# Almost everywhere convergence of convolution powers on compact Abelian groups 

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#### Abstract

It is well-known that a probability measure $\mu$ on the circle $\mathbb{T}$ satisfies $\left\|\mu^{n} * f-\int f \mathrm{~d} m\right\|_{p} \rightarrow 0$ for every $f \in L_{p}$, every (some) $p \in[1, \infty$ ), if and only if $|\hat{\mu}(n)|<1$ for every non-zero $n \in \mathbb{Z}$ ( $\mu$ is strictly aperiodic). In this paper we study the a.e. convergence of $\mu^{n} * f$ for every $f \in L_{p}$ whenever $p>1$. We prove a necessary and sufficient condition, in terms of the FourierStieltjes coefficients of $\mu$, for the strong sweeping out property (existence of a Borel set $B$ with $\lim \sup \mu^{n} * 1_{B}=1$ a.e. and $\lim \inf \mu^{n} * 1_{B}=0$ a.e.). The results are extended to general compact Abelian groups $G$ with Haar measure $m$, and as a corollary we obtain the dichotomy: for $\mu$ strictly aperiodic, either $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $p>1$ and every $f \in L_{p}(G, m)$, or $\mu$ has the strong sweeping out property.

Résumé. Il est connu qu'une mesure de probabilité $\mu$ sur le cercle $\mathbb{T}$ satisfait $\left\|\mu^{n} * f-\int f \mathrm{~d} m\right\|_{p} \rightarrow 0$ pour toute fonction $f \in L_{p}$ et pour tout $p \in[1, \infty$ ) (ou pour un $p \in[1, \infty$ ), si et seulement si $\mu$ est strictement apériodique (i.e. $|\hat{\mu}(n)|<1$ pour tout $n$ non nul dans $\mathbb{Z}$ ). Nous étudions ici la convergence presque partout de $\mu^{n} * f$ pour $f \in L_{p}, p>1$. Nous montrons une condition nécessaire et suffisante portant sur les coefficients de Fourier-Stieltjes de $\mu$ pour la propriété de "balayage fort" (existence d'un borélien $B$ tel que $\lim \sup \mu^{n} * 1_{B}=1$ p.p. et $\lim \inf \mu^{n} * 1_{B}=0$ p.p.). Les résultats sont étendus aux groupes abéliens compacts généraux $G$ de mesure de Haar $m$. Comme corollaire nous obtenons la dichotomie suivante : pour $\mu$ strictement apériodique, soit $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ p.p. pour tout $p>1$ et toute fonction $f \in L_{p}(G, m)$, soit $\mu$ vérifie la propriété de balayage fort.


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## 1. Introduction

Let $(X, \mathcal{B})$ be a measurable space and $P(x, A): X \times \mathcal{B} \longrightarrow[0,1]$ a transition probability, with Markov operator $P f(x)=\int f(y) P(x, \mathrm{~d} y)$ defined for bounded $f$. When $m$ is a probability on $\mathcal{B}$ which is $P$-invariant, the operator $P$ can be extended to a contraction of $L_{1}(X, m)$. Moreover $P$ becomes a contraction in each $L_{p}(X, m)$ space, $1 \leq p \leq \infty$ [13].

Hopf's pointwise ergodic theorem yields that for $f \in L_{1}(m)$ the Cesàro averages $\frac{1}{n} \sum_{k=1}^{n} P^{k} f$ converge a.e. and in $L_{p}$-norm when $f \in L_{p}(m), 1 \leq p<\infty$. The limit is $\int f \mathrm{~d} m$ if $P$ is ergodic in $L_{1}$, i.e. when $P f=f$ a.e. for $f \in L_{1}$ holds only for $f$ constant a.e.

It is therefore a natural question to study the convergence of the unaveraged sequence $\left\{P^{n} f\right\}$, in norm or a.e. The following general results for a.e. convergence are known:

1. If $P^{*}=P$ and -1 is not an eigenvalue, then $P^{n} f$ converges a.e. for every $f \in L_{p}, p>1$ (Stein-Rota theorem [23,25]; Rota's proof yields the convergence also for $f \in L \log ^{+} L$ [6], but in general convergence may fail for $f \in L_{1}$ [17]).
2. If $P$ is an aperiodic Harris recurrent operator, then $P^{n} f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{1}(X, m)$ ( $m$ is assumed finite), by S. Horowitz [11].

Often, a.e. convergence of $\left\{P^{n} f\right\}_{n \geq 1}$ for every bounded measurable function fails in a very strong manner expressed in the following definition, which seems to have been introduced for operator sequences $\left\{P_{n}\right\}$ in the study of a.e. convergence of averages along subsequences.

Definition 1.1. We say that $\left\{P^{n}\right\}$ (or simply $P$ ) has the strong sweeping out property (SSO property) if there exists in


In this paper we study the strong sweeping out property for the convolution operator $P_{\mu}$ defined by a strictly aperiodic probability measure $\mu$ on a compact Abelian group $G, P_{\mu} f(x)=\mu * f(x)=\int_{G} f(x+y) \mathrm{d} \mu(y)$. We obtain a necessary and sufficient condition for the strong sweeping out property for $P_{\mu}$ (we will say simply "for $\mu$ "), in terms of the Fourier-Stieltjes transform of $\mu$, and deduce the dichotomy: either $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{p}(G, m)$, every $p>1$, or $\mu$ has the strong sweeping out property.

For the sake of clarity, we prove the results first for convolutions on the unit circle (Section 3), and after some examples for discrete probabilities on the circle (Section 4), we add the necessary ingredients to prove the result in the general case (Section 5).

## 2. Convolution powers on compact Abelian groups

In this section we look at the problem of almost everywhere convergence (to the integral) of convolution powers of a probability $\mu$ on a compact Abelian group $G$, with Borel $\sigma$-algebra $\mathcal{B}$ and dual group $\hat{G}$. Characters on $G$ will be denoted by $\gamma$. The Markov transition is $P(x, A)=\mu(A-x)$, with invariant probability the normalized Haar measure $m$, and the corresponding Markov operator is Pf:= $P_{\mu} f=\mu * f$. The dual Markov operator is $P_{\mu}^{*}=P_{\check{\mu}}$, where $\check{\mu}$ is the reflected probability given by $\check{\mu}(A)=\mu(-A)$. By commutativity of $G$, the operator $P_{\mu}$ is normal in $L_{2}(G, m)$. We note that the Markov chain $\left\{Y_{n}\right\}$ on $\Omega=G^{\mathbb{N}}$ induced by $P_{\mu}$ is the random walk on $G$ of law $\mu$, and $\mu^{n} * 1_{A}(x)=\mathbb{P}_{x}\left\{Y_{n} \in A\right\}$, where $\mathbb{P}_{x}$ is the probability on $\Omega$ for the chain started at $x$ (initial distribution $\delta_{x}$ ).

The Fourier-Stieltjes coefficients $\hat{\mu}(\gamma)$ are eigenvalues of $P_{\mu}$ with continuous eigenfunctions, so a necessary condition for a.e. convergence of $\left\{\mu^{n} * f\right\}$ to the integral for all continuous functions is that $|\hat{\mu}(\gamma)|<1$ for every $\gamma \neq 0$, i.e. $\mu$ is strictly aperiodic. We recall the well-known properties equivalent to strict aperiodicity of a probability $\mu$ on a compact Abelian group $G$ :

## Proposition 2.1. The following are equivalent:

(i) $|\hat{\mu}(\gamma)|<1$ for every character $0 \neq \gamma \in \hat{G}$;
(ii) $\mu^{n} * f \rightarrow \int_{G} f \mathrm{~d} m$ uniformly for every continuous function on $G$;
(iii) the support of $\mu$ is not contained in a class of a proper closed subgroup;
(iv) $\left\|\mu^{n} * f-\int_{G} f \mathrm{~d} m\right\|_{2} \rightarrow 0$ for every $f \in L_{2}(G, m)$.

It follows that $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ pointwise for $f$ in a dense subspace of $L_{p}, 1 \leq p<\infty$, namely $C(G)$. However, a result of J . Rosenblatt [20] yields that for $\alpha=\mathrm{e}^{2 \pi i \theta}$ with $\theta \in(0,1)$ irrational, the strictly aperiodic $\mu=\frac{1}{2}\left(\delta_{1}+\delta_{\alpha}\right)$ is strongly sweeping out on $\mathbb{T}$.

Some of the general results of a.e. convergence cited in the Introduction can be improved for the powers of the convolution operator $P_{\mu}$ in several particular cases:

1. If $\mu$ is symmetric and strictly aperiodic, then for every $f \in L \log L$ we have $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. [23] (see also [18]).
2. In any compact group $G$ (not necessarily Abelian), if some power $\mu^{k}$ is non-singular with respect to the Haar measure $m$, then $\left\|\mu^{n}-m\right\| \rightarrow 0$ in total variation norm with exponential rate (cf. [2], Theorem 3, for $G$ connected, [1], Theorem 4.1, for $G$ not necessarily connected and for the precise rate); see [22], Theorem 4.1, for a list of equivalent conditions. In this case, for every $f$ in $L_{1}(G, m)$ the series $\sum_{n=1}^{\infty} \mu^{n} *\left(f-\int f \mathrm{~d} m\right)$ converges a.e. It follows that any $\mu$ with the strong sweeping out property has all its convolution powers singular (cf. Proposition 3.4).
3. Another sufficient condition for a.e. convergence is $\sup _{\gamma \neq 0}|\hat{\mu}(\gamma)|<1$. It implies $\sup _{\|f\|_{p} \leq 1} \| \mu^{n} * f-$ $\int f \mathrm{~d} m \|_{p} \rightarrow 0$ (exponentially fast) for $p \in(1, \infty)$ by [21], p. 202 ff (see also [9], Proposition 4.1); so for $p \in(1, \infty)$, the series $\sum_{n=1}^{\infty} \mu^{n} *\left(f-\int f \mathrm{~d} m\right)$ converges a.e. for any $f \in L_{p}(G, m)$.

In particular, on the circle $\mathbb{T}$, the condition $\sup _{\gamma \neq 0}|\hat{\mu}(\gamma)|<1$ holds when $\hat{\mu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$ [9]. Note that the above norm convergence needs not hold for $p=1$ or $p=\infty$ [9].
4. There exists on $\mathbb{T}$ a continuous probability $\mu$ with all its convolution powers singular, such that $\sup _{n \neq 0}|\hat{\mu}(n)|<$ 1 (and then, by [21], $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $\left.f \in L_{p}(\mathbb{T}, m), p>1\right)$. See [9], Proposition 4.7, for a construction along classical lines. A result of Varopoulos [26] shows that we can find $\mu$ continuous with all its convolution powers singular satisfying $\hat{\mu}(n) \rightarrow 0$ as $|n| \rightarrow \infty$ (see also the first lines following [19], Theorem 4.2). (The result of Varopoulos is that in any non-discrete compact Abelian group there is a probability $\mu$ with all its convolution powers singular to the Haar measure such that $\hat{\mu}(\gamma)$ vanishes at infinity.)

Notations. The spectrum of the operator $P f=\mu * f$ in $L_{2}(\mathbb{T}, m)$ is denoted by $\sigma(P)$. It is the closure of $\{\hat{\mu}(\gamma): \gamma \in$ $\hat{G}\}$. In the sequel we study the peripheral spectrum of $P$, i.e. the unimodular complex numbers in $\sigma(P)$. It will be useful to distinguish notationally $\mathbb{T}$, the group on which the convolution operates (the state space of the Markov chain generated), from $\mathbb{S}=\mathbb{S}^{1}$, the boundary of the closed unit disk which contains the spectrum.

Since the spectral radius of an operator is bounded by its norm, and using the spectral theorem, one easily proves the equivalence between the following conditions:

$$
\begin{align*}
& \text { (i) }\left\|P^{n}(I-P)\right\|_{2} \rightarrow 0, \quad \text { (ii) } \sup _{\lambda \in \sigma(P)}\left|\lambda^{n}(1-\lambda)\right| \rightarrow 0,  \tag{1}\\
& \text { (iii) } \sup _{\gamma \in \hat{G}}\left|\hat{\mu}^{n}(\gamma)(1-\hat{\mu}(\gamma))\right| \rightarrow 0, \quad \text { (iv) } \sigma(P) \cap \mathbb{S}=\{1\} .
\end{align*}
$$

Remarks. 1. If $\left\|\mu^{n+1}-\mu^{n}\right\|_{1} \rightarrow 0$, then for $1 \leq p<\infty$ continuity of the representation of $G$ in $L_{p}(G, m)$ by translations yields $\left\|P^{n+1}-P^{n}\right\|_{p} \rightarrow 0$.
2. If $\left\|P^{n+1}-P^{n}\right\|_{2} \rightarrow 0$, then by the Riesz-Thorin theorem $\left\|P^{n+1}-P^{n}\right\|_{p} \rightarrow 0$ for every $1<p<\infty$. However, for $p=1$ we may still have $\left\|\mu^{n+1}-\mu^{n}\right\|_{1}=2$ for every $n$ (e.g., [12], Remark 2.16(b)).

