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Euler hydrodynamics for attractive particle systems in random environment

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Abstract. We prove quenched hydrodynamic limit under hyperbolic time scaling for bounded attractive particle systems on $\mathbb{Z}$ in random ergodic environment. Our result is a strong law of large numbers, that we illustrate with various examples.

Résumé. Nous obtenons la limite hydrodynamique trempée, sous un changement d'échelle hyperbolique, pour un système de particules attractif sur $\mathbb{Z}$ en milieu aléatoire ergodique, avec un nombre borné de particules par site. Notre résultat est une loi forte des grands nombres. Nous l'illustrons sur différents exemples.

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1. Introduction

Hydrodynamic limit describes the time evolution (usually governed by a limiting PDE, called the hydrodynamic equation) of empirical density fields in interacting particle systems (*IPS*). For usual models, such as the simple exclusion process, the limiting PDE is a nonlinear diffusion equation or hyperbolic conservation law (see [20] and references therein). In this context, a random environment leads to homogeneization-like effects, where an effective diffusion matrix or flux function is expected to capture the effect of inhomogeneity. Hydrodynamic limit in random environment has been widely addressed and robust methods have been developed in the diffusive case ([11,12,14,16,18,21,25,26]).

In the hyperbolic setting, due to non-existence of strong solutions and non-uniqueness of weak solutions, the key issue is to establish convergence to the so-called entropy solution (see e.g. [30]) of the Cauchy problem. The first such result without restrictive assumptions is due to [27] for spatially homogeneous attractive systems with product invariant measures. In random environment, the few available results depend on particular features of the investigated models. In [6], the authors consider the asymmetric zero-range process with site disorder on $\mathbb{Z}^d$, extending a model introduced in [10]. They prove a quenched hydrodynamic limit given by a hyperbolic conservation law with an effective homogeneized flux function. To this end, they use in particular the existence of explicit product invariant measures for the disordered zero-range process below some critical value of the disorder parameter. In [29], extension to the supercritical case is carried out in the totally asymmetric case with constant jump rate. In [28], under a strong

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mixing assumption, the author establishes a quenched hydrodynamic limit for the totally asymmetric nearest-neighbor K-exclusion process on $\mathbb{Z}$ with site disorder, for which explicit invariant measures are not known. The last two results rely on a microscopic version of the Lax–Hopf formula. However, the simple exclusion process beyond the totally asymmetric nearest-neighbor case, or more complex models with state-dependent jump rates, remain outside the scope of the above approaches.

In this paper, we prove quenched hydrodynamics for attractive particle systems in random environment on $\mathbb{Z}$ with a bounded number of particles per site. Our method is quite robust with respect to the model and disorder. We only require the environment to be ergodic. Besides, we are not restricted to site or bond disorder. However, for simplicity we treat in detail the misanthropes' process with site disorder, and explain in the last section how our method applies to various other models. An essential difficulty for the disordered system is the simultaneous loss of translation invariance *and* lack of knowledge of explicit invariant measures. Note that even if the system without disorder has explicit invariant measures, the disordered system in general does not, with the above exception of the zero-range process. In particular, one does not have an effective characterization theorem for invariant measures of the quenched process. Our strategy is to prove hydrodynamic limit for a joint disorder-particle process which, unlike the quenched process, is translation invariant. The idea is that hydrodynamic limit for the joint process should imply quenched hydrodynamic limit. This is false for limits in the usual (weak) sense, but becomes true if a *strong* hydrodynamic limit is proved for the joint process. We are able to do it by characterizing the extremal invariant and translation invariant measures of the joint process, and by adapting the tools developed in [5].

The paper is organized as follows. In Section 2, we define the model and state our main result. Section 3 is devoted to the study of the joint disorder-particle process and characterization of its invariant measures. The hydrodynamic limit is proved in Section 4. Finally, in Section 5 we consider models other than the misanthropes' process: We detail generalizations of misanthropes and k-step exclusion processes, as well as a traffic model.

2. Notation and results

Throughout this paper $\mathbb{N} = \{1, 2, \dots\}$ will denote the set of natural numbers, and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ the set of non-negative integers. The integer part $\lfloor x \rfloor \in \mathbb{Z}$ of $x \in \mathbb{R}$ is uniquely defined by $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. We consider particle configurations on $\mathbb{Z}$ with at most K particles per site, $K \in \mathbb{N}$. Thus the state space of the process is $\mathbf{X} = \{0, 1, \dots, K\}^{\mathbb{Z}}$, which we endow with the product topology, that makes $\mathbf{X}$ a compact metrisable space, with the product (partial) order.

The set $\mathbf{A}$ of environments is a compact metric space endowed with its Borel σ-field. A function f defined on $\mathbf{A} \times \mathbf{X}$ (resp. g on $\mathbf{A} \times \mathbf{X}^2$) is called *local* if there is a finite subset Λ of $\mathbb{Z}$ such that $f(\alpha, \eta)$ depends only on α and $(\eta(x), x \in \Lambda)$ (resp. $g(\alpha, \eta, \xi)$ depends only on α and $(\eta(x), \xi(x), x \in \Lambda)$). We denote by τ_x either the spatial translation operator on the real line for $x \in \mathbb{R}$, defined by $\tau_x y = x + y$, or its restriction to $x \in \mathbb{Z}$. By extension, if f is a function defined on $\mathbb{Z}$ (resp. $\mathbb{R}$), we set $\tau_x f = f \circ \tau_x$ for $x \in \mathbb{Z}$ (resp. $\mathbb{R}$). In the sequel this will be applied to particle configurations $\eta \in \mathbf{X}$, disorder configurations $\alpha \in \mathbf{A}$, or joint disorder-particle configurations $(\alpha, \eta) \in \mathbf{A} \times \mathbf{X}$. In the latter case, unless mentioned explicitely, τ_x applies simultaneously to both components.

If τ_x acts on some set and μ is a measure on this set, $\tau_x \mu = \mu \circ \tau_x^{-1}$. We let $\mathcal{M}^+(\mathbb{R})$ denote the set of nonnegative measures on $\mathbb{R}$ equipped with the metrizable topology of vague convergence, defined by convergence on continuous test functions with compact support. The set of probability measures on $\mathbf{X}$ is denoted by $\mathcal{P}(\mathbf{X})$. If η is an $\mathbf{X}$-valued random variable and $\nu \in \mathcal{P}(\mathbf{X})$, we write $\eta \sim \nu$ to specify that η has distribution ν. Similarly, for $\alpha \in \mathbf{A}$, $Q \in \mathcal{P}(\mathbf{A})$, $\alpha \sim Q$ means that α has distribution Q.

A sequence $(\nu_n, n \in \mathbb{N})$ of probability measures on $\mathbf{X}$ converges weakly to some $\nu \in \mathcal{P}(\mathbf{X})$, if and only if $\lim_{n\to\infty} \int f \, d\nu_n = \int f \, d\nu$ for every continuous function f on $\mathbf{X}$. The topology of weak convergence is metrizable and makes $\mathcal{P}(\mathbf{X})$ compact. A partial stochastic order is defined on $\mathcal{P}(\mathbf{X})$; namely, for $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$, we write $\mu_1 \le \mu_2$ if the following equivalent conditions hold (see e.g. [23,31]):

(i) For every nondecreasing nonnegative function f on $\mathbf{X}$, $\int f \, d\mu_1 \le \int f \, d\mu_2$.
(ii) There exists a coupling measure $\overline{\mu}$ on $\mathbf{X} \times \mathbf{X}$ with marginals μ_1 and μ_2, such that $\overline{\mu}\{(\eta, \xi)\colon \eta \le \xi\} = 1$.

In the following model, we fix a constant $c > 0$ and define $\mathbf{A} = [c, 1/c]^{\mathbb{Z}}$ to be the set of environments (or disorders) $\alpha = (\alpha(x)\colon x \in \mathbb{Z})$ such that

$$\forall x \in \mathbb{Z}, \quad c \le \alpha(x) \le c^{-1}. \tag{1}$$

For each realization $\alpha \in \mathbf{A}$ of the disorder, the *quenched process* $(\eta_t)_{t\geq 0}$ is a Feller process on $\mathbf{X}$ with generator given by, for any local function f on $\mathbf{X}$,

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} \alpha(x)p(y-x)b\big(\eta(x),\eta(y)\big)\big[f\big(\eta^{x,y}\big)-f(\eta)\big], \tag{2}$$

where $\eta^{x,y}$ denotes the new state after a particle has jumped from x to y (that is $\eta^{x,y}(x)=\eta(x)-1, \eta^{x,y}(y)=\eta(y)+1, \eta^{x,y}(z)=\eta(z)$ otherwise), the particles' jump kernel p is a probability distribution on $\mathbb{Z}$, and $b:\mathbb{Z}^+\times\mathbb{Z}^+\to\mathbb{R}^+$ is the jump rate. We assume that p and b satisfy:

(A1) Irreducibility: For every $z\in\mathbb{Z}$, $\sum_{n\in\mathbb{N}}[p^{*n}(z)+p^{*n}(-z)]>0$, where $*n$ denotes nth convolution power;
(A2) finite mean: $\sum_{z\in\mathbb{Z}}|z|p(z)<+\infty$;
(A3) K-exclusion rule: $b(0,\cdot)=0, b(\cdot,K)=0$;
(A4) non-degeneracy: $b(1,K-1)>0$;
(A5) attractiveness: b is nondecreasing (nonincreasing) in its first (second) argument.

For the graphical construction of the system given by (2) (see [5] and references therein), let us introduce a general framework that applies to a larger class of models (see Section 5 below). Given a measurable space $(\mathcal{V},\mathcal{F}_\mathcal{V},m)$, for m a finite nonnegative measure, we consider the probability space $(\Omega,\mathcal{F},\mathbb{P})$ of locally finite point measures $\omega(\mathrm{d}t,\mathrm{d}x,\mathrm{d}v)$ on $\mathbb{R}^+\times\mathbb{Z}\times\mathcal{V}$, where $\mathcal{F}$ is generated by the mappings $\omega\mapsto\omega(S)$ for Borel sets S of $\mathbb{R}^+\times\mathbb{Z}\times\mathcal{V}$, and $\mathbb{P}$ makes ω a Poisson process with intensity

$$M(\mathrm{d}t,\mathrm{d}x,\mathrm{d}v)=\lambda_{\mathbb{R}^+}(\mathrm{d}t)\lambda_\mathbb{Z}(\mathrm{d}x)m(\mathrm{d}v)$$

denoting by λ either the Lebesgue or the counting measure. We write $\mathbb{E}$ for expectation with respect to $\mathbb{P}$. For the particular model (2) we take

$$\mathcal{V}:=\mathbb{Z}\times[0,1],\quad v=(z,u)\in\mathcal{V},\quad m(\mathrm{d}v)=c^{-1}\|b\|_\infty p(\mathrm{d}z)\lambda_{[0,1]}(\mathrm{d}u). \tag{3}$$

Thanks to assumption (A2), for $\mathbb{P}$-a.e. ω, there exists a unique mapping

$$(\alpha,\eta_0,t)\in\mathbf{A}\times\mathbf{X}\times\mathbb{R}^+\mapsto\eta_t=\eta_t(\alpha,\eta_0,\omega)\in\mathbf{X} \tag{4}$$

satisfying:

(a) $t\mapsto\eta_t(\alpha,\eta_0,\omega)$ is right-continuous;
(b) $\eta_0(\alpha,\eta_0,\omega)=\eta_0$;
(c) the particle configuration is updated at points $(t,x,v)\in\omega$ (and only at such points; by $(t,x,v)\in\omega$ we mean $\omega\{(t,x,v)\}=1$) according to the rule

$$\eta_t(\alpha,\eta_0,\omega)=\mathcal{T}^{\alpha,x,v}\eta_{t-}(\alpha,\eta_0,\omega), \tag{5}$$

where, for $v=(z,u)\in\mathcal{V}$, $\mathcal{T}^{\alpha,x,v}$ is defined by

$$\mathcal{T}^{\alpha,x,v}\eta=\begin{cases}\eta^{x,x+z}, & \text{if } u<\alpha(x)\dfrac{b(\eta(x),\eta(x+z))}{c^{-1}\|b\|_\infty},\\ \eta, & \text{otherwise.}\end{cases} \tag{6}$$

Notice the shift commutation property

$$\mathcal{T}^{\tau_x\alpha,y,v}\tau_x=\tau_x\mathcal{T}^{\alpha,y+x,v}, \tag{7}$$

where τ_x on the right-hand side acts only on η. By assumption (A5),

$$\mathcal{T}^{\alpha,x,v}:\mathbf{X}\to\mathbf{X}\quad\text{is nondecreasing.} \tag{8}$$

Hence,

$$(\alpha, \eta_0, t) \mapsto \eta_t(\alpha, \eta_0, \omega) \quad \text{is nondecreasing w.r.t. } \eta_0. \tag{9}$$

For every $\alpha \in \mathbf{A}$, under $\mathbb{P}$, $(\eta_t(\alpha, \eta_0, \omega))_{t\geq 0}$ is a Feller process with generator

$$L_\alpha f(\eta) = \sum_{x\in\mathbb{Z}} \int_{\mathcal{V}} \big[f\big(\mathcal{T}^{\alpha,x,v}\eta\big) - f(\eta)\big] m(\mathrm{d}v). \tag{10}$$

With (6), (10) reduces to (2). Thus for any $t \in \mathbb{R}^+$ and continuous function f on $\mathbf{X}$, $\mathbb{E}[f(\eta_t(\alpha, \eta_0, \omega))] = S_\alpha(t) f(\eta_0)$, where S_α denotes the semigroup generated by L_α. From (9), for $\mu_1, \mu_2 \in \mathcal{P}(\mathbf{X})$,

$$\mu_1 \leq \mu_2 \quad \Rightarrow \quad \forall t \in \mathbb{R}^+, \quad \mu_1 S_\alpha(t) \leq \mu_2 S_\alpha(t). \tag{11}$$

Property (11) is usually called *attractiveness*. Condition (8) implies the stronger *complete monotonicity* property [9, 13], that is, existence of a monotone Markov coupling for an arbitrary number of processes with generator (2), see (27) below; we also say that the process is *strongly attractive*.

