left hand side is less than ε for all M sufficiently large, by first choosing N_0 large depending on ε and then M large depending on N_0 and ε it is enough to show the following.

Lemma 5.1. *If $d \geq 6$ and $\vec{\delta} = (\delta_i)_{i\in A}$ with A as in Theorem 2.3,*

$$\sup_{\vec{\delta}}\sum_{N=N_0+1}^{\infty}\sum_{m=3}^{\infty}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \to 0 \quad \text{as } N_0\to\infty$$

and for any $N\in\mathbb{N}$,

$$\sup_{\vec{\delta}}\sum_{m=M+1}^{\infty}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \to 0 \quad \text{as } M\to\infty.$$

Proof. By Proposition 4.2 we have that

$$\sum_{N=N_0+1}^{\infty}\sum_{m=3}^{\infty}\Bigg|\sum_{x,y}(y-x)\pi_m^{(N)}(x,y)\Bigg| \leq \sum_{N=N_0+1}^{\infty} C_d(2a_d)^{N-2}. \tag{5.7}$$

In particular this bound does not depend on $\mathbb{Q}$ at all (so in particular not on $\vec{\delta}$). The first claim follows provided that $C_d < \infty$ and $2a_d < 1$, which holds for $d \geq 6$. The second claim follows immediately from (4.23). □

Similarly for the exchange of limit and derivative it is sufficient (e.g. [16], Theorem 7.17) to show the previous result and that $V_M^{\langle k\rangle}$ converges uniformly. Since

$$\begin{aligned}\sup_{\vec\delta}&\left|\sum_{m=M+1}^{\infty}\sum_{N=1}^{m-1}\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)\right|\\&\le\sum_{N=1}^{N_0}\sup_{\vec\delta}\sum_{m=M+1}^{\infty}\left|\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)\right|+\sup_{\vec\delta}\sum_{N=N_0+1}^{\infty}\sum_{m=3}^{\infty}\left|\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)\right|,\end{aligned}\tag{5.8}$$

for uniform convergence of $V_M^{\langle k\rangle}$ it is sufficient to show the following.

Lemma 5.2. *If $d\ge 8$ and $\vec\delta=(\delta_i)_{i\in A}$ with A as in Theorem 2.3*

$$\sup_{\vec\delta}\sum_{N=N_0+1}^{\infty}\sum_{m=3}^{\infty}\left|\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)\right|\to 0\quad \text{as } N_0\to\infty$$

and for any $N\in\mathbb{N}$

$$\sup_{\vec\delta}\sum_{m=M+1}^{\infty}\left|\sum_{x,y}(y-x)\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)\right|\to 0\quad \text{as } M\to\infty.$$

Before proving this lemma and proving Theorem 2.3, we need bounds of the form of Proposition 4.2 that arise after taking derivatives. These bounds appear in Proposition 5.3 below.

Let $k\in A$ and define $\varphi_m^{(N)}(x,y)=\frac{\partial}{\partial\delta_k}\pi_m^{(N)}(x,y)$. It follows from (3.9) that

$$\varphi_m^{(N)}(x,y)=\varphi_m^{(N,1)}(x,y)+\varphi_m^{(N,2)}(x,y)+\varphi_m^{(N,3)}(x,y),\tag{5.9}$$

where $\varphi_m^{(N,i)}(x,y)$, $i=1,2,3$, arise from differentiating $p^o(o,x_1^{(0)})$, $\prod_{n=1}^N\prod_{i_n=0}^{j_n-1}p^{\vec x_{j_{n-1}+1}^{(n-1)}\circ\vec x_{i_n}^{(n)}}(x_{i_n}^{(n)},x_{i_n+1}^{(n)})$, and $\prod_{n=1}^N\Delta_n$ respectively. Assuming that Lemma 5.2 holds, the derivative of the velocity exists and is given by

$$\frac{\partial v^{[1]}}{\partial\delta_k}=\frac{2}{d}I_{\{k=1\}}+\sum_{t=1}^{3}\sum_{m=2}^{\infty}\sum_{N=1}^{\infty}\sum_{x,y}(y-x)^{[1]}\varphi_m^{(N,t)}(x,y).\tag{5.10}$$

Note that it follows from (5.4) and (5.5) that

$$\left|\frac{\partial}{\partial\delta_k}p^{\vec x_i}(x_i,x_{i+1})\right|\le\frac{1}{d}I_{\{L(\vec x_i)=k\}}I_{\{x_{i+1}=x_i\pm e_1\}}\tag{5.11}$$

and

$$\left|\frac{\partial}{\partial\delta_k}\Delta_n\right|\le\frac{1}{d}I_{\{L(\vec x_{j_n}^{(n)})=k\}}I_{\{x_{j_n}^{(n)}\in\vec x_{j_{n-1}}^{(n-1)}\}}I_{\{x_{j_n+1}=x_{j_n}\pm e_1\}}\le\frac{1}{d}\sum_{r_{n-1}=0}^{j_{n-1}}I_{\{x_{j_n}^{(n)}=x_{r_{n-1}}^{(n-1)}\}}I_{\{x_{j_n+1}=x_{j_n}\pm e_1\}}.\tag{5.12}$$

Let $\rho^{(N)}$ be obtained by replacing $p^o(o,x_1^{(0)})$ in (4.24) with $\frac{1}{d}I_{\{L(o)=k\}}I_{\{x_1^{(0)}=\pm e_1\}}$ corresponding to the right hand side of (5.11) for the first step. Note that these quantities are zero unless $k=1$. For $t=1,\dots,N$ and $M\in\mathbb{N}$, let $\rho^{(N)}(M,t)$ be obtained from $\rho^{(N)}$ by restricting the summation over j_t to $j_t\ge\frac{M}{N}-1$.

For $s=1,\dots,N$, let $\gamma_s^{(N)}$ be obtained from (4.24) by replacing $\prod_{i_s=0}^{j_s-1} p^{\vec{x}^{(s-1)}_{j_{s-1}+1}\circ\vec{x}^{(s)}_{i_s}}(x^{(s)}_{i_s},x^{(s)}_{i_s+1})$ with the following bound on its derivative (obtained from (5.11))

$$\sum_{t=0}^{j_s-1}\frac{I_{\{x^{(s)}_{t+1}-x^{(s)}_t=\pm e_1\}}I_{\{L(\vec{x}^{(s)}_t)=k\}}}{d}\prod_{\substack{i_s=0\\ i_s\neq t}}^{j_s-1} p^{\vec{x}^{(s-1)}_{j_{s-1}+1}\circ\vec{x}^{(s)}_{i_s}}\bigl(x^{(s)}_{i_s},x^{(s)}_{i_s+1}\bigr).$$

Define $\gamma_s^{(N)}(M,t)$ by restricting the summation over j_t to $j_t\geq\frac{M}{N}-1$. Similarly, let $\chi_s^{(N)}$ be obtained by replacing $|\Delta_s|$ in (4.24) by the following bound on the derivative of Δ_s (see (5.12))

$$\frac{1}{d}\sum_{r_{s-1}=0}^{j_{s-1}} I_{\{x^{(s)}_{j_s}=x^{(s-1)}_{r_{s-1}}\}}I_{\{x_{j_s+1}=x_{j_s}\pm e_1\}}.$$

Define $\chi_s^{(N)}(M,t)$ by restricting the summation over j_t to $j_t\geq\frac{M}{N}-1$.

Letting $\gamma^{(N)}=\sum_{s=1}^N\gamma_s^{(N)}$ and $\chi^{(N)}=\sum_{s=1}^N\chi_s^{(N)}$, we obtain that

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,1)}(x,y)\bigr|\leq\rho^{(N)},\qquad\sum_{m=2}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,2)}(x,y)\bigr|\leq\gamma^{(N)},\quad\text{and}\\&\sum_{m=2}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,3)}(x,y)\bigr|\leq\chi^{(N)}.\end{aligned}\tag{5.13}$$

Since $1+\sum_{i=1}^N(j_i+1)=m$, we must have $j_t\geq\frac{m-1-N}{N}$ for some t, so that with $\gamma^{(N)}(M,t)=\sum_{s=1}^N\gamma_s^{(N)}(M,t)$ and $\chi^{(N)}(M,t)=\sum_{s=1}^N\chi_s^{(N)}(M,t)$ we have

$$\begin{aligned}\sum_{m=M+1}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,1)}(x,y)\bigr|&\leq\sum_{t=1}^N\rho^{(N)}(M,t),\\ \sum_{m=M+1}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,2)}(x,y)\bigr|&\leq\sum_{t=1}^N\gamma^{(N)}(M,t),\\ \sum_{m=M+1}^{\infty}\sum_{x,y}\bigl|\varphi_m^{(N,3)}(x,y)\bigr|&\leq\sum_{t=1}^N\chi^{(N)}(M,t).\end{aligned}\tag{5.14}$$

Define

$$\varepsilon(d)=\frac{2d}{(d-1)^4}G_{d-1}G^{*3}_{d-1}+\frac{\mathcal{E}_1(d)}{d(d-1)^2}G^{*2}_{d-1}.\tag{5.15}$$