Proposition 2.2. Let $\mu$ be a strictly aperiodic probability on a compact Abelian group G. If $\sigma(P) \cap \mathbb{S} \neq \mathbb{S}$, then there exists an increasing subsequence $\left\{n_{k}\right\}$ such that $\mu^{n_{k}} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{2}(G, m)$.

Proof. Since $P$ is a positive contraction, the assumption $\sigma(P) \cap \mathbb{S} \neq \mathbb{S}$ implies that this intersection is finite, and for some $j \geq 1$ we have $\sigma\left(P^{j}\right) \cap \mathbb{S}=\{1\}[14]$, Proposition 1 (see Lemma 3.5 below for a direct proof). Therefore (iii) of (2) holds for $\mu^{j}$ and it follows from [12], Proposition 2.15, that there exists an increasing subsequence $\left\{\ell_{k}\right\}$ such that $\mu^{j \ell_{k}} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{2}(G, m)$.

Remark. The probability $\mu=\frac{1}{2}\left(\delta_{1}+\delta_{\alpha}\right)$ on $\mathbb{T}$ has the strong sweeping out property as mentioned above, although $\sigma\left(P_{\mu}\right) \cap \mathbb{S}=\{1\}$ (and also $\left\|\mu^{n+1}-\mu^{n}\right\|_{1} \rightarrow 0$ by Foguel's zero-two law). By Proposition 2.2 there is a subsequence $\left\{n_{k}\right\}$ with $\mu^{n_{k}} * f \rightarrow \int f \mathrm{~d} m$ for every $f \in L_{2}(\mathbb{T}, m)$.

Now we give a variant of a result of [5] (see also [12], Theorem 2.20) which gives a sufficient condition for the a.e. convergence for $f$ in $L_{p}, p>1$. It is based on Theorem 14 of [4], or the following extension of it which does not require normality.

We consider a positive contraction $T$ on $L_{2}(X, m)$ where $(X, m)$ is a probability space. For every integer $r \in[1, n]$, let $\Delta^{r} T^{n}:=T^{n-r}(T-I)^{r}$, where $I$ is the identity on $L_{2}(X, m)$.

Theorem 2.3. Let $T$ be a positive contraction of $L_{2}(X, m)$, with $W$ be a closed $T$-invariant subspace. Let $W=$ $\bigoplus_{j \in J} V_{j}$ be an orthogonal decomposition of $W$ into closed $T$-invariant subspaces such that the restriction $T_{j}$ of $T$ to $V_{j}$ satisfies $\left\|T_{j}\right\|<1$. Let $L(j):=\frac{\left\|I_{j}-T_{j}\right\|}{1-\left\|T_{j}\right\|}$, where $I_{j}$ is the identity on $V_{j}$.

Put $f_{0}^{*}(x):=\sup _{n \geq 0}\left|T^{n} f(x)\right|$ and $f_{r}^{*}(x)=\sup _{n \geq r}\left|n^{r} \Delta^{r} T^{n} f(x)\right|$, for $r \geq 1$. Then:
(i) For $f \in W$, if $f=\sum_{j} \Pi_{j} f$ is its orthogonal decomposition, the maximal function satisfies

$$
\begin{equation*}
\left\|f_{0}^{*}\right\|_{2} \leq 2\|f\|_{2}+\left[\sum_{j}\left\|\Pi_{j} f\right\|_{2}^{2} L(j)^{2}\right]^{1 / 2} \tag{3}
\end{equation*}
$$

and $\sum_{j}\left\|\Pi_{j} f\right\|^{2} L(j)^{2}<\infty$ implies the convergence $\lim _{n} T^{n} f=0 m$-a.e. and in $L_{2}(m)$.
(ii) If $L_{\infty}:=\sup _{j} L(j)<\infty$, there are finite constants $C_{r}$ such that for every $f \in W$,

$$
\begin{equation*}
\left\|f_{r}^{*}\right\|_{2} \leq C_{r}\|f\|_{2}, \quad \forall r \geq 0 \tag{4}
\end{equation*}
$$

and for every $r \geq 0$, the convergence $\lim _{n} n^{r} \Delta^{r} T^{n} f=0$ holds m-a.e. and in $L_{2}(m)$.
Proof. (i) The proof is based on the idea of comparing $T^{n}$ with its Cesàro averages, like the proof of Stein's theorem for self-adjoint positive contractions [25] or that of [4] for positive normal contractions, but does not require the spectral theorem.

For $n \geq 1$ put $A_{n} f=\frac{1}{n} \sum_{k=0}^{n-1} T^{k} f$. By Akcoglu's ergodic theorem [13], p. 190, for every $f \in L_{2}$ the sequence $A_{n} f(x)$ converges a.e. and the maximal function $A^{*} f(x)=\sup _{k \geq 0}\left|A_{k} f\right|(x)$ satisfies $\left\|A^{*} f\right\|_{2} \leq 2\|f\|_{2}$. Using the inequality

$$
\begin{equation*}
\left(\frac{1}{n}\left|\sum_{k=1}^{n} k a_{k}\right|\right)^{2} \leq \sum_{k=1}^{n} k\left|a_{k}\right|^{2} \tag{5}
\end{equation*}
$$

which holds for every complex numbers $a_{1}, \ldots, a_{n}$, we can write for $f \in W$ :

$$
\left|T^{n} f-A_{n} f\right|^{2}=\left(\frac{1}{n}\left|\sum_{k=1}^{n} k\left(T^{k} f-T^{k-1} f\right)\right|\right)^{2} \leq \sum_{k=1}^{n} k\left|T^{k} f-T^{k-1} f\right|^{2} \leq F_{1}(f)^{2},
$$

with $F_{1}(f)^{2}=\sum_{k=1}^{\infty} k\left|T^{k-1}(T-I) f\right|^{2}$.
Observe that for $0 \leq \lambda<1$, we have $(1-\lambda)^{2} \sum_{1}^{\infty} k \lambda^{2(k-1)}=1 /(1+\lambda)^{2} \leq 1$. Therefore,

$$
\begin{aligned}
\left\|F_{1}(f)\right\|_{2}^{2} & =\sum_{k=1}^{\infty} k \int_{X}\left|\sum_{j} T_{j}^{k-1}\left(T_{j}-I_{j}\right) \Pi_{j} f\right|^{2} \mathrm{~d} m=\sum_{k} k \sum_{j}\left\|T_{j}^{k-1}\left(T_{j}-I_{j}\right) \Pi_{j} f\right\|_{2}^{2} \\
& \leq \sum_{j}\left(\sum_{k} k\left\|T_{j}\right\|^{2(k-1)}\right)\left\|T_{j}-I_{j}\right\|^{2}\left\|\Pi_{j} f\right\|_{2}^{2} \leq \sum_{j} \frac{\left\|T_{j}-I_{j}\right\|^{2}}{\left(1-\left\|T_{j}\right\|\right)^{2}}\left\|\Pi_{j} f\right\|_{2}^{2}=\sum_{j}\left\|\Pi_{j} f\right\|_{2}^{2} L(j)^{2} .
\end{aligned}
$$

This proves (3), since we have $\left|T^{n} f\right| \leq A^{*} f+F_{1}(f), \forall n \geq 1$; hence $f_{0}^{*} \leq A^{*} f+F_{1}(f)$. The convergence in $L^{2}$-norm follows from the assumption $\left\|T_{j}\right\|<1$.

Let $K$ be a finite subset of $J$. Write $f=\varphi_{K}+\rho_{K}$, with $\varphi_{K}=\sum_{j \in K} \Pi_{j} f$ and $\rho_{K}=\sum_{j \neq K} \Pi_{j} f$. For $\varepsilon>0$, take $K$ such that $\left\|\rho_{K}\right\|_{2} \leq \varepsilon$ and $\sum_{j \notin K}\left\|\Pi_{j} f\right\|_{2}^{2} L(j)^{2} \leq \varepsilon^{2}$.

Clearly $\lim \sup _{n}\left|T^{n} \varphi_{K}\right|=0$, since $\lim _{n}\left\|T^{n} \Pi_{j} f\right\|=0, \forall j \in J$. Hence $\lim \sup _{n}\left|T^{n} f\right| \leq \lim \sup _{n}\left|T^{n} \varphi_{K}\right|+$ $\lim \sup _{n}\left|T^{n} \rho_{K}\right| \leq\left(\rho_{K}\right)_{0}^{*}$. Applying inequality (3) to $\rho_{K}$, we obtain

$$
\left\|\limsup _{n}\left|T^{n} f\right|\right\|_{2} \leq\left\|\left(\rho_{K}\right)_{0}^{*}\right\|_{2} \leq\left\|A^{*} \rho_{K}\right\|_{2}+F_{1}\left(\rho_{K}\right) \leq 2\left\|\rho_{K}\right\|_{2}+\left[\sum_{j \notin K}\left\|\Pi_{j} f\right\|_{2}^{2} L(j)^{2}\right]^{1 / 2} \leq 3 \varepsilon
$$

(ii) For $r \geq 1, x \in\left[0,1\left[\right.\right.$, we have $\sum_{k=r}^{\infty} k^{2 r-1} x^{k-r}=\rho_{r}(x) /(1-x)^{2 r}$, where $\rho_{r}$ is a polynomial. Therefore there exists a finite constant $D_{r}$ such that

$$
\begin{equation*}
(1-\lambda)^{2 r} \sum_{k=r}^{\infty} k^{2 r-1} \lambda^{2(k-r)}=\rho_{r}\left(\lambda^{2}\right) /(1+\lambda)^{2 r} \leq D_{r}^{2}, \quad 0 \leq \lambda<1 . \tag{6}
\end{equation*}
$$

The relation $C_{k}^{r-1}+C_{k}^{r}=C_{k+1}^{r}$, satisfied by the binomial coefficients $C_{k}^{r}$, yields

$$
\begin{equation*}
C_{n+1}^{r+1} \Delta^{r} T^{n}=\sum_{k=r}^{n} C_{k}^{r} \Delta^{r} T^{k}+\sum_{k=r+1}^{n} C_{k}^{r+1} \Delta^{r+1} T^{k}, \quad 0 \leq r \leq n . \tag{7}
\end{equation*}
$$

For $f \in L_{2}(X, m)$, it follows from (5):

$$
\left|\frac{1}{n} \sum_{k=r}^{n} C_{k}^{r} \Delta^{r} T^{k} f\right|^{2} \leq \sum_{k=r}^{n} k\left(C_{k}^{r} / k\right)^{2}\left|T^{k-r}(T-I)^{r} f\right|^{2} \leq F_{r}^{2},
$$

with

$$
F_{r}=\left(\sum_{k=r}^{\infty} k^{-1}\left(C_{k}^{r}\right)^{2}\left|T^{k-r}(T-I)^{r} f\right|^{2}\right)^{1 / 2}
$$

From the decomposition in the orthogonal subspaces $V_{j}$, we get:

$$
\begin{equation*}
\left\|F_{r}\right\|_{2}^{2} \leq \sum_{j}\left[\sum_{k=r}^{\infty} k^{-1}\left(C_{k}^{r}\right)^{2}\left\|T_{j}\right\|^{2(k-r)}\right]\left\|T_{j}-I_{j}\right\|^{2 r}\left\|\Pi_{j} f\right\|^{2} . \tag{8}
\end{equation*}
$$

According to (6) with $\lambda=\left\|T_{j}\right\|$, the series in [] in (8) is less than $D_{r}^{2} /\left(1-\left\|T_{j}\right\|\right)^{2 r}$, so

$$
\left\|F_{r}\right\|_{2}^{2} \leq D_{r}^{2} \sum_{j} L(j)^{2 r}\left\|\Pi_{j} f\right\|^{2} \leq D_{r}^{2}\left(\sup _{j} L(j)\right)^{2 r} \sum_{j}\left\|\Pi_{j} f\right\|^{2}=D_{r}^{2} L_{\infty}^{2 r}\|f\|^{2} .
$$

For $n \geq r \geq 1$ the identity (7) implies the inequality

$$
\left|\frac{1}{n+1} C_{n+1}^{r+1} \Delta^{r} T^{n} f\right| \leq \frac{1}{n+1}\left|\sum_{k=r+1}^{n} C_{k}^{r+1} \Delta^{r+1} T^{k} f\right|+\frac{1}{n+1}\left|\sum_{k=r}^{n} C_{k}^{r} \Delta^{r} T^{k} f\right| \leq F_{r+1}+F_{r} ;
$$

since $C_{n+1}^{r+1}=\frac{n+1}{r+1} \prod_{j=0}^{r-1} \frac{n-j}{r-j} \geq \frac{n+1}{r+1}\left(\frac{n}{r}\right)^{r}$ for $n \geq r \geq 1$, putting $B_{r}=(r+1) r^{r}$ we obtain

$$
\begin{aligned}
\left\|f_{r}^{*}\right\|_{2} & =\left\|\sup _{n \geq r}\left|n^{r} \Delta^{r} T^{n} f\right|\right\|_{2} \leq B_{r}\left\|\sup _{n \geq r}\left|\frac{1}{n+1} C_{n+1}^{r+1} \Delta^{r} T^{n} f\right|\right\|_{2} \\
& \leq B_{r}\left(\left\|F_{r+1}\right\|_{2}+\left\|F_{r}\right\|_{2}\right) \leq B_{r}\left(D_{r+1} L_{\infty}^{r+1}+D_{r} L_{\infty}^{r}\right)\|f\|_{2} .
\end{aligned}
$$