Let $N \in \mathbb{N}$ be the scaling parameter for the hydrodynamic limit, that is, the inverse of the macroscopic distance between two consecutive sites. The empirical measure of a configuration η viewed on scale N is given by

$$\pi^N(\eta)(\mathrm{d}x) = N^{-1} \sum_{y\in\mathbb{Z}} \eta(y)\delta_{y/N}(\mathrm{d}x) \in \mathcal{M}^+(\mathbb{R}),$$

where, for $x \in \mathbb{R}$, δ_x denotes the Dirac measure at x.

Our main result is:

Theorem 2.1. *Assume $p(\cdot)$ has finite third moment. Let Q be an ergodic probability distribution on $\mathbf{A}$. Then there exists a Lipschitz-continuous function G^Q on $[0, K]$ defined below (depending only on $p(\cdot)$, $b(\cdot,\cdot)$ and Q) such that the following holds. Let $(\eta_0^N, N \in \mathbb{N})$ be a sequence of $\mathbf{X}$-valued random variables on a probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ such that*

$$\lim_{N\to\infty} \pi^N\big(\eta_0^N\big)(\mathrm{d}x) = u_0(\cdot)\,\mathrm{d}x \quad \mathbb{P}_0\text{-a.s.} \tag{12}$$

for some measurable $[0, K]$-valued profile $u_0(\cdot)$. Then for Q-a.e. $\alpha \in \mathbf{A}$, the $\mathbb{P}_0 \otimes \mathbb{P}$-a.s. convergence

$$\lim_{N\to\infty} \pi^N\big(\eta_{Nt}\big(\alpha, \eta_0^N(\omega_0), \omega\big)\big)(\mathrm{d}x) = u(\cdot, t)\,\mathrm{d}x$$

holds uniformly on all bounded time intervals, where $(x, t) \mapsto u(x, t)$ denotes the unique entropy solution with initial condition u_0 to the conservation law

$$\partial_t u + \partial_x\big[G^Q(u)\big] = 0. \tag{13}$$

We refer the reader (for instance) to [30] for the definition of entropy solutions. To define the *macroscopic flux* G^Q, let the *microscopic flux* through site 0 be

$$\begin{aligned} j(\alpha,\eta) &= j^+(\alpha,\eta) - j^-(\alpha,\eta), \\ j^+(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\ y\leq 0<y+z} \alpha(y)p(z)b\big(\eta(y), \eta(y+z)\big), \\ j^-(\alpha,\eta) &= \sum_{y,z\in\mathbb{Z}:\ y+z\leq 0<y} \alpha(y)p(z)b\big(\eta(y), \eta(y+z)\big). \end{aligned} \tag{14}$$

We will show in Corollary 3.1 below that there exists a closed subset $\mathcal{R}^Q$ of $[0, K]$, a subset $\widetilde{\mathbf{A}}^Q$ of $\mathbf{A}$ with Q-probability 1 (both depending also on $p(\cdot)$ and $b(\cdot,\cdot)$), and a family of probability measures $(\nu_\alpha^{Q,\rho}\colon \alpha \in \widetilde{\mathbf{A}}^Q, \rho \in \mathcal{R}^Q)$ on $\mathbf{X}$, such that, for every $\rho \in \mathcal{R}^Q$:

(B1) For every $\alpha \in \widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$ is an invariant measure for L_α.
(B2) For every $\alpha \in \widetilde{\mathbf{A}}^Q$, $\nu_\alpha^{Q,\rho}$-a.s.,

$$\lim_{l\to\infty}(2l+1)^{-1}\sum_{x\in\mathbb{Z}:\ |x|\le l}\eta(x)=\rho.$$

(B3) The quantity

$$G_\alpha^Q(\rho):=\int j(\alpha,\eta)\nu_\alpha^{Q,\rho}(\mathrm{d}\eta) \tag{15}$$

does not depend on $\alpha \in \widetilde{\mathbf{A}}^Q$. Hence we define $G^Q(\rho)$ as (15) for $\rho \in \mathcal{R}^Q$ and extend it by linear interpolation on the complement of $\mathcal{R}^Q$, which is a finite or countably infinite union of disjoint open intervals.

The function G^Q is Lipschitz continuous (see Remark 3.3 below), which is the minimum regularity required for the classical theory of entropy solutions. We cannot say more about G^Q in general, because the measures $\nu_\alpha^{Q,\rho}$ are not explicit. This is true even in the spatially homogeneous case $\alpha(x)\equiv 1$, unless b satisfies additional algebraic relations introduced in [8]. In the absence of disorder, for the exclusion process and a few simple models satisfying these relations (see for instance [3], Section 4), we have an explicit flux function. Nevertheless, invariant measures are no longer computable when introducing disorder, so that the effect of the latter on the flux function is difficult to evaluate. However, in the special case $b(n,m)=\mathbf{1}_{\{n>0\}}\mathbf{1}_{\{m<K\}}$, $p(1)=1$, G^Q is shown to be concave in [28], as a consequence of the variational approach used to derive hydrodynamic limit. But this approach does not apply to the models we consider in the present paper.

3. The disorder-particle process

In this section we study invariant measures for the markovian *joint process* $(\alpha_t,\eta_t)_{t\ge 0}$ on $\mathbf{A}\times\mathbf{X}$ with generator given by, for any local function f on $\mathbf{A}\times\mathbf{X}$,

$$Lf(\alpha,\eta)=\sum_{x\in\mathbb{Z}}\int_{\mathcal{V}}\big[f\big(\alpha,\mathcal{T}^{\alpha,x,v}\eta\big)-f(\alpha,\eta)\big]m(\mathrm{d}v) \tag{16}$$

that is, for the particular model (6),

$$Lf(\alpha,\eta)=\sum_{x,y\in\mathbb{Z}}\alpha(x)p(y-x)b\big(\eta(x),\eta(y)\big)\big[f\big(\alpha,\eta^{x,y}\big)-f(\alpha,\eta)\big]. \tag{17}$$

We denote by $(S(t), t\in\mathbb{R}^+)$ the semigroup generated by L. Given $\alpha_0=\alpha$, this dynamics simply means that $\alpha_t=\alpha$ for all $t\ge 0$, while $(\eta_t)_{t\ge 0}$ is a Markov process with generator L_α given by (2). Note that L is *translation invariant*, that is

$$\tau_x L = L\tau_x, \tag{18}$$

where τ_x acts jointly on (α,η). This is equivalent to a *shift commutation property* for the quenched dynamics:

$$L_\alpha\tau_x=\tau_x L_{\tau_x\alpha}, \tag{19}$$

where, since L_α is a Markov generator on $\mathbf{X}$, the first τ_x on the r.h.s. acts only on η. We need to introduce a conditional stochastic order. For the sequel, we define the set $\overline{\mathcal{O}}=\overline{\mathcal{O}}_+\cup\overline{\mathcal{O}}_-$, where

$$\overline{\mathcal{O}}_+=\big\{(\alpha,\eta,\xi)\in\mathbf{A}\times\mathbf{X}^2:\ \eta\le\xi\big\},$$
$$\overline{\mathcal{O}}_-=\big\{(\alpha,\eta,\xi)\in\mathbf{A}\times\mathbf{X}^2:\ \xi\le\eta\big\}. \tag{20}$$

Lemma 3.1. *For two probability measures* $\mu^1 = \mu^1(\mathrm{d}\alpha, \mathrm{d}\eta)$, $\mu^2 = \mu^2(\mathrm{d}\alpha, \mathrm{d}\eta)$ *on* $\mathbf{A} \times \mathbf{X}$, *the following properties* (*denoted by* $\mu^1 \ll \mu^2$) *are equivalent*:

(i) *For every bounded measurable local function* f *on* $\mathbf{A} \times \mathbf{X}$, *such that* $f(\alpha, \cdot)$ *is nondecreasing for all* $\alpha \in \mathbf{A}$, *we have* $\int f \,\mathrm{d}\mu^1 \le \int f \,\mathrm{d}\mu^2$.
(ii) *The measures* μ^1 *and* μ^2 *have a common* α*-marginal* Q, *and* $\mu^1(\mathrm{d}\eta|\alpha) \le \mu^2(\mathrm{d}\eta|\alpha)$ *for* Q*-a.e.* $\alpha \in \mathbf{A}$.
(iii) *There exists a coupling measure* $\overline{\mu}(\mathrm{d}\alpha, \mathrm{d}\eta, \mathrm{d}\xi)$ *supported on* $\overline{\mathcal{O}}_+$ *under which* $(\alpha, \eta) \sim \mu^1$ *and* $(\alpha, \xi) \sim \mu^2$.

Proof. (ii)$\Rightarrow$(i) follows from conditioning. For (i)$\Rightarrow$(ii), consider $f(\alpha, \eta) = g(\alpha)h(\eta)$, where g is a nonnegative measurable function on $\mathbf{A}$ and h a nondecreasing local function on $\mathbf{X}$. Specializing to $h \equiv 1$, using both f and $-f$ in (i), we obtain

$$\int g(\alpha)\mu^1(\mathrm{d}\alpha, \mathrm{d}\eta) = \int g(\alpha)\mu^2(\mathrm{d}\alpha, \mathrm{d}\eta).$$

Thus μ^1 and μ^2 have a common α-marginal Q. Now with a general h, by conditioning, we have

$$\int g(\alpha)\left(\int h(\eta)\mu^1(\mathrm{d}\eta|\alpha)\right)Q(\mathrm{d}\alpha) \le \int g(\alpha)\left(\int h(\eta)\mu^2(\mathrm{d}\eta|\alpha)\right)Q(\mathrm{d}\alpha).$$

Thus, for any nondecreasing local function h on $\mathbf{X}$,

$$\int h(\eta)\mu^1(\mathrm{d}\eta|\alpha) \le \int h(\eta)\mu^2(\mathrm{d}\eta|\alpha)$$

holds $Q(\mathrm{d}\alpha)$-a.e. Since the set of nondecreasing local functions on $\mathbf{X}$ has a countable dense subset (w.r.t. uniform convergence), we can exchange "for any h" and "Q-a.e." In other words, $\mu^1(\mathrm{d}\eta|\alpha) \le \mu^2(\mathrm{d}\eta|\alpha)$ for Q-a.e. $\alpha \in \mathbf{A}$.

For (ii)$\Rightarrow$(iii), by Strassen's theorem [31], for Q-a.e. $\alpha \in \mathbf{A}$, there exists a coupling measure $\overline{\mu}_\alpha(\mathrm{d}\eta, \mathrm{d}\xi)$ on $\mathbf{X}^2$ under which $\eta \sim \mu^1(\cdot|\alpha)$, $\xi \sim \mu^2(\cdot|\alpha)$, and $\eta \le \xi$ a.s. Then $\overline{\mu}(\mathrm{d}\alpha, \mathrm{d}\eta, \mathrm{d}\xi) := \int_{\mathbf{A}}[\delta_\beta(\mathrm{d}\alpha)\overline{\mu}_\alpha(\mathrm{d}\eta, \mathrm{d}\xi)]Q(\mathrm{d}\beta)$ yields the desired coupling. (iii)$\Rightarrow$(i) is straightforward. $\square$

We now state the main result of this section. Let $\mathcal{I}_L$, $\mathcal{S}$ and $\mathcal{S}^{\mathbf{A}}$ denote the sets of probability measures that are respectively invariant for L, shift-invariant on $\mathbf{A} \times \mathbf{X}$ and shift-invariant on $\mathbf{A}$.

