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Nonequilibrium fluctuations for a tagged particle in one-dimensional sublinear zero-range processes

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Abstract. Nonequilibrium fluctuations of a tagged, or distinguished particle in a class of one dimensional mean-zero zero-range systems with sublinear, increasing rates are derived. In Jara–Landim–Sethuraman (*Probab. Theory Related Fields* **145** (2009) 565–590), processes with at least linear rates are considered.

A different approach to establish a main "local replacement" limit is required for sublinear rate systems, given that their mixing properties are much different. The method discussed also allows to capture the fluctuations of a "second-class" particle in unit rate, symmetric zero-range models.

Résumé. Nous démontrons les fluctuations hors d'équilibre d'une particule marquée pour une classe de systèmes de particules à portée nulle uni-dimensionels de moyenne nulle dont le taux de sauts croit de manière sous-linéaire. Dans Jara–Landim–Sethuraman (*Probab. Theory Related Fields* **145** (2009) 565–590), ce résutat a été démontré pour des processus dont le taux croit au moins linéairement.

La démonstration du lemme de remplacement dans le cas sous-linéaire exige une nouvelle approche en conséquence des différences entre les propriétés de mélanges des deux processus. La méthode présentée permet également de démontrer les fluctuations d'une particule de deuxième classe dans le modèle à portée nulle symmétrique dont le taux de sauts est égal à 1.

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1. Introduction

Characterizing the motion of a distinguished, or tagged particle interacting with others is a long standing concern in statistical physics, and connects with the problem of establishing a rigorous physical basis of Brownian motion (cf. [18], Chapters 8.I, 6.II). Because of the particle interaction, the tagged particle is not usually Markovian with respect to its own history, which complicates analysis. However, despite this difficulty, one expects its position to homogenize to a diffusion with parameters given in terms of the "bulk" hydrodynamic density.

Although fluctuations of Markov processes are much examined in the literature (cf. [9]), and there are many central limit theorems for types of tagged particles when the system is in "equilibrium" (cf. [8,16,17]), much less is understood when particles *both* interact nontrivially, and begin in "nonequilibrium." Virtually, the only fluctuations work in this case takes advantage of special features in types of exclusion and interacting Brownian motion models, namely [4,5],

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and [3]. Note also "propagation of chaos" results yield homogenization limits for the average tagged particle position in simple exclusion [15].

The main goal of this paper is to give a general method, which takes into account the evolution of the hydrodynamic density, to capture the "nonequilibrium" fluctuations of a tagged particle in zero-range interacting particle systems. Such systems are well-established and have long served as models for types of queuing, traffic, fluid, granular flow etc. [2]. Informally, they follow a collection of random walks on a lattice which interact in the following way: A particle at a location with k particles displaces by j with infinitesimal rate $(g(k)/k)p(j)$ where the process rate $g:\mathbb{N}_0 \to \mathbb{R}_+$, where $\mathbb{N}_0 = \{0, 1, \ldots\}$, is a function on the nonnegative integers, and $p(\cdot)$ is a translation-invariant single particle transition probability. The name 'zero-range' derives from the fact that the infinitesimal interaction is only with the particle number at a vertex.

As might be suspected, different behaviors may be found by varying the choice of the rate g. For instance, when p is nearest-neighbor and symmetric, the spectral gap or mixing properties of the zero-range system defined on a cube of width n with k particles depend strongly on the form of g. For a class of models, when g grows linearly, the gap is of order n^{-2} and does not depend on k [10]. However, when g grows at most sublinearly, the gap depends on the number of particles k. In particular, when g is of form $g(x) = x^\gamma$ for $0 < \gamma \le 1$, the gap is of the order $n^{-2}(1+\rho)^{\gamma-1}$ where $\rho = k/n$ [13]. Also, when g is the unit rate, $g(x) = \mathbf{1}\{x \ge 1\}$, the gap is of order $n^{-2}(1+\rho)^{-2}$ [12].

In this context, we prove a "nonequilibrium" scaling limit for a tagged particle in a large class of 'bounded' or 'sublinear' rate one dimensional zero-range interacting particle systems. This article can be thought of as a companion to our previous work [6] which considered the problem in 'linear growth' rate zero-range systems. The proof in [6] relies on an important estimate, a "local" hydrodynamic limit, which however makes strong use of the linear growth assumption on the process rate, in particular, as mentioned above, that spectral gap bounds on a localized cube do not depend on the number of particles in the cube. Unfortunately, this proof does not carry over to the bounded or sublinear rate situation, where the mixing behavior must be more carefully understood.

Our main contribution then is to give a different approach for the "local hydrodynamic replacement" (Theorem 2.6) with respect to a class of sublinear rate zero-range models so that the nonequilibrium limit for the tagged particle can be established (Theorem 2.2). As in [6], a consequence of the argument is that the limit of the empirical density in the reference frame of the tagged particle can be identified as the hydrodynamic density in the frame of the limit tagged particle diffusion (Theorem 2.3).

We remark that the approach taken here with respect to the "local hydrodynamic replacement" is robust enough so that it can apply to determine the nonequilibrium fluctuations of a "second-class" particle, and associated reference frame empirical density, in the symmetric unit rate case, that is when $g(k) = \mathbf{1}\{k \ge 1\}$ (Theorems 2.4, 2.5). This is the first work to address a nonequilibrium central limit theorem for a second-class particle.

We now give a sketch of the results, and discuss afterwards the differences in the argument with [6]. Define configurations $\xi \in \mathbb{N}_0^{\mathbb{Z}}$ of the zero-range process, so that $\xi(x)$, for $x \in \mathbb{Z}$, denotes the number of particles at site x in state ξ. With respect to a scale parameter, $N \ge 1$, we diffusively rescale the process, that is space is scaled by N^{-1} and time is speeded up by N^2. Suppose this system $\{\xi_t^N : t \ge 0\}$ begins from a local equilibrium measure with density profile $\rho_0 : \mathbb{R} \to \mathbb{R}_+$ (cf. before Theorem 2.1). The bulk density evolution is captured by the well-known "hydrodynamic limit" (Theorem 2.1), where the empirical density $\pi_t^{N,0}$, the measure found by assigning mass N^{-1} to each particle at times $t \ge 0$, converges in probability to an absolutely continuous measure $\rho(t,u)\,\mathrm{d}u$, where $\rho(t,u)$ is the solution of a non-linear parabolic equation with initial condition ρ_0.

The main point is to relate the "hydrodynamics" to the tagged particle behavior. Let X_t^N be the position of a tagged particle, initially at the origin, at time t. It isn't difficult to see that the rescaled trajectory $\{X_t^N/N : 0 \le t \le T\}$ is tight in the uniform topology. We now identify the limit points.

It turns out that in mean-zero zero-range processes, as opposed to other models with different interactions, X_t^N is a square integrable martingale with bounded quadratic variation given by

$$\langle X^N \rangle_t = \sigma^2 N^2 \int_0^t \frac{g(\eta_s^N(0))}{\eta_s^N(0)}\,\mathrm{d}s.$$

Here, σ^2 is the variance of the transition probability $p(\cdot)$, $g(\cdot)$ is the process rate already mentioned, and $\eta_s^N = \tau_{X_s^N}\xi_s^N$ is the state of the process as seen in the reference frame of the tagged particle, where $\{\tau_x : x \in \mathbb{Z}\}$ are translations.

We now observe, if the rescaled position of the tagged particle $x_t^N = X_t^N/N$ converges to some trajectory x_t, this process x_t inherits the martingale property from X_t^N. If in addition x_t is continuous, by Levy's characterization, one needs only to examine the asymptotics of its quadratic variation to identify it.

Denote by $\{\nu_\rho\colon \rho \geq 0\}$ the family of invariant measures, indexed by the density, for the process as seen from the tagged particle. Let π_t^N be the associated empirical density $\pi_t^N = \tau_{X_t^N}\pi_t^{N,0}$ and suppose that one can replace the integrand $g(\eta_s^N(0))/\eta_s^N(0)$ by a function of the empirical density. If we assume "conservation of local equilibrium" for the reference process, that is π_t^N converges to a certain density (cf. [7], Chapters I, III, VIII), this function should be in the form $h(\lambda(s,0))$, where $h(\rho)$ is the expected value of $g(\eta(0))/\eta(0)$ under the invariant measure ν_ρ and $\lambda(s,0)$ is the density of particles around the tagged particle.

Since we assume X_t^N/N converges to x_t, $\pi_t^N = \tau_{X_t^N}\pi_t^{N,0}$, and $\pi_t^{N,0}$ converges to $\rho(t,u)\,\mathrm{d}u$, we conclude that $\lambda(s,0) = \rho(s,x_s)$. Therefore, the quadratic variation of X_t^N/N converges to the quadratic variation of x_t, $\langle x\rangle_t = \sigma^2\int_0^t h(\rho(s,x_s))\,\mathrm{d}s$. In particular, by the characterization of continuous martingales, x_t satisfies

$$\mathrm{d}x_t = \sigma\sqrt{h\bigl(\rho(s,x_s)\bigr)}\,\mathrm{d}B_s,$$

where ρ is the solution of the hydrodynamic equation, h is defined above and B is a Brownian motion.

The main difficulty in this outline is to prove the "conservation of local equilibrium" around the tagged particle, or what we have called "local hydrodynamic replacement." A major complication is that since the local function is not a space average, one cannot avoid pathologies, such as large densities near the tagged particle, by 'averaging' them away as is the case with the usual hydrodynamics. It would seem then that the only robust tools available are local central limit theorems, and spectral gap and localized Dirichlet form estimates. Our argument is in three steps.

Step 1, "local 1-block," of our method is to replace the quadratic variation integral $\int_0^t g(\eta_s^N(0))/\eta_s^N(0)\,\mathrm{d}s$ by $\int_0^t H(\mathrm{Av}_\ell\eta_s^N)$. Here, $H(\rho)$ is the mean-value of $g(\eta(0))/\eta(0)$ with respect to the reference frame invariant measure ν_ρ with density ρ, and $\mathrm{Av}_\ell\eta_s^N$ is the local density around the tagged particle in a window of size ℓ, an introduced intermediate scale. In step 2, "local 2-block," we replace the integrand $H(\mathrm{Av}_\ell\eta_s^N)$ by $(N\varepsilon)^{-1}\sum_{x=1}^{N\varepsilon}\tau_x H(\mathrm{Av}_\ell\eta_s^N)$, the average of its translates in a block of size $N\varepsilon$ where ε is small. Finally, in step 3, having now brought in an $N\varepsilon$-block average into the integrand, which is a spatial average over $\mathrm{O}(N)$ translates of a local function on ℓ sites, more usual hydrodynamic techniques can be used to replace it by $(N\varepsilon)^{-1}\sum_{x=1}^{N\varepsilon}\tau_x\bar{H}_l(\mathrm{Av}_{N\kappa}\eta_s^N)$. Here, $\bar{H}_l(\rho)$ is the mean-value of $H_l(\eta) = H(\mathrm{Av}_\ell\eta)$ with respect to the invariant measure μ_ρ in the usual undistinguished particles frame, and $\mathrm{Av}_{N\kappa}\eta_s^N$, with another introduced small parameter $\kappa \ll \varepsilon \ll 1$, is an $\mathrm{O}(N)$ average which can be written in terms of the reference frame empirical density.

In [6], as mentioned, the main assumption is that the process rate g is of "linear order," a condition which ensures a sharp lower bound on the spectral gap in a box of width n, independent of the number of particles k in the box, and also which allows for uniform local central limit theorems. Because of such spectral gap bounds, *uniform* approximations, with respect to local densities, in the scheme above may be performed. Intuitively, there is a lot of 'local' mixing which can be exploited in this model.

However, in the current work, when g is assumed to be 'sublinear,' since the spectral gap depends on k, large local densities slow down the mixing behavior, and need to be estimated. This observation would in fact likely prevent the 'local replacement' if the integrand of the quadratic variation integral were different. However, under the 'sublinear' assumption on g, the function $H(\rho)$ vanishes as the density ρ diverges, which is useful to temper some of the large density effects. [This is not the case in [6] where $H(\rho)$ is bounded above and below by constants.] Even so, more control on large densities is needed in all steps 1, 2, and 3. In particular, in step 3, which performs a "global" replacement, if g is not linearly growing, truncations, which are essential, are untractable in general. For this reason, we assume g is also increasing, or "attractive," so that certain couplings can be used (cf. [1]).

Step 2, in which $H(\mathrm{Av}_\ell\eta_s^N)$ and $\tau_x H(\mathrm{Av}_\ell\eta_s^N)$ for $1 \leq x \leq N\varepsilon$, functions of averages over distant blocks, are compared, is the most difficult. With the idea that the process mixes faster within each block than between the blocks, each term can be replaced by its conditional expectation given the number of particles in its block. This reduces the between block dynamics to a birth–death process whose mixing properties, with some analysis and the assumption $\lim_{k\uparrow\infty} g(k)/k = 0$, can be estimated and found suitable to complete the step. We remark that this last point rules out the case in [6].

2. Notation and results

Let $\xi_t = \{\xi_t(x)\colon\ x \in \mathbb{T}_N\}$ be the zero-range process on the discrete one dimensional torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ with single particle transition probability $p(\cdot)$ and process rate $g : \mathbb{N}_0 \to \mathbb{R}_+$. We will assume that $g(0) = 0$, $g(1) > 0$ and, as is usual when dealing with sublinear rates, that g is increasing (or "attractive"), $g(k+1) \geq g(k)$ for $k \geq 1$. In addition, throughout the paper, and in all results, we impose one of the following set of conditions (B) or (SL):

(B) g is bounded: For $k \geq 1$, there are constants $0 < a_0 \leq a_1$ such that $a_0 \leq g(k) \leq a_1$.

To give the class of sublinear rates considered, let $W(l,k)$ be the inverse of the spectral gap of the process, where p is nearest-neighbor and symmetric, defined on the cube $\Lambda_l = \{-l, \dots, l\}$ with k particles (cf. Section 3 for more definitions).

(SL1) g is sublinear: $\lim_{k\to\infty} g(k) = \infty$, $g(k)/k : \mathbb{N} \to \mathbb{R}_+$ is decreasing, and $\lim_{k\to\infty} g(k)/k = 0$. In particular, since g is increasing, there exists constants $a_0, a_1 > 0$ such that $a_0 \leq g(k)$ and $g(k)/k \leq a_1$, $k \geq 1$.
(SL2) g is *Lipschitz*: There is a constant a_2 such that $|g(k+1) - g(k)| \leq a_2$ for $k \geq 0$.
(SL3) The spectral gap satisfies, for all constants C and $l \geq 1$, that

$$\lim_{N\uparrow\infty} N^{-1} \max_{1\leq k\leq C\log N} k^2 W(l,k) = 0. \tag{2.1}$$

It is proved in Lemma 3.2 that all processes with bounded rates g satisfy (2.1). In addition, by the spectral gap estimate [13], processes with rates $g(k) = k^\gamma$ for $0 < \gamma \leq 1$ satisfy (2.1). In addition, we will assume that p is finite-range, irreducible, and mean-zero, that is:

(MZ) There exists $R > 0$ such that $p(z) = 0$ for $|z| > R$, and $\sum z p(z) = 0$.

We also will take the scaling parameter N larger than the support of $p(\cdot)$.