The proof of the convergence statement in (ii) is similar to that of (i).
Suppose now that $T$ defines a positive contraction of each $L_{p}(m), 1 \leq p \leq \infty$, and $W=L_{2}(m)$ in Theorem 2.3. The inequalities (4) and the classical inequalities $\left\|\sup _{n} \frac{1}{n+1}\left(f+T f+\cdots+T^{n} f\right)\right\|_{p} \leq \frac{p}{p-1}\|f\|_{p}, f \in L_{p}(m), 1<$ $p<\infty$, are the needed properties for Stein's complex interpolation theorem [25] (see also [7] for a detailed presentation of the method applied to iterates of composed conditional expectations). It implies for $1<p<\infty$ the maximal inequality and the a.e. convergence of $T^{n} f$ for $f \in L_{p}(m)$. This applies to convolution powers, i.e. $T=P_{\mu}$, even on non-Abelian (compact) groups. In the Abelian case, it yields:

Theorem 2.4 ([4]). Let $\mu$ be a strictly aperiodic probability on a compact Abelian group G. If

$$
\begin{equation*}
\sup _{\gamma \neq 0} \frac{|1-\hat{\mu}(\gamma)|}{1-|\hat{\mu}(\gamma)|}<\infty, \tag{9}
\end{equation*}
$$

then for $p>1$ there is a constant $C_{p}$ such that for every $f \in L_{p}(G, m)$, the maximal inequality $\left\|f_{0}^{*}\right\|_{p} \leq C_{p}\|f\|_{p}$ holds and $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e.

Proof. Let $P f=\mu * f$ be the normal operator induced on $L_{2}(G, m)$. Then $\sigma(P)$ is the closure of $\{\hat{\mu}(\gamma): \gamma \in \hat{G}\}$, and (9) implies that $\sigma(P)$ contains no unimodular points except 1 and is included in a Stolz region of the closed unit disk. Using part (ii) of Theorem 2.3, when $V_{\gamma}$ is the space of multiples of the character $\gamma \in \hat{G}$, the maximal inequality and the a.e. convergence follow from Stein's theorem as explained above.

As mentioned in the Introduction, failure of a.e. convergence for a sequence of operators may be quite strong. In [8], del Junco and Rosenblatt gave a condition which implies the SSO property (cf. Definition 1.1) and, for powers of convolution operators on $G$, reads:

Theorem 2.5. Let $\mu$ be a probability measure on the compact Abelian group G. Iffor every integer $N_{0}>1$ and every $\varepsilon>0$, there exists a measurable set $A$ such that $m(A)<\varepsilon$ and $m\left\{\sup _{n \geq N_{0}} \mu^{n} * 1_{A}(x) \geq 1-\varepsilon\right\}>1-\varepsilon$, then $\mu$ has the strong sweeping out property.

The next lemma shows that it suffices to fulfill the conditions for $N_{0}=1$.
Lemma 2.6. If for every $\varepsilon>0$ there exists a set $A \in \mathcal{B}$ such that $m(A)<\varepsilon$ and

$$
\begin{equation*}
m\left\{x \in G: \sup _{n \geq 1} \mu^{n} * 1_{A}(x) \geq 1-\varepsilon\right\}>1-\varepsilon, \tag{10}
\end{equation*}
$$

then $\mu$ has the strong sweeping out property on $G$.
Proof. Let be given $N_{0} \geq 1$ and $\varepsilon>0$. By applying (10) with $\varepsilon^{\prime}=\frac{\varepsilon}{2 N_{0}+\varepsilon}$ instead of $\varepsilon$, we obtain a set $A$ with $\mu(A)<\varepsilon^{\prime}$ which satisfies the condition of Theorem 2.5 since

$$
\begin{aligned}
m\left\{\sup _{n \geq N_{0}} \mu^{n} * 1_{A}(x) \geq 1-\varepsilon\right\} & \geq m\left\{\sup _{n \geq N_{0}} \mu^{n} * 1_{A}(x) \geq 1-\varepsilon^{\prime}\right\} \\
& \geq m\left\{\sup _{n} \mu^{n} * 1_{A}(x) \geq 1-\varepsilon^{\prime}\right\}-N_{0} m(A) /\left(1-\varepsilon^{\prime}\right) \\
& \geq 1-\varepsilon^{\prime}-N_{0} m(A) /\left(1-\varepsilon^{\prime}\right) \geq 1-\varepsilon^{\prime}-N_{0} \varepsilon^{\prime} /\left(1-\varepsilon^{\prime}\right) \geq 1-\varepsilon .
\end{aligned}
$$

## 3. Convergence and divergence of convolution powers on the circle

Our aim is to characterize the strong sweeping out property on the circle $\mathbb{T}$ by a property of the Fourier-Stieltjes coefficients of $\mu$, and to study the a.e. convergence of the convolution powers of $\mu$. In order to avoid repetition when dealing with general groups, we denote the characters by $\gamma$ and the Fourier-Stieltjes coefficients by $\hat{\mu}(\gamma)$. For $\gamma(x)=\mathrm{e}^{2 \pi \mathrm{i} n x}$, we write either $\hat{\mu}(\gamma)$ or $\hat{\mu}(n)$.

Theorem 3.1. Let $\mu$ be a strictly aperiodic probability on the unit circle $\mathbb{T}$. If

$$
\begin{equation*}
\limsup _{\hat{\mu}(n) \rightarrow 1, n \neq 0} \frac{|1-\hat{\mu}(n)|}{1-|\hat{\mu}(n)|}=\infty, \tag{11}
\end{equation*}
$$

then $\mu$ has the strong sweeping out property on $\mathbb{T}$.
Proof. We will use Lemma 2.6 to show the SSO property. For a character $\gamma \neq 0$ denote

$$
\begin{equation*}
L(\gamma)=\frac{|1-\hat{\mu}(\gamma)|}{1-|\hat{\mu}(\gamma)|} \quad \text { and } \quad \rho(\gamma)=\left|1-\frac{\hat{\mu}(\gamma)}{|\hat{\mu}(\gamma)|}\right| \tag{12}
\end{equation*}
$$

$L(\gamma)$ is well-defined by strict aperiodicity, $L(\gamma) \geq 1$ and the triangle inequality yields

$$
\begin{equation*}
\rho(\gamma) \geq|1-\hat{\mu}(\gamma)|-\left|\hat{\mu}(\gamma)-\frac{\hat{\mu}(\gamma)}{|\hat{\mu}(\gamma)|}\right|=(1-|\hat{\mu}(\gamma)|)(L(\gamma)-1) . \tag{13}
\end{equation*}
$$

By (11), there exists an infinite sequence $\left\{\gamma_{k}\right\}, \gamma_{k}(x)=\mathrm{e}^{2 \pi \mathrm{i} n_{k} x}$ for $x \in \mathbb{R} / \mathbb{Z}$, such that $L\left(\gamma_{k}\right) \rightarrow \infty$ and $\rho\left(\gamma_{k}\right) \rightarrow 0$.
Let $0<\varepsilon<1$ be given and put $M=\frac{1+\pi}{2 \varepsilon}$. We will construct $A \in \mathcal{B}$ satisfying (10), by adapting the ideas of Losert [16]. From the above sequence $\left\{\gamma_{k}\right\}$, we fix $\gamma_{k}$ such that

$$
\begin{equation*}
\rho\left(\gamma_{k}\right)<\frac{1}{M} \quad \text { and } \quad L\left(\gamma_{k}\right)>\frac{32 M^{2} \pi}{\varepsilon} . \tag{14}
\end{equation*}
$$

Since $\gamma_{k}$ is fixed, we denote $L=L\left(\gamma_{k}\right)$ and $\rho=\rho\left(\gamma_{k}\right)$; in all other quantities defined we will suppress the dependence on $\gamma_{k}$. Let $\xi \in(0,1)$ satisfy $\mathrm{e}^{2 \pi \mathrm{i} \xi}=\hat{\mu}\left(\gamma_{k}\right) /\left|\hat{\mu}\left(\gamma_{k}\right)\right|$. Then $\rho=\left|1-\mathrm{e}^{2 \pi \mathrm{i} \xi}\right|=2|\sin (\pi \xi)| \leq 2 \pi \xi$. Since $L>2$, (13) yields

$$
\begin{equation*}
\frac{\rho}{L} \geq\left(1-\left|\hat{\mu}\left(\gamma_{k}\right)\right|\right) \frac{L-1}{L} \geq \frac{1}{2}\left(1-\left|\hat{\mu}\left(\gamma_{k}\right)\right|\right) . \tag{15}
\end{equation*}
$$

Put $j:=\left[\frac{2 \pi}{\rho}\right]+1$ and $r:=1-\left|\hat{\mu}\left(\gamma_{k}\right)\right|^{j}$. By (15) we have $1-r \geq\left(1-\frac{2 \rho}{L}\right)^{j}>0$. Then

$$
\begin{equation*}
-r \geq \ln (1-r) \geq j \ln \left(1-\frac{2 \rho}{L}\right) \geq-j \frac{4 \rho}{L}, \tag{16}
\end{equation*}
$$

since $\ln (1-t) \geq-2 t$ for $t<1 / 4$, while $\rho<1 / M$ and $L>32 \pi$ imply $\frac{\rho}{L} \leq 1 / 32 \pi$. The estimate (16) and the definition of $j$ yield

$$
\begin{equation*}
r \leq \frac{4 \rho j}{L} \leq \frac{16 \pi}{L} . \tag{17}
\end{equation*}
$$

We now define $\delta:=\max \left\{\frac{\rho}{4}, \sqrt{\frac{r}{2 \varepsilon}}\right\}$. The estimates (14) and (17) yield

$$
\begin{equation*}
\sqrt{\frac{r}{2 \varepsilon}} \leq \sqrt{\frac{8 \pi}{L \varepsilon}}<\sqrt{\frac{8 \pi}{32 M^{2} \pi}}=\frac{1}{2 M}, \tag{18}
\end{equation*}
$$

which shows $\delta<1 / 2 M$ since $\rho<1 / M$.
Since $2|\sin (\pi \xi)|=\rho<1$, we have $\xi<1 / 6$. We saw that $\rho \leq 2 \pi \xi$, so

$$
\begin{equation*}
\frac{1}{j}<\frac{\rho}{2 \pi}<\xi<\frac{1}{2} \sin (\pi \xi)=\frac{\rho}{4}<\delta . \tag{19}
\end{equation*}
$$

Hence the $j$ intervals $[(\ell-1) \xi, \ell \xi) \bmod 1,1 \leq \ell \leq j$, each of length $\xi>1 / j$, cover all the unit interval, so for each $x \in[0,1)$ there exists $\ell_{x}$ with $\left|n_{k} x \bmod 1-\ell_{x} \xi\right|<\xi<\delta\left(\right.$ recall that $\left.\gamma_{k}(x)=\mathrm{e}^{2 \pi \mathrm{i} n_{k} x}\right)$.