Proposition 3.1. *For every* $Q \in \mathcal{S}_e^{\mathbf{A}}$, *there exists a closed subset* $\mathcal{R}^Q$ *of* $[0, K]$ *containing* 0 *and* K, *such that*

$$(\mathcal{I}_L \cap \mathcal{S})_e = \left\{\nu^{Q,\rho}, Q \in \mathcal{S}_e^{\mathbf{A}}, \rho \in \mathcal{R}^Q\right\},$$

where index e denotes the set of extremal elements, *and* $(\nu^{Q,\rho}\colon \rho \in \mathcal{R}^Q)$ *is a family of shift-invariant measures on* $\mathbf{A} \times \mathbf{X}$, *weakly continuous with respect to* ρ, *such that*

$$\int \eta(0)\nu^{Q,\rho}(\mathrm{d}\alpha, \mathrm{d}\eta) = \rho, \tag{21}$$

$$\lim_{l \to \infty} (2l+1)^{-1} \sum_{x \in \mathbb{Z}\colon |x| \le l} \eta(x) = \rho, \quad \nu^{Q,\rho}\text{-a.s.}, \tag{22}$$

$$\rho \le \rho' \quad \Rightarrow \quad \nu^{Q,\rho} \ll \nu^{Q,\rho'}. \tag{23}$$

For $\rho = 0 \in \mathcal{R}^Q$ (resp. $\rho = K \in \mathcal{R}^Q$) we get the invariant distribution $\delta_0^{\otimes\mathbb{Z}}$ (resp. $\delta_K^{\otimes\mathbb{Z}}$), the deterministic distribution of the configuration with no particles (resp. with maximum number of particles K everywhere).

Remark 3.1. *The set* $\mathcal{R}^Q$ *and measures* $\nu^{Q,\rho}$ *also depend on* $p(\cdot)$ *and* $b(\cdot, \cdot)$, *but we did not reflect this in the notation because only* Q *varies in Proposition* 3.1.

Corollary 3.1.

(i) *The family of probability measures* $\nu_\alpha^{Q,\rho}(\cdot) := \nu^{Q,\rho}(\cdot|\alpha)$ *on* $\mathbf{X}$ *satisfies properties* (B1)–(B3) *on page* 407;
(ii) *for* $\rho \in \mathcal{R}^Q$, $G^Q(\rho) = \int j(\alpha,\eta)\nu^{Q,\rho}(\mathrm{d}\alpha,\mathrm{d}\eta)$.

Remark 3.2. *By* (ii) *of Corollary* 3.1, *and shift-invariance of* $\nu^{Q,\rho}(\mathrm{d}\alpha,\mathrm{d}\eta)$,

$$G^Q(\rho) = \int j(\alpha,\eta)\nu^{Q,\rho}(\mathrm{d}\alpha,\mathrm{d}\eta) = \int \widetilde{\jmath}(\alpha,\eta)\nu^{Q,\rho}(\mathrm{d}\alpha,\mathrm{d}\eta) \tag{24}$$

for every $\rho \in \mathcal{R}^Q$, *where*

$$\widetilde{\jmath}(\alpha,\eta) := \alpha(0)\sum_{z\in\mathbb{Z}} zp(z)b\bigl(\eta(0),\eta(z)\bigr). \tag{25}$$

Thus one can alternatively take $\widetilde{\jmath}(\alpha,\eta)$ *as a microscopic flux function* (*we refer to* [4], *p.* 1347 *for an analogous remark in the non-disordered setting*).

Proof of Corollary 3.1. Properties (B1) and (B2) follow from Proposition 3.1 by conditioning (here and after, we proceed as in the proof of Lemma 3.1). By translation invariance of $\nu^{Q,\rho}$ and conditioning we have, for Q-a.e. $\alpha \in \mathbf{A}$,

$$\tau_x \nu_\alpha^{Q,\rho} = \nu_{\tau_x\alpha}^{Q,\rho}, \tag{26}$$

where τ_x on the l.h.s. acts on $\mathbf{X}$. For property (B3) the result will follow from ergodicity of Q once we show that, for every $\rho \in \mathcal{R}^Q$, $G_\alpha^Q(\rho) = G_{\tau_1\alpha}^Q(\rho)$ holds Q-a.s. To this end we note that, as a result of (19),

$$L_\alpha\bigl[\eta(1)\bigr] = j(\alpha,\eta) - j(\tau_1\alpha,\tau_1\eta).$$

Taking expectation w.r.t. invariant measure $\nu_\alpha^{Q,\rho}$, and using (26), we obtain

$$G_\alpha^Q(\rho) = \int j(\alpha,\eta)\nu_\alpha^{Q,\rho}(\mathrm{d}\eta) = \int j(\tau_1\alpha,\tau_1\eta)\nu_\alpha^{Q,\rho}(\mathrm{d}\eta) = \int j(\tau_1\alpha,\eta)\nu_{\tau_1\alpha}^{Q,\rho}(\mathrm{d}\eta) = G_{\tau_1\alpha}^Q(\rho). \qquad \square$$

To prove Proposition 3.1, we need some definitions and lemmas. For every $\alpha \in \mathbf{A}$, we denote by $\overline{L}_\alpha$ the coupled generator on $\mathbf{X}^2$ given by

$$\overline{L}_\alpha f(\eta,\xi) := \sum_{x\in\mathbb{Z}}\int_{\mathcal{V}}\bigl[f\bigl(\mathcal{T}^{\alpha,x,v}\eta,\mathcal{T}^{\alpha,x,v}\xi\bigr) - f(\eta,\xi)\bigr]m(\mathrm{d}v) \tag{27}$$

for any local function f on $\mathbf{X}^2$. For the particular model (6), this is equivalent to the "basic coupling" of L_α defined in [8], namely $\overline{L}_\alpha = \sum_{x,y\in\mathbb{Z}:\, x\neq y}\overline{L}_\alpha^{x,y}$, with $\overline{L}_\alpha^{x,y} f(\eta,\xi)$ given by

$$\begin{aligned}
&\alpha(x)p(y-x)\bigl[b\bigl(\eta(x),\eta(y)\bigr)\wedge b\bigl(\xi(x),\xi(y)\bigr)\bigr]\bigl[f\bigl(\eta^{x,y},\xi^{x,y}\bigr) - f(\eta,\xi)\bigr] \\
&\quad + \alpha(x)p(y-x)\bigl[b\bigl(\eta(x),\eta(y)\bigr) - b\bigl(\xi(x),\xi(y)\bigr)\bigr]^+\bigl[f\bigl(\eta^{x,y},\xi\bigr) - f(\eta,\xi)\bigr] \\
&\quad + \alpha(x)p(y-x)\bigl[b\bigl(\xi(x),\xi(y)\bigr) - b\bigl(\eta(x),\eta(y)\bigr)\bigr]^+\bigl[f\bigl(\eta,\xi^{x,y}\bigr) - f(\eta,\xi)\bigr].
\end{aligned} \tag{28}$$

If (η_t,ξ_t) is a Markov process with generator $\overline{L}_\alpha$, and $\eta_0 \le \xi_0$, then $\eta_t \le \xi_t$ a.s. for every $t > 0$. We indicate this by saying that $\overline{L}_\alpha$ is a *monotone coupling* of L_α. We denote by $\overline{L}$ the coupled generator for the joint process $(\alpha_t,\eta_t,\xi_t)_{t\ge 0}$ on $\mathbf{A}\times\mathbf{X}^2$ defined by

$$\overline{L} f(\alpha,\eta,\xi) = \bigl(\overline{L}_\alpha f(\alpha,\cdot)\bigr)(\eta,\xi) \tag{29}$$

for any local function f on $\mathbf{A}\times\mathbf{X}^2$. Given $\alpha_0=\alpha$, this means that $\alpha_t=\alpha$ for all $t\geq 0$, while $(\eta_t,\xi_t)_{t\geq 0}$ is a Markov process with generator $\overline{L}_\alpha$. Let $\overline{S}(t)$ denote the semigroup generated by $\overline{L}$. We denote by $\overline{\mathcal{S}}$ the set of probability measures on $\mathbf{A}\times\mathbf{X}^2$ that are invariant by space shift $\tau_x(\alpha,\eta,\xi)=(\tau_x\alpha,\tau_x\eta,\tau_x\xi)$. In the following, if $\overline{\nu}(\mathrm{d}\alpha,\mathrm{d}\eta,\mathrm{d}\xi)$ is a probability measure on $\mathbf{A}\times\mathbf{X}^2$, $\overline{\nu}_1$, $\overline{\nu}_2$ and $\overline{\nu}_3$ (resp. $\overline{\nu}_{12}$ and $\overline{\nu}_{13}$) denote marginal distributions of α, η and ξ (resp. (α,η) and (α,ξ)) under $\overline{\nu}$.

Lemma 3.2. *Let $\mu',\mu''\in(\mathcal{I}_L\cap\mathcal{S})_e$ with a common α-marginal Q. Then there exists $\overline{\nu}\in(\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}})_e$ such that $\overline{\nu}_{12}=\mu'$ and $\overline{\nu}_{13}=\mu''$.*

Proof. Let $\overline{\mathcal{M}}(\mu',\mu'')$ denote the set of probability measures $\overline{\nu}\in\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}}$ with $\overline{\nu}_{12}=\mu'$ and $\overline{\nu}_{13}=\mu''$. We show that $\overline{\mathcal{M}}(\mu',\mu'')$ is a nonempty set. Set $\overline{\nu}^0(\mathrm{d}\alpha,\mathrm{d}\eta,\mathrm{d}\xi):=Q(\mathrm{d}\alpha)[\mu'(\mathrm{d}\eta|\alpha)\otimes\mu''(\mathrm{d}\xi|\alpha)]$. Then $\overline{\nu}^0_{12}=\mu'$, $\overline{\nu}^0_{13}=\mu''$ and $\overline{\nu}^0\in\overline{\mathcal{S}}$. Let

$$\overline{\nu}^t:=\frac{1}{t}\int_0^t\overline{\nu}^0\overline{S}(s)\,\mathrm{d}s.$$

The set $\{\overline{\nu}^t,t>0\}$ is relatively compact because $\overline{\nu}^t_1=Q$ is independent of t and, for $i\in\{2,3\}$, $\overline{\nu}^t_i\leq\delta_K^{\otimes\mathbb{Z}}$. Let $\overline{\nu}^\infty$ be any subsequential weak limit of $\overline{\nu}^t$ as $t\to\infty$. Then $\overline{\nu}^\infty$ retains the above properties of $\overline{\nu}^0$, and $\overline{\nu}^\infty\in\mathcal{I}_{\overline{L}}$, thus $\overline{\nu}^\infty\in\overline{\mathcal{M}}(\mu',\mu'')$. Let $\overline{\nu}$ be an extremal element of the compact convex set $\overline{\mathcal{M}}(\mu',\mu'')$. We now prove that $\overline{\nu}\in(\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}})_e$. Assume there exist $\lambda\in(0,1)$ and probability measures $\overline{\nu}^l$, $\overline{\nu}^r$ on $\mathbf{A}\times\mathbf{X}^2$, such that

$$\overline{\nu}=\lambda\overline{\nu}^l+(1-\lambda)\overline{\nu}^r \tag{30}$$

with $\overline{\nu}^i\in\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}}$ for $i\in\{l,r\}$. Since $\overline{\nu}\in\overline{\mathcal{M}}(\mu',\mu'')$, the projections of (30) on (α,η) and (α,ξ) yield

$$\mu'=\lambda\overline{\nu}^l_{12}+(1-\lambda)\overline{\nu}^r_{12}, \tag{31}$$

$$\mu''=\lambda\overline{\nu}^l_{13}+(1-\lambda)\overline{\nu}^r_{13}. \tag{32}$$

For $i\in\{l,r\}$, $\overline{\nu}^i\in\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}}$ implies $\overline{\nu}^i_{1j}\in\mathcal{I}_L\cap\mathcal{S}$ for $j\in\{2,3\}$. Since μ', μ'' belong to $(\mathcal{I}_L\cap\mathcal{S})_e$, $\overline{\nu}^i_{12}=\mu'$, $\overline{\nu}^i_{13}=\mu''$ by (31)–(32), that is, $\overline{\nu}^i\in\overline{\mathcal{M}}(\mu',\mu'')$. Since $\overline{\nu}$ is extremal in $\overline{\mathcal{M}}(\mu',\mu'')$, (30) yields $\overline{\nu}^l=\overline{\nu}^r=\overline{\nu}$. □

Lemma 3.3. *Let ν be a stationary distribution for some Markov transition semigroup, and $(X_t)_{t\geq 0}$ be a Markov process associated to this semigroup with initial distribution ν. Assume A is a subset of E such that, for every $t>0$, $\mathbf{1}_A(X_t)\geq\mathbf{1}_A(X_0)$ almost surely. Then $\nu_A(\mathrm{d}x)=\nu(\mathrm{d}x|x\in A)$ and $\nu_{A^c}(\mathrm{d}x)=\nu(\mathrm{d}x|x\in A^c)$ are stationary for the considered semigroup.*