Denote by $\Omega_N = \mathbb{N}_0^{\mathbb{T}_N}$ the state space and by ξ the configurations of Ω_N so that $\xi(x)$, $x \in \mathbb{T}_N$, stands for the number of particles in the site x for the configuration ξ. The zero-range process is a continuous-time Markov chain generated by

$$(\mathcal{L}_N f)(\xi) = \sum_{x\in\mathbb{T}_N} \sum_z p(z) g\big(\xi(x)\big)\big[f\big(\xi^{x,x+z}\big) - f(\xi)\big], \tag{2.2}$$

where $\xi^{x,y}$ represents the configuration obtained from ξ by displacing a particle from x to y:

$$\xi^{x,y}(z) = \begin{cases} \xi(x) - 1 & \text{for } z = x, \\ \xi(y) + 1 & \text{for } z = y, \\ \xi(z) & \text{for } z \neq x, y. \end{cases}$$

The zero-range process $\xi(t)$ has a well known explicit family product invariant measures $\bar\mu_\varphi$, $0 \leq \varphi < \lim g(k) =: g(\infty)$, on Ω_N defined on the nonnegative integers,

$$\bar\mu_\varphi\big(\xi(x) = k\big) = \frac{1}{Z_\varphi}\frac{\varphi^k}{g(k)!} \quad \text{for } k \geq 1 \quad \text{and} \quad \bar\mu_\varphi\big(\xi(x) = 0\big) = \frac{1}{Z_\varphi},$$

where $g(k)! = g(1)\cdots g(k)$ and Z_φ is the normalization. Denote by $\rho(\varphi)$ the mean of the marginal $\bar\mu_\varphi$, $\rho(\varphi) = \sum_k k\mu_\varphi(\xi(x) = k)$. Since g is increasing, the radius of convergence of Z_φ is $g(\infty)$, and $\lim_{\varphi\uparrow g(\infty)} \rho(\varphi) = \infty$. As $\rho(0) = 0$ and $\rho(\varphi)$ is strictly increasing, for a given $0 \leq \rho < \infty$, there is a unique inverse $\varphi = \varphi(\rho)$. Define then the family in terms of the density ρ as $\mu_\rho = \bar\mu_{\varphi(\rho)}$.

Now consider an initial configuration ξ such that $\xi(0) \geq 1$, and let $\Omega_N^* \subset \Omega_N$ be the set of such configurations. Distinguish, or tag one of the particles initially at the origin, and follow its trajectory X_t, jointly with the evolution of the process ξ_t. It will be convenient for our purposes to consider the process as seen by the tagged particle. This reference process $\eta_t(x) = \xi_t(x + X_t)$ is also Markovian and has generator in form $L_N = L_N^{\mathrm{env}} + L_N^{\mathrm{tp}}$, where L_N^{env}, L_N^{tp}

are defined by

$$\begin{aligned}\big(L_N^{\mathrm{env}} f\big)(\eta) &= \sum_{x\in\mathbb{T}_N\setminus\{0\}} \sum_z p(z) g\big(\eta(x)\big)\big[f\big(\eta^{x,x+z}\big) - f(\eta)\big] \\ &\quad + \sum_y p(y) g\big(\eta(0)\big)\frac{\eta(0)-1}{\eta(0)}\big[f\big(\eta^{0,y}\big) - f(\eta)\big], \qquad (2.3)\\ \big(L_N^{\mathrm{tp}} f\big)(\eta) &= \sum_z p(z)\frac{g(\eta(0))}{\eta(0)}\big[f(\theta_z\eta) - f(\eta)\big].\end{aligned}$$

In this formula, the translation θ_z is defined by

$$(\theta_z\eta)(x) = \begin{cases} \eta(x+z) & \text{for } x\neq 0, -z,\\ \eta(z)+1 & \text{for } x = 0,\\ \eta(0)-1 & \text{for } x = -z.\end{cases}$$

The operator L_N^{tp} corresponds to jumps of the tagged particle, while the operator L_N^{env} corresponds to jumps of the other particles, called environment.

A key feature of the tagged motion is that it can be written as a martingale in terms of the reference process:

$$X_t = \sum_j jN_t(j) = \sum_j jM_t(j) + \mathfrak{m}\int_0^t \frac{g(\eta_s(0))}{\eta_s(0)}\,\mathrm{d}s = \sum_j jM_t(j), \qquad (2.4)$$

where $\mathfrak{m} = \sum_j jp(j) = 0$ is the mean drift, $N_t(j)$ is the counting process of translations of size j up to time t, and $M_t(j) = N_t(j) - p(j)\int_0^t g(\eta_s(0))/\eta_s(0)\,\mathrm{d}s$ is its corresponding martingale. In addition, $M_t^2(j) - p(j)\int_0^t g(\eta_s(0))/\eta_s(0)\,\mathrm{d}s$ are martingales which are orthogonal as jumps are not simultaneous a.s. Hence, the quadratic variation of X_t is $\langle X\rangle_t = \sigma^2\int_0^t g(\eta_s(0))/\eta_s(0)\,\mathrm{d}s$ where $\sigma^2 = \sum j^2 p(j)$.

For the reference process η_t, the "Palm" or origin size biased measures given by $\mathrm{d}\nu_\rho = (\eta(0)/\rho)\,\mathrm{d}\mu_\rho$ are invariant (cf. [14,16]). Note that ν_ρ is also a product measure whose marginal at the origin differs from that at other points $x\neq 0$. Here, we take $\nu_0 = \delta_{\mathfrak{d}_0}$, the Dirac measure concentrated on the configuration $\mathfrak{d}_0$ with exactly one particle at the origin, and note that ν_ρ converges to $\delta_{\mathfrak{d}_0}$ as $\rho\downarrow 0$.

The families $\{\mu_\rho\colon \rho\geq 0\}$ and $\{\nu_\rho\colon \rho\geq 0\}$ are stochastically ordered. Indeed, this follows as the marginals of μ_ρ and ν_ρ are stochastically ordered. Also, since we assume that g is increasing, the system is "attractive," that is by the "basic coupling" (cf. [11]) if $\mathrm{d}R$ and $\mathrm{d}R'$ are initial measures of two processes ξ_t and ξ'_t, and $\mathrm{d}R \ll \mathrm{d}R'$ in stochastic order, then the distributions of ξ_t and ξ'_t are similarly stochastically ordered [11]. We also note, when p is symmetric, that μ_ρ and ν_ρ are reversible with respect to $\mathcal{L}_N$, and L_N and L_N^{env} respectively.

From this point, to avoid uninteresting compactness issues, we define every process in a finite time interval $[0,T]$, where $T<\infty$ is fixed. Let $\mathbb{T}$ be the unit torus and let $\mathcal{M}_+(\mathbb{T})$ be the set of positive Radon measures in $\mathbb{T}$.

For a continuous, positive function $\rho_0\colon\mathbb{T}\to\mathbb{R}_+$, define $\mu^N = \mu^N_{\rho_0(\cdot)}$ as the product measure in Ω_N given by $\mu^N_{\rho_0(\cdot)}(\eta(x)=k) = \mu_{\rho_0(x/N)}(\eta(x)=k)$.

Consider the process $\xi_t^N =: \xi_{tN^2}$, generated by $N^2\mathcal{L}_N$ starting from the initial measure μ^N. Define the process $\pi_t^{N,0}$ in $\mathcal{D}([0,T],\mathcal{M}_+(\mathbb{T}))$, the space of $M_+(\mathbb{T})$ valued right-continuous paths with left limits for times $0\leq t\leq T$ endowed with the Skorohod topology, as

$$\pi_t^{N,0}(\mathrm{d}u) = \frac{1}{N}\sum_{x\in\mathbb{T}_N}\xi_t^N(x)\delta_{x/N}(\mathrm{d}u),$$

where δ_u is the Dirac distribution at the point u.

The next result, "hydrodynamics," under the assumption $p(\cdot)$ is mean-zero, is well known (cf. [1,7]).

Theorem 2.1. *For each $0 \le t \le T$, $\pi_t^{N,0}$ converges in probability to the deterministic measure $\rho(t,u)\,\mathrm{d}u$, where $\rho(t,u)$ is the solution of the hydrodynamic equation*

$$\begin{cases} \partial_t \rho = \sigma^2 \, \partial_x^2 \varphi(\rho), \\ \rho(0,u) = \rho_0(u), \end{cases} \tag{2.5}$$

and $\varphi(\rho) = \int g(\xi(0))\,\mathrm{d}\mu_\rho$.

We now state results for the tagged particle motion. Define the product measure $\nu^N = \nu^N_{\rho_0(\cdot)}$ in Ω_N^* given by $\nu^N_{\rho_0(\cdot)}(\eta(x) = k) = \nu_{\rho_0(x/N)}(\eta(x) = k)$, and let $\eta_t^N =: \eta_{tN^2}$ be the process generated by $N^2 L_N$ and starting from the initial measure ν^N. Define the empirical measure π_t^N in $\mathcal{D}([0,T], \mathcal{M}_+(\mathbb{T}))$ by

$$\pi_t^N(\mathrm{d}u) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t^N(x) \delta_{x/N}(\mathrm{d}u).$$

Let also $X_t^N = X_{N^2 t}$ be the position of the tagged particle at time $N^2 t$.

Define also the continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi(\rho) = \int \frac{g(\eta(0))}{\eta(0)} \,\mathrm{d}\nu_\rho.$$

Note $\psi(\rho) = \varphi(\rho)/\rho$ for $\rho > 0$, and $\psi(0) = g(1)$. The first main result of the article is to identify the scaling limit of the tagged particle as a diffusion process:

Theorem 2.2. *Let $x_t^N = X_t^N / N$ be the rescaled position of the tagged particle for the process ξ_t^N. Then, $\{x_t^N\colon t \in [0,T]\}$ converges in distribution in the uniform topology to the diffusion $\{x_t\colon t \in [0,T]\}$ defined by the stochastic differential equation*

$$\mathrm{d}x_t = \sigma \sqrt{\psi\big(\rho(t, x_t)\big)}\,\mathrm{d}B_t, \tag{2.6}$$

where B_t is a standard Brownian motion on $\mathbb{T}$, and $\rho(t,u)$ is the solution of the hydrodynamic Eq. (2.5) as in Theorem 2.1.

In terms of this characterization, we can describe the evolution of the empirical measure as seen from the tagged particle:

Theorem 2.3. *We have $\{\pi_t^N\colon t \in [0,T]\}$ converges in distribution with respect to the Skorohod topology on $\mathcal{D}([0,T], \mathcal{M}_+(\mathbb{T}))$ to the measure-valued process $\{\rho(t, u + x_t)\,\mathrm{d}u\colon t \in [0,T]\}$, where $\rho(t,u)$ is the solution of the hydrodynamic Eq. (2.5) and x_t is given by (2.6).*

When the rate $g(k) = \mathbf{1}\{k \ge 1\}$, scaling limits of a "second-class" particle $\mathcal{X}_t$ can also be captured. Informally, such a particle must wait until all the other particles, say "first-class" particles, have left its position before it can displace by j with rate $p(j)$. More precisely, its dynamics can be described in terms of its reference frame motion. For an initial configuration ξ such that $\xi(0) \ge 1$, let $\zeta_t(x) = \xi_t(x + \mathcal{X}_t) - \delta_{0,x}$, where $\delta_{a,b}$ is Kronecker's delta, be the system of first-class particles in the reference frame of the second-class particle. The generator $\mathfrak{L}_N$ takes form $\mathfrak{L}_N = \mathfrak{L}_N^{\mathrm{env}} + \mathfrak{L}_N^{\mathrm{tp}}$, where

$$\big(\mathfrak{L}_N^{\mathrm{env}} f\big)(\zeta) = \sum_{x \in \mathbb{T}_N} \sum_z p(z) \mathbf{1}\big\{\zeta(x) \ge 1\big\}\big[f\big(\zeta^{x,x+z}\big) - f(\zeta)\big],$$

$$\big(\mathfrak{L}_N^{\mathrm{tp}} f\big)(\zeta) = \sum_z p(z) \mathbf{1}\big\{\zeta(0) = 0\big\}\big[f(\tau_z \zeta) - f(\zeta)\big],$$

where τ_z is the pure spatial translation by z, $(\tau_z\zeta)(y)=\zeta(y+z)$ for $y\in\mathbb{T}_N$.

Then, as for the regular tagged particle, we have

$$\mathcal{X}_t=\sum_j j\mathcal{N}_t(j)=\sum_j j\mathcal{M}_t(j)+\mathfrak{m}\int_0^t \mathbf{1}\big\{\zeta_s(0)=0\big\}\,\mathrm{d}s=\sum_j j\mathcal{M}_t(j),$$

where $\mathfrak{m}=\sum_j jp(j)=0$ is the mean drift, $\mathcal{N}_t(j)$ is the counting process of translations of size j up to time t, and $\mathcal{M}_t(j)=\mathcal{N}_t(j)-p(j)\int_0^t \mathbf{1}\{\zeta_s(0)=0\}\,\mathrm{d}s$ are the associated martingales. As before, since $\mathcal{M}_t^2(j)-p(j)\int_0^t \mathbf{1}\{\zeta_s(0)=0\}\,\mathrm{d}s$ are orthogonal martingales, the quadratic variation of $\mathcal{X}_t$ is $\langle\mathcal{X}\rangle_t=\sigma^2\int_0^t \mathbf{1}\{\zeta_s(0)=0\}\,\mathrm{d}s$.

For the second-class reference process ζ_t, under the assumption $p(\cdot)$ is symmetric, the family $\mathrm{d}\chi_\rho=(1+\rho)^{-1}\times(\zeta(0)+1)\,\mathrm{d}\mu_\rho$ for $\rho>0$ are invariant. We remark that symmetry of $p(\cdot)$ is needed to show χ_ρ are invariant with respect to the second-class tagged process.

Let $\chi^N=\chi^N_{\rho(\cdot)}$ be the product measure with $\chi^N_{\rho_0(\cdot)}(\zeta(x)=k)=\chi_{\rho_0(x/N)}(\zeta(x)=k)$. Let also $\zeta^N_t=\zeta_{N^2t}$ be the process generated by $N^2\mathfrak{L}_N$ starting from χ^N. Correspondingly, define empirical measure $\pi^{N,1}_t(\mathrm{d}u)=(1/N)\sum_{x\in\mathbb{T}_N}\zeta^N_t(x)\delta_{x/N}(\mathrm{d}u)$. In addition, let

$$\upsilon(\rho)=\int \mathbf{1}\big\{\zeta(0)=0\big\}\,\mathrm{d}\chi_\rho=\frac{1}{1+\rho}\int \mathbf{1}\big\{\zeta(0)=0\big\}\,\mathrm{d}\mu_\rho=\frac{1}{(1+\rho)^2}.$$

In the case $g(k)=\mathbf{1}\{k\geq 1\}$, $\varphi(\rho)=\rho/[1+\rho]$. Denote by ρ^1 the solution of (2.5) with such function φ. We may now state results for the second-class tagged motion.

Theorem 2.4. *Suppose $p(\cdot)$ is symmetric. Let $y^N_t=\mathcal{X}^N_t/N$ be the rescaled position of the second-class tagged particle for the process ζ^N_t. Then, $\{y^N_t\colon t\in[0,T]\}$ converges in distribution in the uniform topology to the diffusion $\{y_t\colon t\in[0,T]\}$ defined by the stochastic differential equation*

$$\mathrm{d}y_t=\sigma\sqrt{\upsilon\big(\rho^1(t,y_t)\big)}\,\mathrm{d}B_t, \tag{2.7}$$

where B_t is a standard Brownian motion on $\mathbb{T}$.