Fix $1 \leq \ell \leq j$. Using the definition of $\xi$ we obtain

$$
\begin{aligned}
\left|\hat{\mu}^{\ell}\left(n_{k}\right)\right| & =\left|\hat{\mu}\left(n_{k}\right)\right|^{\ell}=\mathrm{e}^{-2 \pi \mathrm{i} \ell \xi}\left(\hat{\mu}\left(n_{k}\right)\right)^{\ell}=\mathrm{e}^{-2 \pi \mathrm{i} \ell \xi} \int_{I} \mathrm{e}^{-2 \pi \mathrm{i} n_{k} s} \mathrm{~d} \mu^{\ell}(s) \\
& =\int_{I} \mathrm{e}^{-2 \pi \mathrm{i}\left(n_{k} s+\ell \xi\right)} \mathrm{d} \mu^{\ell}(s)=\int_{I} \cos \left(2 \pi\left(n_{k} s+\ell \xi\right)\right) \mathrm{d} \mu^{\ell}(s)=1-2 \int_{I}\left(\sin \left(\pi\left(n_{k} s+\ell \xi\right)\right)\right)^{2} \mathrm{~d} \mu^{\ell}(s) .
\end{aligned}
$$

But for $\ell \leq j$, we have $\left|\hat{\mu}\left(n_{k}\right)\right|^{\ell} \geq\left|\hat{\mu}\left(n_{k}\right)\right|^{j}=1-r$, so

$$
\int_{I}\left(\sin \left(\pi\left(n_{k} s+\ell \xi\right)\right)\right)^{2} \mathrm{~d} \mu^{\ell}(s)=\frac{1}{2}\left(1-\left|\hat{\mu}\left(n_{k}\right)\right|^{\ell}\right) \leq \frac{r}{2} .
$$

From Tchebyshev's inequality we obtain $\mu^{\ell}\left\{s \in I:\left|\sin \left(\pi\left(n_{k} s+\ell \xi\right)\right)\right| \geq \delta\right\} \leq \frac{r}{2 \delta^{2}}$. Given $x \in I$, we have $\ell_{x} \leq j$ with $\left|n_{k} x \bmod 1-\ell_{x} \xi\right|<\delta$. Hence $\left|\sin \left(\pi\left(n_{k} s+\ell_{x} \xi\right)\right)\right|<\delta$ implies

$$
\left|\sin \left(\pi\left(n_{k} s+n_{k} x\right)\right)\right| \leq\left|\sin \left(\pi\left(n_{k} s+\ell_{x} \xi\right)\right)\right|+\left|\sin \left(\pi\left(n_{k} x-\ell_{x} \xi\right)\right)\right|<(1+\pi) \delta .
$$

Since $\delta^{2} \geq r / 2 \varepsilon$ by the definition of $\delta$, it follows that

$$
\mu^{\ell_{x}}\left\{s \in I:\left|\sin \left(\pi\left(n_{k} s+n_{k} x\right)\right)\right|<(1+\pi) \delta\right\} \geq 1-\frac{r}{2 \delta^{2}} \geq 1-\varepsilon .
$$

Define $\varphi: I \longrightarrow I$ by $\varphi(t)=n_{k} t \bmod 1$. Then $\varphi$ preserves Lebesgue's measure on $I$. Let $A=\varphi^{-1}(B)$, where $B=$ $\left\{s \in I:\left|\sin \left(\pi n_{k} s\right)\right|<(1+\pi) \delta\right\}$. Since $|\sin (\pi y)| \geq|2 y|$ for $|y| \leq \frac{1}{2}$, we have $m(A)=m(B) \leq(1+\pi) \delta<\frac{1+\pi}{2 M}=\varepsilon$ and by the previous estimate

$$
\begin{aligned}
\sup _{1 \leq \ell \leq j} \mu^{\ell} * 1_{A}(x) & =\sup _{1 \leq \ell \leq j} \mu^{\ell}(A-x) \geq \mu^{\ell_{x}}(A-x) \\
& =\mu^{\ell_{x}}\left\{s \in I:\left|\sin \left(\pi\left(n_{k} s+n_{k} x\right)\right)\right|<(1+\pi) \delta\right\} \geq 1-\varepsilon .
\end{aligned}
$$

This yields $m\left\{x \in[0,1): \sup _{n \geq 0} \mu^{n} * 1_{A}(x)>1-\varepsilon\right\}=m([0,1))=1$, so (10) is satisfied.
Corollary 3.2. Let $\mu$ be strictly aperiodic such that $\mathbb{S} \subset \overline{\{\hat{\mu}(n): n \in \mathbb{Z}\}}$. Then (11) holds, and therefore $\mu$ has the strong sweeping out property on the circle.

Proof. Let $P f=\mu * f$ on $L_{2}(\mathbb{T}, m)$ with spectrum $\sigma(P)$; it is easy to show that $\sigma(P)$ is the closure of $\{\hat{\mu}(n): n \in \mathbb{Z}\}$. Let $1 \neq \lambda_{k} \in \mathbb{S}$ with $\lambda_{k} \rightarrow 1$. By assumption there exists a sequence $\hat{\mu}\left(n_{k, j}\right)$ (with $\left|\hat{\mu}\left(n_{k, j}\right)\right|<1$ by strict aperiodicity) converging to $\lambda_{k}$ as $j \rightarrow \infty$. Then $\lim _{j}\left|1-\hat{\mu}\left(n_{k, j}\right)\right| /\left(1-\left|\hat{\mu}\left(n_{k, j}\right)\right|\right)=\infty$, since numerator converges to $\left|1-\lambda_{k}\right| \neq 0$. Call $n_{k}$ a value of $n_{k, j}$ with $j$ large so $\left|1-\hat{\mu}\left(n_{k, j}\right)\right|<2\left|1-\lambda_{k}\right|$ and $L\left(n_{k, j}\right)>k$. Thus (11) holds, and the strong sweeping out property follows from the theorem.

Lemma 3.3 ([15], Lemma 1). Condition (11) is equivalent to

$$
\begin{equation*}
\limsup _{\hat{\mu}(n) \rightarrow 1, n \neq 0} \frac{|\Im m \hat{\mu}(n)|}{1-\Re e \hat{\mu}(n)}=\infty . \tag{20}
\end{equation*}
$$

Proof. For the sake of completeness, we give a proof and show the following equivalence: for every sequence $\left\{z_{n}\right\}$ with $\left|z_{n}\right|<1$,

$$
\limsup _{\left|z_{n}\right|<1, z_{n} \rightarrow 1} \frac{\left|1-z_{n}\right|}{1-\left|z_{n}\right|}=\infty \quad \text { if and only if } \quad \limsup _{\left|z_{n}\right|<1, z_{n} \rightarrow 1} \frac{\left|\Im m z_{n}\right|}{1-\mathfrak{R} e z_{n}}=\infty .
$$

Let $z=\rho \mathrm{e}^{\mathrm{i} \alpha}$ be a complex number with argument $\alpha \in\left(-\frac{1}{2} \pi, \frac{1}{2} \pi\right)$ and modulus $\rho<1$. Put $A(z)=A(\alpha, \rho)=\frac{|1-z|}{1-|z|}$ and $B(z)=B(\alpha, \rho)=\frac{|\xi m z|}{1-\Re i e z}$. We have

$$
A(\alpha, \rho)=\left(1+4 \rho\left(\frac{\sin (\alpha / 2)}{1-\rho}\right)^{2}\right)^{1 / 2}, \quad B(\alpha, \rho)=2 \rho\left|\cos \frac{\alpha}{2}\right| \frac{|\sin (\alpha / 2)|}{1-\rho \cos \alpha} .
$$

Putting $M(z)=M_{\alpha, \rho}=2 \frac{|\sin (\alpha / 2)|}{1-\rho}$, we get

$$
A(\alpha, \rho)=\left(1+\rho M_{\alpha, \rho}^{2}\right)^{1 / 2}, \quad B(\alpha, \rho)=\frac{\rho|\cos (\alpha / 2)|}{|\sin (\alpha / 2)|+\cos \alpha M_{\alpha, \rho}^{-1}} .
$$

Hence the equivalence between $\lim _{z_{n} \rightarrow 1} A\left(z_{n}\right)=\infty, \lim _{z_{n} \rightarrow 1} M\left(z_{n}\right)=\infty, \lim _{z_{n} \rightarrow 1} B\left(z_{n}\right)=\infty$.

Remarks. 1. Even for $\mu$ strictly aperiodic, the condition $\sup _{n \neq 0} \frac{|1-\hat{\mu}(n)|}{1-|\hat{\mu}(n)|}=\infty$ is insufficient for the strong sweeping out (and (9) is not necessary for a.e. convergence). Let $\theta \in(0,1)$ be irrational, $\alpha=\mathrm{e}^{2 \pi \mathrm{i} \theta} \in \mathbb{T}$, and put $\mu=\frac{1}{2}\left(\delta_{\alpha^{-1}}+\right.$ $\delta_{\alpha}$ ). The operator Pf $=\mu * f$ is self-adjoint on $L_{2}(m)$, so by the Stein-Rota theorem $\sup _{n} P^{n}|f| \in L_{2}$ for every $f \in L_{2}$. We have $\hat{\mu}(n)=\cos (2 \pi n \theta)$, so $\mu$ is strictly aperiodic, hence $\mu^{n} * f \rightarrow \int f \mathrm{~d} \mu$ a.e. for every $f \in L_{2}$, and thus $\mu$ does not have the sweeping out property. (Note that the result of [5] does not apply to $\mu$, because $\frac{1}{2}\left(\delta_{1}+\delta_{-1}\right)$ is not strictly aperiodic on $\mathbb{Z}$, but it applies to $\mu^{2}$.) We have $\sup _{n \neq 0} \frac{|1-\hat{\mu}(n)|}{1-|\hat{\mu}(n)|}=\infty$ since $L\left(n_{k}\right) \rightarrow \infty$ when $n_{k} \theta \rightarrow 1 / 2 \bmod 1$, but (11) fails since when $\hat{\mu}(n)$ is close to 1 its values are positive reals and $L(n)=1$.
2. Let $\alpha=\mathrm{e}^{2 \pi \mathrm{i} \theta}$ be as above, and define $\mu=\frac{1}{2}\left(\delta_{1}+\delta_{\alpha}\right)$. Then $\hat{\mu}(n)=\frac{1}{2}\left(1+\alpha^{-n}\right)$, so $\mu$ is strictly aperiodic. For $z=\frac{1}{2}\left(1+\mathrm{e}^{-2 \pi \mathrm{in} \theta}\right)$, we have $|z|=|\cos (\pi n \theta)|$ and $|1-z|=|\sin (\pi n \theta)|$, so $L(n) \rightarrow \infty$ as $n \theta \rightarrow 0 \bmod 1$. Hence (11) holds.
3. Let $\alpha=\mathrm{e}^{2 \pi \mathrm{i} \theta}$ be as above. Theorem 3.1 for $\mu=\sum_{k \in \mathbb{Z}} p_{k} \delta_{\alpha^{k}}$ (where $p_{k} \geq 0$ with $\sum_{k \in \mathbb{Z}} p_{k}=1$ ) follows from Theorem 2 of [16]: We put on $\mathbb{Z}$ the probability $v:=\sum_{k \in \mathbb{Z}} p_{k} \delta_{k}$ and obtain $\hat{\mu}(n)=\hat{v}(\{n \theta\})$. Hence $\hat{\mu}(n) \rightarrow 1$ implies $\{n \theta\} \rightarrow 0$, so (11) implies $\lim \sup _{t \rightarrow 0} \frac{|1-\hat{v}(t)|}{1-|\hat{v}(t)|}=\infty$.

Example 1. Let $\left\{\alpha_{k}=\mathrm{e}^{2 \pi \mathrm{i} \theta_{k}}\right\}_{k=1}^{d} \subset \mathbb{T}$ with $d>1$, and assume that $1, \theta_{1}, \theta_{2}, \ldots, \theta_{d}$ are linearly independent over the rationals. Let $\mu=\sum_{k=1}^{d} p_{k} \delta_{\alpha_{k}}$ (where $0 \leq p_{k}<1$ and $\sum p_{k}=1$ ). Then $\mu$ has the strong sweeping out property on $\mathbb{T}$.

Proof. We have $\hat{\mu}(n)=\sum_{k=1}^{d} p_{k} \alpha_{k}^{-n}$, and $\mu$ is strictly aperiodic since its support has at least two "irrational" points. The linear independence implies, by a result of Kronecker (e.g. [10], p. 382, [13], pp. 12-13), that the powers of $\left(\alpha_{1}, \ldots, \alpha_{d}\right)$ are dense in the $d$-dimensional torus $\mathbb{T}^{d}$; hence for $\lambda \in \mathbb{S}$ there exists a subsequence $\left\{n_{j}\right\}$ such that $\alpha_{k}^{n_{j}} \rightarrow \bar{\lambda}$ for $k=1, \ldots, d$, which yields that $\hat{\mu}\left(n_{j}\right) \rightarrow \lambda$. We conclude that $\mathbb{S} \subset \overline{\{\hat{\mu}(n): n \in \mathbb{Z}\}}$, and Corollary 3.2 applies.

Remark. Since the flow defined on $(\mathbb{T}, m)$ by $T_{t} x=\mathrm{e}^{2 \pi \mathrm{it}} x$ is periodic, for $d=3$ we cannot obtain Example 1 from Theorem 2.18 of [12] (which yields only non-convergence for some $f \in L_{2}(\mathbb{T}, m)$, and not the strong sweeping out property).

Proposition 3.4. There exists a continuous probability $\mu$ on $\mathbb{T}$ which has the strong sweeping out property on $\mathbb{T}$.
Proof. Recall that a closed set $\mathcal{K} \subset \mathbb{T}$ is called a Kronecker set if every continuous $f$ on $\mathcal{K}$ with $|f| \equiv 1$ can be uniformly approximated by continuous characters, i.e., there is a sequence $\left\{n_{j}\right\}$ such that $\alpha^{n_{j}} \rightarrow f(\alpha)$ uniformly for $\alpha \in \mathcal{K}$.

Hence for $\lambda \in \mathbb{S}$ there exists $\left\{n_{j}\right\}$ such that $\alpha^{n_{j}} \rightarrow \bar{\lambda}$ for $\alpha \in \mathcal{K}$ (uniformly). If $\mu$ is a probability supported in $\mathcal{K}$, we obtain that $\hat{\mu}\left(n_{j}\right) \rightarrow \lambda$. When $\mu$ is strictly aperiodic, Corollary 3.2 applies and $\mu$ has the strong sweeping out property on $\mathbb{T}$.