Proposition 5.3. *For cookie environment $\mathbb{Q}$ satisfying the assumptions of Theorem 2.3, and fixed $k\in A$, the following hold*

$$\rho^{(N)}\leq\begin{cases}4d^{-2}\mathcal{E}_0(d), & N=1,\\ 8d^{-2}(d-1)^{-1}G_{d-1}\mathcal{E}_1(d)(2a_d)^{N-2}, & N>1,\end{cases}\tag{5.16}$$

$$\chi^{(N)}\leq\begin{cases}2d^{-1}\mathcal{E}_0(d), & N=1,\\ 4d^{-1}(d-1)^{-1}G_{d-1}\mathcal{E}_1(d)N(2a_d)^{N-2}, & N>1,\end{cases}\tag{5.17}$$

$$\gamma_s^{(N)} \leq \begin{cases} \frac{4G^{*(2)}_{d-1}}{(d-1)^2}, & 1 = s = N, \\ \big[\frac{8\mathcal{E}_1(d)G^{*(2)}_{d-1}}{d(d-1)^2}\big](2a_d)^{N-2}, & 1 < s = N, \\ \big[\frac{16dG_{d-1}G^{*(3)}_{d-1}}{(d-1)^4}\big](2a_d)^{N-2}, & 1 = s < N, \\ \big[\frac{32G_{d-1}G^{*(3)}_{d-1}}{(d-1)^4}\mathcal{E}_1(d)\big](2a_d)^{N-3}, & 1 < s < N, \end{cases} \tag{5.18}$$

$$\sup_{\mathbb{Q}}\big(\rho^{(N)}(M,t) + \gamma^{(N)}(M,t) + \chi^{(N)}(M,t)\big) \to 0 \quad \text{as } M \to \infty \text{ if } d \geq 8. \tag{5.19}$$

The proof of this proposition is essentially the same as that of Proposition 4.2, and is deferred to Section 5.1. Assuming Proposition 5.3, we are ready to prove Lemma 5.2 and Theorem 2.3.

Proof of Lemma 5.2. Observe that if $2a_d < 1$ and $G^{*3}_{d-1} < \infty$ (both hold when $d \geq 8$) then

$$\sup_{\vec{\delta}} \sum_{N=N_0+1}^{\infty} \sum_{m=3}^{\infty} \left| \sum_{x,y} (y-x) \frac{\partial}{\partial \delta_i} \pi_m^{(N)}(x,y) \right| \leq \sum_{N=N_0+1}^{\infty} \big[\rho^{(N)} + \chi^{(N)} + \gamma^{(N)}\big] \to 0,$$

as $N_0 \to \infty$, by Proposition 5.3. This establishes the first claim of the lemma. Similarly if $d \geq 8$ then

$$\sup_{\vec{\delta}} \sum_{m=M+1}^{\infty} \left| \sum_{x,y} (y-x) \frac{\partial}{\partial \delta_i} \pi_m^{(N)}(x,y) \right| \leq \sup_{\mathbb{Q}} \sum_{t=1}^{N} \big[\rho^{(N)}(M,t) + \chi^{(N)}(M,t) + \gamma^{(N)}(M,t)\big] \to 0,$$

as $M \to \infty$ by Proposition 5.3, which establishes the second claim. □

Proof of Theorem 2.3. Fix $A \subset \mathbb{N}$, $\vec{\delta} = (\delta_i)_{i \in A}$, and the distribution of $\omega(o, A^c)$ as in the statement of the theorem. By Corollary 4.3 the velocity is given by

$$v = E[X_1] + \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x,y} (y-x) \pi_m^{(N)}(x,y),$$

where $v^{[j]} = 0$ for $j \neq 1$ and $E[X_1^{[1]}] = \frac{2\delta_1 - 1}{d}$. The first term is continuous in $\vec{\delta}$. The second term is continuous in $\vec{\delta}$ when $d \geq 6$ by Lemma 5.1, (5.6) and the preceeding discussion.

Similarly Lemma 5.2 applies when $d \geq 8$, whence for $k \in A$, $v^{[1]}$ is differentiable with respect to δ_k and (5.10) holds. It therefore remains to verify the final claim of monotonicity in δ_1 when $1 \in A$ and $d \geq 12$. From (5.10) it is sufficient to show that

$$\left| \sum_{t=1}^{3} \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x,y} (y-x)^{[1]} \varphi_m^{(N,t)}(x,y) \right| < \frac{2}{d},$$

since this would imply that $\frac{\partial v^{[1]}}{\partial \delta_1} > 0$. From Proposition 5.3, for all $(\delta_j)_{j \in A}$, and d such that $2a_d < 1$ we have

$$d \sum_{N=1}^{\infty} \rho^{(N)} \leq \left(\frac{2\mathcal{E}_0(d)}{d} + \frac{4G_{d-1}\mathcal{E}_1(d)}{d(d-1)(1-2a_d)} \right), \tag{5.20}$$

$$d \sum_{N=1}^{\infty} \chi^{(N)} \leq \left(2\mathcal{E}_0(d) + \frac{4G_{d-1}\mathcal{E}_1(d)(2-2a_d)}{(d-1)(1-2a_d)^2} \right), \tag{5.21}$$

$$d \sum_{N=1}^{\infty} \gamma^{(N)} \leq \left(\frac{4dG^{*2}_{d-1}}{(d-1)^2} + \frac{8\varepsilon(d)d}{(1-2a_d)} + \frac{32d\mathcal{E}_1(d)G_{d-1}G^{*3}_{d-1}}{(d-1)^4(1-2a_d)^2} \right). \tag{5.22}$$

All of these bounds are decreasing in d, so by (5.13) we need only find a d sufficiently large so that (5.16) + (5.17) + (5.18) < 2. Once again this involves inserting rigorous bounds on the (convolutions of) simple random walk Green's functions. Using the bounds [8,9]:

$$G_{11}(o) \le 1.05314, \qquad G_{11}^{*2}(o) \le 1.18018, \qquad G_{11}^{*3}(o) \le 1.43043$$

gives the desired result. □

5.1. *Proof of Proposition* 5.3

For each of the claimed bounds we verify a result of the form of Proposition 4.2, but adjusted to take into account the different terms appearing due to the derivative.

For the bound on $\rho^{(N)}$ we proceed exactly as in Proposition 4.2, except for the last step. Instead of a bound of the form (4.28) or the corresponding bound on (4.30), the derivative of the first transition probability with respect to δ_k leaves us (see (5.11)) with a bound

$$\frac{2}{d}\sum_{\vec{x}_1^{(0)}} \frac{1}{d} I_{\{k=0\}} I_{\{\vec{x}_1^{(0)}=\pm e_1\}} \sum_{j_1=1}^{\infty} \frac{(j_1+i)!}{j_1!} Q^{\vec{x}_1^{(0)}}(X_{j_1}=o), \tag{5.23}$$

with $i=0$ if $N=1$ and $i=1$ if $N>1$. By (4.4) this is bounded by

$$i!\mathcal{E}_i(d)\frac{2}{d}\sum_{\vec{x}_1^{(0)}} \frac{1}{d} I_{\{\vec{x}_1^{(0)}=\pm e_1\}} = \frac{4}{d^2}\mathcal{E}_i(d), \tag{5.24}$$

instead of the factor $\frac{2}{d}\mathcal{E}_i(d)$ (also with $i=0$ if $N=1$ and $i=1$ if $N>1$) obtained in Proposition 4.2. This gives

$$\rho^{(N)} \le \begin{cases} 4d^{-2}\mathcal{E}_0(d), & N=1, \\ 8d^{-2}(d-1)^{-1}G_{d-1}\mathcal{E}_1(d)(2a_d)^{N-2}, & N>1, \end{cases} \tag{5.25}$$

as claimed.

For the second bound we proceed similarly, except that when estimating the term $n=s$ we replace (4.26) with (5.12). However these are exactly the same bounds used in proving Proposition 4.2, so summing over s from 1 to N we obtain

$$\chi^{(N)} \le \begin{cases} 2d^{-1}\mathcal{E}_0(d), & N=1, \\ 4d^{-1}(d-1)^{-1}G_{d-1}\mathcal{E}_1(d)N(2a_d)^{N-2}, & N>1, \end{cases} \tag{5.26}$$

as claimed.

For the third claim we have to work a bit harder. Starting from $n=N$ we can bound each term up to the term $n=s+1$ as before, using (4.3) with $i=0$ for the term $n=N$ and then with $i=1$ on terms $s<n<N$. We then reach the term $n=s$ in which a transition probability has been replaced with its derivative (by definition of $\gamma_s^{(N)}$). This term is

$$\sum_{j_s=0}^{\infty} \frac{(j_s+i)!}{i!} \sum_{\vec{x}_{j_s+1}^{(s)}} |\Delta_s| \sum_{t=0}^{j_s-1} \frac{I_{\{x_{t+1}^{(s)}-x_t^{(s)}=\pm e_1\}} I_{\{L(\vec{x}_t^{(s)})=k\}}}{d} \prod_{\substack{i_s=0 \\ i_s\neq t}}^{j_s-1} p^{\vec{x}_{j_{s-1}+1}^{(s-1)}\circ\vec{x}_{i_s}^{(s)}}\big(x_{i_s}^{(s)}, x_{i_s+1}^{(s)}\big),$$

with $i=0$ if $s=N$ and $i=1$ otherwise.