Proof. Since $(X_t)_{t\geq 0}$ is stationary, we have $\mathbb{E}[\mathbf{1}_A(X_t)]=\mathbb{E}[\mathbf{1}_A(X_0)]$, thus $\mathbf{1}_A(X_0)=\mathbf{1}_A(X_t)$ and $\mathbf{1}_{A^c}(X_t)=\mathbf{1}_{A^c}(X_0)$ almost surely. Stationarity of ν_A amounts to $\mathbb{E}[f(X_t)|X_0\in A]=\mathbb{E}[f(X_0)|X_0\in A]$ for every bounded f. We conclude with

$$\begin{aligned}\mathbb{E}\big[f(X_t)|X_0\in A\big]&=\frac{\mathbb{E}[f(X_t)\mathbf{1}_A(X_0)]}{\mathbb{E}[\mathbf{1}_A(X_0)]}=\frac{\mathbb{E}[f(X_t)\mathbf{1}_A(X_t)]}{\mathbb{E}[\mathbf{1}_A(X_0)]}\\&=\frac{\mathbb{E}[f(X_0)\mathbf{1}_A(X_0)]}{\mathbb{E}[\mathbf{1}_A(X_0)]}=\mathbb{E}\big[f(X_0)|X_0\in A\big].\end{aligned}$$

□

Lemma 3.4. *Let $\overline{\nu}\in(\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}})_e$. Then $\overline{\nu}(\overline{\mathcal{O}}_+)$ and $\overline{\nu}(\overline{\mathcal{O}}_-)$ belong to $\{0,1\}$.*

Proof. Let $A=\{(\alpha,\eta,\xi)\in\mathbf{A}\times\mathbf{X}^2\colon\ \eta\leq\xi\}$ and assume $\lambda:=\overline{\nu}(A)\in(0,1)$. Since the coupling defined by $\overline{L}$ is monotone, we have $\mathbf{1}_A(\alpha_t,\eta_t,\xi_t)\geq\mathbf{1}_A(\alpha_0,\eta_0,\xi_0)$. By Lemma 3.3,

$$\overline{\nu}_A:=\overline{\nu}\big(\mathrm{d}\alpha,\mathrm{d}\eta,\mathrm{d}\xi|(\alpha,\eta,\xi)\in A\big)\in\mathcal{I}_{\overline{L}}.$$

From $\overline{\nu}=\lambda\overline{\nu}_A+(1-\lambda)\overline{\nu}_{A^c}$, we deduce $\overline{\nu}_{A^c}\in\mathcal{I}_{\overline{L}}$. Since A is shift invariant in $\mathbf{A}\times\mathbf{X}^2$, $\overline{\nu}_A$ and $\overline{\nu}_{A^c}$ lie in $\overline{\mathcal{S}}$. By extremality of $\overline{\nu}$, we must have $\overline{\nu}_A=\overline{\nu}_{A^c}$ which is impossible since these measures are supported on disjoint sets. □

Attractiveness assumption (A5) ensures that an initially ordered pair of coupled configurations remains ordered at later times. Assumptions (A1), (A4) induce a stronger property: pairs of opposite discrepancies between two coupled configurations eventually get killed, so that the two configurations become ordered.

Proposition 3.2. *Every $\overline{\nu}\in\mathcal{I}_{\overline{L}}\cap\overline{\mathcal{S}}$ is supported on $\overline{\mathcal{O}}$.*

Proof. We follow the scheme used in [1,8,15,17,22] for the non-disordered case, and only sketch the arguments needed for the disordered setting.

Step 1. For $x\in\mathbb{Z}$, let $f_x(\eta,\xi)=(\eta(x)-\xi(x))^+$. By translation invariance of $\overline{\nu}$, the shift commutation property (19) and (27), (28),

$$
\begin{aligned}
0&=\int\overline{L}_\alpha f_0(\alpha,\eta,\xi)\overline{\nu}(\mathrm{d}\alpha,\mathrm{d}\eta,\mathrm{d}\xi)\\
&=\sum_{v\in\mathbb{Z}}\int\overline{L}_\alpha^{0,v}[f_0+f_v](\alpha,\eta,\xi)\overline{\nu}(\mathrm{d}\alpha,\mathrm{d}\eta,\mathrm{d}\xi).
\end{aligned}
$$

On the other hand (see [8,15])

$$
\overline{L}_\alpha^{0,v}(f_0+f_v)\le -p(v)\alpha(0)\big|b\big(\eta(0),\eta(v)\big)-b\big(\xi(0),\xi(v)\big)\big|(\mathbf{1}_{\{\eta(0)>\xi(0),\eta(v)<\xi(v)\}}+\mathbf{1}_{\{\eta(0)<\xi(0),\eta(v)>\xi(v)\}}).
$$

Using Assumptions (A4)–(A5), (1) and translation invariance of $\overline{\nu}$, we obtain

$$
\begin{aligned}
&\overline{\nu}\big((\alpha,\eta,\xi)\colon\ \eta(x)>\xi(x),\eta(y)<\xi(y)\big)\\
&\quad+\overline{\nu}\big((\alpha,\eta,\xi)\colon\ \eta(x)<\xi(x),\eta(y)>\xi(y)\big)=0
\end{aligned}\tag{33}
$$

for $x\neq y$ with $p(y-x)+p(x-y)>0$. Whenever one of the events in (33) holds, we say there is *a pair of opposite discrepancies* at (x,y).

Step 2. One proves by induction that, for all $n\in\mathbb{N}$, (33) holds if $x\neq y$ with $p^{*n}(y-x)+p^{*n}(x-y)>0$. The induction step is based on the following idea. Assume (η,ξ) has a pair of opposite discrepancies at (x,y). Then one can find a finite path of coupled transitions (with rates uniformly bounded below thanks to (A4)–(A5) and (1)), leading to a coupled state with a pair of opposite discrepancies, either at (x,z) for some z with $p^{*(n-1)}(z-x)+p^{*(n-1)}(x-z)>0$, or at (z,y) with $p^{*(n-1)}(y-z)+p^{*(n-1)}(z-y)>0$. This part of the argument is insensitive to the presence of disorder so long as $\alpha(x)$ is uniformly bounded below.

Conclusion. By irreducibility assumption (A1), (33) holds for all $(x,y)\in\mathbb{Z}^2$ with $x\neq y$. This implies $\overline{\nu}(\overline{\mathcal{O}})=1$. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. We define

$$
\mathcal{R}^Q:=\left\{\int\eta(0)\nu(\mathrm{d}\alpha,\mathrm{d}\eta)\colon\ \nu\in(\mathcal{I}_L\cap\mathcal{S})_e,\nu\text{ has }\alpha\text{-marginal }Q\right\}.
$$

Let $\nu^i\in(\mathcal{I}_L\cap\mathcal{S})_e$ with α-marginal Q and $\rho^i:=\int\eta(0)\nu^i(\mathrm{d}\alpha,\mathrm{d}\eta)\in\mathcal{R}^Q$ for $i\in\{1,2\}$. Assume $\rho^1\le\rho^2$. Using Lemma 3.1(iii), Lemmas 3.2 and 3.4, and Proposition 3.2, we obtain $\nu^1\ll\nu^2$, that is (23). Existence (22) of an asymptotic particle density can be obtained by a proof analogous to [24], Lemma 14, where the space–time ergodic theorem is applied to the joint disorder-particle process. Then, closedness of $\mathcal{R}^Q$ is established as in [4], Proposition 3.1. We end up proving the weak continuity statement given the rest of the proposition. Let $\rho,\rho'\in\mathcal{R}^Q$ with

$\rho \le \rho'$. By (23) and Lemma 3.1, there exists a coupling $\overline{\nu}^{Q,\rho,\rho'}(d\alpha, d\eta, d\xi)$ of $\nu^{Q,\rho}(d\alpha, d\eta)$ and $\nu^{Q,\rho'}(d\alpha, d\xi)$ supported on $\overline{\mathcal{O}}_+$. Thus, for $x \in \mathbb{Z}$

$$\int \left|\eta(x) - \xi(x)\right| \overline{\nu}^{Q,\rho,\rho'}(d\alpha, d\eta, d\xi) = \left|\rho - \rho'\right| \tag{34}$$

from which weak continuity follows by a coupling argument. □

Remark 3.3. *Since*

$$G^Q(\rho) - G^Q(\rho') = \int \left[\tilde{j}(\alpha, \eta) - \tilde{j}(\alpha, \xi)\right] \overline{\nu}^{Q,\rho,\rho'}(d\alpha, d\eta, d\xi) \tag{35}$$

a Lipschitz constant V for G^Q follows from (25), (34):

$$V = 2c^{-1}\|b\|_\infty \sum_{z\in\mathbb{Z}} |z| p(z). \tag{36}$$

4. Proof of hydrodynamics

In this section, we prove the hydrodynamic limit following the strategy introduced in [3,4] and significantly strengthened in [5]. That is, we reduce general Cauchy data to step initial conditions (the so-called Riemann problem) and use a constructive approach (as in [2]). Some technical details similar to [5] will be omitted. We shall rather focus on how to deal with the disorder, which is the substantive part of this paper. The measure Q being fixed once and for all by Theorem 2.1, we simply write ν^ρ, $\mathcal{R}$, G.

4.1. *Riemann problem*

Let $\lambda, \rho \in [0, K]$ with $\lambda < \rho$ (for $\lambda > \rho$ replace infimum with supremum below), and

$$R_{\lambda,\rho}(x, 0) = \lambda \mathbf{1}_{\{x<0\}} + \rho \mathbf{1}_{\{x\ge 0\}}. \tag{37}$$

The entropy solution to the conservation law (13) with initial condition (37), denoted by $R_{\lambda,\rho}(x, t)$, is given ([4], Proposition 4.1) by a variational formula, and satisfies

$$\int_v^w R_{\lambda,\rho}(x, t)\, dx = t\left[\mathcal{G}_{v/t}(\lambda, \rho) - \mathcal{G}_{w/t}(\lambda, \rho)\right], \quad \text{with} \quad \mathcal{G}_v(\lambda, \rho) := \inf\left\{G(r) - vr\colon r \in [\lambda, \rho] \cap \mathcal{R}\right\} \tag{38}$$

for all $v, w \in \mathbb{R}$. Microscopic states with profile (37) will be constructed using the following lemma, established in Section 4.3 below.

Lemma 4.1. *There exist random variables α and $(\eta^\rho\colon \rho \in \mathcal{R})$ on a probability space $(\Omega_{\mathbf{A}}, \mathcal{F}_{\mathbf{A}}, \mathbb{P}_{\mathbf{A}})$ such that*

$$(\alpha, \eta^\rho) \sim \nu^\rho, \qquad \alpha \sim Q, \tag{39}$$

$$\mathbb{P}_{\mathbf{A}}\text{-a.s.}, \qquad \rho \mapsto \eta^\rho \text{ is nondecreasing}. \tag{40}$$

Let $\overline{\nu}^{\lambda,\rho}$ denote the distribution of $(\alpha, \eta^\lambda, \eta^\rho)$, and $\overline{\nu}_\alpha^{\lambda,\rho}$ the conditional distribution of $(\alpha, \eta^\lambda, \eta^\rho)$ given α. For $(x_0, t_0) \in \mathbb{Z} \times \mathbb{R}^+$, the *space–time shift* θ_{x_0,t_0} is defined for any $\omega \in \Omega$, for any $(t, x, z, u) \in \mathbb{R}^+ \times \mathbb{Z} \times \mathbb{Z} \times [0, 1]$, by

$$(t, x, z, u) \in \theta_{x_0,t_0}\omega \quad \text{if and only if} \quad (t_0 + t, x_0 + x, z, u) \in \omega.$$

By its definition and property (7), the mapping introduced in (4) satisfies, for all $s, t \ge 0$, $x \in \mathbb{Z}$ and $(\alpha, \eta, \omega) \in \mathbf{A} \times \mathbf{X} \times \Omega$:

$$\eta_s\left(\alpha, \eta_t(\alpha, \eta, \omega), \theta_{0,t}\omega\right) = \eta_{t+s}(\alpha, \eta, \omega),$$

$$\tau_x \eta_t(\alpha, \eta, \omega) = \eta_t(\tau_x\alpha, \tau_x\eta, \theta_{x,0}\omega).$$

We now introduce an extended shift θ' on $\Omega' = \mathbf{A} \times \mathbf{X}^2 \times \Omega$. If $\omega' = (\alpha, \eta, \xi, \omega)$ denotes a generic element of Ω', we set

$$\theta'_{x,t}\omega' = \big(\tau_x\alpha, \tau_x\eta_t(\alpha, \eta, \omega), \tau_x\eta_t(\alpha, \xi, \omega), \theta_{x,t}\omega\big). \tag{41}$$

It is important to note that this shift incorporates disorder. Let $T : \mathbf{X}^2 \to \mathbf{X}$ be given by

$$T(\eta, \xi)(x) = \eta(x)\mathbf{1}_{\{x<0\}} + \xi(x)\mathbf{1}_{\{x\geq 0\}}. \tag{42}$$

The main result of this subsection is:

Proposition 4.1. *Set, for $t \geq 0$,*

$$\beta_t^N\big(\omega'\big)(\mathrm{d}x) := \pi^N\big(\eta_t\big(\alpha, T(\eta, \xi), \omega\big)\big)(\mathrm{d}x). \tag{43}$$