Theorem 2.5. *Suppose $p(\cdot)$ is symmetric. Then, $\{\pi^{N,1}_t\colon t\in[0,T]\}$ converges in distribution with respect to the Skorohod topology on $\mathcal{D}([0,T],\mathcal{M}_+(\mathbb{T}))$ to the measure-valued process $\{\rho^1(t,u+y_t)\,\mathrm{d}u\colon t\in[0,T]\}$ where y_t is given by* (2.7).

The outline of the proofs of Theorems 2.2, 2.3, 2.4 and 2.5 are given at the end of this section. We now state the main replacement estimate with respect to the process η^N_t for a (regular) tagged particle. A similar estimate holds with respect to the process ζ^N_t and a "second-class" particle, stated in the proof of Theorem 2.4. As remarked earlier, this replacement estimate is the main ingredient to show Theorems 2.2 and 2.3.

Denote by $\mathbb{P}_\nu$ the probability measure in $\mathcal{D}([0,T],\Omega_N)$ induced by the process η^N_t, starting from the initial measure ν, and by $\mathbb{E}_\nu$ the corresponding expectation. When $\nu=\nu^N$, we abbreviate $\mathbb{P}_{\nu^N}=\mathbb{P}^N$ and $\mathbb{E}_{\nu^N}=\mathbb{E}^N$. With respect to the process ξ^N_t, denote by $\mathbf{P}_\mu$ the probability measure in $\mathcal{D}([0,T],\Omega_N)$ starting from measure μ, and by $\mathbf{E}_\mu$ the associated expectation. Denote also by $E_\mu[h]$ and $\langle h\rangle_\mu$ the expectation of a function $h:\Omega_N\to\mathbb{R}$ with respect to the measure μ; when $\mu=\nu_\rho$, let $E_\rho[h]$, $\langle h\rangle_\rho$ stand for $E_{\nu_\rho}[h]$, $\langle h\rangle_{\nu_\rho}$. Define also the inner product $\langle f,g\rangle_\mu=E_\mu[fg]$, and covariance $\langle f;g\rangle_\mu=E_\mu[fg]-E_\mu[f]E_\mu[g]$ with the same convention when $\mu=\nu_\rho$. To simplify notation, we will drop the superscript N in the speeded-up process η^N_t.

For $l\geq 0$, let

$$\eta^l(x)=\frac{1}{2l+1}\sum_{|y|\leq l}\eta(x+y).$$

Theorem 2.6. *Let $h:\mathbb{N}\to\mathbb{R}_+$ be a nonnegative, bounded, Lipschitz function such that there exists a constant C such that $h(k)\le C[g(k)/k]$ for $k\ge 1$. Then,*

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\limsup_{\kappa\to 0}\limsup_{N\to\infty}\mathbb{E}^N\left[\left|\int_0^t h\big(\eta_s(0)\big)-\frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N}\bar{H}_l\big(\eta_s^{\kappa N}(x)\big)\,\mathrm{d}s\right|\right]=0,$$

where $H(\rho)=E_{\nu_\rho}[h(\eta(0))]$, $H_l(\eta)=H(\eta^l(0))$, and $\bar{H}_l(\rho)=E_{\mu_\rho}[H_l]$.

We now give the outlines of the proof of the main theorems.

Proofs of Theorems 2.2 and 2.3. First, the replacement estimate, Theorem 2.6, applies when $h(k)=g(k)/k$: Under the assumptions on g, clearly h is positive, bounded, and Lipschitz. Given Theorem 2.6, the proof of the main theorems straightforwardly follow the same steps as in [6]. Namely, (1) tightness is proved for $(x_t^N, A_t^N, \pi_t^{N,0}, \pi_t^N)$ where $A_t^N=\langle x_t^N\rangle$ is the quadratic variation of the martingale x_t^N. (2) Using the hydrodynamic limit, Theorem 2.1, one determines the limit points of $\pi_t^{N,0}$, and $\pi_t^N=\tau_{x_t^N}\pi_t^{N,0}$. Limit points of A_t^N are obtained through the replacement estimate, Theorem 2.6. Finally, one obtains that all limits of x_t^N are characterized as continuous martingales with certain quadratic variations. Theorems 2.2 and 2.3 follow now by Levy's theorem. More details on these last points can be found in [6]. □

Proofs of Theorems 2.4 and 2.5. The proofs follow the same scheme as for Theorems 2.2 and 2.3, given a replacement estimate. One can rewrite Theorem 2.6 in terms of ζ_s^N:

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\limsup_{\kappa\to 0}\limsup_{N\to\infty}\mathbb{E}_{\mathrm{sec}}^N\left[\left|\int_0^t h\big(\zeta_s(0)\big)-\frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N}\bar{H}_l\big(\zeta_s^{\kappa N}(x)\big)\,\mathrm{d}s\right|\right]=0.$$

Here, $h(k)=\mathbf{1}\{k=0\}$, $H(\rho)=E_{\chi_\rho}[h(\zeta(0))]$, $H_l(\zeta)=H(\zeta^l(0))$, and $\bar{H}_l(\rho)=E_{\mu_\rho}[H_l]$. Also, $\mathbb{E}_{\mathrm{sec}}^N$ is the process expectation with respect to ζ_s^N.

Given $\mathrm{d}\chi_\rho=(1+\rho)^{-1}\,\mathrm{d}\mu_\rho+\rho(1+\rho)^{-1}\,\mathrm{d}\nu_\rho$, the proof of this replacement follows quite closely the proof of Theorem 2.6 with straightforward modifications. □

The plan of the paper now is to give some spectral gap estimates, "global," "local 1-block" and "local 2-blocks" estimates in Sections 3, 4, 5, and 6, which are used to give the proof of Theorem 2.6 in Section 6.2.

For simplicity in the proofs, we will suppose that $p(\cdot)$ is symmetric, and nearest-neighbor, but our results hold, with straightforward modifications, when $p(\cdot)$ is finite-range, irreducible, and mean-zero, because mean-zero zero-range processes are gradient processes.

3. Spectral gap estimates

We discuss some spectral gap bounds which will be useful in the sequel. For $l\ge 0$, let $\Lambda_l=\{x: |x|\le l\}$ be a cube of length $2l+1$ around the origin, and let $\nu_\rho^{\Lambda_l}$ and $\mu_\rho^{\Lambda_l}$ be the measures ν_ρ and μ_ρ restricted to Λ_l.

For $j\ge 0$, define the sets of configurations $\Sigma_{\Lambda_l,j}=\{\eta\in\mathbb{N}_0^{\Lambda_l}: \sum_{x\in\Lambda_l}\eta(x)=j\}$, and $\Sigma_{\Lambda_l,j}^*=\{\eta\in\mathbb{N}_0^{\Lambda_l}: \eta(0)\ge 1, \sum_{x\in\Lambda_l}\eta(x)=j\}$. Define also the canonical measures $\nu_{\Lambda_l,j}(\cdot)=\nu_\rho^{\Lambda_l}(\cdot|\Sigma_{\Lambda_l,j}^*)$, and $\mu_{\Lambda_l,j}(\cdot)=\mu_\rho^{\Lambda_l}(\cdot|\Sigma_{\Lambda_l,j})$. Note that both $\nu_{\Lambda_l,j}$ and $\mu_{\Lambda_l,j}$ do not depend on ρ.

Denote by $\mathcal{L}_{\Lambda_l}$, $L_{\Lambda_l}^{\mathrm{env}}$ the restrictions of the generators $\mathcal{L}_N$, L_N^{env} on $\Sigma_{\Lambda_l,j}$, $\Sigma_{\Lambda_l,j}^*$, respectively. These generators are obtained by restricting the sums over x,y,z in (2.2) and (2.3) to x, $x+z$, $y\in\Lambda_l$. Clearly, $\nu_{\Lambda_l,j}$, $\mu_{\Lambda_l,j}$ are invariant with respect to $L_{\Lambda_l}^{\mathrm{env}}$, $\mathcal{L}_{\Lambda_l}$, respectively. Denote the Dirichlet forms $D(\mu_{\Lambda_l,j},f)=\langle f,(-\mathcal{L}_{\Lambda_l}f)\rangle_{\mu_{\Lambda_l,j}}$ and $D(\nu_{\Lambda_l,j},f)=\langle f,(-L_{\Lambda_l}^{\mathrm{env}}f)\rangle_{\nu_{\Lambda_l,j}}$. One can compute

$$D(\mu_{\Lambda_l,j},f)=\frac12\sum_{x,y\in\Lambda_l}p(y-x)E_{\mu_{\Lambda_l,j}}\big[g\big(\eta(x)\big)\big(f\big(\eta^{x,y}\big)-f(\eta)\big)^2\big],$$

$$D(\nu_{\Lambda_l,j}, f) = \frac{1}{2}\sum_{x\in\Lambda_l\setminus\{0\}}\sum_{y\in\Lambda_l} p(y-x)E_{\nu_{\Lambda_l,j}}\big[g\big(\eta(x)\big)\big(f\big(\eta^{x,y}\big)-f(\eta)\big)^2\big]$$
$$+\frac{1}{2}\sum_{z\in\Lambda_l} p(z)E_{\nu_{\Lambda_l,j}}\bigg[g\big(\eta(0)\big)\frac{\eta(0)-1}{\eta(0)}\big(f\big(\eta^{0,z}\big)-f(\eta)\big)^2\bigg].$$

Let $W(l,j)$ and $W^{\mathrm{env}}(l,j)$ be the inverse of the spectral gaps of $\mathcal{L}_{\Lambda_l}$ and $L^{\mathrm{env}}_{\Lambda_l}$ with respect to $\Sigma_{\Lambda_l,j}$ and $\Sigma^*_{\Lambda_l,j}$ respectively. In particular, the following Poincaré inequalities are satisfied: For all L^2 functions,

$$\langle f; f\rangle_{\mu_{\Lambda_l,j}} \le W(l,j)D(\mu_{\Lambda_l,j}, f),$$
$$\langle f; f\rangle_{\nu_{\Lambda_l,j}} \le W^{\mathrm{env}}(l,j)D(\nu_{\Lambda_l,j}, f).$$

In the next two lemmas, we do not assume that g is increasing. We first relate the environment spectral gap to the untagged process spectral gap.

Lemma 3.1. *Suppose on $\Sigma^*_{\Lambda_l,j}$ that $a_1^{-1}\le \eta(0)/g(\eta(0))\le ja_0^{-1}$. Then, for $j\ge 1$, we have that $W^{\mathrm{env}}(l,j)\le (a_1a_0^{-1}j)^2W(l,j-1)$.*

Proof. Note $E_{\mu_\rho}[g(\eta(0))f(\eta)] = \varphi(\rho)E_{\mu_\rho}[f'(\eta)]$ with $f'(\eta) = f(\eta+\mathfrak{d}_0)$, where we recall $\mathfrak{d}_0$ is the configuration with exactly one particle at the origin. By a suitable change of variables one can show that $D(\mu_{\Lambda_l,j-1}, f')\le a_1a_0^{-1}jD(\nu_{\Lambda_l,j}, f)$.

By the assumption on g, for every $c\in\mathbb{R}$,

$$E_{\nu_{\Lambda_l,j}}\big[(f-E_{\nu_{\Lambda_l,j}}f)^2\big] \le \frac{E_{\mu_\rho}[\eta(0)(f-c)^2\mathbf{1}\{\Sigma^*_{\Lambda_l,j}\}]}{E_{\mu_\rho}[\eta(0)\mathbf{1}\{\Sigma^*_{\Lambda_l,j}\}]}$$
$$\le a_1a_0^{-1}j\frac{E_{\mu_\rho}[g(\eta(0))(f-c)^2\mathbf{1}\{\Sigma^*_{\Lambda_l,j}\}]}{E_{\mu_\rho}[g(\eta(0))\mathbf{1}\{\Sigma^*_{\Lambda_l,j}\}]}.$$

The change of variables $\eta' = \eta-\mathfrak{d}_0$ and an appropriate choice of the constant c permits to rewrite last expression as

$$a_1a_0^{-1}jE_{\mu_{\Lambda_l,j-1}}\big[\big(f'-E_{\mu_{\Lambda_l,j-1}}f'\big)^2\big]\le a_1a_0^{-1}jW(l,j-1)D\big(\mu_{\Lambda_l,j-1}, f'\big),$$

where the last inequality follows from the spectral gap for the zero range process. By the observation made at the beginning of the proof, this expression is bounded by

$$\big(a_1a_0^{-1}j\big)^2W(l,j-1)D(\nu_{\Lambda_l,j}, f),$$

which concludes the proof of the lemma. □

Lemma 3.2. *Suppose g satisfies $a_0\le g(k)\le a_1$ for $k\ge 1$, and $\lim_{k\uparrow\infty} g(k) = L$. For every $\alpha>0$, there is a constant $B = B_\alpha$ such that $W(l,j)\le B^l(1+\alpha)^j(l+j)^2$.*

Proof. We need only to establish, for all L^2 functions f, that

$$\langle f; f\rangle_{\mu_{\Lambda_l,j}} \le B^l(1+\alpha)^j(l+j)^2D(\mu_{\Lambda_l,j}, f).$$

To argue the bound, we make a comparison with the measure $\mu_{\Lambda_l,j}$ when $g(k) = \mathbf{1}\{k\ge 1\}$. Denote this measure by $\mu^1_{\Lambda_l,j}$, and recall, by conversion to the simple exclusion process (cf. [10], Example 1.1 and [12]), that

$$E_{\mu^1_{\Lambda_l,j}}\big[(f-E_{\mu^1_{\Lambda_l,j}}f)^2\big]\le b_0(l+j)^2D_{\mu^1_{\Lambda_l,j}}(f) \tag{3.1}$$

for some finite constant b_0. Write

$$E_{\mu_{\Lambda_l,j}}\big[(f-E_{\mu_{\Lambda_l,j}}f)^2\big]=\inf_c E_{\mu_{\Lambda_l,j}}\big[(f-c)^2\big]$$
$$=\inf_c \frac{\sum_\eta \prod_{x=-l}^{l}(\varphi^{\eta(x)})/(g(\eta(x))!)(f(\eta)-c)^2\mathbf{1}\{\sum_x \eta(x)=j\}}{\sum_\eta \prod_{x=-l}^{l}(\varphi^{\eta(x)})/(g(\eta(x))!)\mathbf{1}\{\sum_x \eta(x)=j\}}.$$

Without loss of generality, we may now assume that $L=1$ since we can replace g by its scaled version, $g'=g/L$, in the above expression.

For $\beta>0$, let r_0 be so large that $1-\beta\le g(z)\le 1+\beta$ for $z\ge r_0$. Then,

$$a_1^{-r_0}(1+\beta)^{-\eta(x)}\le \frac{1}{g(\eta(x))!}\le a_0^{-r_0}(1-\beta)^{-\eta(x)}.$$

This bound is achieved by overestimating the first r_0 factors by the bound $a_0\mathbf{1}\{z\ge 1\}\le g(z)\le a_1$, and the remaining factors by

$$(1+\beta)^{-\eta(x)}\le \frac{1}{\prod_{z=r_0+1}^{\eta(x)} g(z)}\le (1-\beta)^{-\eta(x)},$$

where by convention an empty product is defined as 1.