Ignoring the indicator $I_{\{L(\vec{x}_t^{(s)})=k\}}$ and using (4.26) (and the fact that if $j_s=0$ there is no term to differentiate here) this is bounded by

$$\frac{1}{d}\sum_{j_s=1}^{\infty}\frac{(j_s+i)!}{i!}\sum_{\vec{x}_{j_s}^{(s)}}\sum_{r=0}^{j_{s-1}}\sum_{t=0}^{j_s-1} I_{\{x_{j_s}^{(s)}=x_r^{(s-1)}\}} Q^{\vec{x}_{j_{s-1}+1}^{(s-1)}}\big(\vec{X}_t=\vec{x}_t^{(s)}\big)\frac{I_{\{x_{t+1}^{(s)}-x_t^{(s)}=\pm e_1\}}}{d}$$
$$\times Q^{\vec{x}_{j_{s-1}+1}^{(s-1)}\circ \vec{x}_{t+1}^{(s)}}\big(\vec{X}_{j_s-t-1}=\big(x_{t+1}^{(s)},\dots,x_{j_s}^{(s)}\big)\big)\sum_{x_{j_s+1}^{(s)}} I_{\{x_{j_s+1}^{(s)}-x_{j_s+1}^{(s)}=\pm e_1\}}, \tag{5.27}$$

with $i=0$ if $s=N$ and $i=1$ otherwise.

The summation over $x_{j_s+1}^{(s)}$ is equal to 2. Split the summation over $\vec{x}_{j_s}^{(s)}$ into the sum of the first $t+1$ steps and the remaining steps, and then perform the latter of the two summations to see that (5.27) is equal to

$$\frac{4}{d^2}\sum_{j_s=1}^{\infty}\frac{(j_s+i)!}{i!}\sum_{t=0}^{j_s-1}\sum_{\vec{x}_{t+1}^{(s)}}\sum_{r=0}^{j_{s-1}} Q^{\vec{x}_{j_{s-1}+1}^{(s-1)}}\big(\vec{X}_t=\vec{x}_t^{(s)}\big)\frac{I_{\{x_{t+1}^{(s)}-x_t^{(s)}=\pm e_1\}}}{2} Q^{\vec{x}_{j_{s-1}+1}^{(s-1)}\circ \vec{x}_{t+1}^{(s)}}\big(X_{j_s-t-1}=x_r^{(s-1)}\big). \tag{5.28}$$

Recall the definition of the law $Q^{\leftrightarrow_t,\vec{x}_{j_{s-1}+1}^{(s-1)}}$ (following (4.2)), under which any given path has the same probability as under $Q^{\vec{x}_{j_{s-1}+1}^{(s-1)}}$ except that the $(t+1)$st step is chosen uniformly from $\pm e_1$, independent of both the history of the walk and $\mathbb{Q}$. Then the term $\frac{I_{\{x_{t+1}^{(s)}-x_t^{(s)}=\pm e_1\}}}{2}$ is the transition kernel for the $(t+1)$st step of the walk under $Q^{\leftrightarrow_t,\vec{x}_{j_{s-1}+1}^{(s-1)}}$ and (5.28) is equal to

$$=\frac{4}{d^2}\sum_{j_s=1}^{\infty}\frac{(j_s+i)!}{i!}\sum_{t=0}^{j_s-1}\sum_{r=0}^{j_{s-1}} Q^{\leftrightarrow_t,\vec{x}_{j_{s-1}+1}^{(s-1)}}\big(X_{j_s}=x_r^{(s-1)}\big)$$
$$\le \frac{4}{d^2}(j_{s-1}+1)\sup_u\sum_{j_s=1}^{\infty}\frac{(j_s+i)!}{i!}\sum_{t=0}^{j_s-1} Q^{\leftrightarrow_t,\vec{x}_{j_{s-1}+1}^{(s-1)}}(X_{j_s}=u). \tag{5.29}$$

Applying (4.6), (5.29) is bounded by

$$\frac{4}{d^2}(j_{s-1}+1)(i+1)!q_d^{-(i+2)}G_{d-1}^{*(i+2)}. \tag{5.30}$$

If $N=s=1$, we use the bound (5.30) with $i=0$ and $j_0=0$ to bound the summation over j_1, followed by $\sum_{x_1^{(0)}} p^o(o,x_1^{(0)})=1$ to get the bound $\frac{4}{d^2}q_d^{-2}G_{d-1}^{*2}$ as claimed. If $s=N>1$ we use (5.30) with $i=0$ on the term $n=N$ to get $\frac{4}{d^2}(j_{N-1}+1)q_d^{-2}G_{d-1}^{*2}$ for this piece, instead of $\frac{2(j_{N-1}+1)}{d-1}G_{d-1}$ and then proceed as in Proposition 4.2 to get the claimed bound.

When $s=1<N$ we proceed as in Proposition 4.2, except that we use (5.30) with $i=1$ and $j_0=0$ to bound the summation over j_1, followed by $\sum_{x_1^{(0)}} p^o(o,x_1^{(0)})=1$. This yields a factor of $\frac{8}{d^2}q_d^{-3}G_{d-1}^{*3}$ instead of $\frac{2\mathcal{E}_1}{d}$, giving the claimed bound.

When $1<s<N$ we proceed as in Proposition 4.2, except that we use (5.30) with $i=1$ as a bound on the summation over j_s. This yields a factor $\frac{8}{d^2}q_d^{-3}G_{d-1}^{*3}$ instead of $2a_d$, giving the claimed bound.

It remains to prove (5.19), i.e. that

$$\sup_{\mathbb{Q}}\big(\rho^{(N)}(M,t)+\gamma^{(N)}(M,t)+\chi^{(N)}(M,t)\big)\to 0 \quad \text{as } M\to\infty \text{ if } G_{d-1}^{*3}<\infty.$$

This is proved exactly as in the proof of (4.23), except that certain estimates (e.g. using $G^{*3} < \infty$) require $d \geq 8$ instead of $d \geq 6$. In particular (4.31) remains valid. Along with (4.32), one of the contributions is the case $s = t$, which is a quantity of the form

$$\frac{4}{d^2}(j_{s-1}+1)\sup_u \sum_{j_s=\lceil M/N\rceil -1}^{\infty} \frac{(j_s+i)!}{i!} \sum_{t=0}^{j_s-1} Q^{\leftrightarrow_t, \vec{x}^{(s-1)}_{j_{s-1}+1}}(X_{j_s}=u), \tag{5.31}$$

to which we apply (4.7) (with $i = 1$, which requires $d \geq 8$, or with $i = 0$) instead of (4.5) to give the result. □

6. Excited against the tide

In this section we consider the special class of EAT walks defined in terms of three parameters $(\lambda, \beta, \mu) \in [0,1] \times [-1,1]^2$ by (2.4). In this case all cookies at a site are μ-cookies, except possibly the first cookie, which is a β cookie with probability λ. In other words,

$$p^{\vec{x}_i}(x_i, x_{i+1}) = \frac{1 + [(\lambda\beta + (1-\lambda)\mu) I_{\{x_i \notin \vec{x}_{i-1}\}} + \mu I_{\{x_i \in \vec{x}_{i-1}\}}][I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]}{2d}. \tag{6.1}$$

From our point of view the situation of most interest is when $\beta > 0$ and $\mu < 0$, i.e. the drifts oppose each other. Although for fixed (λ, β, μ) this model can be reparametrised in terms of δ_1 and $\delta_2 = \delta_{\geq 1}$ (in the annealed setting), we are interested in the effect of changing these three parameters individually, so it is informative to keep them in the notation.

Of course Theorem 2.1 applies to the EAT walk for every (λ, β, μ). For any $\mu \in [-1, 1]$ and $d \geq 9$, by Theorem 2.2 and the fact that $\delta_1 = \lambda(1+\beta)/2 + (1-\lambda)(1+\mu)/2$, we can find $\lambda\beta$ sufficiently close to 1 so that $v^{[1]} > 0$.

The following is a version of Theorem 2.3 for the EAT walks. Since the proof is very similar, we give only a sketch proof, highlighting the main differences.