For all $t > 0$, $s_0 \geq 0$ and $x_0 \in \mathbb{R}$, we have that, for Q-a.e. $\alpha \in \mathbf{A}$,

$$\lim_{N\to\infty} \beta_{Nt}^N\big(\theta'_{\lfloor Nx_0\rfloor, Ns_0}\omega'\big)(\mathrm{d}x) = R_{\lambda,\rho}(\cdot, t)\,\mathrm{d}x, \qquad \overline{\nu}_\alpha^{\lambda,\rho} \otimes \mathbb{P}\text{-}a.s.$$

Proposition 4.1 will follow from a law of large numbers for currents. Let $x_. = (x_t, t \geq 0)$ be a $\mathbb{Z}$-valued *cadlag* random path, with $|x_t - x_{t^-}| \leq 1$, independent of the Poisson measure ω. We define the particle current seen by an observer travelling along this path by

$$\varphi_t^{x.}(\alpha, \eta_0, \omega) = \varphi_t^{x.,+}(\alpha, \eta_0, \omega) - \varphi_t^{x.,-}(\alpha, \eta_0, \omega) + \widetilde{\varphi}_t^{x.}(\alpha, \eta_0, \omega), \tag{44}$$

where $\varphi_t^{x.,\pm}(\alpha, \eta_0, \omega)$ count the number of rightward/leftward crossings of $x.$ due to particle jumps, and $\widetilde{\varphi}_t^{x.}(\alpha, \eta_0, \omega)$ is the current due to the self-motion of the observer. We shall write φ_t^v in the particular case $x_t = \lfloor vt\rfloor$. Set $\phi_t^v(\omega') := \varphi_t^v(\alpha, T(\eta, \xi), \omega)$. Note that for $(v, w) \in \mathbb{R}^2$, $\beta_{Nt}^N(\omega')([v, w]) = t(Nt)^{-1}(\phi_{Nt}^{v/t}(\omega') - \phi_{Nt}^{w/t}(\omega'))$. By (38), Proposition 4.1 is reduced to:

Proposition 4.2. *For all $t > 0$, $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $v \in \mathbb{R}$,*

$$\lim_{N\to\infty} (Nt)^{-1}\phi_{Nt}^v\big(\theta'_{\lfloor bN\rfloor, aN}\omega'\big) = \mathcal{G}_v(\lambda, \rho), \qquad \overline{\nu}^{\lambda,\rho} \otimes \mathbb{P}\text{-}a.s. \tag{45}$$

To prove Proposition 4.2, we introduce a probability space Ω^+, whose generic element is denoted by ω^+, on which is defined a Poisson process $(N_t(\omega^+))_{t\geq 0}$ with intensity $|v|$ ($v \in \mathbb{R}$). Denote by $\mathbb{P}^+$ the associated probability. Set

$$x_s^N\big(\omega^+\big) := \big(\mathrm{sgn}(v)\big)\big[N_{aN+s}\big(\omega^+\big) - N_{aN}\big(\omega^+\big)\big], \tag{46}$$

$$\widetilde{\eta}_s^N\big(\alpha, \eta_0, \omega, \omega^+\big) := \tau_{x_s^N(\omega^+)}\eta_s(\alpha, \eta_0, \omega), \tag{47}$$

$$\widetilde{\alpha}_s^N\big(\alpha, \omega^+\big) := \tau_{x_s^N(\omega^+)}\alpha. \tag{48}$$

Thus $(\widetilde{\alpha}_s^N, \widetilde{\eta}_s^N)_{s\geq 0}$ is a Feller process with generator

$$L^v = L + S^v, \qquad S^v f(\alpha, \zeta) = |v|\big[f(\tau_{\mathrm{sgn}(v)}\alpha, \tau_{\mathrm{sgn}(v)}\zeta) - f(\alpha, \zeta)\big]$$

for f local and $\alpha \in \mathbf{A}$, $\zeta \in \mathbf{X}$. Since any translation invariant measure on $\mathbf{A} \times \mathbf{X}$ is stationary for the pure shift generator S^v, we have $\mathcal{I}_L \cap \mathcal{S} = \mathcal{I}_{L^v} \cap \mathcal{S}$. Define the time and space–time empirical measures (where $\varepsilon > 0$) by

$$m_{tN}\big(\omega', \omega^+\big) := (Nt)^{-1}\int_0^{tN} \delta_{(\widetilde{\alpha}_s^N(\alpha,\omega^+), \widetilde{\eta}_s^N(\alpha, T(\eta,\xi), \omega, \omega^+))}\,\mathrm{d}s, \tag{49}$$

$$m_{tN,\varepsilon}\big(\omega', \omega^+\big) := \big|\mathbb{Z} \cap [-\varepsilon N, \varepsilon N]\big|^{-1} \sum_{x\in\mathbb{Z}:\ |x|\leq\varepsilon N} \tau_x m_{tN}\big(\omega', \omega^+\big). \tag{50}$$

Notice that there is a disorder component we cannot omit in the empirical measure, although ultimately we are only interested in the behavior of the η-component. Let $\mathcal{M}_{\lambda,\rho}$ denote the compact set of probability measures $\mu(\mathrm{d}\alpha, \mathrm{d}\eta) \in \mathcal{I}_L \cap \mathcal{S}$ such that μ has α-marginal Q, and $\nu^\lambda \ll \mu \ll \nu^\rho$. By Proposition 3.1,

$$\mathcal{M}_{\lambda,\rho} = \left\{\nu(\mathrm{d}\alpha, \mathrm{d}\eta) = \int \nu^r(\mathrm{d}\alpha, \mathrm{d}\eta)\gamma(\mathrm{d}r)\colon\ \gamma \in \mathcal{P}\big([\lambda, \rho] \cap \mathcal{R}\big)\right\}. \tag{51}$$

The key ingredients for Proposition 4.2 are the following lemmas, proved in Section 4.3 below.

Lemma 4.2. *The function $\phi_t^v(\alpha, \eta, \xi, \omega)$ is increasing in η, decreasing in ξ.*

Lemma 4.3. *With $\overline{\nu}^{\lambda,\rho} \otimes \mathbb{P} \otimes \mathbb{P}^+$-probability one, every subsequential limit as $N \to \infty$ of $m_{tN,\varepsilon}(\theta'_{\lfloor bN \rfloor, aN}\omega', \omega^+)$ lies in $\mathcal{M}_{\lambda,\rho}$.*

Proof of Proposition 4.2. We will show that

$$\liminf_{N\to\infty}(Nt)^{-1}\phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}\big(\omega'\big) \geq \mathcal{G}_v(\lambda, \rho), \qquad \overline{\nu}^{\lambda,\rho} \otimes \mathbb{P}\text{-a.s.}, \tag{52}$$

$$\limsup_{N\to\infty}(Nt)^{-1}\phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}\big(\omega'\big) \leq \mathcal{G}_v(\lambda, \rho), \qquad \overline{\nu}^{\lambda,\rho} \otimes \mathbb{P}\text{-a.s.} \tag{53}$$

Step one: proof of (52).

Setting $\varpi_{aN} = \varpi_{aN}(\omega') := T(\tau_{\lfloor bN \rfloor}\eta_{aN}(\alpha, \eta, \omega), \tau_{\lfloor bN \rfloor}\eta_{aN}(\alpha, \xi, \omega))$ we have

$$(Nt)^{-1}\phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}\big(\omega'\big) = (Nt)^{-1}\varphi_{tN}^v(\tau_{\lfloor bN \rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN \rfloor, aN}\omega). \tag{54}$$

Let, for every $(\alpha, \zeta, \omega, \omega^+) \in \mathbf{A} \times \mathbf{X} \times \Omega \times \Omega^+$ and $x_{\cdot}^N(\omega^+)$ given by (46),

$$\psi_{tN}^{v,\varepsilon}\big(\alpha, \zeta, \omega, \omega^+\big) := \big|\mathbb{Z} \cap [-\varepsilon N, \varepsilon N]\big|^{-1} \sum_{y \in \mathbb{Z}\colon |y| \leq \varepsilon N} \varphi_{tN}^{x_{\cdot}^N(\omega^+)+y}(\alpha, \zeta, \omega). \tag{55}$$

Note that $\lim_{N\to\infty}(Nt)^{-1}x_{tN}^N(\omega^+) = v$, $\mathbb{P}^+$-a.s., and that for two paths $y_{\cdot}, z_{\cdot}$ (see (44)),

$$\big|\varphi_{tN}^{y_{\cdot}}(\alpha, \eta_0, \omega) - \varphi_{tN}^{z_{\cdot}}(\alpha, \eta_0, \omega)\big| \leq K\big(|y_{tN} - z_{tN}| + |y_0 - z_0|\big).$$

Hence the proof of (52) reduces to that of the same inequality where we replace $(Nt)^{-1}\phi_{tN}^v \circ \theta'_{\lfloor bN \rfloor, aN}(\omega')$ by $(Nt)^{-1}\psi_{tN}^{v,\varepsilon}(\tau_{\lfloor bN \rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN \rfloor, aN}\omega, \omega^+)$ and $\overline{\nu}^{\lambda,\rho} \otimes \mathbb{P}$ by $\overline{\nu}^{\lambda,\rho} \otimes \mathbb{P} \otimes \mathbb{P}^+$. By definitions (14), (44) of flux and current, for any $\alpha \in \mathbf{A}$, $\zeta \in \mathbf{X}$,

$$\begin{aligned} M_{tN}^{x,v}\big(\alpha, \zeta, \omega, \omega^+\big) := {}& \varphi_{tN}^{x_{\cdot}^N(\omega^+)+x}(\alpha, \zeta, \omega) \\ & - \int_0^{tN} \tau_x\big\{j\big(\widetilde{\alpha}_s^N\big(\alpha, \omega^+\big), \widetilde{\eta}_s^N\big(\alpha, \zeta, \omega, \omega^+\big)\big) \\ & - v\big(\widetilde{\eta}_s^N\big(\alpha, \zeta, \omega, \omega^+\big)\big)(\mathbf{1}_{\{v>0\}})\big\}\,\mathrm{d}s \end{aligned}$$

is a mean 0 martingale under $\mathbb{P} \otimes \mathbb{P}^+$. Let

$$\begin{aligned} R_{tN}^{\varepsilon,v} &:= \big(Nt\big|\mathbb{Z} \cap [-\varepsilon N, \varepsilon N]\big|\big)^{-1} \sum_{x \in \mathbb{Z}\colon |x| \leq \varepsilon N} M_{tN}^{x,v}\big(\tau_{\lfloor bN \rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN \rfloor, aN}\omega, \omega^+\big) \\ &= (Nt)^{-1}\psi_{tN}^{v,\varepsilon}\big(\tau_{\lfloor bN \rfloor}\alpha, \varpi_{aN}, \theta_{\lfloor bN \rfloor, aN}\omega, \omega^+\big) \\ &\quad - \int \big[j(\alpha, \eta) - v\eta(\mathbf{1}_{\{v>0\}})\big]m_{tN,\varepsilon}\big(\theta'_{\lfloor bN \rfloor, aN}\omega', \omega^+\big)(\mathrm{d}\alpha, \mathrm{d}\eta), \end{aligned} \tag{56}$$

where the last equality comes from (50), (55). The exponential martingale associated with $M_{tN}^{x,v}$ yields a Poissonian bound, uniform in (α,ζ), for the exponential moment of $M_{tN}^{x,v}$ with respect to $\mathbb{P}\otimes\mathbb{P}^+$. Since ϖ_{aN} is independent of $(\theta_{\lfloor bN\rfloor,aN}\omega,\omega^+)$ under $\overline{\nu}^{\lambda,\rho}\otimes\mathbb{P}\otimes\mathbb{P}^+$, the bound is also valid under this measure, and Borel–Cantelli's lemma implies $\lim_{N\to\infty} R_{tN}^{\varepsilon,v}=0$. From (56), Lemma 4.3 and Corollary 3.1(ii) imply (52), as well as

$$\limsup_{N\to\infty}(Nt)^{-1}\phi_{tN}^{v}\circ\theta'_{\lfloor bN\rfloor,aN}(\omega')\leq\sup_{r\in[\lambda,\rho]\cap\mathcal{R}}\big[G(r)-vr\big],\qquad \overline{\nu}^{\lambda,\rho}\otimes\mathbb{P}\text{-a.s.}\tag{57}$$

Step two: proof of (53). Let $r\in[\lambda,\rho]\cap\mathcal{R}$. We define $\overline{\nu}^{\lambda,r,\rho}$ as the distribution of $(\alpha,\eta^\lambda,\eta^r,\eta^\rho)$. By (52) and (57),

$$\lim_{N\to\infty}(Nt)^{-1}\phi_{tN}^{v}\circ\theta'_{\lfloor bN\rfloor,aN}\big(\alpha,\eta^r,\eta^r,\omega\big)=G(r)-vr.$$

By Lemma 4.2,

$$\phi_{tN}^{v}\circ\theta'_{\lfloor bN\rfloor,aN}\big(\omega'\big)\leq\phi_{tN}^{v}\circ\theta'_{\lfloor bN\rfloor,aN}\big(\alpha,\eta^r,\eta^r,\omega\big).$$

The result follows by continuity of G and minimizing over r. □

4.2. *Cauchy problem*

For two measures $\mu,\nu\in\mathcal{M}^+(\mathbb{R})$ with compact support, we define

$$\Delta(\mu,\nu):=\sup_{x\in\mathbb{R}}\big|\nu\big((-\infty,x]\big)-\mu\big((-\infty,x]\big)\big|\tag{58}$$

which satisfies: (P1) For a sequence $(\mu_n)_{n\geq 0}$ of measures with uniformly bounded support, $\mu_n\to\mu$ vaguely is equivalent to $\lim_{n\to\infty}\Delta(\mu_n,\mu)=0$; (P2) the macroscopic stability property [7,24] states that Δ is, with high probability, an "almost" nonincreasing function of two coupled particle systems; (P3) correspondingly, there is Δ-stability for (13), that is, Δ is a nonincreasing function along two entropy solutions ([5], Proposition 4.1(iii), (b)).