As there are $2l+1$ sites, we bound the right hand side of the displayed expression appearing just below (3.1) by

$$\big(a_0^{-1}a_1\big)^{(2l+1)r_0}\big[(1+\beta)(1-\beta)^{-1}\big]^j E_{\mu^1_{\Lambda_l,j}}\big[(f-E_{\mu^1_{\Lambda_l,j}}f)^2\big].$$

By the spectral gap estimate (3.1) and the same bounds on the Radon–Nikodym derivative of $\mathrm{d}\mu^1_{\Lambda_l,j}/\mathrm{d}\mu_{\Lambda_l,j}$, the previous expression is less than or equal to

$$\big(a_0^{-1}a_1\big)^{2(2l+1)r_0}\big[(1+\beta)(1-\beta)^{-1}\big]^{2j} b_0(l+j)^2 D_{\mu_{\Lambda_l,j}}(f).$$

We may now choose $\beta=\beta(\alpha)$ appropriately to finish the proof. □

We claim that for any constant $C>0$,

$$\lim_{N\to\infty}\frac{1}{N}\max_{1\le j\le Cl\log N} W^{\mathrm{env}}(l,j)=0. \tag{3.2}$$

Indeed, under the conditions (SL) this follows by Lemma 3.1 and by assumption (SL3). On the other hand, under the condition (B), by Lemma 3.2 we may choose α appropriately to have $\max_{1\le j\le Cl\log N} W(l,j)\le C_2(\ell)N^{1/2}$. This proves (3.2) in view of Lemma 3.1.

4. "Global" replacement

In this section, we replace the full, or "global" empirical average of a local, bounded and Lipschitz function, with respect to the process η_s, in terms of the density field π^N_s. By a local function $r:\Omega^*_N\to\mathbb{R}$, we mean a function r supported on a finite number of occupation variables. In addition, we say that a local function r, supported on coordinates $A\subset\mathbb{Z}$, is Lipschitz if there exists a finite constant C_0 such that

$$\big|r(\eta)-r\big(\eta'\big)\big|\le C_0\sum_{x\in A}\big|\eta(x)-\eta'(x)\big|$$

for all configurations η, η' of Ω^*_N, and N larger than the support size $|A|$.

The proof involves only a few changes to the hydrodynamics proof of [1], Theorem 3.2.1, and is similar to that in [6]. However, since the rate g is bounded, some details with respect to the "2-blocks" lemma below are different.

Proposition 4.1 ("Global" replacement). *Let $r:\Omega_N^*\to\mathbb{R}$ be a local, bounded and Lipschitz function. Then, for every $\delta>0$,*

$$\limsup_{\kappa\to\infty}\limsup_{N\to\infty}\mathbb{P}^N\left[\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x\mathcal{V}_{\kappa N}(\eta_s)\,\mathrm{d}s\geq\delta\right]=0,$$

where

$$\mathcal{V}_l(\eta)=\left|\frac{1}{2l+1}\sum_{|y|\leq l}\tau_y r(\eta)-\bar r\big(\eta^l(0)\big)\right|\quad\textit{and}\quad \bar r(a)=E_{\mu_a}[r].$$

Denote by $\mathcal{H}(\mu|\nu)$ the entropy of μ with respect to ν:

$$\mathcal{H}(\mu|\nu)=\sup_f\left\{\int f\,\mathrm{d}\mu-\log\int \mathrm{e}^f\,\mathrm{d}\nu\right\},$$

where the supremum is over bounded continuous functions f.

We may compute, with respect to the product measures $\nu^N_{\rho_0(\cdot)}$ and ν_ρ, that the initial entropy $\mathcal{H}(\nu^N_{\rho_0(\cdot)}|\nu_\rho)\leq C_0N$ for some finite constant C_0 depending only on $\rho_0(\cdot)$ and g. Let $f_t^N(\eta)$ be the density of η_t under $\mathbb{P}^N$ with respect to a reference measure ν_ρ for $\rho>0$, and let $\hat f_t^N(\eta)=t^{-1}\int_0^t f_s^N(\eta)\,\mathrm{d}s$. By usual arguments (cf. Section V.2 in [7]),

$$\mathcal{H}_N\big(\hat f_t^N\big):=\mathcal{H}\big(\hat f_t^N\,\mathrm{d}\nu_\rho|\nu_\rho\big)\leq C_0N\quad\text{and}\quad \mathcal{D}_N\big(\hat f_t^N\big):=\Big\langle\sqrt{\hat f_t^N},\big(-L_N\sqrt{\hat f_t^N}\big)\Big\rangle_\rho\leq\frac{C_0}{N},$$

where $\langle u,v\rangle_\rho$ stands for the scalar product in $L^2(\nu_\rho)$, as defined in the first section.

Consequently, to prove Proposition 4.1 it is enough to show, for any finite constant C, that

$$\limsup_{\kappa\to 0}\limsup_{N\to\infty}\sup_{\substack{\mathcal{H}_N(f)\leq CN\\ \mathcal{D}_N(f)\leq C/N}}\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x\mathcal{V}_{\kappa N}(\eta)f(\eta)\,\mathrm{d}\nu_\rho=0, \tag{4.1}$$

where the supremum is with respect to ν_ρ-densities f.

We may remove from the sum in (4.1) the integers x close to the origin, say $|x|\leq 2\kappa N$, as $\mathcal{V}_{\kappa N}$ is bounded. Now, the underlying reference measure ν_ρ may be treated as homogeneous, and a standard strategy may be employed as follows.

Proposition 4.1 now follows from the two standard lemmas below. In this context, see also [1], and [7] where the same method is used to prove [1], Theorem 3.2.1 and [7], Lemma V.1.10 respectively.

Lemma 4.2 (Global 1-block estimate).

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\mathbb{E}^N\left[\int_0^T\frac{1}{N}\sum_{|x|>2\kappa N}\tau_x\mathcal{V}_k(\eta_s)\,\mathrm{d}s\right]=0.$$

The proof of Lemma 4.2 is the same as for [6], Lemma 5.2, and follows the scheme of [7], Lemma V.3.1, using that g has "sub-linear growth (SLG)." Details are omitted here.

Lemma 4.3 (Global 2-blocks estimate).

$$\limsup_{k\to\infty}\limsup_{\kappa\to 0}\limsup_{N\to\infty}\mathbb{E}^N\left[\int_0^T\frac{1}{2N\kappa+1}\sum_{|y|\leq N\kappa}\frac{1}{N}\sum_{|x|>3\kappa N}\big|\eta_s^k(x+y)-\eta_s^k(x)\big|\,\mathrm{d}s\right]=0.$$

Proof. We discuss in terms of modifications to the argument in [7], Section V.4. The first step is to cut-off high densities. We claim that

$$\limsup_{A\to\infty}\limsup_{k\to\infty}\limsup_{N\to\infty}\mathbb{E}^N\left[\int_0^T \frac{1}{N}\sum_{|x|>3\kappa N}\eta_s^k(x)\mathbf{1}\{\eta_s^k(x)>A\}\,\mathrm{d}s\right]=0.$$

To prove this assertion, we first replace the sum over x by a sum over all sites of $\mathbb{T}_N$. At this point, since the environment at time t is obtained from the system by a shift, we may replace the variable η_t by ξ_t. We need therefore to estimate

$$\mathbf{E}_{\mu^N_{\rho_0(\cdot)}}\left[\frac{\xi_0(0)}{\rho_0(0)}\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}_N}\xi_s^k(x)\mathbf{1}\{\xi_s^k(x)>A\}\,\mathrm{d}s\right].$$

Let $\bar\rho=\|\rho_0\|_{L^\infty}$, and note that $\mu_{\rho_0(\cdot)}$ is stochastically dominated by $\mu_{\bar\rho}$. By attractiveness we may replace $\mu^N_{\rho_0(\cdot)}$ by $\mu_{\bar\rho}$ in the previous expression and bound this expectation by

$$\mathbf{E}_{\mu_{\bar\rho}}\left[\frac{\xi_0(0)}{\rho_0(0)}\int_0^T \frac{1}{AN}\sum_{x\in\mathbb{T}_N}\big(\xi_s^k(x)\big)^2\,\mathrm{d}s\right].$$

By Schwarz inequality, and noting that $\mu_{\bar\rho}$ is invariant with respect to the untagged process ξ_s, the last expression is of order A^{-1}, which proves the claim.

In view of the truncation just proved and the entropy calculations presented at the beginning of this section, to prove the lemma it is enough to show that for every $A>0$,

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\sup_{\substack{\mathcal{H}_N(f)\le CN\\ \mathcal{D}_N(f)\le C/N}}\int\frac{1}{2N\kappa+1}\sum_{|y|\le N\kappa}\frac{1}{N}\sum_{|x|>3\kappa N}W^{k,A}_{x,y}(\eta)f(\eta)\,\mathrm{d}\nu_\rho=0,$$

where

$$W^{k,A}_{x,y}(\eta)=\big|\eta^k(x+y)-\eta^k(x)\big|\mathbf{1}\big\{\max\{\eta^k(x),\eta^k(x+y)\}\le A\big\}.$$

The argument is now the same as in the proof of Lemma 5.3 in [6] following [7], Section V.5. □

5. "Local" one-block estimate

We now detail a "local" one-block limit. Let $h:\mathbb{N}_0\to\mathbb{R}$ be a bounded, Lipschitz function, and $H(a)=E_{\nu_a}[h(\eta(0))]$. Define also

$$V_l(\eta)=h\big(\eta(0)\big)-H\big(\eta^l(0)\big).$$

Lemma 5.1 (One-block estimate). *For every* $0\le t\le T$,

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\mathbb{E}^N\left[\left|\int_0^t V_l(\eta_s)\,\mathrm{d}s\right|\right]=0.$$

Proof. The proof is in four steps.

Step 1. The first step is to introduce a truncation. Since the dynamics is not attractive, we cannot bound $\eta(0)>A$ for some constant A in a simple way. However, by considering the maximum of such quantities over the torus, we may rewrite the maximum in terms of the original system ξ_s, which is attractive:

$$\max_{x\in\mathbb{T}_N}\eta_s(x)=\max_{x\in\mathbb{T}_N}\xi_s(x).$$

Also, by simple estimates, recalling $\bar\rho = \|\rho_0\|_L^\infty$, we have that

$$\mathbf{P}_{\nu_{\rho_0(\cdot)}}\Big[\max_x \xi_s(x) \ge C\log N\Big] = \mathbf{E}_{\mu_{\rho_0(\cdot)}}\bigg[\frac{\xi_0(0)}{\rho_0(0)}\mathbf{1}\Big\{\max_x \xi_s(x) \ge C\log N\Big\}\bigg]$$
$$\le \mathbf{E}_{\mu_{\bar\rho}}\bigg[\frac{\xi_0(0)}{\rho_0(0)}\mathbf{1}\Big\{\max_x \xi_s(x) \ge C\log N\Big\}\bigg].$$

Under the stationary measure $\mu_{\bar\rho}$, the variables $\xi_s(x)$ are independent and identically distributed, with finite exponential moments of some order. Hence, by Chebychev's inequality, the last expression vanishes as $N\uparrow\infty$ for a well chosen constant $C = C_1$. Therefore, as $\eta^l(0) \le \max_x \eta(x)$, it is enough to estimate

$$\mathbb{E}^N\bigg[\bigg|\int_0^t V_l(\eta_s)\mathbf{1}\{G_{N,l}\}(\eta_s)\,\mathrm{d}s\bigg|\bigg],$$

where $G_{N,l} = \{\eta\colon\ \eta^l(0) \le C_1\log N\}$.

Step 2. Since the initial entropy $\mathcal{H}(\nu^N_{\rho_0(\cdot)}|\nu_\rho)$ is bounded by $C_0 N$, by the entropy inequality,

$$\mathbb{E}^N\bigg[\bigg|\int_0^t V_l(\eta_s)\mathbf{1}\{G_{N,l}\}(\eta_s)\,\mathrm{d}s\bigg|\bigg]$$
$$\le \frac{C_0}{\gamma} + \frac{1}{\gamma N}\log\mathbb{E}_{\nu_\rho}\bigg[\exp\bigg\{\gamma N\bigg|\int_0^t V_l(\eta_s)\mathbf{1}\{G_{N,l}\}(\eta_s)\,\mathrm{d}s\bigg|\bigg\}\bigg].$$

We can get rid of the absolute value in the previous integral, using the inequality $\mathrm{e}^{|x|} \le \mathrm{e}^x + \mathrm{e}^{-x}$. By the Feynman–Kac formula, the second term on the right hand side is bounded by $(\gamma N)^{-1}T\lambda_{N,l}$, where $\lambda_{N,l}$ is the largest eigenvalue of $N^2 L_N + \gamma N V_l \mathbf{1}\{G_{N,l}\}$. Therefore, to prove the lemma, it is enough to show that $(\gamma N)^{-1}\lambda_{N,l}$ vanishes, as $N\uparrow\infty$ and then $l\uparrow\infty$, for every $\gamma > 0$.

Step 3. By the variational formula for $\lambda_{N,l}$,

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f\big\{\big\langle V_l\mathbf{1}\{G_{N,l}\}, f^2\big\rangle_\rho - \gamma^{-1}N\big\langle f, (-L_N f)\big\rangle_\rho\big\}, \tag{5.1}$$

where the supremum is carried over all densities f^2 with respect to ν_ρ. As the Dirichlet forms satisfy $\langle f, (-L^{\mathrm{env}}_{\Lambda_l} f)\rangle_\rho \le \langle f, (-L_N f)\rangle_\rho$ (cf. [17], equation (3.1)), we may bound the previous expression by a similar one where L_N is replaced by $L^{\mathrm{env}}_{\Lambda_l}$.

Denote by $\hat f_l^2$ the conditional expectation of f^2 given $\{\eta(z)\colon\ z\in\Lambda_l\}$. Since $V_l\mathbf{1}\{G_{N,l}\}$ depends on the configuration η only through $\{\eta(z)\colon\ z\in\Lambda_l\}$ and since the Dirichlet form is convex, the expression inside braces in (5.1) is less than or equal to

$$\int V_l\mathbf{1}\{G_{N,l}\}\hat f_l^2\,\mathrm{d}\nu_\rho^{\Lambda_l} - \gamma^{-1}N\int \hat f_l\big(-L^{\mathrm{env}}_{\Lambda_l}\hat f_l\big)\,\mathrm{d}\nu_\rho^{\Lambda_l}. \tag{5.2}$$

The first term in this formula, decomposing in terms of canonical measures $\nu_{\Lambda_l,j}$, is equal to

$$\sum_{j=1}^{C_1 l\log N} c_{l,j}(f)\int V_l\mathbf{1}\{G_{N,l}\}\hat f_{l,j}^2\,\mathrm{d}\nu_{\Lambda_l,j},$$

where the value of the constant C_1 changed and

$$c_{l,j}(f) = \int_{\Sigma_{\Lambda_l,j}}\hat f_l^2\,\mathrm{d}\nu_\rho^{\Lambda_l},\qquad \hat f_{l,j}^2(\eta) = c_{l,j}(f)^{-1}\nu_\rho^{\Lambda_l}(\Sigma_{\Lambda_l,j})\hat f_l^2(\eta).$$

The sum starts at $j=1$ because there is always a particle at the origin. Note also that $\sum_{j\ge1}c_{l,j}(f) = 1$ and that $\hat f_{l,j}^2(\cdot)$ is a density with respect to $\nu_{\Lambda_l,j}$.

Also, the Dirichlet form term of (5.2) can be written as

$$\gamma^{-1}N\sum_{1\le j\le C_1 l\log N} c_{l,j}(f)\int \hat f_{l,j}\big(-L^{\mathrm{env}}_{\Lambda_l}\hat f_{l,j}\big)\,\mathrm{d}\nu_{\Lambda_l,j}.$$

In view of this decomposition, (5.1) is bounded above by

$$\sup_{1\le j\le C_1 l\log N}\sup_f\bigg\{\int V_l f^2\,\mathrm{d}\nu_{\Lambda_l,j}-\gamma^{-1}N\int f\big(-L^{\mathrm{env}}_{\Lambda_l}f\big)\,\mathrm{d}\nu_{\Lambda_l,j}\bigg\},$$

where the second supremum is over all densities f^2 with respect to $\nu_{\Lambda_l,j}$.