Lemma 6.1. *For an EAT walk $v^{[1]}$ is a continuous function of the triple (λ, β, μ) when $d \geq 6$ and is differentiable in λ, β, μ when $d \geq 8$. When $d \geq 12$, $v^{[1]}$ is*:

- *strictly increasing in β when $\lambda > 0$*,
- *strictly increasing (resp. decreasing) in λ when $\beta > \mu$ (resp. $\beta < \mu$)*,
- *strictly increasing in μ when λ is sufficiently small.*

Proof. (Sketch.) Fixing λ and μ, $\omega(o, 1)$ is independent of $(\omega(o,i))_{i>1}$. Moreover

$$\frac{\partial v}{\partial \beta} = \frac{\partial v}{\partial \delta_1}\frac{\partial \delta_1}{\partial \beta} = \frac{\lambda}{2}\frac{\partial v}{\partial \delta_1}.$$

Since $\frac{\partial v}{\partial \delta_1} > 0$ when $d \geq 12$, $v^{[1]}$ is strictly increasing in β when $\lambda > 0$. Similarly, fixing β and μ,

$$\frac{\partial v}{\partial \lambda} = \frac{\partial v}{\partial \delta_1}\frac{\partial \delta_1}{\partial \lambda} = \frac{\beta - \mu}{2}\frac{\partial v}{\partial \delta_1},$$

we see that for $d \geq 12$, v is increasing (resp. decreasing) in λ if $\beta > \mu$ (resp. $\beta < \mu$).

For the third monotonicity claim, and the continuity in (λ, β, μ) we cannot use such a simple argument since changing μ affects δ_i for every i. Clearly

$$E\left[X_1^{[1]}\right] = \frac{2\delta_1 - 1}{d} = \frac{\lambda(1+\beta) + (1-\lambda)(1+\mu) - 1}{d} = \frac{\lambda\beta + (1-\lambda)\mu}{d}, \tag{6.2}$$

is a continuous function of the triple (λ, β, μ), and

$$\frac{\partial E[X_1^{[1]}]}{\partial \beta} = \frac{\lambda}{d}, \qquad \frac{\partial E[X_1^{[1]}]}{\partial \lambda} = \frac{\beta - \mu}{d}, \quad \text{and} \quad \frac{\partial E[X_1^{[1]}]}{\partial \mu} = \frac{1-\lambda}{d}. \tag{6.3}$$

Observe that (6.1) is a continuous function of the triple (λ, β, μ) since the functions $(\lambda, \beta, \mu) \mapsto \lambda\beta + (1-\lambda)\mu$ and $(\lambda, \beta, \mu) \mapsto \mu$ are. Moreover,

$$\begin{aligned} \frac{\partial p^{\vec{x}_i}(x_i, x_{i+1})}{\partial \beta} &= \frac{\lambda I_{\{x_i \notin \vec{x}_{i-1}\}}[I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]}{2d}, \\ \frac{\partial p^{\vec{x}_i}(x_i, x_{i+1})}{\partial \lambda} &= \frac{(\beta-\mu) I_{\{x_i \notin \vec{x}_{i-1}\}}[I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]}{2d}, \\ \frac{\partial p^{\vec{x}_i}(x_i, x_{i+1})}{\partial \mu} &= \frac{[(1-\lambda) I_{\{x_i \notin \vec{x}_{i-1}\}} + I_{\{x_i \in \vec{x}_{i-1}\}}][I_{\{x_{i+1}=x_i+e_1\}} - I_{\{x_{i+1}=x_i-e_1\}}]}{2d}. \end{aligned} \tag{6.4}$$

By reproducing the proof of Theorem 2.3 for this 3-parameter model (with a sup over (λ, β, μ) instead of $\vec{\delta}$) we get that $v^{[1]}$ is continuous in (λ, β, μ) for $d \geq 6$, and differentiable in λ, β, μ for $d \geq 8$. Indeed we can recover the monotonicity results for β and λ in this fashion by extracting a factor λ or $\beta - \mu$ from (6.4) to match the corresponding constant in (6.3).

The derivative of $v^{[1]}$ with respect to μ is given by

$$\frac{\partial v^{[1]}}{\partial \mu} = \frac{1-\lambda}{d} + \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x,y} (y-x)^{[1]} \frac{\partial}{\partial \mu} \pi_m^{(N)}(x, y). \tag{6.5}$$

From (6.4) there is no multiplicative factor of $(1-\lambda)$ that we can extract from the derivative with respect to μ in (6.4) to match that in (6.3). However for $d \geq 12$, and all λ sufficiently small, $\frac{1-\lambda}{d} \approx 1/d$ does dominate the remainder term, as in the proof of Theorem 2.3. □

Thus, in high dimensions this simple model exhibits some of the intuitive properties one expects: increasing the intensity (β) or occurrence (λ) of strong cookies increases the velocity. The derivative with respect to μ is not so easy, essentially because when $\lambda \gg 0$, the effect of μ is a *second-order* effect, whereas β and λ give first-order effects. When $\lambda \approx 0$, μ does indeed give a first order effect.

For any given δ_1 we believe that this class produces both the minimum and maximum $v^{[1]}$ (when $\mu = -1$ and $\mu = +1$ respectively) obtainable for any excited random walk with this δ_1. One might imagine that if the cookie drifts at each site are non-increasing, then the excited random walker tends to slow down over time, in the sense that $\limsup_{n\to\infty} \frac{X_n^{[1]}}{n} \leq \mathbb{E}[\frac{X_k^{[1]}}{k}]$, with the right side non-increasing in k. Then we might also expect that when the cookie drifts at each site are non-decreasing, the excited random walker tends to speed up over time in the sense that $\liminf_{n\to\infty} \frac{X_n^{[1]}}{n} \geq \mathbb{E}[\frac{X_k^{[1]}}{k}]$ (non-decreasing in k). Combining these would imply that in the random walk in a random environment case ($\omega(x, i) = \omega(x, 1)$ for all i almost surely), $\lim_{n\to\infty} \frac{X_n^{[1]}}{n} = \mathbb{E}[\frac{X_k^{[1]}}{k}]$ for all k, which is not true. Perhaps such monotonicity results do hold when $(\omega(o, i))_{i\in\mathbb{N}}$ are independent and δ_i are increasing or decreasing. Weak results can be obtained by an easy coupling argument that is equivalent to regenerating the environment after every step ($k = 1$). This argument gives $\mathbb{E}[X_1^{[1]}] = (2\delta_1 - 1)/d$ as an upper bound for the speed (and hence $\sum_m \sum_x x^{[1]} \pi_m(x) < 0$) when $\{\omega(x, i)\}_{i\in\mathbb{N}}$ are independent with δ_i decreasing in i. Likewise in the corresponding non-decreasing cases such an argument should give $(2\delta_1 - 1)/d$ as a lower bound for the speed.

We now direct our attention to a cookie-replacement/regeneration argument (with $k = 3$) for the EAT walks. This will enable us to give examples of non-trivial walks in high dimensions with zero speed. The intuition for this result is as follows. If $\mu < \beta$ then at any previously visited site you have only weaker (in terms of right drift) cookies, so to increase your speed to the right, you would be better off regenerating the environment to (possibly) get a stronger cookie at some previously visited sites. At previously unvisited sites, regeneration makes no difference (you can generate the environment at each site at your first visit). In the proof we regenerate the environment after every third step. It would be interesting to know how far this kind of argument can be extended, e.g. to environmental regeneration after every k steps (for what k?) for excited random walks with $\omega(x,i)$ non-increasing in i almost surely or where $\{\omega(x,i)\}_{i\in\mathbb{N}}$ are independent with δ_i decreasing in i.

Lemma 6.2. *Fix $d \geq 2$. For the (λ,β,μ)-EAT walk with $\mu \leq \beta$, $\mathbb{Q}$-almost surely,*

$$Q_{o,\omega}\Big(\limsup_{n\to\infty} n^{-1}X_n^{[1]} < \frac{1}{3}E\big[X_3^{[1]}\big]\Big).$$

Proof. We give a coupling argument as follows. Let $\Omega^* = \Omega^{\mathbb{Z}_+} \times [0,1]^{\mathbb{N}}$ and write $(\vec{\omega},\vec{U})$ for an element of Ω^*. Define a probability measure $\mathbb{P}$ on Ω^* so that under $\mathbb{P}$, $\vec{\omega} = (\omega^{(i)})_{i\in\mathbb{Z}_+}$ are independent copies of the EAT-environment (i.e. $\mathbb{P}$ includes $\mathbb{Q}$-product measure) and $\vec{U} = (U_n)_{n\in\mathbb{N}}$ are independent and uniformly distributed on $[0,1]$ and such that $\vec{U}$ and $\vec{\omega}$ are also mutually independent. In what follows we will often make statements that are true only up to sets of measure zero.