By Theorem 5.2.2(a) in [24], $\mathbb{T}$ contains a Cantor set $\mathcal{K}$ which is a Kronecker set. Hence $\mathcal{K}$ supports a continuous probability $\mu$ with uncountable support (so strictly aperiodic), and by the above $\mu$ has the strong sweeping out property.

By Theorem 3.1, a.e. convergence of $\mu^{n} * f$ for every $f \in L_{2}(\mathbb{T}, m)$ implies

$$
\begin{equation*}
\limsup _{\hat{\mu}(n) \rightarrow 1, n \neq 0} \frac{|1-\hat{\mu}(n)|}{1-|\hat{\mu}(n)|}<\infty \tag{21}
\end{equation*}
$$

for the Fourier-Stieltjes coefficients of $\mu$. We will prove the converse in Theorem 3.6 below.
First, let us observe that the proof of Corollary 3.2 shows that, if there is a sequence $\left\{\lambda_{k}\right\} \subset \mathbb{S} \cap \sigma(P)$ with $1 \neq$ $\lambda_{k} \rightarrow 1$, then (11) holds, contradicting (21). Hence, if (21) holds, then $\sigma(P) \cap \mathbb{S} \neq \mathbb{S}$ and therefore this intersection is finite, with $\sigma\left(P^{j}\right) \cap \mathbb{S}=\{1\}$ for some $j \geq 1$, since $P$ is a positive contraction [14], Proposition 1. We give below a simple proof of this last fact for the convolution operators treated in this paper.

Lemma 3.5. Let $\mu$ be a probability on a compact Abelian group $G$. Then the peripheral spectrum $\sigma\left(P_{\mu}\right) \cap \mathbb{S}$ is a closed subgroup of the multiplicative group of the circle.

Proof. Let $\lambda_{1}, \lambda_{2} \in \sigma\left(P_{\mu}\right) \cap \mathbb{S}$. Since the spectrum is the closure of the Fourier coefficients, also $\bar{\lambda}_{1}=\lambda_{1}^{-1}$ and $\bar{\lambda}_{2}=\lambda_{2}^{-1}$ are in the peripheral spectrum, so there are sequences of characters with $\hat{\mu}\left(\gamma_{k}^{\prime}\right) \rightarrow \bar{\lambda}_{1}$ and $\hat{\mu}\left(\gamma_{j}^{\prime \prime}\right) \rightarrow \bar{\lambda}_{2}$.

$$
\begin{aligned}
\left|\int_{G} \gamma_{j}^{\prime \prime} \gamma_{k}^{\prime} \mathrm{d} \mu-\lambda_{2} \lambda_{1}\right| & \leq\left|\int_{G} \gamma_{j}^{\prime \prime} \gamma_{k}^{\prime} \mathrm{d} \mu-\lambda_{2} \int_{G} \gamma_{k}^{\prime} \mathrm{d} \mu\right|+\left|\lambda_{2} \int_{G} \gamma_{k}^{\prime} \mathrm{d} \mu-\lambda_{1} \lambda_{2}\right| \\
& \leq \int_{G}\left|\gamma_{j}^{\prime \prime}-\lambda_{2}\right| \mathrm{d} \mu+\left|\int_{G} \gamma_{k}^{\prime} \mathrm{d} \mu-\lambda_{1}\right| \leq\left[\int_{G}\left|\gamma_{j}^{\prime \prime}-\lambda_{2}\right|^{2} \mathrm{~d} \mu\right]^{1 / 2}+\left|\int_{G} \gamma_{k}^{\prime} \mathrm{d} \mu-\lambda_{1}\right|
\end{aligned}
$$

The last term tends to zero, and we have the estimate

$$
\left[\int_{G}\left|\gamma_{j}^{\prime \prime}-\lambda_{2}\right|^{2} \mathrm{~d} \mu\right]^{1 / 2}=\sqrt{2}\left[1-\Re e\left(\bar{\lambda}_{2} \int_{G} \gamma_{j}^{\prime \prime} \mathrm{d} \mu\right)\right]^{1 / 2} \leq \sqrt{2}\left|1-\bar{\lambda}_{2} \int_{G} \gamma_{j}^{\prime \prime} \mathrm{d} \mu\right|^{1 / 2} \rightarrow 0
$$

Hence the peripheral spectrum is a subgroup, closed since the spectrum is closed.
Remark also that, if in addition to (21) we have $\limsup _{\hat{\mu}(n) \mid \rightarrow 1}|1-\hat{\mu}(n)|<\sqrt{3}$, then $\sigma(P) \cap \mathbb{S}=\{1\}$ (see proof of Theorem 5 in [14]). When $\sigma(P) \cap \mathbb{S}=\{1\}$, (21) implies (9); hence Theorem 2.4 applies.

Theorem 3.6. The following are equivalent for a strictly aperiodic probability $\mu$ on $\mathbb{T}$ :
(i) for every $f \in L_{p}(\mathbb{T}, m), p>1$, we have $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e.;
(ii) for every $A \in \mathcal{B}$ we have $\mu^{n} * 1_{A} \rightarrow m(A)$ a.e.;
(iii) for some $j \geq 1$ the Fourier-Stieltjes coefficients of $\mu$ satisfy $\sup _{n \neq 0} \frac{\left|1-\hat{\mu}(n)^{j}\right|}{1-|\hat{\mu}(n)|^{j}}<\infty$;
(iv) for every $f \in L_{p}(\mathbb{T}, m), p>1$, we have $\sup \mu^{n} *|f| \in L_{p}(\mathbb{T}, m)$;
(v) the Fourier-Stieltjes coefficients of $\mu$ satisfy (21).

Proof. Obviously (i) implies (ii).
We now assume (ii). By Corollary 3.2 we must have that the peripheral spectrum $\sigma(P) \cap \mathbb{S}$ of the convolution operator $P f=\mu * f$ on $L_{2}$ is not all of $\mathbb{S}$, so by Lemma 3.5 it is a finite group, with $\sigma\left(P^{j}\right) \cap \mathbb{S}=\{1\}$ for some $j \geq 1$. Put $\eta=\mu^{j}$, which is clearly also strictly aperiodic. (ii) implies $\eta^{n} * 1_{A} \rightarrow m(A)$ a.e. for every $A \in \mathcal{B}$. By Theorem 3.1 we must have

$$
\begin{equation*}
\limsup _{\hat{\eta}(n) \rightarrow 1, n \neq 0} \frac{|1-\hat{\eta}(n)|}{1-|\hat{\eta}(n)|}<\infty \tag{22}
\end{equation*}
$$

and $\eta$ then satisfies (9) since $\sigma\left(P^{j}\right) \cap \mathbb{S}=\{1\}$. Thus (iii) holds.
Assume (iii), and put $\eta=\mu^{j}$. Theorem 2.4 applied to $\eta$ yields $\eta^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{p}(\mathbb{T}, m)$, $p>1$. Applying this to $\mu^{k} * f, 0 \leq k<j$, we obtain (i).
(iii) also implies $\sup \eta^{n} *|f| \in L_{p}(\mathbb{T}, m)$, for every $f \in L_{p}(\mathbb{T}, m), p>1$ (Theorem 2.4). Hence (iv) is satisfied.
(iv) implies (i) by the Banach principle, since pointwise convergence holds for $f$ continuous.

As (i) implies (v) by Theorem 3.1, it remains to show that (v) implies (iii).
Assume (v). By Lemma $3.5 \sigma(P) \cap \mathbb{S}$ is a closed subgroup of $\mathbb{S}$, which is not all of $\mathbb{S}$ by Corollary 3.2, so it is a finite group of roots of unity, say of order $r$. When $r=1$, (9) holds and (iii) of Theorem 3.6 holds with $j=r$. If $r>1$, then the peripheral spectrum of $P_{\mu^{r}}=P^{r}$ contains only the point 1 . By the assumption (v) and Proposition 3.7 below,

$$
\limsup _{\hat{\mu}(\gamma) \rightarrow \mathrm{e}^{2 \pi i e} / r} \frac{\left|1-\hat{\mu}(\gamma)^{r}\right|}{1-|\hat{\mu}(\gamma)|^{r}}<\infty
$$

for every $0 \leq \ell \leq r-1$, hence (iii) holds with $j=r$.

Proposition 3.7. Let $\mu$ be a strictly aperiodic probability on a compact Abelian group $G$. Let $r \geq 1$ be an integer and $\mathrm{e}^{2 \pi \mathrm{i} \alpha}=\mathrm{e}^{2 \pi \mathrm{i} / / r}, 0 \leq \ell \leq r-1$, a root of unity of order $r$. The condition

$$
\begin{equation*}
\limsup _{\hat{\mu}(\gamma) \rightarrow \mathrm{e}^{2 \pi i \alpha}} \frac{\left|1-\hat{\mu}(\gamma)^{r}\right|}{1-|\hat{\mu}(\gamma)|^{r}}=\infty \tag{23}
\end{equation*}
$$

implies

$$
\begin{equation*}
\limsup _{\gamma \neq 0, \hat{\mu}(\gamma) \rightarrow 1} \frac{|1-\hat{\mu}(\gamma)|}{1-|\hat{\mu}(\gamma)|}=\infty . \tag{24}
\end{equation*}
$$

Proof. Let $\left\{\gamma_{k}\right\} \subset \hat{G}$ be a sequence of characters on $G$ such that $\lim _{k} \hat{\mu}\left(\gamma_{k}\right)=\mathrm{e}^{2 \pi \mathrm{i} \alpha}$ and

$$
\begin{equation*}
\lim _{k} \frac{\left|1-\hat{\mu}\left(\gamma_{k}\right)^{r}\right|}{1-\left|\hat{\mu}\left(\gamma_{k}\right)\right|^{r}}=\infty . \tag{25}
\end{equation*}
$$

We write $\hat{\mu}\left(\gamma_{k}\right)=\left(1-\varepsilon_{k}\right) \mathrm{e}^{2 \pi \mathrm{i}\left(\ell / r+\delta_{k}\right)}$, with $\varepsilon_{k}>0$ and $\lim _{k} \varepsilon_{k}=\lim _{k} \delta_{k}=0$. The quotient in (23) for $\gamma_{k}$ reads

$$
\begin{aligned}
\frac{\left|1-\left(1-\varepsilon_{k}\right)^{r} \mathrm{e}^{2 \pi \mathrm{i} r \delta_{k}}\right|}{1-\left(1-\varepsilon_{k}\right)^{r}} & =\left|\frac{r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)-\left(1-r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)\right)\left(2 \pi \mathrm{i} r \delta_{k}+\mathrm{o}\left(\delta_{k}\right)\right)}{r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)}\right| \\
& =\left|1-\frac{\left(1-r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)\right)\left(2 \pi \mathrm{i} r \delta_{k}+\mathrm{o}\left(\delta_{k}\right)\right)}{r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)}\right| .
\end{aligned}
$$

Therefore by (25) we have

$$
\lim _{k} \frac{\left|\left(1-r \varepsilon_{k}+\mathrm{o}\left(\varepsilon_{k}\right)\right)\left(2 \pi \mathrm{i} \delta_{k}+r^{-1} \mathrm{o}\left(\delta_{k}\right)\right)\right|}{\varepsilon_{k}+r^{-1} \mathrm{o}\left(\varepsilon_{k}\right)}=\infty,
$$

i.e.

$$
\begin{equation*}
\lim _{k} \frac{\left|\delta_{k}\right|}{\varepsilon_{k}}=\infty . \tag{26}
\end{equation*}
$$

Now let us consider two characters $\gamma_{j}, \gamma_{k}$ from our sequence, where $k$ and $j$ are two indices which will be chosen later. The computations in the proof of Lemma 3.5 yield

$$
\left|\int_{G} \bar{\gamma}_{j} \gamma_{k} \mathrm{~d} \mu-\mathrm{e}^{2 \pi \mathrm{i} \alpha} \int_{G} \gamma_{k} \mathrm{~d} \mu\right| \leq \sqrt{2}\left|\mathrm{e}^{-2 \pi \mathrm{i} \alpha}-\int_{G} \gamma_{j} \mathrm{~d} \mu\right|^{1 / 2} .
$$

Since $\int_{G} \bar{\gamma}_{j} \mathrm{~d} \mu=\hat{\mu}\left(\gamma_{j}\right) \rightarrow \mathrm{e}^{2 \pi \mathrm{i} \alpha}$, we can find $j$, independently of $k$, such that $\int_{G} \bar{\gamma}_{j} \gamma_{k} \mathrm{~d} \mu$ is arbitrarily close to $\mathrm{e}^{2 \pi \mathrm{i} \alpha} \int_{G} \gamma_{k} \mathrm{~d} \mu$ for every $k$. This implies that for each $k$ we have

$$
\lim _{j \rightarrow \infty} \frac{\left|1-\int_{G} \bar{\gamma}_{j} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \bar{\gamma}_{j} \gamma_{k} \mathrm{~d} \mu\right|}=\frac{\left|1-\mathrm{e}^{2 \pi \mathrm{i} \alpha} \int_{G} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \gamma_{k} \mathrm{~d} \mu\right|} .
$$