Proposition 4.3. *Assume* (η_0^N) *is a sequence of configurations such that*:

(i) *there exists* $C>0$ *such that for all* $N\in\mathbb{N}$, η_0^N *is supported on* $\mathbb{Z}\cap[-CN,CN]$;
(ii) $\pi^N(\eta_0^N)\to u_0(\cdot)\,\mathrm{d}x$ *as* $N\to\infty$, *where* u_0 *has compact support, is a.e.* $\mathcal{R}$*-valued and has finite space variation.*

Let $u(\cdot,t)$ *denote the unique entropy solution to* (13) *with Cauchy datum* $u_0(\cdot)$. *Then,* $Q\otimes\mathbb{P}$*-a.s. as* $N\to\infty$,

$$\Delta^N(t):=\Delta\big(\pi^N\big(\eta_{Nt}^N\big(\alpha,\eta_0^N,\omega\big)\big),u(\cdot,t)\,\mathrm{d}x\big)$$

converges uniformly to 0 *on* $[0,T]$ *for every* $T>0$.

Theorem 2.1 follows for general initial data u_0 by coupling and approximation arguments (see [5], Section 4.2.2).

Proof of Proposition 4.3. By initial assumption (12), $\lim_{N\to\infty}\Delta^N(0)=0$. Let $\varepsilon>0$, and $\varepsilon'=\varepsilon/(2V)$, for V given by (36). Set $t_k=k\varepsilon'$ for $k\leq\kappa:=\lfloor T/\varepsilon'\rfloor$, $t_{\kappa+1}=T$. Since the number of steps is proportional to ε, if we want to bound the total error, the main step is to prove

$$\limsup_{N\to\infty}\sup_{k=0,\dots,\mathcal{K}-1}\big[\Delta^N(t_{k+1})-\Delta^N(t_k)\big]\leq 3\delta\varepsilon,\qquad Q\otimes\mathbb{P}\text{-a.s.},\tag{59}$$

where $\delta:=\delta(\varepsilon)$ goes to 0 as ε goes to 0; the gaps between discrete times are filled by an estimate for the time modulus of continuity of $\Delta^N(t)$ (see [5], Lemma 4.5).

Proof of (59). Since $u(\cdot,t_k)$ has locally finite variation, by [5], Lemma 4.2, for all $\varepsilon>0$ we can find functions

$$v_k=\sum_{l=0}^{l_k}r_{k,l}\mathbf{1}_{[x_{k,l},x_{k,l+1})}\tag{60}$$

with $-\infty = x_{k,0} < x_{k,1} < \cdots < x_{k,l_k} < x_{k,l_k+1} = +\infty$, $r_{k,l} \in \mathcal{R}$, $r_{k,0} = r_{k,l_k} = 0$, such that $x_{k,l} - x_{k,l-1} \geq \varepsilon$, and

$$\Delta\bigl(u(\cdot, t_k)\,\mathrm{d}x, v_k\,\mathrm{d}x\bigr) \leq \delta\varepsilon. \tag{61}$$

For $t_k \leq t < t_{k+1}$, we denote by $v_k(\cdot, t)$ the entropy solution to (13) at time t with Cauchy datum $v_k(\cdot)$. The configuration $\xi^{N,k}$ defined on $(\Omega_{\mathbf{A}} \otimes \Omega, \mathcal{F}_{\mathbf{A}} \otimes \mathcal{F}, \mathbb{P}_{\mathbf{A}} \otimes \mathbb{P})$ (see Lemma 4.1) by

$$\xi^{N,k}(\omega_{\mathbf{A}}, \omega)(x) := \eta_{Nt_k}\bigl(\alpha(\omega_{\mathbf{A}}), \eta^{r_{k,l}}(\omega_{\mathbf{A}}), \omega\bigr)(x), \quad \text{if } \lfloor N x_{k,l} \rfloor \leq x < \lfloor N x_{k,l+1} \rfloor$$

is a microscopic version of $v_k(\cdot)$, since by Proposition 4.1 with $\lambda = \rho = r^{k,l}$,

$$\lim_{N\to\infty} \pi^N\bigl(\xi^{N,k}(\omega_{\mathbf{A}}, \omega)\bigr)(\mathrm{d}x) = v_k(\cdot)\,\mathrm{d}x, \qquad \mathbb{P}_{\mathbf{A}} \otimes \mathbb{P}\text{-a.s.} \tag{62}$$

We denote by $\xi_t^{N,k}(\omega_{\mathbf{A}}, \omega) = \eta_t(\alpha(\omega_{\mathbf{A}}), \xi^{N,k}(\omega_{\mathbf{A}}, \omega), \theta_{0,Nt_k}\omega)$ evolution starting from $\xi^{N,k}$. By triangle inequality,

$$\begin{aligned} \Delta^N(t_{k+1}) - \Delta^N(t_k) \leq{}& \Delta\bigl[\pi^N\bigl(\eta^N_{Nt_{k+1}}\bigr), \pi^N\bigl(\xi^{N,k}_{N\varepsilon'}\bigr)\bigr] - \Delta^N(t_k) && (63)\\ &+ \Delta\bigl[\pi^N\bigl(\xi^{N,k}_{N\varepsilon'}\bigr), v_k\bigl(\cdot, \varepsilon'\bigr)\,\mathrm{d}x\bigr] && (64)\\ &+ \Delta\bigl(v_k\bigl(\cdot, \varepsilon'\bigr)\,\mathrm{d}x, u(\cdot, t_{k+1})\,\mathrm{d}x\bigr). && (65) \end{aligned}$$

To conclude, we rely on properties (P1)–(P3) of Δ: Since $\varepsilon' = \varepsilon/(2V)$, finite propagation property for (13) and for the particle system (see [5], Proposition 4.1(iii), (a) and Lemma 4.3) and Proposition 4.1 imply

$$\lim_{N\to\infty} \pi^N\bigl(\xi^{N,k}_{N\varepsilon'}(\omega_{\mathbf{A}}, \omega)\bigr) = v_k\bigl(\cdot, \varepsilon'\bigr)\,\mathrm{d}x, \qquad \mathbb{P}_{\mathbf{A}} \otimes \mathbb{P}\text{-a.s.}$$

Hence, the term (64) converges a.s. to 0 as $N \to \infty$. By Δ-stability for (13), the term (65) is bounded by $\Delta(v_k(\cdot)\,\mathrm{d}x, u(\cdot, t_k)\,\mathrm{d}x) \leq \delta\varepsilon$. We now consider the term (63). By macroscopic stability ([24], Theorem 2, Equation (4) and Remark 1), outside probability $\mathrm{e}^{-CN\delta\varepsilon}$,

$$\Delta\bigl[\pi^N\bigl(\eta^N_{Nt_{k+1}}\bigr), \pi^N\bigl(\xi^{N,k}_{N\varepsilon'}\bigr)\bigr] \leq \Delta\bigl[\pi^N\bigl(\eta^N_{Nt_k}\bigr), \pi^N\bigl(\xi^{N,k}\bigr)\bigr] + \delta\varepsilon. \tag{66}$$

Thus the event (66) holds a.s. for N large enough. By triangle inequality,

$$\begin{aligned} &\Delta\bigl[\pi^N\bigl(\eta^N_{Nt_k}\bigr), \pi^N\bigl(\xi^{N,k}\bigr)\bigr] - \Delta^N(t_k)\\ &\quad\leq \Delta\bigl(u(\cdot, t_k)\,\mathrm{d}x, v_k(\cdot)\,\mathrm{d}x\bigr) + \Delta\bigl[v_k(\cdot)\,\mathrm{d}x, \pi^N\bigl(\xi^{N,k}\bigr)\bigr] \end{aligned}$$

for which (61), (62) yield as $N \to \infty$ an upper bound $2\delta\varepsilon$, hence $3\delta\varepsilon$ for the term (63). □

4.3. *Proofs of lemmas*

Proof of Lemma 4.1. Let $\mathcal{R}_d$ be a countable dense subset of $\mathcal{R}$ that contains all the isolated points of $\mathcal{R}$. We denote by $\mathcal{R}_d^+$, resp. $\mathcal{R}_d^-$, the set of $\rho \in [0, K]$ that lie in the closure of $[0, \rho) \cap \mathcal{R}_d$, resp. $(\rho, K] \cap \mathcal{R}_d$. Because $\mathcal{R}$ is closed, we have $\mathcal{R} = \mathcal{R}_d \cup \mathcal{R}_d^+ \cup \mathcal{R}_d^-$. By (23) there exists a subset $\mathbf{A}'$ of $\mathbf{A}$ with Q-probability 1, such that $\nu_\alpha^\rho \leq \nu_\alpha^{\rho'}$ for all $\alpha \in \mathbf{A}'$ and $\rho, \rho' \in \mathcal{R}_d$. By [19], Theorem 6, for every $\alpha \in \mathbf{A}'$, there exists a family of random variables $(\eta_\alpha^\rho\colon \rho \in \mathcal{R}_d)$ on a probability space $(\Omega_\alpha, \mathcal{F}_\alpha, \mathbb{P}_\alpha)$, such that (39)–(40) hold for $\rho \in \mathcal{R}_d$. Let $\Omega_{\mathbf{A}} = \{(\alpha, \omega_\alpha)\colon \alpha \in \mathbf{A}', \omega_\alpha \in \Omega_\alpha\}$, $\mathcal{F}_{\mathbf{A}}$ be the σ-field generated by mappings $(\alpha, \omega_\alpha) \mapsto \eta^\rho(\alpha, \omega_\alpha) := \eta_\alpha^\rho(\omega_\alpha)$ for $\rho \in \mathcal{R}_d$, and $\mathbb{P}_{\mathbf{A}}(\mathrm{d}\alpha, \mathrm{d}\omega_\alpha) = Q(\mathrm{d}\alpha) \otimes \mathbb{P}_\alpha(\mathrm{d}\omega_\alpha)$. Now consider $\rho \in \mathcal{R} \setminus \mathcal{R}_d$. Since η_α^r is a nondecreasing function of r, for every $\alpha \in \mathbf{A}'$ and $\omega_\alpha \in \Omega_\alpha$, $\eta^{\rho+}(\alpha, \omega_\alpha) := \lim_{r\to\rho, r<\rho, r\in\mathcal{R}_d} \eta_\alpha^r(\omega_\alpha)$ exists if $\rho \in \mathcal{R}_d^+$, and $\eta^{\rho-}(\alpha, \omega_\alpha) := \lim_{r\to\rho, r>\rho, r\in\mathcal{R}_d} \eta_\alpha^r(\omega_\alpha)$ exists if $\rho \in \mathcal{R}_d^-$. We set $\eta^\rho(\alpha, \omega_\alpha) = \eta^{\rho+}(\alpha, \omega_\alpha)$ if $\rho \in \mathcal{R}_d^+$, $\eta^\rho(\alpha, \omega_\alpha) = \eta^{\rho-}(\alpha, \omega_\alpha)$ otherwise. Suppose for instance $\rho \in \mathcal{R}_d^+$. Since $\eta^{\rho+}$ is a $\mathbb{P}_{\mathbf{A}}$-a.s. limit of η^r as $r \to \rho$, $r < \rho$, $r \in \mathcal{R}_d$, it is a limit in distribution. Weak continuity of ν^ρ then implies (39). Property (40) on $\mathcal{R}$ follows from the property on $\mathcal{R}_d$ and definitions of $\eta^{\rho\pm}$. □

To prove Lemma 4.3, we need the following uniform upper bound (proved in [5], Lemma 3.4).