We will define two $\mathbb{Z}^d$ nearest-neighbour walks $(X_n)_{n\in\mathbb{Z}_+}$ and $(\tilde{X}_n)_{n\in\mathbb{Z}_+}$ and two sequences of events $A_n, \tilde{A}_n$ recursively. Firstly we define

$$\begin{aligned} &A_1 = \Big\{U_1 < \frac{\omega^{(0)}(o,1)}{d}\Big\} \quad \text{and} \quad \text{for } n \geq 1, \\ &A_{n+1} = \Big(\Big\{U_{n+1} < \frac{\omega^{(0)}(X_n,1)}{d}\Big\} \cap \{X_n \notin \vec{X}_{n-1}\}\Big) \cup \Big(\Big\{U_{n+1} < \frac{1+\mu}{2d}\Big\} \cap \{X_n \in \vec{X}_{n-1}\}\Big). \end{aligned} \tag{6.6}$$

We also define $\tilde{A}_1 = A_1$ and

$$\begin{aligned} \tilde{A}_{3n+k} = {} & \Big(\Big\{U_{3n+k} < \frac{\omega^{(0)}(X_{3n+k-1},1)}{d}\Big\} \cap \{X_{3n+k-1} \notin \vec{X}_{3n+k-2}\}\Big) \\ & \cup \Big(\Big\{U_{3n+k} < \frac{\omega^{(n)}(\tilde{X}_{3n+k-1},1)}{d}\Big\} \cap \{X_{3n+k-1} \in \vec{X}_{3n+k-2}\}\Big), \quad k=1,2, \end{aligned} \tag{6.7}$$

$$\begin{aligned} \tilde{A}_{3n+3} = {} & \Big(\Big\{U_{3n+3} < \frac{\omega^{(0)}(X_{3n+2},1)}{d}\Big\} \cap \{X_{3n+2} \notin \vec{X}_{3n+1}\} \cap \{\tilde{X}_{3n+2} \neq \tilde{X}_{3n}\}\Big) \\ & \cup \Big(\Big\{U_{3n+3} < \frac{\omega^{(n)}(\tilde{X}_{3n+2},1)}{d}\Big\} \cap \{X_{3n+2} \in \vec{X}_{3n-k-2}\} \cap \{\tilde{X}_{3n+2} \neq \tilde{X}_{3n}\}\Big) \\ & \cup \Big(\Big\{U_{3n+3} < \frac{1+\mu}{2d}\Big\} \cap \{\tilde{X}_{3n+2} = \tilde{X}_{3n}\}\Big). \end{aligned} \tag{6.8}$$

The unions in (6.6)–(6.8) are over disjoint events. Since $\frac{\omega^{(i)}(x,j)}{d} \geq \frac{1+\mu}{2d}$ for all i,x,j, $A_{3n+k} \subset \tilde{A}_{3n+k}$ for $k=1,2$. The events $\tilde{A}_{3n+k}$ are defined so that the environments $\omega^{(i)}(x,\cdot)$ on which the next step is based are not environments that have been seen before unless $k=3$ and $\tilde{X}_{3n+2} = \tilde{X}_{3n}$. Note that A_{n+1} is defined in terms of $\vec{X}_n$, and $\tilde{A}_{n+1}$ is defined in terms of $\vec{X}_n, \vec{\tilde{X}}_n$. We now define $X_{n+1}, \tilde{X}_{n+1}$ in terms of $A_{n+1}, \tilde{A}_{n+1}$ respectively, so that we have a recursive definition of all of these events and random variables (i.e. they are not ill-defined).

Set $\mathbb{P}(X_0 = o) = 1$ and given $\omega^{(0)}$ and $U_1,\dots,U_{n+1}$ and $\vec{X}_n$ we define for $n \in \mathbb{Z}_+$

$$X_{n+1} - X_n = \begin{cases} e_1 & \text{if } A_{n+1} \text{ occurs}, \\ e_j & \text{if } \frac{2j-2}{2d} \leq U_{n+1} < \frac{2j-1}{2d},\ j \in \{2,3,\dots,d\}, \\ -e_j & \text{if } \frac{2j-1}{2d} \leq U_{n+1} < \frac{2j}{2d},\ j \in \{2,3,\dots,d\}, \\ -e_1 & \text{otherwise}. \end{cases} \tag{6.9}$$

Set $\mathbb{P}(\tilde{X}_0 = o) = 1$ and given $\vec{\omega}$, $U_1, \dots, U_{n+1}$ and $\vec{X}_n$ we define for $n \in \mathbb{Z}_+$

$$\tilde{X}_{n+1} - \tilde{X}_n = \begin{cases} e_1 & \text{if } \tilde{A}_{n+1} \text{ occurs,} \\ e_j & \text{if } \frac{2j-2}{2d} \le U_{n+1} < \frac{2j-1}{d}, j \in \{2,3,\dots,d\}, \\ -e_j & \text{if } \frac{2j-1}{2d} \le U_{n+1} < \frac{2j}{2d}, j \in \{2,3,\dots,d\} \\ -e_1 & \text{otherwise.} \end{cases} \tag{6.10}$$

Since $A_{3n+k} \subset \tilde{A}_{3n+k}$ for $k = 1, 2$, whenever $X_{3n+k} - X_{3n+k-1} = e_1$ also $\tilde{X}_{3n+k} - \tilde{X}_{3n+k-1} = e_1$ for $k = 1, 2$. Similarly if $\tilde{X}_{3n+2} \neq \tilde{X}_{3n}$ then $X_{3n+3} - X_{3n+2} = e_1$ implies that $\tilde{X}_{3n+3} - \tilde{X}_{3n+2} = e_1$. Suppose that $\tilde{X}_{3n+2} = \tilde{X}_{3n}$ and $\tilde{X}_{3n+1} = \tilde{X}_{3n} + u$. If $u \neq \pm e_1$ then under this coupling we also have $X_{3n+1} = X_{3n} + u$, $X_{3n+2} = X_{3n}$ and $X_{3n+3} - X_{3n+2} = \tilde{X}_{3n+3} - \tilde{X}_{3n+2}$. If $u = -e_1$ then also $X_{3n+1} = X_{3n} - e_1$ and either: $X_{3n+2} = X_{3n+1} - e_1$ so that $X^{[1]}_{3n+3} - X^{[1]}_{3n} \le \tilde{X}^{[1]}_{3n+3} - \tilde{X}^{[1]}_{3n}$; or $X_{3n+2} = X_{3n+1} + e_1$ so that $X_{3n+2} = X_{3n}$ and $X_{3n+3} - X_{3n} = \tilde{X}^{[1]}_{3n+3} - \tilde{X}^{[1]}_{3n}$. In all cases we have that $X^{[1]}_{3n+3} - X^{[1]}_{3n} \le \tilde{X}^{[1]}_{3n+3} - \tilde{X}^{[1]}_{3n}$ for all n, $\mathbb{P}$-almost surely.

We claim that given $\omega^{(0)}$, the sequence $(X_n)_{n\in\mathbb{Z}_+}$ has the distribution of an EAT walk in environment $\omega^{(0)}$. To see this note that $\vec{X}_n$ above depends only on $\omega^{(0)}$ and $U_1, \dots, U_n$. Furthermore, since U_{n+1} is independent of $\vec{X}_n$ and $\vec{\omega}$, (6.9) gives

$$\begin{aligned} \mathbb{P}\big(X_{n+1} - X_n = e_1 | \vec{X}_n, \omega^{(0)}\big) &= \mathbb{P}\big(A_{n+1} | \vec{X}_n, \omega^{(0)}\big) = \frac{\omega^{(0)}(X_n, 1)}{d} I_{\{X_n \notin \vec{X}_{n-1}\}} + \frac{1+\mu}{2d} I_{\{X_n \in \vec{X}_{n-1}\}} \\ &= Q_{\omega^{(0)}, o}(X_{n+1} - X_n = e_1 | \vec{X}_n) \end{aligned}$$

for the EAT-walk (for almost every $\omega^{(0)}$), by comparison with (2.2) and (2.4).

Now observe that $\tilde{X}_{3n+3} - \tilde{X}_{3n}$ depends on $\{U_{3n+i}\}_{i\in\{1,2,3\}}$, $\{\omega^{(0)}(X_{3n+i}, 1)\colon X_{3n+i} \notin \vec{X}_{3n+i-1}, i = 0, 1, 2\}$, and $\{\omega^{(n)}(\tilde{X}_{3n+i}, 1)\colon X_{3n+i} \in \vec{X}_{3n+i-1}, i = 0, 1, 2\}$. If $n > m$ and for some $i \in \{0, 1, 2\}$ we have $X_{3n+i} \notin \vec{X}_{3n+i-1}$, then also $X_{3n+i} \notin \vec{X}_{3m+2}$ and so the increment $X_{3n+i+1} - X_{3n+i}$ depends on a piece of environment $\omega^{(0)}(X_{3n+i}, \cdot)$ that has never been encountered before. On the other hand, if $X_{3n+i} \in \vec{X}_{3n+i-1}$ then by construction $X_{3n+i+1} - X_{3n+i}$ depends on $\omega^{(n)}$, which is independent of the environments $\omega^{(m)}$ for $m < n$, and hence of all environments encountered before time $3n$. It follows that $Y_{n+1} := \tilde{X}_{3n+3} - \tilde{X}_{3n}$ and $Y_{m+1} = \tilde{X}_{3m+3} - \tilde{X}_{3m}$ depend on disjoint collections of $\omega^{(\cdot)}(\cdot, 1)$, $U_\cdot$ and hence $\{Y_n\}_{n\ge 1}$ is an independent sequence of random variables under $\mathbb{P}$. We now show that $\{Y^{[1]}_n\}_{n\ge1}$ are also identically distributed. Note that $Y^{[1]}_n$ takes values in $\{-3, -2, -1, 0, 1, 2, 3\}$.