Step 4. Recall that $V_l(\eta)=h(\eta(0))-H(\eta^l(0))$. Let

$$V_{l,j}(\eta)=V_l-E_{\nu_{\Lambda_l,j}}[V_l].$$

By (3.2), $N^{-1}\max_{1\le j\le C_1 l\log N}W^{\mathrm{env}}(l,j)$ vanishes as $N\uparrow\infty$. Then, as h is bounded, by Rayleigh expansion [7], Theorem A3.1.1, for $j\le C_1 l\log N$ and sufficiently large N,

$$\begin{aligned}
&\int V_l f^2\,\mathrm{d}\nu_{\Lambda_l,j}-\gamma^{-1}N\int f\big(-L^{\mathrm{env}}_{\Lambda_l}f\big)\,\mathrm{d}\nu_{\Lambda_l,j}\\
&\quad\le\int V_l\,\mathrm{d}\nu_{\Lambda_l,j}+\frac{\gamma N^{-1}}{1-2\|V_l\|_{L^\infty}W^{\mathrm{env}}(l,j)\gamma N^{-1}}\int V_{l,j}\big(-L^{\mathrm{env}}_{\Lambda_l}\big)^{-1}V_{l,j}\,\mathrm{d}\nu_{\Lambda_l,j}\\
&\quad\le\int V_l\,\mathrm{d}\nu_{\Lambda_l,j}+2\gamma N^{-1}\int V_{l,j}\big(-L^{\mathrm{env}}_{\Lambda_l}\big)^{-1}V_{l,j}\,\mathrm{d}\nu_{\Lambda_l,j}.
\end{aligned}$$

The second term is bounded as follows. By the spectral theorem, the second term is less than or equal to

$$2W^{\mathrm{env}}(l,j)\gamma N^{-1}\int V_{l,j}^2\,\mathrm{d}\nu_{\Lambda_l,j}\le 8\|h\|^2_{L^\infty}W^{\mathrm{env}}(l,j)\gamma N^{-1}. \tag{5.3}$$

This expression vanishes as $N\uparrow\infty$ in view of (3.2).

On the other hand, the first term is written as

$$\int V_l\,\mathrm{d}\nu_{\Lambda_l,j}=\int h\big(\eta(0)\big)\,\mathrm{d}\nu_{\Lambda_l,j}-H(j/2l+1).$$

By Lemma 5.2 below, this difference vanishes uniformly in j as $l\uparrow\infty$. This proves that (5.1) vanishes as $N\uparrow\infty$ and then $l\uparrow\infty$, finishing the proof. □

Lemma 5.2. *Let $h:\mathbb{N}_0\to\mathbb{R}$ be a bounded Lipschitz function which vanishes at infinity. Then, we have*

$$\limsup_{l\to\infty}\sup_{k\ge1}\big|E_{\nu_{\Lambda_l,k}}\big[h\big(\eta(0)\big)\big]-E_{\nu_{k/|\Lambda_l|}}\big[h\big(\eta(0)\big)\big]\big|=0.$$

Proof. The argument is in three steps.

Step 1. We first consider the case $1\le k\le K_0$. By adding and subtracting $h(1)$, we need only to estimate

$$\big|E_{\nu_{\Lambda_l,k}}\big[h\big(\eta(0)\big)\big]-h(1)\big|\quad\text{and}\quad\big|E_{\nu_{k/|\Lambda_l|}}\big[h\big(\eta(0)\big)\big]-h(1)\big|. \tag{5.4}$$

The first term is bounded by $2\|h\|_{L^\infty}\nu_{\Lambda_l,k}\{\eta(0)\ge2\}$. To show that it vanishes as $l\uparrow\infty$, note that $\eta(0)\le k$ and that $E_{\mu_{\Lambda_l,k}}[\eta(0)]=k/(2l+1)$ to write

$$\begin{aligned}
\nu_{\Lambda_l,k}\big\{\eta(0)\ge2\big\}&=\frac{1}{E_{\mu_{\Lambda_l,k}}[\eta(0)]}E_{\mu_{\Lambda_l,k}}\big[\eta(0)\mathbf{1}\big\{\eta(0)\ge2\big\}\big]\\
&\le(2l+1)\mu_{\Lambda_l,k}\big\{\eta(0)\ge2\big\}.
\end{aligned}$$

For $2 \le s \le k$, we may write the canonical measure in terms of the grand canonical:

$$\mu_{\Lambda_l,k}\{\eta(0)=s\} = \mu_\rho\{\eta(0)=s\}\frac{\mu_\rho\{\sum_{0<|x|\le l}\eta(x)=k-s\}}{\mu_\rho\{\sum_{|x|\le l}\eta(x)=k\}}$$

for any choice of the parameter ρ. Recall $\mu^1_{\Lambda_l,j}$ is the canonical measure when $g(k)=\mathbf{1}\{k\ge 1\}$. In the numerator and the denominator, at least $2\ell-k$ sites receive no particles. We may therefore replace in these sites the rate g by the rate constant equal to one with no cost. Since $a_0 \le g(\ell) \le a_1\ell$, in the remaining sites we have that $C(k)^{-1} \le a_0^n \le g(n)! \le a_1^n n! \le C(k)$ if $n \le k$. The previous expression is thus bounded above by

$$C(k)\mu^1_{\Lambda_l,k}\{\eta(0)=s\} = C(k)\binom{2l}{k-s}\bigg/\binom{2l+1}{k} = \mathrm{O}(l^{-s}).$$

To bound the second term in (5.4), we proceed in a similar way. The absolute value of the difference $E_{\nu_\rho}[h(\eta(0))] - h(1)$ is bounded by $2\|h\|_\infty\nu_\rho\{\eta(0)\ge 2\}$. Last probability is equal to $\rho^{-1}E_{\mu_\rho}[\eta(0)\mathbf{1}\{\eta(0)\ge 2\}]$. Since $g(n)\ge a_0$, change of variables $\eta' = \eta - 2\mathfrak{d}_0$ permits to bound the previous expression by $C_0\varphi(\rho)^2[\rho+2]/\rho$ for some finite constant C_0. Since $g(n)\le a_1 n$, $\varphi(\rho)\le a_1\rho$. In conclusion, the second term in (5.4) is bounded above by $C_0\|h\|_\infty(k/l)^2$, which concludes the proof of Step 1.

Step 2. Next, we consider the case in which $K_0 \le k \le B|\Lambda_l|$ for some $B<\infty$. By definition of the Palm measure, the difference $E_{\nu_{\Lambda_l,k}}[h(\eta(0))] - E_{\nu_{k/|\Lambda_l|}}[h(\eta(0))]$ is equal to

$$\frac{|\Lambda_l|}{k}\big\{E_{\mu_{\Lambda_l,k}}\big[\eta(0)h\big(\eta(0)\big)\big] - E_{\mu_{k/|\Lambda_l|}}\big[\eta(0)h\big(\eta(0)\big)\big]\big\}.$$

By [7], Corollary 1.7, Appendix 2.1, this expression is bounded above by C_0k^{-1} for some finite constant C_0. This expression can be made as small as need by choosing K_0 large.

Step 3. Finally, we consider the case $k\ge B|\Lambda_l|$. We shall take advantage of the fact that h vanishes at infinity. Fix $A>0$ to bound $E_{\nu_{\Lambda_l,k}}[h(\eta(0))]$ by

$$E_{\nu_{\Lambda_l,k}}\big[h\big(\eta(0)\big)\mathbf{1}\big\{\eta(0)\le A\big\}\big] + \sup_{x\ge A} h(x).$$

By definition of the Palm measure and since the density $k/|\Lambda_l|$ is bounded below by B, the first term is less than or equal to

$$\frac{\|h\|_{L^\infty}|\Lambda_l|}{k}E_{\mu_{\Lambda_l,k}}\big[\eta(0)\mathbf{1}\big\{\eta(0)\le A\big\}\big] \le \frac{A\|h\|_{L^\infty}}{B}.$$

In view of the previous estimates, we see that the expectation $E_{\nu_{\Lambda_l,k}}[h(\eta(0))]$ can be made arbitrarily small by choosing A and B sufficiently large. The expectation $E_{\nu_{k/|\Lambda_l|}}[h(\eta(0)]$ can be estimated similarly. □

6. Local two-blocks estimate

In this section we show how to go from a box of size l to a box of size εN.

Lemma 6.1 (Two-blocks estimate). *Let $H:\mathbb{R}_+\to\mathbb{R}$ be a bounded, Lipschitz function, which vanishes at infinity,* $\lim_{x\to\infty}H(x)=0$. *Then, for every $t>0$,*

$$\limsup_{l\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N\left[\left|\int_0^t\left\{H\big(\eta_s^l(0)\big) - \frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N}H\big(\eta_s^l(x)\big)\right\}\mathrm{d}s\right|\right] = 0. \tag{6.1}$$

Proof. The proof is handled in several steps.

Step 1. As H is bounded, the expectation in (6.1) is bounded by

$$\frac{1}{\varepsilon N}\sum_{x=4l+2}^{\varepsilon N}\mathbb{E}^N\left[\left|\int_0^t\left\{H\left(\eta_s^l(0)\right)-H\left(\eta_s^l(x)\right)\right\}\mathrm{d}s\right|\right]+\frac{C_0\|H\|_{L^\infty}l}{\varepsilon N}$$

for some finite constant C_0. Hence, we need to estimate, uniformly over $4l+2\le x\le\varepsilon N$,

$$\mathbb{E}^N\left[\left|\int_0^t\left\{H\left(\eta_s^l(0)\right)-H\left(\eta_s^l(x)\right)\right\}\mathrm{d}s\right|\right].$$

Step 2. Write

$$H\left(\eta^l(0)\right)-H\left(\eta^l(x)\right)=H\left(\eta^l(0)\right)-H\left(\eta^l(2l+1)\right)+H\left(\eta^l(2l+1)\right)-H\left(\eta^l(x)\right).$$

We now claim that

$$\lim_{l\to\infty}\lim_{N\to\infty}E_N\left[\left|\int_0^t\left\{H\left(\eta_s^l(0)\right)-H\left(\eta_s^l(2l+1)\right)\right\}\mathrm{d}s\right|\right]=0.$$

Indeed, since $H(\eta^l(0))-H(\eta^l(2l+1))$ is a function of $\hat{\Lambda}_l=\{-l,\dots,3l+1\}$, we may apply the "local 1-block" argument for Lemma 5.1 up to (5.3), with respect to $V_l'=H(\eta^l(0))-H(\eta^l(2l+1))$. Now, in the last line of the proof of Lemma 5.1, instead of using Lemma 5.2, we use Lemma 6.4 to show the expectation under the canonical measure $\nu_{\hat{\Lambda}_l,k}$ vanishes, $\lim_{l\uparrow\infty}\sup_{k\ge1}E_{\hat{\Lambda}_l,k}[V_l']=0$.

Step 3. Therefore, we need only estimate when the integrand is $H(\eta^l(2l+1))-H(\eta^l(x))$. As for the "local 1-block" development (Lemma 5.1), we may introduce a truncation, and restrict to the set $G_{N,l,x}=\{\eta\colon\ \eta^l(2l+1)+\eta^l(x)\le 2C_1\log N\}$. That is, we need only bound, uniformly over x

$$\mathbb{E}^N\left[\left|\int_0^t\left[H\left(\eta_s^l(2l+1)\right)-H\left(\eta_s^l(x)\right)\right]\mathbf{1}\{G_{N,l,x}\}\,\mathrm{d}s\right|\right].$$

Step 4. Following the first part of the proof of Lemma 5.1, appealing to entropy estimates and eigenvalue estimates, we need only to bound, uniformly in $4l+2\le x\le\varepsilon N$,

$$\sup_f\left\{\left\langle\left[H\left(\eta^l(2l+1)\right)-H\left(\eta^l(x)\right)\right]\mathbf{1}\{G_{N,l,x}\},f^2\right\rangle_\rho-N\gamma^{-1}\left\langle f,(-L_Nf)\right\rangle_\rho\right\},\tag{6.2}$$

where the supremum is over all density functions f^2 with $\int f^2\,\mathrm{d}\nu_\rho=1$.

Since $V_{l,x}(\eta)=H(\eta^l(2l+1))-H(\eta^l(x))$ does not involve the origin, we can avoid details involving the inhomogeneity at point 0 in the following. Define disjoint blocks $\Lambda_l'=\{l+1,\dots,3l+1\}$ and $\Lambda_l(x)=\{x-l,\dots,x+l\}$. Let $L_{\Lambda_{l,x}}$ be the restriction of L_N^{env} to the set $\Lambda_{l,x}=\Lambda_l'\cup\Lambda_l(x)$, and define also $L_{l,x}$ by

$$\begin{aligned}L_{l,x}f(\eta)&=\frac12 g\left(\eta(x-l)\right)\left[f\left(\eta^{x-l,3l+1}\right)-f(\eta)\right]\\&\quad+\frac12 g\left(\eta(3l+1)\right)\left[f\left(\eta^{3l+1,x-l}\right)-f(\eta)\right].\end{aligned}$$

The operator $L_{l,x}$ corresponds to zero-range dynamics where particles jump between endpoints $3l+1$ and $x-l$.

As $x\le\varepsilon N$, by adding and subtracting at most εN terms (cf. [7], pp. 94–95 and [17], equation (3.1)), we have that

$$\left\langle f,(-L_{l,x}f)\right\rangle_\rho\le\varepsilon N\left\langle f,\left(-L_N^{\mathrm{env}}f\right)\right\rangle_\rho.$$

Hence,

$$\left\langle f,-\left(N\gamma^{-1}L_{\Lambda_{l,x}}+\varepsilon^{-1}\gamma^{-1}L_{l,x}\right)f\right\rangle_\rho\le 2N\gamma^{-1}\left\langle f,(-L_Nf)\right\rangle_\rho,$$

and we may replace $N\gamma^{-1}L_N$ in (6.2) by $(1/2)(N\gamma^{-1}L_{\Lambda_{l,x}} + \varepsilon^{-1}\gamma^{-1}L_{l,x})$.

Step 5. To simplify notation, we shift the indices so that the blocks are to the left and right of the origin. In particular, let $\Lambda_l^- = \{-(2l+1), \dots, -1\}$, $\Lambda_l^+ = \{1, \dots, (2l+1)\}$ and $\Lambda_l^* = \Lambda_l^- \cup \Lambda_l^+$. Configurations of $\mathbb{N}_0^{\Lambda_l^-}$ will be denoted by the Greek letter η, while configurations of $\mathbb{N}_0^{\Lambda_l^+}$ are denoted by the Greek letter ζ. Recall $\mathfrak{d}_z$ stands for the configuration with no particles but one at z.