Lemma 4.4. *Let $\mathbf{P}_\nu^v$ denote the law of a Markov process $(\widetilde{\alpha}_., \widetilde{\xi}_.)$ with generator L^v and initial distribution ν. For $\varepsilon > 0$, let*

$$\pi_{t,\varepsilon} := \left|\mathbb{Z} \cap [-\varepsilon t, \varepsilon t]\right|^{-1} \sum_{x\in\mathbb{Z}\cap[-\varepsilon t,\varepsilon t]} t^{-1} \int_0^t \delta_{(\tau_x \widetilde{\alpha}_s, \tau_x \widetilde{\xi}_s)} \, \mathrm{d}s. \tag{67}$$

Then, there exists a functional $\mathcal{D}_v$ which is nonnegative, l.s.c., and satisfies $\mathcal{D}_v^{-1}(0) = \mathcal{I}_{L^v}$, such that, for every closed subset F of $\mathcal{P}(\mathbf{A} \times \mathbf{X})$,

$$\limsup_{t\to\infty} t^{-1} \log \sup_{\nu\in\mathcal{P}(\mathbf{A}\times\mathbf{X})} \mathbf{P}_\nu^v\big(\pi_{t,\varepsilon}(\widetilde{\xi}_.) \in F\big) \le -\inf_{\mu\in F} \mathcal{D}_v(\mu). \tag{68}$$

Proof of Lemma 4.3. We give a brief sketch of the arguments (details are similar to [5], Lemma 3.3). Spatial averaging in (67) implies that any subsequential limit μ lies in $\mathcal{S}$. Lemma 4.4 and Borel–Cantelli's lemma imply that μ lies in $\mathcal{I}_{L^v}$ (uniformity in (68) is important because θ-shifts make the initial distribution of the process unknown). Finally, the inequality $\nu^\lambda \ll \mu \ll \nu^\rho$ is obtained by coupling the initial distribution with η^λ and η^ρ, using attractiveness and space–time ergodicity for the equilibrium processes. □

Proof of Lemma 4.2. Assume for instance $\eta \le \eta'$. Let $\gamma := T(\eta, \xi)$ and $\gamma' := T(\eta', \xi)$, $\gamma_t = \eta_t(\alpha, \gamma, \omega)$ and $\gamma_t' = \eta_t(\alpha, \gamma', \omega)$. By (8), $\gamma_t \le \gamma_t'$ for all $t \ge 0$. By definition of the current, $\phi_t^v(\alpha, \eta', \xi, \omega) - \phi_t^v(\alpha, \eta, \xi, \omega) = \sum_{x>vt}[\gamma_t'(x) - \gamma_t(x)] \ge 0$. □

5. Other models

For the proof of Theorem 2.1 we have not used the particular form of L_α in (2), but the following properties.

1) The set of environments is a probability space $(\mathbf{A}, \mathcal{F}_\mathbf{A}, Q)$, where $\mathbf{A}$ is a compact metric space and $\mathcal{F}_\mathbf{A}$ its Borel σ-field. On $\mathbf{A}$ we have a group of space shifts $(\tau_x\colon x \in \mathbb{Z})$, with respect to which Q is ergodic. For each $\alpha \in \mathbf{A}$, L_α is the generator of a Feller process on $\mathbf{X}$ that satisfies (19). The latter should be viewed as the assumption on "how the disorder enters the dynamics." It is equivalent to L satisfying (18), that is being a translation-invariant generator on $\mathbf{A} \times \mathbf{X}$.

2) For L_α we can define a graphical construction (5) on a space–time Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$ such that L_α coincides with (10), for some mapping $\mathcal{T}^{\alpha,z,v}$ satisfying the shift commutation and strong attractiveness properties (7) and (8). The existence of this graphical construction for the infinite-volume system follows from assumption (A2), which controls the rate of faraway jumps. This assumption is also responsible for the finite propagation property of discrepancies in the particle system, and its macroscopic counterpart, the Lipschitz continuity of the flux function (see (15), Remarks 3.2 and 3.3).

3) Irreducibility and non-degeneracy assumptions (A1), (A4) (combined with attractiveness assumption (A5)) imply Proposition 3.2.

In the sequel we consider other models satisfying 1) and 2), for which appropriate assumptions replacing (A1)–(A5) imply existence of a graphical construction, and Proposition 3.2 as in 3). In these examples, the transition defined by $\mathcal{T}^{\alpha,z,v}$ in (5) is a particle jump, that is of the form $\mathcal{T}^{\alpha,z,v}\eta = \eta^{x(\alpha,z,v),y(\alpha,z,v)}$. It follows that (10) yields (in replacement of (2))

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} c_\alpha(x, y, \eta)\big[f\big(\eta^{x,y}\big) - f(\eta)\big], \tag{69}$$

where

$$c_\alpha(x, y, \eta) = \sum_{z\in\mathbb{Z}} m\big(\big\{v \in \mathcal{V}\colon \mathcal{T}^{\alpha,z,v}\eta = \eta^{x,y}\big\}\big) \tag{70}$$

and the shift-commutation property (7) implies

$$c_\alpha(x, y, \eta) = c_{\tau_x\alpha}(0, y - x, \tau_x\eta) \tag{71}$$

which, for (69), is equivalent to (19). Microscopic fluxes (14) and (25) more generally write

$$\begin{aligned} j^+(\alpha, \eta) &= \sum_{y,z\in\mathbb{Z}:\ y\le 0<y+z} c_\alpha\big(\eta(y), \eta(y+z)\big), \\ j^-(\alpha, \eta) &= \sum_{y,z\in\mathbb{Z}:\ y+z\le 0<y} c_\alpha\big(\eta(y), \eta(y+z)\big), \\ \widetilde{\jmath}(\alpha, \eta) &= \sum_{z\in\mathbb{Z}} z c_\alpha(0, z, \eta). \end{aligned} \tag{72}$$

5.1. *Generalized misanthropes' process*

Let $c \in (0, 1)$, and $p(\cdot)$ (resp. $P(\cdot)$), be a probability distribution on $\mathbb{Z}$ satisfying assumption (A1) (resp. (A2)). Define $\mathbf{A}$ to be the set of functions $B : \mathbb{Z}^2 \times \{0, \dots, K\}^2 \to \mathbb{R}^+$ such that for all $(x, z) \in \mathbb{Z}^2$, $B(x, z, \cdot, \cdot)$ satisfies assumptions (A3)–(A5) and

$$B(x, z, 1, K-1) \ge cp(z), \tag{73}$$

$$B(x, z, K, 0) \le c^{-1} P(z). \tag{74}$$

The shift operator τ_y on $\mathbf{A}$ is defined by $(\tau_y B)(x, z, n, m) = B(x + y, z, n, m)$. We generalize (2) by setting

$$L_\alpha f(\eta) = \sum_{x,y\in\mathbb{Z}} B\big(x, y - x, \eta(x), \eta(y)\big)\big[f\big(\eta^{x,y}\big) - f(\eta)\big], \tag{75}$$

where we assume that the distribution Q of $B(\cdot, \cdot, \cdot, \cdot)$ is ergodic with respect to the above spatial shift (we kept the notation L_α to be consistent with the rest of the paper, but we should have written L_B). Assumption (73) replaces (A1) and implies Proposition 3.2. Assumption (74) replaces (A2) and implies existence of the infinite volume dynamics given by the following graphical construction. For $v = (z, u)$, set $m(\mathrm{d}v) = c^{-1} P(\mathrm{d}z)\lambda_{[0,1]}(\mathrm{d}u)$ in (3), and replace (6) with

$$\mathcal{T}^{\alpha,x,v}\eta = \begin{cases} \eta^{x,x+z}, & \text{if } u < \dfrac{B(x, z, \eta(x), \eta(x+z))}{c^{-1}P(z)}, \\ \eta, & \text{otherwise.} \end{cases} \tag{76}$$

Here the microscopic flux (72) writes

$$\widetilde{\jmath}(\alpha, \eta) = \sum_{z\in\mathbb{Z}} zB\big(0, z, \eta(0), \eta(z)\big)$$

and the Lipschitz constant $V = 2c^{-1}\sum_{z\in\mathbb{Z}} |z|P(z)$ for G^Q follows as in (36) from (34)–(35). The basic model (2) is recovered with $B(x, z, n, m) = \alpha(x)p(z)b(n, m)$, for $p(\cdot)$ a probability distribution on $\mathbb{Z}$ satisfying (A1)–(A2), $\alpha(\cdot)$ an ergodic $(c, 1/c)$-valued random field, and $b(\cdot, \cdot)$ a function satisfying (A3)–(A5). In this case (73)–(74) hold with $P(\cdot) = p(\cdot)$. Here are two other examples.

Example 5.1. *This is the bond-disorder version of* (2): *we have* $B(x, z, n, m) = \alpha(x, x + z)b(n, m)$, *where* $\alpha = (\alpha(x, y) : x, y \in \mathbb{Z})$ *is a positive random field on* $\mathbb{Z}^2$, *bounded away from* 0, *ergodic with respect to the space shift* $\tau_z\alpha = \alpha(\cdot + z, \cdot + z)$. *Sufficient assumptions replacing* (A1) *and* (A2) *are*

$$cp(y - x) \le \alpha(x, y) \le c^{-1}P(y - x) \tag{77}$$

for some constant $c > 0$, *and probability distributions* $p(\cdot)$ *and* $P(\cdot)$ *on* $\mathbb{Z}$, *respectively satisfying* (A1) *and* (A2).

Example 5.2. *This is a model that switches between two rate functions according to the environment*: *we have* $B(x,z,n,m) = p(z)[(1-\alpha(x))b_0(n,m) + \alpha(x)b_1(n,m)]$, *where* $(\alpha(x), x \in \mathbb{Z})$ *is an ergodic* $\{0,1\}$*-valued field*, $p(\cdot)$ *satisfies assumption* (A1), *and* b_0, b_1 *assumptions* (A3)–(A5).

5.2. *Generalized* k*-step* K*-exclusion process*

We first recall the definition of the k-step exclusion process, introduced in [17]. Let $K = 1$, $k \in \mathbb{N}$, and $p(\cdot)$ be a jump kernel on $\mathbb{Z}$ satisfying assumptions (A1)–(A2). A particle at x performs a random walk with kernel $p(\cdot)$ and jumps to the first vacant site it finds along this walk, unless it returns to x or does not find an empty site within k steps, in which case it stays at x.

To generalize this, let $K \geq 1$, $k \geq 1$, $c \in (0,1)$, and $\mathcal{D}$ denote the set of functions $\beta = (\beta^1, \dots, \beta^k)$ from $\mathbb{Z}^k$ to $(0,1]^k$ such that

$$\beta^1(\cdot) \in [c,1], \tag{78}$$

$$\beta^i(\cdot) \geq \beta^{i+1}(\cdot), \quad \forall i \in \{1, \dots, k-1\}. \tag{79}$$

In the sequel, an element of $\mathbb{Z}^k$ is denoted by $\underline{z} = (z_1, \dots, z_k)$. Let q be a probability distribution on $\mathbb{Z}^k$, and $\beta \in \mathcal{D}$. We define the (q,β)-k-step K-exclusion process as follows. A particle at x (if some) picks a q-distributed random vector $\underline{Z} = (Z_1, \dots, Z_k)$, and jumps to the first site $x + Z_i$ ($i \in \{1, \dots, k\}$) with strictly less than K particles along the path $(x + Z_1, \dots, x + Z_k)$, if such a site exists, with rate $\beta^i(\underline{Z})$. Otherwise, it stays at x. The k-step exclusion process corresponds to the particular case where $K = 1$, q is the distribution (hereafter denoted by $q^k_{RW}(p)$) of the first k-steps of a random walk with kernel $p(\cdot)$ absorbed at 0, and $\beta^i(\underline{z}) = 1$. Outside the fact that K can take values ≥ 1, our model extends k-step exclusion in different directions:

1) The random path followed by the particle need not be a Markov process.
2) The distribution q is not necessarily supported on paths absorbed at 0.
3) Different rates can be assigned to jumps according to the number of steps, and the collection of these rates may depend on the path realization.

Next, disorder is introduced: the environment is a field $\alpha = ((q_x, \beta_x)\colon x \in \mathbb{Z}) \in \mathbf{A} := (\mathcal{P}(\mathbb{Z}^k) \times \mathcal{D})^{\mathbb{Z}}$. For a given realization of the environment, the distribution of the path $\underline{Z}$ picked by a particle at x is q_x, and the rate at which it jumps to $x + Z_i$ is $\beta^i_x(\underline{Z})$. The corresponding generator is given by (69) with $c_\alpha = \sum_{i=1}^k c^i_\alpha$, where (with the convention that an empty product is equal to 1)

$$c^i_\alpha(x,y,\eta) = \mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(y)<K\}} \int \left[\beta^i_x(\underline{z})\mathbf{1}_{\{x+z_i=y\}} \prod_{j=1}^{i-1} \mathbf{1}_{\{\eta(x+z_j)=K\}}\right] \mathrm{d}q_x(\underline{z}).$$

The distribution Q of the environment on $\mathbf{A}$ is assumed ergodic with respect to the space shift τ_y, where $\tau_y\alpha = ((q_{x+y}, \beta_{x+y})\colon x \in \mathbb{Z})$.