Let $O_n = \{\tilde{X}_n - \tilde{X}_{n-1} \notin \pm e_1\}$ be the event that the nth step of $\tilde{X}$ is in any one of the directions other than $\pm e_1$ and $O_{n,m} = O_n \cap \{\tilde{X}_m - \tilde{X}_{m-1} = -(\tilde{X}_n - \tilde{X}_{n-1})\}$ denote the event that the nth step is in one of these directions and the mth step is in the reverse direction to the nth step. Let $\tilde{A}^-_n = \{\tilde{X}_n - \tilde{X}_{n-1} = -e_1\}$. Then we have that

$$\mathbb{P}\big(Y^{[1]}_n = 3\big) = \mathbb{P}(\tilde{A}_{3n+1}, \tilde{A}_{3n+2}, \tilde{A}_{3n+3}), \tag{6.11}$$

$$\mathbb{P}\big(Y^{[1]}_n = 2\big) = \mathbb{P}(\tilde{A}_{3n+1}, \tilde{A}_{3n+2}, O_{3n+3}) + \mathbb{P}(\tilde{A}_{3n+1}, O_{3n+2}, \tilde{A}_{3n+3}) + \mathbb{P}(O_{3n+1}, \tilde{A}_{3n+2}, \tilde{A}_{3n+3}), \tag{6.12}$$

$$\mathbb{P}\big(Y^{[1]}_n = 1\big) = \mathbb{P}(\tilde{A}_{3n+1}, O_{3n+2}, O_{3n+3}) + \mathbb{P}(O_{3n+1}, \tilde{A}_{3n+2}, O_{3n+3}) \tag{6.13}$$

$$+ \mathbb{P}\big(O_{3n+1}, O_{3n+2}, O^c_{3n+1,3n+2}, \tilde{A}_{3n+3}\big) + \mathbb{P}\big(\tilde{A}_{3n+1}, \tilde{A}_{3n+2}, \tilde{A}^-_{3n+3}\big) \tag{6.14}$$

$$+ \mathbb{P}\big(O_{3n+1,3n+2}, \tilde{A}_{3n+3}\big) + \mathbb{P}\big(\tilde{A}^-_{3n+1}, \tilde{A}_{3n+2}, \tilde{A}_{3n+3}\big) + \mathbb{P}\big(\tilde{A}_{3n+1}, \tilde{A}^-_{3n+2}, \tilde{A}_{3n+3}\big). \tag{6.15}$$

Similarly

$$\mathbb{P}\big(Y^{[1]}_n = -3\big) = \mathbb{P}\big(\tilde{A}^-_{3n+1}, \tilde{A}^-_{3n+2}, \tilde{A}^-_{3n+3}\big), \tag{6.16}$$

$$\mathbb{P}\big(Y^{[1]}_n = -2\big) = \mathbb{P}\big(\tilde{A}^-_{3n+1}, \tilde{A}^-_{3n+2}, O_{3n+3}\big) + \mathbb{P}\big(\tilde{A}^-_{3n+1}, O_{3n+2}, \tilde{A}^-_{3n+3}\big) + \mathbb{P}\big(O_{3n+1}, \tilde{A}^-_{3n+2}, \tilde{A}^-_{3n+3}\big), \tag{6.17}$$

$$\mathbb{P}\big(Y^{[1]}_n = -1\big) = \mathbb{P}\big(\tilde{A}^-_{3n+1}, O_{3n+2}, O_{3n+3}\big) + \mathbb{P}\big(O_{3n+1}, \tilde{A}^-_{3n+2}, O_{3n+3}\big) \tag{6.18}$$

$$+ \mathbb{P}\big(O_{3n+1}, O_{3n+2}, O^c_{3n+1,3n+2}, \tilde{A}^-_{3n+3}\big) + \mathbb{P}\big(\tilde{A}^-_{3n+1}, \tilde{A}^-_{3n+2}, \tilde{A}_{3n+3}\big) \tag{6.19}$$

$$+ \mathbb{P}\big(O_{3n+1,3n+2}, \tilde{A}^-_{3n+3}\big) + \mathbb{P}\big(\tilde{A}_{3n+1}, \tilde{A}^-_{3n+2}, \tilde{A}^-_{3n+3}\big) + \mathbb{P}\big(\tilde{A}^-_{3n+1}, \tilde{A}_{3n+2}, \tilde{A}^-_{3n+3}\big). \tag{6.20}$$

We will show that the above do not depend on n (hence neither does $\mathbb{P}(Y_n^{[1]}=0)$) and thus $\{Y_n\}_{n\in\mathbb{N}}$ are indeed identically distributed. To be precise, the following equalities for the terms appearing in (6.11)–(6.14) hold:

$$\begin{aligned}
&\mathbb{P}(\tilde{A}_{3n+1},\tilde{A}_{3n+2},\tilde{A}_{3n+3})=\left(\frac{\delta_1}{d}\right)^3,\\
&\mathbb{P}(\tilde{A}_{3n+1},\tilde{A}_{3n+2},O_{3n+3})=\left(\frac{\delta_1}{d}\right)^2\left(\frac{2d-2}{2d}\right)=\mathbb{P}(\tilde{A}_{3n+1},O_{3n+2},\tilde{A}_{3n+3})=\mathbb{P}(O_{3n+1},\tilde{A}_{3n+2},\tilde{A}_{3n+3}),\\
&\mathbb{P}(\tilde{A}_{3n+1},O_{3n+2},O_{3n+3})=\frac{\delta_1}{d}\left(\frac{2d-2}{2d}\right)^2=\mathbb{P}(O_{3n+1},\tilde{A}_{3n+2},O_{3n+3}),\\
&\mathbb{P}\big(O_{3n+1},O_{3n+2},O^c_{3n+1,3n+2},\tilde{A}_{3n+3}\big)=\left(\frac{2d-2}{2d}\right)\left(\frac{2d-3}{2d}\right)\frac{\delta_1}{d},\\
&\mathbb{P}\big(\tilde{A}_{3n+1},\tilde{A}_{3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{\delta_1}{d}\right)^2\left(\frac{1-\delta_1}{d}\right).
\end{aligned}\tag{6.21}$$

The following equalities for the terms appearing in (6.16)–(6.19) also hold

$$\begin{aligned}
&\mathbb{P}\big(\tilde{A}^-_{3n+1},\tilde{A}^-_{3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)^3,\\
&\mathbb{P}\big(\tilde{A}^-_{3n+1},\tilde{A}^-_{3n+2},O_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)^2\left(\frac{2d-2}{2d}\right)=\mathbb{P}\big(\tilde{A}^-_{3n+1},O_{3n+2},\tilde{A}^-_{3n+3}\big)=\mathbb{P}\big(O_{3n+1},\tilde{A}^-_{3n+2},\tilde{A}^-_{3n+3}\big),\\
&\mathbb{P}\big(\tilde{A}^-_{3n+1},O_{3n+2},O_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)\left(\frac{2d-2}{2d}\right)^2=\mathbb{P}\big(O_{3n+1},\tilde{A}^-_{3n+2},O_{3n+3}\big),\\
&\mathbb{P}\big(O_{3n+1},O_{3n+2},O^c_{3n+1,3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{2d-2}{2d}\right)\left(\frac{2d-3}{2d}\right)\left(\frac{1-\delta_1}{d}\right),\\
&\mathbb{P}\big(\tilde{A}^-_{3n+1},\tilde{A}^-_{3n+2},\tilde{A}_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)^2\left(\frac{\delta_1}{d}\right).
\end{aligned}\tag{6.22}$$

Finally, the following equalities for the terms appearing in (6.15) and (6.20) hold:

$$\begin{aligned}
&\mathbb{P}\big(O_{3n+1,3n+2},\tilde{A}_{3n+3}\big)=\left(\frac{2d-2}{2d}\right)\left(\frac{1+\mu}{2d}\right),\\
&\mathbb{P}\big(O_{3n+1,3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{2d-2}{2d}\right)\left(\frac{1-\mu}{2d}\right),\\
&\mathbb{P}\big(\tilde{A}^-_{3n+1},\tilde{A}_{3n+2},\tilde{A}_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)\left(\frac{\delta_1}{d}\right)\left(\frac{1+\mu}{2d}\right),\\
&\mathbb{P}\big(\tilde{A}_{3n+1},\tilde{A}^-_{3n+2},\tilde{A}_{3n+3}\big)=\left(\frac{\delta_1}{d}\right)\left(\frac{1-\delta_1}{d}\right)\left(\frac{1+\mu}{2d}\right),\\
&\mathbb{P}\big(\tilde{A}_{3n+1},\tilde{A}^-_{3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{\delta_1}{d}\right)\left(\frac{1-\delta_1}{d}\right)\left(\frac{1-\mu}{2d}\right),\\
&\mathbb{P}\big(\tilde{A}^-_{3n+1},\tilde{A}_{3n+2},\tilde{A}^-_{3n+3}\big)=\left(\frac{1-\delta_1}{d}\right)\left(\frac{\delta_1}{d}\right)\left(\frac{1-\mu}{2d}\right).
\end{aligned}\tag{6.23}$$