Consider the generator $L_{N,\kappa,l}$ with respect to $\mathbb{N}_0^{\Lambda_l^*}$, $L_{N,\kappa,l} = NL_l^- + NL_l^+ + \kappa^{-1}L_l^0$. Here,

$$\begin{aligned}
\big(L_l^- f\big)(\eta,\zeta) &= \sum_{x,y\in\Lambda_l^-} p(y-x)g\big(\eta(x)\big)\big[f\big(\eta^{x,y},\zeta\big) - f(\eta,\zeta)\big],\\
\big(L_l^+ f\big)(\eta,\zeta) &= \sum_{x,y\in\Lambda_l^+} p(y-x)g\big(\zeta(x)\big)\big[f\big(\eta,\zeta^{x,y}\big) - f(\eta,\zeta)\big],\\
\big(L_l^0 f\big)(\eta,\zeta) &= (1/2)g\big(\eta(-1)\big)\big[f(\eta-\mathfrak{d}_{-1},\zeta+\mathfrak{d}_1) - f(\eta,\zeta)\big]\\
&\quad + (1/2)g\big(\zeta(1)\big)\big[f(\eta+\mathfrak{d}_{-1},\zeta-\mathfrak{d}_1) - f(\eta,\zeta)\big].
\end{aligned}$$

Note that inside each set $\Lambda_l^\pm$ particles jump at rate N while jumps between sets are performed at rate κ^{-1}.

Recall $\mu_\rho^{\Lambda_l^-}$, $\mu_\rho^{\Lambda_l^+}$, $\mu_\rho^{\Lambda_l^*}$ are the restrictions of μ_ρ to $\mathbb{N}_0^{\Lambda_l^-}$, $\mathbb{N}_0^{\Lambda_l^+}$, $\mathbb{N}_0^{\Lambda_l^*}$, respectively. The Dirichlet forms associated to the generators L_l^-, L_l^+, L_l^0 are given by

$$\begin{aligned}
&D_{\Lambda_l^-}\big(\mu_\rho^{\Lambda_l^*}, f\big) = \big\langle f, \big(-L_l^- f\big)\big\rangle_{\mu_\rho^{\Lambda_l^*}}, \qquad D_{\Lambda_l^+}\big(\mu_\rho^{\Lambda_l^*}, f\big) = \big\langle f, \big(-L_l^+ f\big)\big\rangle_{\mu_\rho^{\Lambda_l^*}},\\
&D_0\big(\mu_\rho^{\Lambda_l^*}, f\big) = \big\langle f, \big(-L_l^0 f\big)\big\rangle_{\mu_\rho^{\Lambda_l^*}}.
\end{aligned}\tag{6.3}$$

A simple computation shows that the Dirichlet form can be written as

$$\begin{aligned}
D_{\Lambda_l^-}\big(\mu_\rho^{\Lambda_l^*}, f\big) &= \frac{\varphi(\rho)}{2}\sum_{x=-(2l+1)}^{-2}\int\big\{f(\eta+\mathfrak{d}_{x+1},\zeta) - f(\eta+\mathfrak{d}_x,\zeta)\big\}^2\mu_\rho^{\Lambda_l^*}(\mathrm{d}\eta,\mathrm{d}\zeta),\\
D_0\big(\mu_\rho^{\Lambda_l^*}, f\big) &= \frac{\varphi(\rho)}{2}\int\big\{f(\eta+\mathfrak{d}_{-1},\zeta) - f(\eta,\zeta+\mathfrak{d}_1)\big\}^2\mu_\rho^{\Lambda_l^*}(\mathrm{d}\eta,\mathrm{d}\zeta).
\end{aligned}$$

In this notation, it will be enough, with respect to Eq. (6.2), to bound for $a > 0$ the quantity

$$\sup_f\big\{\big\langle\big[H\big(\eta^l\big) - H\big(\zeta^l\big)\big]\mathbf{1}\big\{G'_{N,l}\big\}, f^2\big\rangle_\rho - a\big\langle f, (-L_{N,\kappa,l}f)\big\rangle_\rho\big\},\tag{6.4}$$

where $\eta^l = (2l+1)^{-1}\sum_{x\in\Lambda_l^-}\eta(x)$, $\zeta^l = (2l+1)^{-1}\sum_{x\in\Lambda_l^+}\zeta(x)$, and $G'_{N,l} = \{(\eta,\zeta)\colon \eta^l + \zeta^l \le 2C_1\log N\}$. By convexity of the Dirichlet form, as in the proof of Lemma 5.1, the supremum may be taken over functions f on $\mathbb{N}_0^{\Lambda_l^*}$ such that $\langle f^2\rangle_{\mu_\rho^{\Lambda_l^*}} = 1$, and the measure μ_ρ in (6.4) may be replaced by $\mu_\rho^{\Lambda_l^*}$.

Step 6. This quantity is estimated in three parts. The first part restricts to the set $S_{N,l}^1 = \{(\eta,\zeta)\colon \eta^l + \zeta^l \le B\}$ for some B fixed. In this case, where we have truncated at a fixed level B, we can use the “local 1-block” method of Lemma 5.1 to show that

$$\sup_f\big\{\big\langle\big[H\big(\eta^l\big) - H\big(\zeta^l\big)\big]\mathbf{1}\big\{S_{N,l}^1\big\}, f^2\big\rangle_{\mu_\rho^{\Lambda_l^*}} - a\big\langle f, (-L_{N,\kappa,l}f)\big\rangle_{\mu_\rho^{\Lambda_l^*}}\big\}$$

vanishes as $N\uparrow\infty$, $\varepsilon\downarrow 0$ and then $l\uparrow\infty$.

Indeed, by convexity considerations, we can decompose the expression in braces in terms of canonical measures $\mu_{\Lambda_l^*,k}$ concentrating on k particles in Λ_l^*. Since for the generator $L_{N,\kappa,l}$ jumps are speeded up by N inside each cube,

$$\begin{aligned}&\langle [H(\eta^l) - H(\zeta^l)]\mathbf{1}\{S^1_{N,l}\}, f^2\rangle_{\mu_{\Lambda_l^*,k}} - a\langle f, (-L_{N,\kappa,l} f)\rangle_{\mu_{\Lambda_l^*,k}} \\ &\quad \le \langle [H(\eta^l) - H(\zeta^l)]\mathbf{1}\{S^1_{N,l}\}, f^2\rangle_{\mu_{\Lambda_l^*,k}} - a\varepsilon^{-1}\langle f, -(L_l^- + L_l^+ + L_l^0 f)\rangle_{\mu_{\Lambda_l^*,k}}.\end{aligned}$$

Let $\tilde V_l = [H(\eta^l) - H(\zeta^l)]\mathbf{1}\{S^1_{N,l}\}$. Note that $\tilde V_l$ has mean-zero with respect to $\mu_{\Lambda_l^*,k}$ and that $\|\tilde V_l\|_{L^\infty} \le 2\|H\|_{L^\infty}$. By the Rayleigh estimate [7], Theorem A3.1.1 and by the spectral gap, for $k \le 2(2l+1)B$, the previous expression is bounded above by

$$\begin{aligned}&\frac{a^{-1}\varepsilon}{1 - 4\|H\|_{L^\infty} W^*(l,k) a^{-1}\varepsilon} \int \tilde V_l (-L_l^- - L_l^+ - L_l^0)^{-1} \tilde V_l \, d\mu_{\Lambda_l^*,k} \\ &\quad \le 2a^{-1}\varepsilon W^*(l,k) \int \tilde V_l^2 \, d\mu_{\Lambda_l^*,k},\end{aligned}$$

where $W^*(l,k)$ is the inverse of the spectral gap of $L^- + L^+ + L_l^0$ with respect to the process on Λ_l^* with k particles. As $\varepsilon \downarrow 0$, the previous expression vanishes.

Step 7. The second part now restricts to $S^2_{N,l} = \{(\eta,\zeta)\colon \zeta^l \ge A, \eta^l \ge A\}$ for some constant A. On this event, the sum $H(\eta^l) + H(\zeta^l)$ is absolutely bounded by $2\sup_{z\ge A}|H(z)|$ so that

$$\langle [H(\eta^l) - H(\zeta^l)]\mathbf{1}\{S^2_{N,l}\}, f^2\rangle_{\mu_\rho^{\Lambda_l^*}} - a\langle f, (-L_{N,\kappa,l} f)\rangle_{\mu_\rho^{\Lambda_l^*}} \le 2\sup_{z\ge A}|H(z)|.$$

Since $H(n)$ vanishes as $n \uparrow \infty$, the right hand side can be made arbitrarily small.

Step 8. Let now $S^3_{N,l} = A_l \cap R_{N,l}$ where $A_l = \{\eta\colon \eta^l \le A\}$ and $R_{N,l} = \{(\eta,\zeta)\colon B \le \eta^l + \zeta^l \le 2C_1 \log N\}$. This case is the difficult part of the proof and is treated in Lemma 6.2 below. □

The proof of Lemma 6.2 is reserved to the next subsection.

Lemma 6.2. *Suppose that $B > 4A$. Then, for every $a > 0$,*

$$\lim_{l\to\infty} \limsup_{\kappa\to 0} \limsup_{N\to\infty} \sup_f \left\{ \int \mathbf{1}\{R_{N,l}\}\mathbf{1}\{A_l\} f(\eta,\zeta)^2 \, d\mu_\rho^{\Lambda_l^*} - a\langle f, (-L_{N,\kappa,l} f)\rangle_{\mu_{\Lambda_l^*,\rho}} \right\} \le 0, \tag{6.5}$$

where the supremum over f is over functions f on $\mathbb{N}_0^{\Lambda_l^*}$ such that $\langle f, f\rangle_{\mu_\rho^{\Lambda_l^*}} = 1$.

Lemma 6.3. *For $s, r \ge 0$, we have $\mu_{\Lambda_s,r} \ll \mu_{\Lambda_s,r+1}$, and $\nu_{\Lambda_s,r} \ll \nu_{\Lambda_s,r+1}$.*

Proof. The first estimate is [10], Lemma 4.4. The second bound has a similar argument: Note $\nu_{\Lambda_s,r}$ is the unique invariant measure for the Markov chain on $\Sigma^*_{\Lambda_s,r} = \{\eta\colon \eta(0) \ge 1, \sum_{|x|\le s} \eta(x) = r\}$ generated by $L^{\mathrm{env}}_{\Lambda_s}$.

Since g is increasing, we can couple two systems starting from configurations $\eta^1 \in \Sigma^*_{\Lambda_s,r}$ and $\eta^2 \in \Sigma^*_{\Lambda_s,r+1}$ such that $\eta^1 \le \eta^2$ coordinatewise, so that the ordering is preserved at later times. Hence, in the limit we obtain $\lim_{t\uparrow\infty} \eta^1_t = \nu_{\Lambda_s,r}$, $\lim_{t\uparrow\infty} \eta^2_t = \nu_{\Lambda_s,r+1}$, and $\nu_{\Lambda_s,r} \ll \nu_{\Lambda_s,r+1}$. □

Recall the set $\hat\Lambda_l = \{-l, \dots, 3l+1\}$.

Lemma 6.4. *Let $H\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a nonnegative, bounded, Lipschitz function which vanishes at infinity. Then, we have*

$$\limsup_{l\to\infty} \sup_{k\ge 0} \left| E_{\nu_{\hat\Lambda_l,k}}\left[H(\eta^l(0)) - H(\eta^l(2l+1))\right]\right| = 0.$$

Proof. The argument is in three parts.

Step 1. Fix $\varepsilon > 0$ and consider (k, l) such that $k/|\hat{\Lambda}_l| \le \varepsilon$. Add and subtract $H(0)$ in the absolute value. Then, the expectation is less than $2\max_{0\le x\le 2\varepsilon}|H(x) - H(0)| = \mathrm{O}(\varepsilon)$ given that H is Lipschitz.

Step 2. Assume now that $\varepsilon \le k/|\hat{\Lambda}_l| \le B_1$. The proof is the same as in Lemma 6.6 in [6] for this case. For the convenience of the reader, we give it here. By definition of $\nu_{\hat{\Lambda}_l,k}$, the expectation appearing in the display of the lemma equals

$$\frac{1}{E_{\mu_{\hat{\Lambda}_l,k}}[\eta(0)]} E_{\mu_{\hat{\Lambda}_l,k}}\big[\eta(0)\big\{H\big(\eta^l(0)\big) - H\big(\eta^l(2l+1)\big)\big\}\big].$$

Since the measure is space homogeneous, the denominator is equal to $\rho_{l,k} = k/|\hat{\Lambda}_l|$ which is bounded below by ε. In the numerator, $\eta(0)$ can be replaced by $\eta^l(0)$. The numerator is then

$$E_{\mu_{\hat{\Lambda}_l,k}}\big[\big\{\eta^l(0) - \rho_{l,k}\big\}\big\{H\big(\eta^l(0)\big) - H\big(\eta^l(2l+1)\big)\big\}\big] + \rho_{l,k} E_{\mu_{\hat{\Lambda}_l,k}}\big[H\big(\eta^l(0)\big) - H\big(\eta^l(2l+1)\big)\big].$$

The second term vanishes because the measure $\mu_{\hat{\Lambda}_l,k}$ is space homogeneous. The first term, as H is bounded, is absolutely dominated by $2\|H\|_{L^\infty} E_{\mu_{\hat{\Lambda}_l,k}}[|\eta^l(0) - \rho_{l,k}|]$. By [7], Appendix II.1, Corollary 1.4, this expression is less than or equal to

$$C_0 E_{\mu^{\hat{\Lambda}_l}_{\rho_{l,k}}}\big[\big|\eta^l(0) - \rho_{l,k}\big|\big] \le C_0\sigma(\rho_{l,k}) l^{-1/2}$$

for some constant C_0 where $\sigma(\rho)$ stands for the variance of $\xi(0)$ under μ_ρ. Since $\varepsilon \le \rho_{l,k} \le B_1$, $\sigma(\rho_{l,k})$ is bounded. Hence, this expression vanishes as $l \uparrow \infty$.

Step 3. Suppose now $k/|\hat{\Lambda}_l| \ge B_1$. We shall prove that in this range both expectations are small because $H(x)$ vanishes as $x \uparrow \infty$. Fix $A > 0$. Introducing the indicator of the set $\eta^l(0) \le A$ and replacing the Palm measure $\nu_{\hat{\Lambda}_l,k}$ by the homogeneous measure $\mu_{\hat{\Lambda}_l,k}$, we get that

$$\begin{aligned} E_{\nu_{\hat{\Lambda}_l,k}}\big[H\big(\eta^l(0)\big)\big] &\le E_{\nu_{\hat{\Lambda}_l,k}}\big[H\big(\eta^l(0)\big)\mathbf{1}\big\{\eta^l(0) \le A\big\}\big] + \sup_{x\ge A} H(x) \\ &= \frac{1}{\rho_{l,k}} E_{\mu_{\hat{\Lambda}_l,k}}\big[\eta(0)H\big(\eta^l(0)\big)\mathbf{1}\big\{\eta^l(0) \le A\big\}\big] + \sup_{x\ge A} H(x) \end{aligned}$$

because $E_{\mu_{\hat{\Lambda}_l,k}}[\eta(0)] = \rho_{l,k}$. In the last expectation, we may replace $\eta(0)$ by $\eta^l(0)$ which is bounded by A. We may also estimate H by $\|H\|_{L^\infty}$ and bound below the density $\rho_{l,k}$ by B_1. The previous expression is thus less than or equal to

$$\frac{A\|H\|_{L^\infty}}{B_1} + \sup_{x\ge A} H(x),$$

which can be made arbitrarily small if A is chosen large enough and then B_1.