We can therefore for each $k$ choose $j=p(k)$ such that

$$
\begin{equation*}
\lim _{k} \hat{\mu}\left(\gamma_{p(k)} \bar{\gamma}_{k}\right)=\lim _{k} \int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu=\mathrm{e}^{2 \pi \mathrm{i} \alpha} \lim _{k} \int_{G} \gamma_{k} \mathrm{~d} \mu=1 \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\frac{\left|1-\int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu\right|}-\frac{\left|1-\mathrm{e}^{2 \pi \mathrm{i} \alpha} \int_{G} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \gamma_{k} \mathrm{~d} \mu\right|}\right| \leq 1 . \tag{28}
\end{equation*}
$$

Using the expression of $\hat{\mu}\left(\gamma_{k}\right)=\int_{G} \bar{\gamma}_{k} \mathrm{~d} \mu$ we get

$$
\begin{aligned}
\frac{\left|1-\mathrm{e}^{2 \pi \mathrm{i} \alpha} \int_{G} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \gamma_{k} \mathrm{~d} \mu\right|} & =\frac{\left|1-\mathrm{e}^{-2 \pi \mathrm{i} \alpha} \int_{G} \bar{\gamma}_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \bar{\gamma}_{k} \mathrm{~d} \mu\right|} \\
& =\frac{\left|\varepsilon_{k}-\left(1-\varepsilon_{k}\right)\left(2 \pi \mathrm{i} \delta_{k}+\mathrm{o}\left(\delta_{k}\right)\right)\right|}{\varepsilon_{k}}=\left|1-\left(1-\varepsilon_{k}\right)\left(2 \pi \mathrm{i} \frac{\delta_{k}}{\varepsilon_{k}}+\frac{\mathrm{o}\left(\delta_{k}\right)}{\varepsilon_{k}}\right)\right|
\end{aligned}
$$

This, put together with (26) and (28), implies:

$$
\lim _{k} \frac{\left|1-\hat{\mu}\left(\gamma_{p(k)} \bar{\gamma}_{k}\right)\right|}{1-\left|\hat{\mu}\left(\gamma_{p(k)} \bar{\gamma}_{k}\right)\right|}=\lim _{k} \frac{\left|1-\int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu\right|}{1-\left|\int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu\right|}=+\infty .
$$

Since $\lim _{k} \int_{G} \bar{\gamma}_{p(k)} \gamma_{k} \mathrm{~d} \mu=1$ by (27), (24) is proved.
Corollary 3.8. Let $\mu$ be a strictly aperiodic probability on $\mathbb{T}$. Then either for every $f \in L_{p}(\mathbb{T}, m), p>1, \mu^{n} * f \rightarrow$ $\int f \mathrm{~d} m$ a.e., or $\mu$ has the strong sweeping out property.

Proof. Assume that $\mu$ does not have the strong sweeping out property. Then (21) holds, so by Theorem 3.6 we have the convergence.

Remark. The corollary yields that if for some $f \in L_{\infty}$ the sequence $\mu^{n} * f$ does not converge a.e., then $\mu$ has the strong sweeping out property on $\mathbb{T}$. The general theory, given in Theorem 1 and Corollary 2 of [3], yields only that $\mu$ on $\mathbb{T}$ is $\delta$-sweeping out for some $\delta>0$.

Problem. Does condition (21) imply $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{1}(\mathbb{T}, m)$ ?
Remarks. 1. In view of Theorem 3.6, the problem is equivalent to the question whether condition (9) implies $\mu^{n} * f \rightarrow$ $\int f \mathrm{~d} m$ a.e. for every $f \in L_{1}(\mathbb{T}, m)$.
2. A particular case of the problem is whether symmetry of $\mu$ implies $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e.for every $f \in L_{1}(\mathbb{T}, m)$. In general, the Stein-Rota result for self-adjoint Markov operators may fail in $L_{1}$ [17].
3. Another special case is whether the condition $\sup _{n \neq 0}|\hat{\mu}(n)|<1$, which is strictly stronger than (9), implies $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{1}(\mathbb{T}, m)$.

Proposition 3.9. If $\mu$ and $v$ are two probabilities on $\mathbb{T}$ satisfying the conditions of Theorem 3.6, so do $v * \mu$ and convex combinations $a \mu+b v$.

Proof. Remark that if $\mu$ is strictly aperiodic, so are $\nu * \mu$ and proper convex combinations $a \mu+b v$. One easily checks that $v * \mu$ and $a \mu+b v$ satisfy condition (iv) of Theorem 3.6.

In the following propositions the normalized continuous and discrete parts of a probability $\mu$ are denoted by $\mu_{c}$ and $\mu_{d}$.

Proposition 3.10. There exists a strictly aperiodic probability $\mu$ on $\mathbb{T}$ with all its powers singular, such that
(i) $\mu_{c}$ is non-zero and satisfies $\mu_{c}^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{p}, p>1$;
(ii) $\mu_{d}$ is strictly aperiodic and has the strong sweeping out property on $\mathbb{T}$;
(iii) $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{p}, p>1$.

Proof. Let $\mu_{1}$ be the probability constructed in [9], which is continuous with all its powers singular, and satisfies $\sup _{n \neq 0}|\hat{\mu}(n)|=c<1$. Let $\mu_{2}$ be a discrete probability as in Example 1 (or any discrete probability supported in a Kronecker set), and put $\mu=\frac{1}{2}\left(\mu_{1}+\mu_{2}\right)$. Then (i) and (ii) are satisfied by the construction. For $n \neq 0$ we have

$$
|\hat{\mu}(n)| \leq \frac{1}{2}\left(\left|\hat{\mu}_{1}(n)\right|+\left|\hat{\mu}_{2}(n)\right|\right) \leq(c+1) / 2<1
$$

Hence (iii) holds by Theorem 2.4.
Proposition 3.11. There exists a strictly aperiodic probability $\mu$ on $\mathbb{T}$ with all its powers singular, such that
(i) $\mu_{c}$ has the $S S O$ property on $\mathbb{T}$;
(ii) $\mu_{d}$ is non-zero strictly aperiodic and satisfies $\mu_{c}^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for every $f \in L_{p}, p>1$;
(iii) $\mu$ has the strong sweeping out property on $\mathbb{T}$.

Proof. Take a continuous measure $\mu_{1}$ supported on a Kronecker set $\mathcal{K}$. By Proposition 3.4 it has the SSO property on $\mathbb{T}$. Let $\mu_{2}$ be the discrete measure $\frac{1}{2}\left(\delta_{\mathrm{e}^{2 \pi i} \theta}+\delta_{\mathrm{e}^{-2 \pi \mathrm{i} \theta} \theta}\right)$, with $\mathrm{e}^{2 \pi \mathrm{i} \theta} \in \mathcal{K}$. Let $\lambda_{k}=\mathrm{e}^{2 \pi \mathrm{i} \beta_{k}}, k \geq 1$, be a sequence in $\mathbb{S}$ with $\lim _{k} \lambda_{k}=1$. By the construction in Proposition 3.4, for each $k$ there is a sequence $\left\{n_{k, j}\right\}$ such that $\alpha^{n_{k, j}} \rightarrow \bar{\lambda}_{k}$ uniformly on $\mathcal{K}$. Hence $\lim _{j} \hat{\mu}_{1}\left(n_{k, j}\right)=\lambda_{k}$, and $\lim _{j} \hat{\mu}_{2}\left(n_{k, j}\right)=\lim _{j} \cos \left(2 \pi n_{k, j} \theta\right)=\cos \left(2 \pi \beta_{k}\right)$.

For the barycenter $\mu:=\frac{1}{2}\left(\mu_{1}+\mu_{2}\right)$ we have, when $j$ tends to $\infty$ :

$$
\hat{\mu}\left(n_{k, j}\right)=\frac{1}{2}\left(\hat{\mu}_{1}\left(n_{k, j}\right)+\hat{\mu}_{2}\left(n_{k, j}\right)\right) \rightarrow \frac{1}{2}\left(\mathrm{e}^{2 \pi \mathrm{i} \beta_{k}}+\cos \left(2 \pi \beta_{k}\right)\right)=\cos \left(2 \pi \beta_{k}\right)+\frac{1}{2} \mathrm{i} \sin \left(2 \pi \beta_{k}\right) .
$$

Now we use the method of Corollary 3.2. Let $n_{k}$ be a value for $j$ large of the sequence $\left\{n_{k, j}, j \geq 1\right\}$ such that

$$
\left|1-\hat{\mu}_{1}\left(n_{k}\right)\right|<2\left|1-\lambda_{k}\right|, \quad\left|1-\cos \left(2 \pi n_{k} \theta\right)\right|<2\left|1-\cos \left(2 \pi \beta_{k}\right)\right|
$$

We have $\lim _{k} \hat{\mu}\left(n_{k}\right)=1$. Using the criterion (20), let us consider:

$$
\begin{aligned}
\frac{\left|\Im m \hat{\mu}\left(n_{k}\right)\right|}{1-\mathfrak{R e} \hat{\mu}\left(n_{k}\right)} & =\frac{\left|\Im m \hat{\mu}_{1}\left(n_{k}\right)\right|}{1-\Re e \hat{\mu}_{1}\left(n_{k}\right)-\cos \left(2 \pi n_{k} \theta\right)} \\
& =\frac{\sin \left(2 \pi \beta_{k}\right)+\varepsilon_{k}}{2-2 \cos \left(2 \pi \beta_{k}\right)+\varepsilon_{k}^{\prime}}=\frac{\cos \left(\pi \beta_{k}\right)+\left(2 \sin \left(\pi \beta_{k}\right)\right)^{-1} \varepsilon_{k}}{2 \sin \left(\pi \beta_{k}\right)+\left(2 \sin \left(\pi \beta_{k}\right)\right)^{-1} \varepsilon_{k}^{\prime}}
\end{aligned}
$$

with $\varepsilon_{k}, \varepsilon_{k}^{\prime}$ small. Since we can take $n_{k}$ such that the errors $\varepsilon_{k}, \varepsilon_{k}^{\prime}$ are much smaller than $\sin \left(\pi \beta_{k}\right)$, we obtain that the limsup in the previous quotient is infinite, when $k$ tends to $\infty$. By (20), the probability measure $\mu$ has the SSO property; however, for its discrete part $\mu_{2}$, almost everywhere convergence of $\mu_{2}^{n} * f$ holds since $\mu_{2}$ is symmetric.

Problem. Let $\mu$ be a non-discrete probability with the SSO property on $\mathbb{T}$. Must its continuous component have the SSO property? At least one component must, by Proposition 3.9(ii).

If $\mu$ has singular powers and both its discrete and continuous components are non-zero with the SSO property, must $\mu$ have it?

## 4. Convolution powers of discrete probabilities on the circle

In this section we study the a.e. convergence of convolution powers of discrete probabilities, and use condition (21) to check the a.e. convergence of $\mu^{n} * f$ for every $f \in L_{p}, p>1$.

Theorem 4.1. Let $\left\{1, \tau_{1}, \ldots, \tau_{s}\right\}$ be linearly independent over $\mathbb{Q}$. Let $\left\{\alpha_{k}=\mathrm{e}^{2 \pi \mathrm{i} \theta_{k}}\right\}$ be a finite or countably infinite set of different points in $\mathbb{T}$, with

$$
\theta_{k}=\sum_{j=1}^{s} r_{k, j} \tau_{j} \bmod 1, \quad r_{k, j} \in \mathbb{Q}, \forall k
$$

Let $\mu=\sum_{k} p_{k} \delta_{\alpha_{k}}$, with $0 \leq p_{k}<1$ and $\sum_{k} p_{k}=1$. We suppose that $\mu$ is strictly aperiodic (i.e. $\theta_{j}-\theta_{k}$ irrational for some $j, k$ with $p_{j} \cdot p_{k}>0$ ). Then $\mu$ has the strong sweeping out property on $\mathbb{T}$ if and only if

$$
\begin{equation*}
\limsup _{0 \neq \vec{x}=\left(x_{1}, \ldots, x_{s}\right) \rightarrow 0} \frac{\left|\sum_{k} p_{k} \sin \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)\right|}{1-\sum_{k} p_{k} \cos \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)}=\infty . \tag{29}
\end{equation*}
$$

Proof. Note that we may assume for the proof that all $p_{k}$ are positive. The Fourier-Stieltjes coefficients of $\mu$ are

$$
\hat{\mu}(n)=\sum_{k} p_{k} \hat{\delta}_{\alpha_{k}}(n)=\sum_{k} p_{k} \mathrm{e}^{2 \pi \mathrm{i} n \theta_{k}}=\sum_{k} p_{k} \mathrm{e}^{2 \pi \mathrm{i} \sum_{j=1}^{s} r_{k, j} n \tau_{j}} .
$$

The spectrum of $P f=\mu * f$ in $L_{2}(\mathbb{T}, m)$ is the closure in the unit disk of the set $\{\hat{\mu}(n): n \in \mathbb{Z}\}$, which yields, by Kronecker's theorem,

$$
\sigma(P)=\left\{F\left(x_{1}, \ldots, x_{s}\right):\left(x_{1}, \ldots, x_{s}\right) \in \mathbb{R}^{s}\right\}
$$

where

$$
F\left(x_{1}, \ldots, x_{s}\right):=\sum_{k} p_{k} \mathrm{e}^{2 \pi \mathrm{i} \sum_{j=1}^{s} r_{k, j} x_{j}}=\sum_{k} p_{k}\left(\cos \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)+\mathrm{i} \sin \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)\right) .
$$

Condition (20) is therefore equivalent to

$$
\limsup _{0 \neq \vec{x}=\left(x_{1}, \ldots, x_{s}\right) \rightarrow 0} \frac{\left|\sum_{k} p_{k} \sin \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)\right|}{1-\sum_{k} p_{k} \cos \left(2 \pi \sum_{j=1}^{s} r_{k, j} x_{j}\right)}=\limsup _{0 \neq \vec{x} \rightarrow 0} \frac{|\Im m F(\vec{x})|}{1-\Re e F(\vec{x})}=\infty .
$$

In view of the equivalence of conditions (11) and (20), we obtain the equivalence of (29) and the strong sweeping out property.