These relationships are verified by standard conditioning arguments (see below). For the moment, assuming that these relations hold the $Y_n^{[1]}$ are i.i.d. as claimed. The law of large numbers then gives

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}Y_i^{[1]}=\mathbb{E}\big[Y_1^{[1]}\big]\right)=1.$$

Since $X_{3n+3}^{[1]}-X_{3n}^{[1]}\le \tilde{X}_{3n+3}^{[1]}-\tilde{X}_{3n}^{[1]}$ almost surely we also have

$$\mathbb{P}\left(\limsup_{n\to\infty}\frac{X_n^{[1]}}{n}\le\lim_{n\to\infty}\frac{\tilde{X}_n^{[1]}}{n}=\frac{\mathbb{E}[Y_1^{[1]}]}{3}\right)=1.$$

Lemma 6.2 then follows from the fact that $X_3=X_3-X_0=\tilde{X}_3-\tilde{X}_0=Y_1$ (set $n=0$ in the A, $\tilde{A}$ events to see that $A_i=\tilde{A}_i$ for $i=1,2,3$).

The remainder of the argument consists of verifying (6.21)–(6.23). These calculations are somewhat tedious, so we will not show all of them. We begin with the first equality of (6.21), which is the quantity (6.11).

Fix $n\ge 0$. If $\tilde{A}_{3n+1}$ and $\tilde{A}_{3n+2}$ both occur then $\tilde{X}_{3n+2}=\tilde{X}_{3n}+2e_1$ so in particular $\tilde{X}_{3n+2}\ne\tilde{X}_{3n}$. It follows that $\tilde{A}_{3n+1}\cap\tilde{A}_{3n+2}\cap\tilde{A}_{3n+3}=\tilde{A}_{3n+1}\cap\tilde{A}_{3n+2}\cap\tilde{A}^*_{3n+3}$ where

$$\begin{aligned}\tilde{A}^*_{3n+3}&=\left(\left\{U_{3n+3}<\frac{\omega^{(0)}(X_{3n+2},1)}{d}\right\}\cap\{X_{3n+2}\notin\vec{X}_{3n+1}\}\right)\\&\quad\cup\left(\left\{U_{3n+3}<\frac{\omega^{(n)}(\tilde{X}_{3n+2},1)}{d}\right\}\cap\{X_{3n+2}\in\vec{X}_{3n+1}\}\right)\\&=Z_3\cup W_3,\end{aligned}\tag{6.24}$$

where $Z_3=E_3\cap G_3$ and $W_3=F_3\cap G_3^c$ are defined to be the events (whose dependence on n is suppressed for notational convenience) appearing in the above disjoint union. Thus we can replace $\tilde{A}_{3n+3}$ with $\tilde{A}^*_{3n+3}$ in (6.11). Similarly in all terms in (6.12)–(6.14) we can make the same replacement, while in terms (6.16)–(6.19) we can replace $\tilde{A}^-_{3n+3}$ with $(\tilde{A}^*_{3n+3}\cup O_{3n+3})^c$.

We wish to evaluate (6.11) by repeated conditioning, beginning with the term $\mathbb{P}(\tilde{A}^*_{3n+3}|\tilde{A}_{3n+1},\tilde{A}_{3n+2})$. Observe that

$$\begin{aligned}\mathbb{P}(E_3\cap G_3|\tilde{A}_{3n+1},\tilde{A}_{3n+2})&=\sum_x\mathbb{P}\left(U_{3n+3}<\frac{\omega^{(0)}(x,1)}{d}\middle|\tilde{A}_{3n+1},\tilde{A}_{3n+2},G_3,X_{3n+2}=x\right)\\&\quad\times\mathbb{P}(G_3,X_{3n+2}=x|\tilde{A}_{3n+1},\tilde{A}_{3n+2}).\end{aligned}\tag{6.25}$$

The conditioning $G_3, X_{3n+2}=x$ implies that $x\notin\vec{X}_{3n+1}$, whence the conditioning contains no information about $\omega^{(0)}(x,1)$ (nor U_{3n+3}). Thus the first conditional probability on the right of (6.25) is equal to $\mathbb{P}(U_{3n+3}<\frac{\omega^{(0)}(x,1)}{d})=\frac{\delta_1}{d}$ and preforming the sum over x we get that (6.25) is equal to

$$\frac{\delta_1}{d}\mathbb{P}(G_3|\tilde{A}_{3n+1},\tilde{A}_{3n+2}).$$

Similarly

$$\begin{aligned}\mathbb{P}\big(F_3\cap G_3^c|\tilde{A}_{3n+1},\tilde{A}_{3n+2}\big)&=\sum_x\mathbb{P}\left(U_{3n+3}<\frac{\omega^{(n)}(x,1)}{d}\middle|\tilde{A}_{3n+1},\tilde{A}_{3n+2},G_3^c,\tilde{X}_{3n+2}=x\right)\\&\quad\times\mathbb{P}\big(G_3^c,\tilde{X}_{3n+2}=x|\tilde{A}_{3n+1},\tilde{A}_{3n+2}\big).\end{aligned}\tag{6.26}$$

The conditioning $\tilde{A}_{3n+1}, \tilde{A}_{3n+2}, \tilde{X}_{3n+2} = x$ implies that $x \notin \{\tilde{X}_{3n}, \tilde{X}_{3n+1}\}$, whence the conditioning contains no information about $\omega^{(n)}(x,1)$ (nor U_{3n+3}). Thus the conditional probability in (6.26) is equal to $\mathbb{P}(U_{3n+3} < \frac{\omega^{(n)}(x,1)}{d}) = \frac{\delta_1}{d}$ and preforming the sum over x we get that (6.26) is equal to

$$\frac{\delta_1}{d}\mathbb{P}\big(G_3^c|\tilde{A}_{3n+1}, \tilde{A}_{3n+2}\big).$$

Thus we obtain

$$\mathbb{P}(\tilde{A}^*_{3n+3}|\tilde{A}_{3n+1}, \tilde{A}_{3n+2}) = \mathbb{P}(E_3 \cap G_3|\tilde{A}_{3n+1}, \tilde{A}_{3n+2}) + \mathbb{P}\big(F_3 \cap G_3^c|\tilde{A}_{3n+1}, \tilde{A}_{3n+2}\big) = \frac{\delta_1}{d}. \tag{6.27}$$

Now note that

$$\mathbb{P}(\tilde{A}_{3n+2}|\tilde{A}_{3n+1}) = \mathbb{P}(E_2 \cap G_2|\tilde{A}_{3n+1}) + \mathbb{P}\big(F_2 \cap G_2^c|\tilde{A}_{3n+1}\big),$$

where E_i, G_i, F_i, G_i^c are the four events appearing in (6.7) in sequential order. Proceeding as before,

$$\begin{aligned}
\mathbb{P}(E_2 \cap G_2|\tilde{A}_{3n+1}) &= \sum_x \mathbb{P}\bigg(U_{3n+2} < \frac{\omega^{(0)}(x,1)}{d}\bigg|\tilde{A}_{3n+1}, G_2, X_{3n+1} = x\bigg)\mathbb{P}(G_2, X_{3n+1} = x|\tilde{A}_{3n+1}) \\
&= \frac{\delta_1}{d}\mathbb{P}(G_2|\tilde{A}_{3n+1}), \\
\mathbb{P}\big(F_2 \cap G_2^c|\tilde{A}_{3n+1}\big) &= \sum_x \mathbb{P}\bigg(U_{3n+2} < \frac{\omega^{(n)}(x,1)}{d}\bigg|\tilde{A}_{3n+1}, G_2^c, \tilde{X}_{3n+1} = x\bigg)\mathbb{P}\big(G_2^c, \tilde{X}_{3n+1} = x|\tilde{A}_{3n+1}\big) \\
&= \frac{\delta_1}{d}\mathbb{P}\big(G_2^c|\tilde{A}_{3n+1}\big).
\end{aligned}$$

Thus

$$\mathbb{P}(\tilde{A}_{3n+2}|\tilde{A}_{3n+1}) = \frac{\delta_1}{d}\big[\mathbb{P}(G_2|\tilde{A}_{3n+1}) + \mathbb{P}\big(G_2^c|\tilde{A}_{3n+1}\big)\big] = \frac{\delta_1}{d}. \tag{6.28}$$

Also

$$\begin{aligned}
\mathbb{P}(\tilde{A}_{3n+1}) &= \mathbb{P}(E_1 \cap G_1) + \mathbb{P}\big(F_1 \cap G_1^c\big) \\
&= \sum_x \mathbb{P}\bigg(U_{3n+1} < \frac{\omega^{(0)}(x,1)}{d}\bigg|G_1, X_{3n+1} = x\bigg)\mathbb{P}(X_{3n+1} = x, G_1) + \cdots \\
&= \frac{\delta_1}{d}\big[\mathbb{P}(G_1) + \mathbb{P}\big(G_1^c\big)\big] = \frac{\delta_1}{d}.
\end{aligned} \tag{6.29}$$

Combining (6.27)–(6.29) we obtain

$$\mathbb{P}\big(\tilde{A}^*_{3n+3}, \tilde{A}_{3n+1}, \tilde{A}_{3n+2}\big) = \mathbb{P}(\tilde{A}_{3n+1})\mathbb{P}(\tilde{A}_{3n+2}|\tilde{A}_{3n+1})\mathbb{P}\big(\tilde{A}^*_{3n+3}|\tilde{A}_{3n+1}, \tilde{A}_{3n+2}\big) = \bigg(\frac{\delta_1}{d}\bigg)^3.$$

We can calculate all of the expressions in (6.21) and (6.22) in a similar fashion.