It remains to prove that the second expectation appearing in the statement of the lemma is small in this range of densities. Introducing the indicator of the set $\{\eta^l(2l+1) \le A\}$ we get that

$$E_{\nu_{\hat{\Lambda}_l,k}}\big[H\big(\eta^l(2l+1)\big)\big] \le \|H\|_{L^\infty}\nu_{\hat{\Lambda}_l,k}\big\{\eta^l(2l+1) \le A\big\} + \sup_{x\ge A} H(x).$$

Since the event $\{\eta^l(2l+1) \le A\}$ is decreasing and $k \ge B_1|\hat{\Lambda}_l|$, by Lemma 6.4, we may bound the previous probability by $\nu_{\hat{\Lambda}_l,K}\{\eta^l(2l+1) \le A\}$, where $K = B_1|\hat{\Lambda}_l|$. At this point, by the same reasons argued above, we obtain that

$$E_{\nu_{\hat{\Lambda}_l,k}}\big[H\big(\eta^l(2l+1)\big)\big] \le \frac{\|H\|_{L^\infty}}{B_1} E_{\mu_{\hat{\Lambda}_l,K}}\big[\eta^l(0)\mathbf{1}\big\{\eta^l(2l+1) \le A\big\}\big] + \sup_{x\ge A} H(x).$$

Note that $\eta^l(0) \le B_1|\hat{\Lambda}_l|/(2l+1) = 2B_1$. Hence, by [7], Corollary 1.4, Appendix 2.1, the previous expectation is bounded by

$$2\|H\|_{L^\infty}\mu_{B_1}\{\eta^l(2l+1) \le A\} + \frac{C_0}{l}$$

for some finite constant C_0. This expression vanishes as $l \uparrow \infty$ by the law of large numbers provided $B_1 > A$. This concludes the proof of the lemma. □

6.1. *Proof of Lemma* 6.2

Fix $B > 4A$. The proof is divided in three steps. Recall the notation developed in Step 5 of the proof of Lemma 6.1.

Step 1. The first integral in (6.5) can be rewritten as

$$\sum_{j,k} \mu_{\rho,l}(j)\mu_{\rho,l}(k) \int\int f(\eta,\zeta)^2 \mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta),$$

where the sum is performed over all indices j, k such that $0 \le j \le A(2l+1)$, $k \ge 0$, $B(2l+1) \le j+k \le \theta_{N,l} := 2C_1(2l+1)\log N$, $\mu_{\rho,l}(m) = \mu_\rho(\sum_{x\in\Lambda_l}\eta(x) = m)$ and $\mu_{\Lambda_l^\pm,m}$ is the canonical measure on the cube $\Lambda_l^\pm$ concentrated on configurations with m particles.

Fix two integers $j,k \ge 0$ such that $B(2l+1) \le j+k \le \theta_{N,l}$. We claim that there exists a function $W_N(l)$ such that $W_N(l) = \mathrm{o}(N)$ for fixed l and

$$\begin{aligned}&\int\int f(\eta,\zeta)^2\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta) - \left\{\int\int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\right\}^2\\&\quad \le W_N(l)\left\{D_{\Lambda_l^-}(\mu_{\Lambda_l^*,j,k},f) + D_{\Lambda_l^+}(\mu_{\Lambda_l^*,j,k},f)\right\},\end{aligned} \tag{6.6}$$

where $\mu_{\Lambda_l^*,j,k}$ represents the measure $\mu_{\Lambda_l^-,j}\mu_{\Lambda_l^+,k}$ and $D_{\Lambda_l^\pm}(\mu_{\Lambda_l^*,j,k},f)$ is the Dirichlet form defined in (6.3) with the canonical measure $\mu_{\Lambda_l^*,j,k}$ in place of the grand canonical measure $\mu_\rho^{\Lambda_l^*}$.

To prove the claim (6.6), recall that $W(l,j)$ is the inverse of the spectral gap of the generator of the zero range process in which j particles move on a cube of length $2l+1$. By definition of $W(l,j)$, for each configuration ζ,

$$\begin{aligned}&\int f(\eta,\zeta)^2\mu_{\Lambda_l^-,j}(\mathrm{d}\eta) - \left\{\int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\right\}^2\\&\quad \le W(l,j)\sum_{x=-l}^{l-1}\int g\big(\eta(x)\big)\{f\big(\eta^{x,x+1},\zeta\big) - f(\eta,\zeta)\}^2\mu_{\Lambda_l^-,j}(\mathrm{d}\eta).\end{aligned}$$

Integrating with respect to $\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)$ we get that

$$\begin{aligned}&\int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\int f(\eta,\zeta)^2\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\\&\quad \le \int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\left\{\int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\right\}^2 + W(l,j)D_{\Lambda_l^-}(\mu_{\Lambda_l^*,j,k},f).\end{aligned}$$

Let

$$h(\zeta) = \int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta).$$

By definition of the spectral gap,

$$\int h(\zeta)^2 \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta) - \left\{\int h(\zeta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\right\}^2 \le W(l,k) D_{\Lambda_l^+}(\mu_{\Lambda_l^+,k}, h).$$

By Schwarz inequality

$$D_{\Lambda_l^+}(\mu_{\Lambda_l^+,k}, h) \le D_{\Lambda_l^+}(\mu_{\Lambda_l^*,j,k}, f).$$

This proves (6.6), applying the estimate on the spectral gap in Lemma 3.2 when g satisfies (B), or by assumption when g satisfies (SL).

Multiplying both sides of (6.6) by $\mu_{\rho,l}(j)\mu_{\rho,l}(k)$ and summing over j and k such that $0 \le j \le A(2l+1)$, $k \ge 0$, $B(2l+1) \le j+k \le \theta_{N,l}$, we see that to prove the lemma it is enough to show that for every $a > 0$

$$\sum_{j,k} \mu_{\rho,l}(j)\mu_{\rho,l}(k)\left\{\int\int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\right\}^2 - a\langle f, (-L_{N,\kappa,l} f)\rangle_{\mu_{\Lambda_l^*,\rho}} \tag{6.7}$$

vanishes as $N \uparrow \infty$, $\kappa \downarrow 0$, $l \uparrow \infty$.

Step 2. To estimate (6.7), let

$$F(j,k) = \int\int f(\eta,\zeta)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)\mu_{\Lambda_l^+,k}(\mathrm{d}\zeta).$$

We now claim that there exists $W_N(l)$, where $W_N(l) = \mathrm{o}(N)$ for fixed l, and a finite constant C_0 such that

$$\begin{aligned} &\sum_{j,k} \mu_{\rho,l}(j)\mu_{\rho,l}(k)\big[F(j+1,k-1) - F(j,k)\big]^2 \\ &\quad \le W_N(l)\big\{D_{\Lambda_l^-}\big(\mu_\rho^{\Lambda_l^*}, f\big) + D_{\Lambda_l^+}\big(\mu_\rho^{\Lambda_l^*}, f\big)\big\} + C_0 l^5 D_0\big(\mu_\rho^{\Lambda_l^*}, f\big), \end{aligned} \tag{6.8}$$

where the sum is over all j and k such that $j \ge 0$, $k \ge 1$, $B(2l+1) \le j+k \le \theta_{N,l}$.

To prove (6.8), note that since $\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)$ is the canonical measure,

$$F(j+1,k-1) = \int \mu_{\Lambda_l^+,k-1}(\mathrm{d}\zeta)\int f(\eta,\zeta)\frac{1}{j+1}\sum_{x\in\Lambda_l^-}\eta(x)\mu_{\Lambda_l^-,j+1}(\mathrm{d}\eta).$$

Changing variables $\eta' = \eta - \mathfrak{d}_x$, the previous expression becomes

$$\frac{1}{2l+1}\sum_{x\in\Lambda_l^-}\int \mu_{\Lambda_l^+,k-1}(\mathrm{d}\zeta)\int f(\eta+\mathfrak{d}_x,\zeta)h_{l,j}\big(\eta(x)\big)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta),$$

where

$$h_{l,j}\big(\eta(x)\big) = \frac{2l+1}{j+1}\frac{\varphi(\rho)\mu_{\rho,l}(j)}{\mu_{\rho,l}(j+1)}\frac{1+\eta(x)}{g(1+\eta(x))}.$$

Note that $h_{l,j}(\eta(x))$ has mean equal to 1 with respect to $\mu_{\Lambda_l^-,j}(\mathrm{d}\eta)$.

Changing variables $\zeta' = \zeta + \mathfrak{d}_y$, the previous integral becomes

$$\frac{1}{(2l+1)^2}\sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}}\int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta)\int f(\eta+\mathfrak{d}_x,\zeta-\mathfrak{d}_y)h_{l,j}\big(\eta(x)\big)e_{l,k}\big(\zeta(y)\big)\mu_{\Lambda_l^-,j}(\mathrm{d}\eta),$$

where

$$e_{l,k}\big(\zeta(y)\big) = \frac{\mu_{\rho,l}(k)}{\mu_{\rho,l}(k-1)} \frac{g(\zeta(y))}{\varphi(\rho)}$$

has mean 1 with respect to $\mu_{\Lambda_l^+,k}$.

This identity permits to write $F(j+1,k-1) - F(j,k)$ as the sum of two terms

$$\frac{1}{(2l+1)^2} \sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}} \int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta) \int \big\{ f(\eta+\mathfrak{d}_x, \zeta-\mathfrak{d}_y) - f(\eta,\zeta)\big\} h_{l,j}\big(\eta(x)\big) e_{l,k}\big(\zeta(y)\big) \mu_{\Lambda_l^-,j}(\mathrm{d}\eta)$$

$$+ \int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta) \int f(\eta,\zeta) \frac{1}{(2l+1)^2} \sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}} \big[h_{l,j}\big(\eta(x)\big) e_{l,k}\big(\zeta(y)\big) - 1\big] \mu_{\Lambda_l^-,j}(\mathrm{d}\eta). \tag{6.9}$$

Since $(a+b)^2 \le 2a^2 + 2b^2$, $[F(j+1,k-1) - F(j,k)]^2$ is bounded above by the sum of two terms. One term, corresponding to the last line of (6.9), is equal to

$$2\Bigg(\Bigg\langle f; \frac{1}{(2l+1)^2} \sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}} h_{l,j}\big(\eta(x)\big) e_{l,k}\big(\zeta(y)\big)\Bigg\rangle_{l,j,k}\Bigg)^2, \tag{6.10}$$

where $\langle F; G\rangle_{l,j,k}$ denotes the covariance of F and G with respect to $\mu_{\Lambda_l^+,k}\mu_{\Lambda_l^-,j}$.

Since

$$\frac{\varphi(\rho)\mu_{\rho,l}(r)}{\mu_{\rho,l}(r+1)} = E_{\mu_{\Lambda_l^+,r+1}}\big[g\big(\eta(1)\big)\big], \tag{6.11}$$

we have that

$$\begin{aligned} h_{l,j}\big(\eta(x)\big) e_{l,k}\big(\zeta(y)\big) &= \frac{2l+1}{j+1} \frac{\varphi(\rho)\mu_{\rho,l}(j)}{\mu_{\rho,l}(j+1)} \frac{1+\eta(x)}{g(1+\eta(x))} \frac{\mu_{\rho,l}(k) g(\zeta(y))}{\mu_{\rho,l}(k-1)\varphi(\rho)} \\ &= \frac{2l+1}{j+1} \frac{1+\eta(x)}{g(1+\eta(x))} g(\zeta(y)) \frac{E_{\mu_{\Lambda_l^-,j+1}}[g(\eta(-1))]}{E_{\mu_{\Lambda_l^+,k}}[g(\zeta(1))]}. \end{aligned}$$

We claim that under the measure $\mu_{\Lambda_l^+,k}\mu_{\Lambda_l^-,j}$,

$$h_{l,j}\big(\eta(x)\big) e_{l,k}\big(\zeta(y)\big) \le C_0 l \frac{g(\zeta(y))}{E_{\mu_{\Lambda_l^+,k}}[g(\zeta(1))]} \tag{6.12}$$

for some finite constant C_0 depending only on a_0, a_1. This bound is simple to derive when when g fulfills assumption (B). On the other hand, under the assumptions (SL), since g is increasing, $E_{\mu_{\Lambda_l^-,j+1}}[g(\eta(-1))] \le g(j+1)$, and since $g(k)/k$ is decreasing, under the measure $\mu_{\Lambda_l^-,j}$, $[1+\eta(x)]/g(1+\eta(x))$ is less than or equal to $(j+1)/g(j+1)$. This proves (6.12). This is the only place where we use that $g(k)/k$ is decreasing in k in the condition (SL).

Therefore, by Schwarz inequality, (6.10) is bounded above by

$$C_0 l \langle f; f\rangle_{l,j,k} \sum_{y\in\Lambda_l^+} \frac{E_{\mu_{\Lambda_l^+,k}}[g(\zeta(y))^2]}{E_{\mu_{\Lambda_l^+,k}}[g(\zeta(1))]^2}$$

for some finite constant C_0. In view of (6.11), the fact that $g(m+1) - g(m) \le a_2$, which follows from assumption (B) or from assumption (SL2), and Lemma 6.3,

$$\begin{aligned}
E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(y)\big)^2\big] &= E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(y)\big)\big]E_{\mu_{\Lambda_l^+,k-1}}\big[g\big(\zeta(y)+1\big)\big] \\
&\le E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(y)\big)\big]\big\{a_2 + E_{\mu_{\Lambda_l^+,k-1}}\big[g\big(\zeta(y)\big)\big]\big\} \\
&\le E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(y)\big)\big]\big\{a_2 + E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(y)\big)\big]\big\}.
\end{aligned}$$

As g is increasing, by Lemma 6.3, $E_{\mu_{\Lambda_l^+,1}}[g(\zeta(1))] \le E_{\mu_{\Lambda_l^+,k}}[g(\zeta(1))]$. Hence, since $a_0\mathbf{1}\{r \ge 1\} \le g(r)$, and since $E_{\mu_{\Lambda_l^+,1}}[\mathbf{1}\{\zeta(1) \ge 1\}] = E_{\mu_{\Lambda_l^+,1}}[\zeta(1)] = (2l+1)^{-1}$, we have that

$$\frac{a_0}{2l+1} = a_0 E_{\mu_{\Lambda_l^+,1}}\big[\mathbf{1}\big\{\zeta(1) \ge 1\big\}\big] \le E_{\mu_{\Lambda_l^+,k}}\big[g\big(\zeta(1)\big)\big]. \tag{6.13}$$

It follows from this estimate and from the previous bound that (6.10) is less than or equal to

$$C_0 l^3 \langle f; f\rangle_{l,j,k}.$$

Multiply this expression by $\mu_{\rho,l}(j)\mu_{\rho,l}(k)$, recall the bound (6.6), and sum over j and k such that $j \ge 0$, $k \ge 1$, $B(2l+1) \le j+k \le \theta_{N,l}$, to get that (6.10) is bounded by

$$W_N(l)\big\{D_{\Lambda_l^-}\big(\mu_\rho^{\Lambda_l^*}, f\big) + D_{\Lambda_l^+}\big(\mu_\rho^{\Lambda_l^*}, f\big)\big\},$$

where $W_N(l) = \mathrm{o}(N)$ for fixed l.