Corollary 4.2. Let $\theta \in(0,1)$ be irrational and $\alpha=\mathrm{e}^{2 \pi \mathrm{i} \theta}$. Let $\mu=\sum_{k \in \mathbb{Z}} p_{k} \delta_{\alpha^{k}}$ with $0 \leq p_{k}<1$ and $\sum_{k \in \mathbb{Z}} p_{k}=1$. Then $\mu$ has the SSO property on $\mathbb{T}$ if and only if

$$
\limsup _{0 \neq x \rightarrow 0} \frac{\left|\sum_{k} p_{k} \sin (2 \pi k x)\right|}{1-\sum_{k} p_{k} \cos (2 \pi k x)}=\infty .
$$

Remark. When we assume that $\left\{k \in \mathbb{Z}: p_{k}>0\right\}$ is not in a class of a subgroup of $\mathbb{Z}$, the corollary is actually a particular case of the characterization of SSO obtained by combining results of [5] and [16]. We do not make such an assumption, and our result covers, for example, the case of $\frac{1}{2}\left(\delta_{\alpha^{-1}}+\delta_{\alpha}\right)$. We note that if $\sum_{k \in \mathbb{Z}}|k| p_{k}<\infty$ and $\sum_{k \in \mathbb{Z}} k p_{k} \neq 0$, then we have SSO [5]. See also Proposition 4.3 below.

Proposition 4.3. Let $\left\{\alpha_{k}=\mathrm{e}^{2 \pi \mathrm{i} \theta_{k}}\right\}$ be a set of d different points in $\mathbb{T}$, which generate an $s$-dimensional subspace of $\mathbb{R}$ over $\mathbb{Q}$, and let $\left\{\tau_{1}, \ldots, \tau_{s}\right\}$ be linearly independent over $\mathbb{Q}$ with

$$
\theta_{k}=\sum_{j=1}^{s} r_{k, j} \tau_{j} \bmod 1, \quad r_{k, j} \in \mathbb{Q}, \forall k
$$

Let $\mu=\sum_{k} p_{k} \delta_{\alpha_{k}}$, with $0<p_{k}<1$ and $\sum_{k} p_{k}=1$. We suppose that $\mu$ is strictly aperiodic (i.e. $\theta_{j}-\theta_{k}$ irrational for some $j, k$ ). Then $\mu^{n} * f$ converges a.e. for every $f \in L_{p}, p>1$, if and only if $\sum_{k=1}^{d} p_{k} r_{k, j}=0, \forall j \in\{1, \ldots, s\}$.

Proof. Assume first that we have the convergence, so by Theorem 4.1 (29) fails - the lim sup is finite. Let $A(\vec{x}), B(\vec{x})$ be the numerator and denominator in the left-hand side of (29). By taking the approximation of order 1 or 2 at 0 , we write:

$$
\begin{equation*}
A(\vec{x})=\sum_{k=1}^{d} p_{k} \sum_{j=1}^{s} r_{k, j} x_{j}+\mathrm{o}(\|\vec{x}\|), \quad 2 B(\vec{x})=\sum_{k=1}^{d} p_{k}\left(\sum_{j=1}^{s} r_{k, j} x_{j}\right)^{2}+\mathrm{o}\left(\|\vec{x}\|^{2}\right) . \tag{30}
\end{equation*}
$$

Assume there is $j_{0} \in\{1, \ldots, s\}$ such that $\sum_{k=1}^{d} p_{k} r_{k, j_{0}} \neq 0$, and put $x_{j}=0$ for every $j \neq j_{0}$. Then the quotient of the right-hand sides of (30), for $x_{j_{0}} \neq 0$, is

$$
\frac{\sum_{k=1}^{d} p_{k} r_{k, j_{0}} x_{j_{0}}+\mathrm{o}\left(\left|x_{j_{0}}\right|\right)}{\sum_{k=1}^{d} p_{k} r_{k, j_{0}}^{2} x_{j_{0}}^{2}+\mathrm{o}\left(\left|x_{j_{0}}\right|^{2}\right)}=\frac{\sum_{k=1}^{d} p_{k} r_{k, j_{0}}+\mathrm{o}(1)}{\sum_{k=1}^{d} p_{k} r_{k, j_{0}}^{2}+\mathrm{o}(1)} \frac{1}{x_{j_{0}}} \underset{x_{j_{0} \rightarrow 0}}{\longrightarrow} \infty .
$$

Then (29) holds. Since we assume that (29) fails, $\sum_{k=1}^{d} p_{k} r_{k, j}=0$ for $1 \leq j \leq s$.
By Theorem 4.1, the a.e. convergence will follow from

$$
\begin{equation*}
\limsup _{0 \neq \vec{x} \rightarrow 0} \frac{\left|\sum_{k=1}^{d} p_{k} \sin \left(2 \pi \sum_{j} r_{k, j} x_{j}\right)\right|}{1-\sum_{k=1}^{d} p_{k} \cos \left(2 \pi \sum_{j} r_{k, j} x_{j}\right)}<\infty . \tag{31}
\end{equation*}
$$

By the assumption that $\sum_{k=1}^{d} p_{k} r_{k, j}=0$ for every $j$, we see that for the numerator we have to take the third order approximation. Then, up to a constant factor, for $\vec{x}$ tending to 0 the ratio can be written:

$$
\begin{equation*}
\frac{\left|\sum_{k} p_{k}\left(\sum_{j} r_{k, j} x_{j}\right)^{3}\right|}{\sum p_{k}\left(\sum_{j} r_{k, j} x_{j}\right)^{2}}(1+\mathrm{o}(|\vec{x}|)) . \tag{32}
\end{equation*}
$$

To show that (32) is well defined and its lim sup is finite, it suffices to prove that the quadratic form

$$
Q(\vec{x}):=\sum_{k=1}^{d} p_{k}\left(\sum_{j=1}^{s} r_{k, j} x_{j}\right)^{2}
$$

is positive definite. Since it is non-negative, this is equivalent to: $Q\left(x_{1}, \ldots, x_{s}\right)=0 \Rightarrow\left(x_{1}, \ldots, x_{s}\right)=0$. But $Q\left(x_{1}, \ldots, x_{s}\right)=0$ implies $\sum_{j=1}^{s} r_{k, j} x_{j}=0$ for $k=1, \ldots, d$. If there is a non-null solution $\left(x_{1}, \ldots, x_{s}\right)$ for this system of linear equations, then the rank of the system is less than $s$, which contradicts the fact that the dimension of the space generated by $\left(\theta_{1}, \ldots, \theta_{d}\right)$ over $\mathbb{Q}$ is $s$. Hence $Q(\vec{x})$ is positive definite.

Example 2. A probability $\mu$ with finite support satisfying $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e.
Let $\tau_{1}, \tau_{2}$ be two reals in $(0,1)$, such that $\left(1, \tau_{1}, \tau_{2}\right)$ are linearly independent over $\mathbb{Q}$. Put $\alpha_{1}=\mathrm{e}^{2 \pi \mathrm{i}\left((1 / 2) \tau_{1}-\tau_{2}\right)}$, $\alpha_{2}=\mathrm{e}^{2 \pi \mathrm{i}\left(-(1 / 3) \tau_{1}+\tau_{2}\right)}$ and $\alpha_{3}=\mathrm{e}^{2 \pi \mathrm{i}\left(-\tau_{2}\right)}$. Let $\mu$ be the discrete probability measure on $\mathbb{T}$ defined by

$$
\mu=\frac{1}{3} \delta_{\alpha_{1}}+\frac{1}{2} \delta_{\alpha_{2}}+\frac{1}{6} \delta_{\alpha_{3}} .
$$

Here $d=3$ and $s=2$, and we have $r_{1,1}=\frac{1}{2}, r_{1,2}=-1, r_{2,1}=-\frac{1}{3}, r_{2,2}=1, r_{3,1}=0, r_{3,2}=-1$, so there is no index $j_{0} \in\{1,2\}$ such that $\sum_{k=1}^{3} p_{k} r_{k, j_{0}} \neq 0$. We can therefore apply Proposition 4.3. An elementary computation shows that $\sigma(P) \cap \mathbb{S}$ is precisely the set of roots of unity of order 6 : $\left\{\mathrm{e}^{2 \pi i \ell / 6}, \ell=0,1, \ldots, 5\right\}$. Hence $P$ is not self-adjoint.

In the previous results and examples, the discrete probabilities were supported by "irrational points" (in $\mathbb{R} / \mathbb{Z}$ ), i.e. points in $\mathbb{T}$ of the form $\alpha=\mathrm{e}^{2 \pi i \theta}$ with $\theta$ irrational. It is easy to construct strictly aperiodic discrete probabilities $\mu$ supported on rational points only, such that $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e. for $f \in L_{p}, p>1$; take an infinite sequence of rationals $\left\{\theta_{k}\right\}_{k \geq 1} \subset(0,1)$ which has an irrational limit point (this is not really necessary, see below), a sequence $\left\{p_{k}\right\}_{k \geq 1}$ with $p_{k}>0$ and $\sum_{k=1}^{\infty} p_{k}=1$, and put $\mu:=\frac{1}{2} \sum_{k=1}^{\infty} p_{k}\left(\delta_{\mathrm{e}^{2 \pi i \theta_{k}}}+\delta_{\mathrm{e}^{-2 \pi i \theta_{k}}}\right)$. The a.e. convergence follows from the symmetry [25].

Example 3. A discrete probability supported on $\mathbb{Q}$ which has the SSO property.
Let $0<t<1$. Using additive notations, we take on $\mathbb{R} / \mathbb{Z}$ the following measure $\mu=(1-t) t^{-1} \sum_{k \geq 1} t^{k} \delta_{1 / k}$.

In order to obtain the SSO property, we want to prove (20), so we need to find a sequence $\left(\ell_{n}\right)$ such that

$$
\begin{equation*}
\lim _{n} \frac{\left|\sum_{k=1}^{\infty} p_{k} \sin \left(2 \pi \ell_{n} / k\right)\right|}{\sum_{k=1}^{\infty} p_{k}\left(\sin \left(\pi \ell_{n} / k\right)\right)^{2}}=\infty . \tag{33}
\end{equation*}
$$

Let $n$ be prime and $\ell_{n}=(n-1)!(n+1) \cdots(n+n-1)\left(n+r_{n}\right)=\frac{(2 n-1)!\left(n+r_{n}\right)}{n}$, where $r_{n}$ is a positive integer which will be chosen below.

Since $n$ is prime, we have $(n-1)!\neq 0 \bmod n$. We choose $r_{n}, 1 \leq r_{n}<n$, such that $\ell_{n}=1 \bmod n$. We have $\ell_{n}=$ $0 \bmod (n+j)$, for $j=1, \ldots, n-1$.