The remaining terms are those appearing in (6.23) (see (6.15) and (6.20)). These are the terms where the first two steps are in opposite directions (whence $\tilde{X}_{3n+2} = \tilde{X}_{3n}$) and the third is $\pm e_1$. For all terms in (6.15) we may replace the event $\tilde{A}_{3n+3}$ with the event $\{U_{3n+3} < (1+\mu)/2d\}$, while for all terms in (6.20) we may replace $\tilde{A}^-_{3n+3}$ with the event $\{(1+\mu)/2d \le U_{3n+3} < 1/d\}$. Both of these events are independent of all other events appearing in these terms so can be factored out immediately, e.g.

$$\mathbb{P}(O_{3n+1,3n+2}, \tilde{A}_{3n+3}) = \mathbb{P}\big(O_{3n+1,3n+2}, U_{3n+3} < (1+\mu)/2d\big) = \left(\frac{2d-2}{2d}\right)\left(\frac{1+\mu}{2d}\right),$$

$$\mathbb{P}\big(O_{3n+1,3n+2}, \tilde{A}^{-}_{3n+3}\big) = \mathbb{P}\big(O_{3n+1,3n+2}, (1+\mu)/2d \le U_{3n+3} < 1/d\big) = \left(\frac{2d-2}{2d}\right)\left(\frac{1-\mu}{2d}\right).$$

Using this fact and one conditioning step we can obtain the remaining equalities of (6.23). □

Proof of Lemma 2.4. Fix $\mu \in [-1, 0)$. In view of Lemma 6.2, we only need to show that $\mathbb{E}[X_3] = \mathbb{E}[Y_1^{[1]}] < 0$ with appropriate choice of the other parameter values. By collecting and matching terms of the form $\mathbb{P}(Y_1 = i)$ and $\mathbb{P}(Y_1 = -i)$ we have that

$$\begin{aligned}
\mathbb{E}[Y_1] &= 3 \times \left[\left(\frac{\delta_1}{d}\right)^3 - \left(\frac{1-\delta_1}{d}\right)^3\right] + 2 \times 3\left(\frac{2d-2}{2d}\right)\left[\left(\frac{\delta_1}{d}\right)^2 - \left(\frac{1-\delta_1}{d}\right)^2\right] \\
&\quad + 1 \times \left[2\left(\frac{2d-2}{2d}\right)^2\left(\frac{\delta_1}{d} - \frac{1-\delta_1}{d}\right) + \left(\frac{2d-2}{2d}\right)\left(\frac{2d-3}{2d}\right)\left(\frac{\delta_1}{d} - \frac{1-\delta_1}{d}\right)\right. \\
&\quad + \left(\frac{\delta_1}{d}\right)^2\left(\frac{1-\delta_1}{d}\right) - \left(\frac{\delta_1}{d}\right)\left(\frac{1-\delta_1}{d}\right)^2 \\
&\quad \left. + \left(\frac{2d-2}{2d}\right)\left(\frac{1+\mu}{2d} - \frac{1-\mu}{2d}\right) + 2\left(\frac{1-\delta_1}{d}\right)\left(\frac{\delta_1}{d}\right)\left(\frac{1+\mu}{2d} - \frac{1-\mu}{2d}\right)\right] \\
&= 3\left[\left(\frac{\delta_1}{d}\right)^3 - \left(\frac{1-\delta_1}{d}\right)^3\right] + 6\left(\frac{2d-2}{2d}\right)\left[\left(\frac{\delta_1}{d}\right)^2 - \left(\frac{1-\delta_1}{d}\right)^2\right] \\
&\quad + \left[\left(\frac{2\delta_1 - 1}{d}\right)\left(2\left(\frac{2d-2}{2d}\right)^2 + \left(\frac{2d-2}{2d}\right)\left(\frac{2d-3}{2d}\right) + \left(\frac{\delta_1}{d}\right)\left(\frac{1-\delta_1}{d}\right)\right)\right. \\
&\quad \left. + \frac{\mu}{d}\left(\left(\frac{2d-2}{2d}\right) + 2\left(\frac{1-\delta_1}{d}\right)\left(\frac{\delta_1}{d}\right)\right)\right].
\end{aligned} \tag{6.30}$$

Since $\mu < 0$, from (6.30) we see that there exists $\gamma > 0$ depending on μ, d such that $\mathbb{E}[Y_1^{[1]}] < 0$ as soon as $\delta_1 \le \frac{1}{2} + \gamma$ (whence $\delta_1 - (1-\delta_1) < 2\gamma$ as well). Now recall that $\delta_1 = \lambda(1+\beta)/2 + (1-\lambda)(1+\mu)/2$. Simple arithmetic gives us that $\delta_1 \le \frac{1}{2} + \gamma$ when

$$\lambda\beta \le -\mu(1-\lambda) + 2\gamma,$$

and the result follows. □

Note that one can get more explicit conditions for negative speed under certain assumptions. For example, when $\mu < 0$ and $\lambda < 1$ the condition $\frac{\lambda\beta}{1-\lambda} \le -\mu$ is sufficient to make $\delta_1 \le \frac{1}{2}$ and hence make the speed negative. As a final note to this section, by comparing $\mathbb{E}[X_3^{[1]}]/3$ as given by (6.30) with $\mathbb{E}[X_1^{[1]}] = (2\delta_1 - 1)/d$, one might be able to show that $\mathbb{E}[X_1^{[1]}] > \mathbb{E}[X_3^{[1]}]/3$ when $\mu < \beta$ so that $\sum_m \sum_x x\pi_m(x) < 0$ and $v^{[1]} < \mathbb{E}[X_1^{[1]}]$, which is another version of the "slowdown" effect when $\mu < \beta$.

7. Generalisations

We now turn to a brief discussion of possible extensions of the presented approach to the more general context, and their limitations.

- The environment is not i.i.d. between sites: The lace expansion is still valid but bounding the quantities appearing in the expansion breaks down since the terms Δ_n can be non-zero even when $x_{j_n}^{(n)} \notin \vec{x}_{j_{n-1}}^{(n-1)}$ (the environment encountered at a site x tells you something about the environment at y). It may be possible to recover some of the results (for example the positive speed for sufficiently large δ_1 in high dimensions) if the dependence between sites is very weak and decays quickly with the distance between sites.
- Non-nearest-neighbour steps: It should be possible to reproduce most of the results of this paper in the case where the steps are not nearest-neighbour, however many of these results require explicit bounds on the associated random walk Green's functions, and we are unaware of such results at the present time.
- The simple random walk step component is not uniform: As for the non-nearest-neighbour step case, if the step distribution was symmetric in each component we would require Green's functions estimates on the associated random walks. If the underlying random walk distribution has a drift, we could repeat much of the analysis in this paper if explicit bounds on return probabilities (uniformly in the history) were available. See for example the analysis of the reinforced random walk with drift in [10].
- Random walk on percolation clusters, random walk in random environment, and reinforced random walk: The methods in this paper cannot (at the present time) be used to study these models in their usual settings. The difficulties arise because there are no bounds (uniform in the history) on return probabilities or Green's functions, and in the case of monotonicity, when there is no tractable expression for the derivative of a transition probability $p^{\vec{x}_n}(x_n, x_{n+1})$. It is likely that major advances in the analysis of the recursion equation obtained using the lace expansion for self-interacting random walks of [10] would be required to make a significant contribution to these types of models in the general setting. See [13] for an application to RWRE when only the first few coordinates of the environment are random.

Acknowledgements

This work was supported in part by a FRDF grant from the University of Auckland. The author would like to thank Takashi Hara for providing the SRW Green's functions upper bounds, Martin Zerner for helpful discussions, and Duncan Temple Lang and Stephen Cope for assistance with the simulations. We also thank an anonymous referee for many helpful suggestions and for asking about possible extensions of the results (in a previous version of this work) that has given rise to the present version of the paper.

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