We now estimate the first term in the decomposition (6.9). By Schwarz inequality and by the bounds (6.12), (6.13), the square of this expression is less than or equal to

$$C_0 l^3 \sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}} \int \mu_{\Lambda_l^+,k}(\mathrm{d}\zeta) \int g\big(\zeta(y)\big)\big\{f(\eta+\mathfrak{d}_x, \zeta-\mathfrak{d}_y) - f(\eta,\zeta)\big\}^2 \mu_{\Lambda_l^-,j}(\mathrm{d}\eta)$$

for some finite constant C_0. The sum over $j \ge 0$, $k \ge 1$ of this expression, when multiplied by $\mu_{\rho,l}(k)\mu_{\rho,l}(j)$, is bounded by

$$C_0 l^3 \sum_{\substack{x\in\Lambda_l^- \\ y\in\Lambda_l^+}} \int \mu_{\Lambda_l^+,\rho}(\mathrm{d}\zeta) \int g\big(\zeta(y)\big)\big\{f(\eta+\mathfrak{d}_x, \zeta-\mathfrak{d}_y) - f(\eta,\zeta)\big\}^2 \mu_{\Lambda_l^-,\rho}(\mathrm{d}\eta).$$

Changing variables $\zeta' = \zeta - \mathfrak{d}_y$, adding and subtracting in the expression inside braces the terms $f(\eta+\mathfrak{d}_{-1},\zeta)$, $f(\eta,\zeta+\mathfrak{d}_1)$, we estimate the previous expression by

$$C_0 l^5 D_0(\mu_{\Lambda_l^*,\rho}, f) + C_0 l^5\big\{D_{\Lambda_l^-}(\mu_{\Lambda_l^*,\rho}, f) + D_{\Lambda_l^+}(\mu_{\Lambda_l^*,\rho}, f)\big\}$$

for some constant C_0. This proves claim (6.8).

Step 3. In view of (6.7) and of (6.8), to prove the lemma it is enough to show that for every $a > 0$

$$\begin{aligned}
&\lim_{l\to\infty}\limsup_{\kappa\to 0}\limsup_{N\to\infty}\sup_F \Big\{\sum_{j,k} F(j,k)^2\mu_{\rho,l}(j)\mu_{\rho,l}(k) \\
&\quad - a\kappa^{-1}\sum_{j,k}\big[F(j+1,k-1) - F(j,k)\big]^2\mu_{\rho,l}(j)\mu_{\rho,l}(k)\Big\} = 0,
\end{aligned}$$

where the first sum is carried over all $0 \le j \le A(2l+1)$, $k \ge 0$, $B(2l+1) \le j+k \le \theta_{N,l}$, the second sum is carried over all $j \ge 0$, $k \ge 1$, $B(2l+1) \le j+k \le \theta_{N,l}$ and where the supremum is carried over all functions $F : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ such that $\sum_{j,k\ge 0} F(j,k)^2 \mu_{\rho,l}(j)\mu_{\rho,l}(k) = 1$.

The expression inside braces can be bounded by

$$\sum_{M=B(2l+1)}^{\theta_{N,l}} \mu_{\rho,\Lambda_l^*}(M) Z_M(F) \left\{ \sum_{j=0}^{A(2l+1)} G(j)^2 \mu_{\Lambda_l^*,M}(j) \right.$$
$$\left. - a\kappa^{-1} \sum_{j=0}^{B(2l+1)-1} \big[G(j+1) - G(j)\big]^2 \mu_{\Lambda_l^*,M}(j) \right\}, \tag{6.14}$$

where

$$\mu_{\rho,\Lambda_l^*}(M) = \sum_{j=0}^{B(2l+1)} \mu_{\rho,l}(j)\mu_{\rho,l}(M-j), \qquad \mu_{\Lambda_l^*,M}(j) = \frac{\mu_{\rho,l}(j)\mu_{\rho,l}(M-j)}{\mu_{\rho,\Lambda_l^*}(M)},$$
$$Z_M(F) = \sum_{j=0}^{B(2l+1)} F(j, M-j)^2 \mu_{\Lambda_l^*,M}(j), \qquad Z_M(F)G(j)^2 = F(j, M-j)^2.$$

Note that we omitted the dependence on B of the variables $\mu_{\rho,\Lambda_l^*}(M)$, $\mu_{\Lambda_l^*,M}(j)$, $Z_M(F)$ and that

$$\sum_{0\le j\le B(2l+1)} G(j)^2 \mu_{\Lambda_l^*,M}(j) = 1.$$

The expression inside braces in (6.14) can be interpreted in terms of a random walk on an interval of length $B(2l+1)$ where the total number of particles M becomes a parameter. In fact, the second term in braces corresponds to the Dirichlet form of a random walk on $\{0, \dots, B(2l+1)\}$ which jumps from j to $j+1$, $0 \le j \le B(2l+1) - 1$, at rate 1 and from $j+1$ to j at rate $r_M(j+1, j) = \mu_{\Lambda_l^*,M}(j)/\mu_{\Lambda_l^*,M}(j+1)$. By (6.11),

$$r_M(j+1, j) = \frac{\mu_{\rho,l}(j)}{\mu_{\rho,l}(j+1)} \frac{\mu_{\rho,l}(M-j)}{\mu_{\rho,l}(M-j-1)} = \frac{E_{\mu_{\Lambda_l^+,j+1}}[g(\eta(1))]}{E_{\mu_{\Lambda_l^+,M-j}}[g(\eta(1))]}.$$

We claim that this random walk has a spectral gap

$$\hat{\lambda}_{l,B} \text{ which depends on } B \text{ and } l \text{ but is uniform over } M. \tag{6.15}$$

Assume first that g satisfies (B). In this case, by (6.13), the previous ratio is bounded above by $a_1 a_0^{-1}(2l+1)$ and below by $a_0 a_1^{-1}(2l+1)^{-1}$. The jump rates are therefore bounded below and above by finite constants independent of M, and claim (6.15) follows easily.

Assume now that g satisfies (SL). We claim that for l large enough,

$$\lim_{M\to\infty} \max_{0\le j\le B(2l+1)-1} r_M(j+1, j) = 0. \tag{6.16}$$

Indeed, by Lemma 6.3, $E_{\mu_{\Lambda_l^+,j+1}}[g(\eta(1))] \le g(B(2l+1))$. On the other hand, for every $D \le D'(2l+1) \le M - B(2l+1) \le M - j$

$$E_{\mu_{\Lambda_l^+,M-j}}\big[g\big(\eta(1)\big)\big] \ge E_{\mu_{\Lambda_l^+,M-j}}\big[g\big(\eta(1)\big)\mathbf{1}\big\{\eta(1) \ge D\big\}\big]$$
$$\ge g(D)\mu_{\Lambda_l^+,D'(2l+1)}\big\{\eta(1) \ge D\big\} \ge g(D)\big(\mu_{D'}\big\{\eta(1) \ge D\big\} - C_0/l\big),$$

where the last inequality follows from the equivalence of ensembles [7], Appendix 2.1, Corollary 1.7 and C_0 is a finite constant. The right-side can be made arbitrarily large since $\lim_{D'\uparrow\infty}\mu_{D'}\{\eta(1)\ge D\}=1$ and $\lim_{D\uparrow\infty}g(D)=\infty$. This proves claim (6.16).

Fix M_0 and l large enough so that $r_M(j+1,j)<1$ for all $0\le j\le B(2l+1)-1=Q$, $M\ge M_0$. The stationary probabilities for the corresponding birth–death chain on the interval can be expressed as

$$\pi_j=\frac{(\prod_{s=0}^{j-1}r_M(s+1,s))^{-1}}{1+\sum_{t=1}^{Q}(\prod_{s=0}^{t-1}r_M(s+1,s))^{-1}},$$

where the empty product in the numerator is taken to be 1 when $j=0$. We note by construction that $\pi_j\le\pi_{j+1}$. We have the Poincaré inequality:

$$\begin{aligned}\operatorname{Var}_\pi(f)&\le\sum_{x,y}\pi_x\pi_y\big(f(x)-f(y)\big)^2\le 2Q\sum_{y>x}\pi_x\pi_y\sum_{z=x}^{y-1}\big(f(z)-f(z+1)\big)^2\\&\le 2Q\sum_{y>x}\pi_y\sum_{z=x}^{y-1}\pi_z\big(f(z)-f(z+1)\big)^2\le 2Q^2\sum_{z=0}^{Q-1}\pi_z\big(f(z)-f(z+1)\big)^2.\end{aligned}$$

Hence, for large M, the inverse of the spectral gap, $\hat\lambda_{l,B}^{-1}$, is bounded by C_0l^2 for some constant C_0 depending only on B. For $M\le M_0$, recalling (6.12), we may obtain a lower and an upper bound on $r_M(j+1,j)$ which depend only on a_0, a_1, a_2, B, l and M_0. It is easy to show that in this case the inverse gap, $\hat\lambda_{l,B}^{-1}$, is bounded by a constant which depends only on B, l and M_0. This concludes the proof of claim (6.15).

At this point, we may apply the Rayleigh bound ([7], Theorem A3.1.1) to estimate the expression in braces in (6.14). Let $V_0=\mathbf{1}\{0,\dots,A(2l+1)\}$ and let $\bar V_0=V_0-E_{\mu_{\Lambda_l^*,M}}[V_0]$ so that

$$\sum_{j=0}^{A(2l+1)}G(j)^2\mu_{\Lambda_l^*,M}(j)=\sum_{j=0}^{B(2l+1)}V_0(j)G(j)^2\mu_{\Lambda_l^*,M}(j).$$

Since $\|V_0\|_{L^\infty}\le 1$, by the Rayleigh expansion and by the spectral gap, the display in braces in (6.14) is bounded by

$$\sum_{j=0}^{A(2l+1)}\mu_{\Lambda_l^*,M}(j)+\frac{a^{-1}\kappa\hat\lambda_{l,B}^{-1}}{1-2a^{-1}\kappa\hat\lambda_{l,B}^{-1}}.$$

The second term vanishes as $\kappa\downarrow 0$. To bound the first term, let $\alpha_j=\mu_{\rho,l}(j)\mu_{\rho,l}(M-j)$, $0\le j\le M$. Since $B>2A$, the first term in the last formula is equal to

$$\frac{\sum_{j=0}^{A(2l+1)}\alpha_j}{\sum_{j=0}^{B(2l+1)}\alpha_j}\le\frac{\sum_{j=0}^{A(2l+1)}\alpha_j}{\sum_{j=A(2l+1)}^{2A(2l+1)}\alpha_j}\le\max_{0\le j\le A(2l+1)}\frac{\alpha_j}{\alpha_{j+A(2l+1)}}.$$

As above in calculating $r_M(j+1,j)$, since g is increasing and $M\ge B(2l+1)$, if $R=A(2l+1)$, $S=(B-2A)(2l+1)$, by Lemma 6.3,

$$\frac{\alpha_j}{\alpha_{j+R}}=\prod_{k=j}^{j+R-1}\frac{E_{\mu_{\Lambda_l^+,k+1}}[g(\eta(1))]}{E_{\mu_{\Lambda_l^+,M-k}}[g(\eta(1))]}\le\left\{\frac{E_{\mu_{\Lambda_l^+,2R}}[g(\eta(1))]}{E_{\mu_{\Lambda_l^+,S}}[g(\eta(1))]}\right\}^R.$$

Since $B>4A$, by [7], Corollary 1.6, Appendix 2.1, for all large l, we have

$$E_{\mu_{\Lambda_l^+,2R}}\big[g\big(\eta(1)\big)\big]\le\varphi(2A)+\frac{C_0}{l}<\varphi(B-2A)-\frac{C_0}{l}\le E_{\mu_{\Lambda_l^+,S}}\big[g\big(\eta(1)\big)\big]$$

for some finite constant C_0. Hence, the expression appearing in the previous displayed formula vanishes exponentially fast as $l\uparrow\infty$. This concludes the proof of the lemma.

6.2. *Proof of Theorem* 2.6

Given Lemmas 5.1 and 6.1, the argument is similar to that in [6]. Recall $H(\rho) = E_{\nu_\rho}[h]$, $H_l(\eta) = H(\eta^l(0))$, and $\bar{H}_l(\rho) = E_{\mu_\rho}[H_l]$. Then, we have that

$$\begin{aligned}
&\mathbb{E}^N\left[\left|\int_0^t \left\{h(\eta_s) - \frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N} \bar{H}_l\big(\eta_s^{\kappa N}(x)\big)\right\} \mathrm{d}s\right|\right] \\
&\quad \le \mathbb{E}^N\left[\left|\int_0^t \left\{h(\eta_s) - H\big(\eta_s^l(0)\big)\right\} \mathrm{d}s\right|\right] \\
&\qquad + \mathbb{E}^N\left[\left|\int_0^t \left\{H\big(\eta_s^l(0)\big) - \frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N} H\big(\eta_s^l(x)\big)\right\} \mathrm{d}s\right|\right] \\
&\qquad + \mathbb{E}^N\left[\left|\int_0^t \left\{\frac{1}{\varepsilon N}\sum_{x=1}^{\varepsilon N}\big(H\big(\eta_s^l(x)\big) - \bar{H}_l\big(\eta_s^{\kappa N}(x)\big)\big)\right\} \mathrm{d}s\right|\right].
\end{aligned}$$

As h and H are bounded, Lipschitz by Lemma 6.5, the first and second terms vanish by Lemmas 5.1 and 6.1. The third term is recast as

$$\mathbb{E}^N\left[\left|\int_0^t \left\{\frac{1}{N}\sum_{x\in\mathbb{T}_N} \iota_\varepsilon(x/N)\big(\tau_x H_l(\eta_s) - \bar{H}_l\big(\eta_s^{\kappa N}(x)\big)\big)\right\} \mathrm{d}s\right|\right],$$

where $\iota_\varepsilon(\cdot) = \varepsilon^{-1}1\{(0,\varepsilon]\}$. It vanishes by Proposition 4.1 as $N \uparrow \infty$, and $\kappa \downarrow 0$.

Lemma 6.5. *Let $h : \mathbb{N} \to \mathbb{R}_+$ be a nonnegative, Lipschitz function for which there is a constant C such that $kh(k) \le Cg(k)$ for $k \ge 1$. Then, $H(\rho) = E_{\nu_\rho}[h(\eta(0))]$ is also nonnegative, bounded and Lipschitz, and vanishes at infinity.*

Proof. It follows from the assumptions of the lemma that h is bounded, as $g(k) \le a_1 k$, and that h vanishes at infinity. Hence, H, which is clearly nonnegative, is also bounded. We claim that H vanishes at infinity since

$$H(\rho) \le \sup_{x\ge A} h(x) + \frac{1}{\rho} E_{\mu_\rho}\big[\eta(0)h\big(\eta(0)\big)\mathbf{1}\big\{\eta(0) \le A\big\}\big] \le \sup_{x\ge A} h(x) + \frac{A\|h\|_{L^\infty}}{\rho}.$$

To show H is Lipschitz, it is enough to show H' is absolutely bounded. Compute

$$H'(\rho) = \frac{\varphi'(\rho)}{\rho\varphi(\rho)}\big\langle h\big(\eta(0)\big)\eta(0)^2\big\rangle_{\mu_\rho} - \left\{\frac{1}{\rho^2} + \frac{\varphi'(\rho)}{\varphi(\rho)}\right\}\big\langle h\big(\eta(0)\big)\eta(0)\big\rangle_{\mu_\rho}.$$

We first examine this expression for ρ large. The second term, by the assumption $kh(k) \le Cg(k)$, is bounded by $C\{\varphi(\rho)/\rho^2 + \varphi'(\rho)\}$. A coupling argument shows that $\varphi'(\rho) \le a_2$ where we recall that a_2 is the Lipschitz constant of the function g. On the other hand, $\varphi(\rho)/\rho^2 \le a_1/\rho$ because $g(k) \le a_1 k$.

Since $kh(k) \le Cg(k)$ and since $E_{\mu_\rho}[g(\eta(0))\eta(0)] = \varphi(\rho)(1+\rho)$, the first term is bounded by $Ca_2(1+\rho)/\rho$. This proves that H' is absolutely bounded for ρ large.

It is also not difficult to see that $H'(\rho)$ is bounded for ρ close to 0. □

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