By the choice of $\ell_{n}$ and definition of $p_{k}$, the left-hand side of (33) reads:

$$
\begin{aligned}
& \frac{\left|\sin \left(2 \pi \ell_{n} / n\right)+p_{n}^{-1} \sum_{k=2 n}^{\infty} p_{k} \sin \left(2 \pi \ell_{n} / k\right)\right|}{\left(\sin \left(\pi \ell_{n} / n\right)\right)^{2}+p_{n}^{-1} \sum_{k=2 n}^{\infty} p_{k}\left(\sin \left(\pi \ell_{n} / k\right)\right)^{2}} \\
& \quad=\frac{\left|\sin \left(2 \pi \ell_{n} / n\right)+t^{n} \sum_{k=2 n}^{\infty} t^{k-2 n} \sin \left(2 \pi \ell_{n} / k\right)\right|}{\left(\sin \left(\pi \ell_{n} / n\right)\right)^{2}+t^{n} \sum_{k=2 n}^{\infty} t^{k-2 n}\left(\sin \left(\pi \ell_{n} / k\right)\right)^{2}} \\
& \quad \geq \frac{\left|\sin \left(2 \pi \ell_{n} / n\right)\right|-t^{n}(1-t)^{-1}}{\left(\sin \left(\pi \ell_{n} / n\right)\right)^{2}+t^{n}(1-t)^{-1}}=\frac{|\sin (2 \pi / n)|-t^{n}(1-t)^{-1}}{(\sin (\pi / n))^{2}+t^{n}(1-t)^{-1}} \underset{n \rightarrow \infty}{\sim} 2 n .
\end{aligned}
$$

Example 4. A discrete probability supported on $\mathbb{Q}$ for which a.e. convergence holds.
Now we give an example of a non-symmetric probability measure supported on an infinite set of rationals, for which the a.e. convergence holds. Let $0<t<1$. We take the following discrete measure $\mu=(1-t) t^{-1} \sum_{k \geq 1} t^{k} \delta_{1 / 2^{k}}$ on $\mathbb{R} / \mathbb{Z}$. We have to prove that

$$
\begin{equation*}
\limsup _{\hat{\mu}(n) \rightarrow 1, n \neq 0} \frac{|\Im m \hat{\mu}(n)|}{1-\Re e \hat{\mu}(n)}<\infty . \tag{34}
\end{equation*}
$$

Let us write $n=2^{r_{n}} u_{n}$, with $u_{n}$ odd. The quotients in (34) read

$$
\begin{aligned}
\frac{\left|\sum_{k=1}^{\infty} p_{k} \sin \left(2 \pi u_{n} / 2^{k-r_{n}}\right)\right|}{\sum_{k=1}^{\infty} p_{k}\left(\sin \left(\pi u_{n} / 2^{k-r_{n}}\right)\right)^{2}} & =\frac{\left| \pm p_{2+r_{n}}+\sum_{k=3}^{\infty} p_{k+r_{n}} \sin \left(2 \pi u_{n} / 2^{k}\right)\right|}{p_{2+r_{n}}+(1 / 2) p_{2+r_{n}}+\sum_{k=3}^{\infty} p_{k+r_{n}}\left(\sin \left(\pi u_{n} / 2^{k}\right)\right)^{2}} \\
& =\frac{\left| \pm 1+t^{-\left(2+r_{n}\right)} \sum_{k=3}^{\infty} t^{k+r_{n}} \sin \left(2 \pi u_{n} / 2^{k}\right)\right|}{t^{-1}+1 / 2+t^{-\left(2+r_{n}\right)} \sum_{k=3}^{\infty} t^{k+r_{n}}\left(\sin \left(\pi u_{n} / 2^{k}\right)\right)^{2}} \leq 2+2 t(1-t)^{-1}
\end{aligned}
$$

Therefore Condition (34) holds.

## 5. Convergence of convolution powers on compact Abelian groups

Let $G$ be a compact Abelian group with Borel $\sigma$-algebra $\mathcal{B}$. We denote by $m$ its normalized Haar measure and by $\hat{G}$ its dual group (which is discrete since $G$ is compact). The elements of $\hat{G}$, i.e. the characters on $G$, will be denoted by $\gamma$. We have the following generalization of Theorem 3.1, which as we will see has a content only for $G$ not of bounded order.

Theorem 5.1. Let $\mu$ be a strictly aperiodic probability on G. If

$$
\begin{equation*}
\limsup _{\hat{\mu}(\gamma) \rightarrow 1, \gamma \neq 0} \frac{|1-\hat{\mu}(\gamma)|}{1-|\hat{\mu}(\gamma)|}=\infty \tag{35}
\end{equation*}
$$

then $\mu$ has the strong sweeping out property on $G$.

Proof. We will use Lemma 2.6 to show the strong sweeping out property on $G$. The proof starts using (35) like the proof of Theorem 3.1, up to equation (19).

We fix the character $\gamma_{k}$ defined by (14). Let $\tilde{H}$ be the subgroup of $\hat{G}$ generated by $\gamma_{k}$ and $H$ the closed subgroup of $G$ of all elements $x$ such that $\gamma_{k}(x)=1$. By Pontryagin duality, $G / H$ is isomorphic to the dual of the discrete group $\tilde{H}$ [24], Theorem 2.1.2. Since $\tilde{H}$ is either $\mathbb{Z}$ or a finite cyclic group, say $\mathbb{Z} / p \mathbb{Z}$, for some integer $p \geq 1$, the quotient $G / H$ is isomorphic to a group $G_{0}$ which is $\mathbb{T}$ in the first case, $\mathbb{Z} / p \mathbb{Z}$ identified with $\left\{\frac{s}{p}, s=0, \ldots, p-1\right\}$ in the second case (when $G$ is connected, $\tilde{H}=\mathbb{Z}$ and $G_{0}=\mathbb{T}$ ). There is a canonical homomorphism $\Pi_{0}$ from $G$ onto $G_{0}$ and the push-forward measure of the normalized Haar measure $m$ on $G$ by $\Pi_{0}$ is the uniform measure $m_{0}$ on $G_{0}$ (i.e. $m_{0}(A)=m\left(\Pi_{0}^{-1} A\right)$ for Borel sets of $\left.G_{0}\right)$. We denote by $\mu_{0}$ the measure on $G_{0}$ obtained from $\mu$ by $\Pi_{0}$. In the second case $\mu_{0}$ is a discrete probability measure on $\left\{\frac{s}{p}: s=0, \ldots, p-1\right\}$.

The character $\gamma_{k}$ can be written as $\gamma_{k}(x)=\zeta\left(\Pi_{0} x\right)$, where $\zeta$ is the character on $G_{0}$, defined by $\zeta(y)=\mathrm{e}^{2 \pi \mathrm{i} y}, y \in$ $\mathbb{R} / \mathbb{Z}$, in the first case, and $\zeta(s)=\mathrm{e}^{2 \pi i s / p}, s \in \mathbb{Z} / p \mathbb{Z}$, in the second case. By construction we have the formula:

$$
\int_{G} \gamma_{k}(s) \mathrm{d} \mu^{\ell}(s)=\int_{G_{0}} \zeta(y) \mathrm{d} \mu_{0}^{\ell}(y),
$$

and the value of the integral is either $\int_{0}^{1} \mathrm{e}^{2 \pi \mathrm{i} y} \mathrm{~d} \mu_{0}^{\ell}(y)$ or $\sum_{s=0}^{p-1} \mathrm{e}^{2 \pi \mathrm{i} s / p} \mu_{0}^{\ell}(s)$.
We now prove the theorem by adapting the second part of the proof of Theorem 3.1. As in the proof of Theorem 3.1, with the same definition of $j, r$ and $\delta$, we obtain

$$
\mu_{0}^{\ell}\left\{y \in G_{0}:|\sin (\pi(y+\ell \xi))| \geq \delta\right\} \leq \frac{r}{2 \delta^{2}}, \quad 1 \leq \ell \leq j .
$$

Given $z \in G_{0}$, there is $\ell_{z} \leq j$ with $\left|z-\ell_{z} \xi\right|<\delta$. Hence $\left|\sin \left(\pi\left(y+\ell_{z} \xi\right)\right)\right|<\delta$ implies

$$
|\sin (\pi(y+z))| \leq\left|\sin \left(\pi\left(y+\ell_{z} \xi\right)\right)\right|+\left|\sin \left(\pi\left(z-\ell_{z} \xi\right)\right)\right|<(1+\pi) \delta,
$$

so, since $\delta^{2} \geq \frac{r}{2 \varepsilon}$ by the definition of $\delta$,

$$
\begin{aligned}
& \mu_{0}^{\ell_{z}}\left\{y \in G_{0}:|\sin (\pi(y+z))|<(1+\pi) \delta\right\} \\
& \quad \geq \mu_{0}^{\ell_{z}}\left\{y \in G_{0}:\left|\sin \left(\pi\left(y+\ell_{z} \xi\right)\right)\right|<\delta\right\} \geq 1-\frac{r}{2 \delta^{2}} \geq 1-\varepsilon .
\end{aligned}
$$

Let $B=\left\{y \in G_{0}:|\sin (\pi y)|<(1+\pi) \delta\right\}$, and define a subset $A$ of $G$ by $A=\Pi_{0}^{-1} B$. Since $|\sin (\pi y)| \geq 2|y|$ for $|y| \leq \frac{1}{2}$, we have $m(A)=m_{0}(B) \leq(1+\pi) \delta \leq \frac{1+\pi}{2 M}=\varepsilon$. Note that when $G_{0}$ is finite, this implies that $\varepsilon \geq 1 / p$ (which is impossible for small $\varepsilon$ if $G$ has a bounded order). By the previous estimate, for any $x \in G$,

$$
\begin{aligned}
\sup _{1 \leq \ell \leq j} \mu^{\ell} * 1_{A}(x) & =\sup _{1 \leq \ell \leq j} \mu^{\ell}(A-x) \geq \mu^{\ell \Pi_{0} x}(A-x) \\
& =\mu_{0}^{\ell \Pi_{0} x}\left\{y \in G_{0}:\left|\sin \left(\pi\left(y+\Pi_{0} x\right)\right)\right|<2 \delta\right\} \geq 1-\varepsilon .
\end{aligned}
$$

This yields $m\left\{x \in G: \sup _{n \geq 0} \mu^{n} * 1_{A}(x)>1-\varepsilon\right\}=m(G)=1$, so (10) is satisfied.
The proof of the following corollary is similar to that of Corollary 3.2.
Corollary 5.2. Let $\mu$ be strictly aperiodic on $G$ such that $\mathbb{S} \subset\{\overline{\{\hat{\mu}(\gamma): \gamma \in \hat{G}\}}$. Then (35) holds, and therefore $\mu$ has the strong sweeping out property on $G$.

Using Theorem 5.1 and Proposition 3.7, we can obtain the following analogue of Theorem 3.6, with practically the same proof.

Theorem 5.3. The following are equivalent for a strictly aperiodic probability $\mu$ on a compact Abelian group $G$ :
(i) for every $f \in L_{p}(G, m), p>1$, we have $\mu^{n} * f \rightarrow \int f \mathrm{~d} m$ a.e.;
(ii) for every $A \in \mathcal{B}$ we have $\mu^{n} * 1_{A} \rightarrow m(A)$ a.e.;
(iii) for some $j \geq 1$ the Fourier-Stieltjes coefficients of $\mu$ satisfy $\sup _{\gamma \neq 0} \frac{\left|1-\hat{\mu}(\gamma)^{j}\right|}{1-|\hat{\mu}(\gamma)|^{j}}<\infty$;
(iv) for every $f \in L_{p}(G, m), p>1$, we have $\sup \mu^{n} *|f| \in L_{p}(G, m)$;

Corollary 5.4. Let $\mu$ be a strictly aperiodic probability on $G$. Then either for every $f \in L_{p}(G, m), p>1, \mu^{n} * f \rightarrow$ $\int f \mathrm{~d} m$ a.e. or $\mu$ has the strong sweeping out property.

Theorem 5.5. Let $G$ be a compact Abelian group.
(i) If $G$ is of bounded order (i.e. there exists $q \in \mathbb{N}$ such that $q x=0$ for every $x \in G$, where $q x$ is the sum of $x$ with itself $q$ times), then for any strictly aperiodic probability measure $\mu$ on $G$, we have $\lim _{n} \mu^{n} * f(x)=\int f \mathrm{~d} m$ a.e. for every $p>1$ and $f \in L_{p}(G, m)$.
(ii) If $G$ is not of bounded order, then there exists a continuous probability measure $\mu$ on $G$ which has the strong sweeping out property on $G$.

Proof. (i) Under the assumption of (i), $\gamma(x)^{q} \equiv 1$ for every $\gamma$ character of $G$ and for any probability $\mu$ we can write $\mu(\gamma)$ as a convex combination of the $q$ th roots of unity. This shows that all the values of $\hat{\mu}(\gamma)$, and therefore also $\sigma\left(P_{\mu}\right)$, are inside the polygon with vertices the $q$ th roots of unity $\left\{\mathrm{e}^{2 \pi i k / q}: 0 \leq k \leq q-1\right\}$. It follows that for $\mu$ strictly aperiodic we have $\lim \sup _{\hat{\mu}(\gamma) \rightarrow 1, \gamma \neq 0} \frac{|1-\hat{\mu}(\gamma)|}{1-|\hat{\mu}(\gamma)|}<\infty$ and we can apply Theorem 5.3.
(ii) Since $G$ is not of bounded order, it has a dense set of elements of infinite order [24], Theorem 2.5.3. Hence by [24], Theorem 5.2.2(a), $G$ contains a Cantor set which is a Kronecker set. Using Corollary 5.2 the proof is now similar to that of Proposition 3.4.

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