For the existence of the process and graphical construction below, and for Proposition 3.2, sufficient assumptions to replace (A1)–(A2) are

$$\inf_{x\in\mathbb{Z}} q^1_x(\cdot) \geq cp(\cdot), \tag{80}$$

$$\sup_{i=1,\dots,k} \sup_{x\in\mathbb{Z}} q^i_x(\cdot) \leq c^{-1}P(\cdot) \tag{81}$$

for some constant $c > 0$, where q^i_x denotes the ith marginal of q_x, and $p(\cdot)$, resp. $P(\cdot)$, are probability distributions satisfying (A1), resp. (A2). To write the microscopic flux and define a graphical construction, we introduce the following notation: for $(x, \underline{z}, \eta) \in \mathbb{Z} \times \mathbb{Z}^k \times \mathbf{X}$, $\beta \in \mathcal{D}$ and $u \in [0,1]$,

$$N(x,\underline{z},\eta) = \inf\bigl\{i \in \{1,\dots,k\}\colon \eta(x+z_i) < K\bigr\} \quad \text{with } \inf\varnothing = +\infty,$$

$$Y(x,\underline{z},\eta)=\begin{cases}x+z_{N(x,\underline{z},\eta)}, & \text{if } N(x,\underline{z},\eta)<+\infty,\\ x, & \text{if } N(x,\underline{z},\eta)=+\infty,\end{cases}$$

$$\mathcal{T}_0{}^{x,\underline{z},\beta,u}\eta=\begin{cases}\eta^{x,Y(x,\underline{z},\eta)}, & \text{if } \eta(x)>0 \quad\text{and}\quad u<\beta^{N(x,\underline{z},\eta)}(\underline{z}),\\ \eta, & \text{otherwise}\end{cases}$$

(where the definition of $\beta^{+\infty}(\underline{z})$ has no importance). With these notations, we have

$$c_\alpha(x,y,\eta)=\mathbf{1}_{\{\eta(x)>0\}}\mathbb{E}_{q_0}\big[\beta_0^{N(x,\underline{Z},\eta)}\mathbf{1}_{\{Y(x,\underline{Z},\eta)=y\}}\big], \tag{82}$$

$$\widetilde{j}(\alpha,\eta)=\mathbf{1}_{\{\eta(0)>0\}}\mathbb{E}_{q_0}\big[\beta_0^{N(0,\underline{Z},\eta)}Y(0,\underline{Z},\eta)\big], \tag{83}$$

where expectation is with respect to $\underline{Z}$. Since

$$\big|\beta_0^{N(0,\underline{Z},\eta)}Y(0,\underline{Z},\eta)-\beta_0^{N(0,\underline{Z},\xi)}Y(0,\underline{Z},\xi)\big|\le 2\sum_{i=1}^k|Z_i|\sum_{i=1}^k\big|\eta(Z_i)-\xi(Z_i)\big|$$

(34)–(35) yield for G^Q the Lipschitz constant $V=2k^2c^{-1}\sum_{z\in\mathbb{Z}}|z|P(z)$.

Let $\mathcal{V}=[0,1]\times[0,1]$, $m=\lambda_{[0,1]}\otimes\lambda_{[0,1]}$. For each probability distribution q on $\mathbb{Z}^k$, there exists a mapping $F_q:[0,1]\to\mathbb{Z}^k$ such that $F_q(V_1)$ has distribution q if V_1 is uniformly distributed on $[0,1]$. Then the transformation $\mathcal{T}$ in (5) is defined by (with $v=(v_1,v_2)$ and $\alpha=((q_x,\beta_x)\colon x\in\mathbb{Z})$)

$$\mathcal{T}^{\alpha,x,v}\eta=\mathcal{T}_0^{x,F_{q_x}(v_1),\beta_x(F_{q_x}(v_1)),v_2}\eta. \tag{84}$$

Strong attractiveness of our process will follow from:

Lemma 5.1. *For every $(x,\underline{z},u)\in\mathbb{Z}\times\mathbb{Z}^k\times[0,1]$, $\mathcal{T}_0^{x,\underline{z},\beta,u}$ is an increasing mapping from $\mathbf{X}$ to $\mathbf{X}$.*

Proof. Let $(\eta,\xi)\in\mathbf{X}^2$ with $\eta\le\xi$. To prove that $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta\le\mathcal{T}_0^{x,\underline{z},\beta,u}\xi$, since η and ξ can only possibly change at sites x, $y:=Y(x,\underline{z},\eta)$ and $y':=Y(x,\underline{z},\xi)$, it is sufficient to verify the inequality at these sites.

If $\xi(x)=0$, then by (84), η and ξ are both unchanged by $\mathcal{T}_0^{x,\underline{z},\beta,u}$. If $\eta(x)=0<\xi(x)$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(y')\ge\xi(y')\ge\eta(y')=\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(y')$.

Now assume $\eta(x)>0$. Then $\eta\le\xi$ implies $N(x,\underline{z},\eta)\le N(x,\underline{z},\xi)$. If $N(x,\underline{z},\eta)=+\infty$, η and ξ are unchanged. If $N(x,\underline{z},\eta)<N(x,\underline{z},\xi)=+\infty$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta=\eta^{x,y}$ and $\xi(y)=K$. Thus, $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(x)=\eta(x)-1\le\xi(x)=\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(x)$ and $\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(y)=\xi(y)=K\ge\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(y)$. If $N(x,\underline{z},\eta)=N(x,\underline{z},\xi)<+\infty$, then $\beta^{N(x,\underline{z},\eta)}=\beta^{N(x,\underline{z},\xi)}=:\beta$. If $u\ge\beta$ both η and ξ are unchanged. Otherwise $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta=\eta^{x,y}$ and $\mathcal{T}_0^{x,\underline{z},\beta,u}\xi=\xi^{x,y}$, whence the conclusion. Finally, assume $N(x,\underline{z},\eta)<N(x,\underline{z},\xi)<+\infty$, hence $\beta:=\beta^{N(x,\underline{z},\eta)}\ge\beta^{N(x,\underline{z},\xi)}=:\beta'$ by (79) and $\eta(y)<\xi(y)=K$. If $u\ge\beta$, η and ξ are unchanged. If $u<\beta'$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(y)=\eta(y)+1\le\xi(y)=\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(y)=K$ and $\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(y')=\xi(y')+1\ge\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(y')$. If $\beta'\le u<\beta$, then $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(x)=\eta(x)-1\le\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(x)$ and $\mathcal{T}_0^{x,\underline{z},\beta,u}\eta(y)=\eta(y)+1\le\mathcal{T}_0^{x,\underline{z},\beta,u}\xi(y)=\xi(y)=K$. □

We now describe a few examples.

Example 5.3. *Let $K=1$, $(\alpha_x\colon x\in\mathbb{Z})$ be an ergodic $[c,1/c]$-valued random field, and $r(\cdot)$ be a probability measure on $\mathbb{Z}$ satisfying* (A1)–(A2). *A disordered version of the k-step exclusion process with jump kernel r is obtained by multiplying the rate of any jump starting from x by α_x. This means that the random field $(q_x,\beta_x)_{x\in\mathbb{Z}}$ is defined by $q_x=q_{RW}^k(r)$, and $\beta_x(\underline{z})=(\alpha_x,\dots,\alpha_x)$ for every $\underline{z}\in\mathbb{Z}^k$.*

Example 5.4. *Let* $(\gamma_x, \iota_x)_{x\in\mathbb{Z}}$ *be an ergodic* $[c,1]^{2k}$*-valued random field, where* $\gamma_x = (\gamma_x^n, 1 \le n \le k)$ *and* $\iota_x = (\iota_x^n, 1 \le n \le k)$. *The random field* $(q_x, \beta_x)_{x\in\mathbb{Z}}$ *is defined by*

$$q_x = \frac{1}{2}\delta_{(1,2,\dots,k)} + \frac{1}{2}\delta_{(-1,-2,\dots,-k)},$$
$$\beta_x^i(1,2,\dots,k) = 2\gamma_x^i, \qquad \beta_x^i(-1,-2,\dots,-k) = 2\iota_x^i.$$

Hence the rates are disordered but not the distribution of the random path followed by particles: *the stationary random field* $(q_x)_{x\in\mathbb{Z}}$ *is deterministic and uniform. Here*, *the jump rate and microscopic flux* (82)–(83) *have a fairly explicit form*:

$$c_\alpha(x,y,\eta) = \gamma_x^{y-x}\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(y)<K\}}\prod_{z=x+1}^{y-1}\mathbf{1}_{\{\eta(z)=K\}} \quad \text{if } y > x, \tag{85}$$

$$c_\alpha(x,y,\eta) = \iota_x^{x-y}\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(y)<K\}}\prod_{z=y+1}^{x-1}\mathbf{1}_{\{\eta(z)=K\}} \quad \text{if } y < x, \tag{86}$$

$$\widetilde{\jmath}(\alpha,\eta) = \eta(0)\sum_{n=1}^{k} n\gamma_0^n\bigl(1-\eta(n)\bigr)\prod_{j=1}^{n-1}\eta(j) - \eta(0)\sum_{n=1}^{k} n\iota_0^n\bigl(1-\eta(-n)\bigr)\prod_{j=1}^{n-1}\eta(-j). \tag{87}$$

Example 5.5. *Set* $q_x = q_{RW}^k(r_x)$, *for* $(r_x)_{x\in\mathbb{Z}}$ *an ergodic random field with values in the probability measures on* $\mathbb{Z}$ *satisfying* (A1)–(A2). *The simplest case is nearest-neighbor jumps, that is,* $r_x = p_x\delta_1 + (1-p_x)\delta_{-1}$, *where, for some* $c \in (0,1)$, $(p_x)_{x\in\mathbb{Z}}$ *is an ergodic* $[c, 1/c]$*-random field. Due to the nearest-neighbor assumption, a particle starting from* x *can only jump to* $y > x$ (*resp.* $y < x$) *if* y *is not full and all sites between* x *and* y (*resp.* y *and* x) *are full. Hence*, *the jump rate* (82) *is identical* (*see example below*) *to the one obtained by taking in* (85)–(86)

$$\gamma_x^n = \sum_{l=0}^{\lfloor (k-n)/2\rfloor} p_x^{n+l}(1-p_x)^l C_n(n+l,l), \qquad \iota_x^n = \sum_{l=0}^{\lfloor (k-n)/2\rfloor}(1-p_x)^{n+l}p_x^l C_n(n+l,l)$$

for $n \in \{1,\dots,k\}$, *where* $C_n(i,j)$, *for* $i, j \in \mathbb{Z}^+$ *and* $i + j > 0$, *is the number of paths* $(z_0 = 0, \dots, z_{i+j})$ *such that* $0 < z_m < n$ *for* $m = 1,\dots,i+j-1$, $|z_{m+1} - z_m| = 1$ *for* $m = 1,\dots,i+j$, *and* $\operatorname{Card}\{m \in \{1,\dots,i+j\}\colon z_m - z_{m-1} = 1\} = i$. *With this choice of* γ_x^n *and* ι_x^n, *the microscopic flux is given by* (87). *For instance if* $k = 5$, *we obtain, for* $n \in \{1,\dots,k\}$:

$$c_\alpha(x,x+n,\eta) = p_x^n\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(x+n)<K\}}\prod_{j=1}^{n-1}\mathbf{1}_{\{\eta(x+j)=K\}} \quad \text{if } n \ne 3, \tag{88}$$

$$c_\alpha(x,x+3,\eta) = p_x^3\bigl[1+p_x(1-p_x)\bigr]\eta(x)\eta(x+1)\eta(x+2)\bigl(1-\eta(x+3)\bigr), \tag{89}$$

$$c_\alpha(x,x-n,\eta) = (1-p_x)^n\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(x-n)<K\}}\prod_{j=1}^{n-1}\mathbf{1}_{\{\eta(x-j)=K\}} \quad \text{if } n \ne 3,$$

$$c_\alpha(x,x-3,\eta) = (1-p_x)^3\bigl[1+p_x(1-p_x)\bigr]\eta(x)\eta(x-1)\eta(x-2)\bigl(1-\eta(x-3)\bigr).$$

Indeed, for $n > 0$ *and* $n \ne 3$, *the only path from* x *to* $x+n$ *that reaches* $x+n$ *in at most* k*-steps before returning to* 0 *is* $x, x+1, \dots, x+n$. *For* $n = 3$, *the additional path* $x, x+1, x+2, x+1, x+2, x+3$ *yields the factor* $p_x(1-p_x)$ *in* (89). *For* $n < 0$, *we change* p_x *to* $1 - p_x$.

Note that in this process a given particle does not follow a random walk in random environment (RWRE) before it finds a nonfull site, but a homogeneous random walk depending (randomly) on its initial location. For instance, in a 3-step process, a particle initially at $x \in \mathbb{Z}$ will follow the path $x, x+1, x+2, x+1$ with probability $p_x^2(1-p_x)$.

Example 5.6. *The same random field $(p_x)_{x\in\mathbb{Z}}$ gives a different model if, at each transition, the selected particle follows a RWRE $(X_n)_{n\geq 0}$ with transition probabilities*

$$\mathbb{P}(X_{n+1}=x+1|X_n=x)=p_x, \qquad \mathbb{P}(X_{n+1}=x-1|X_n=x)=1-p_x. \tag{90}$$

That is, we let q_x be the distribution of $(X_1^x-x,\dots,X_k^x-x)$, for $(X_n^x, 1\leq n\leq k)$ a length k Markov chain starting at x with transition probabilities (90). There, unlike in Example 5.5 above, a particle initially at $x\in\mathbb{Z}$ follows the path $x, x+1, x+2, x+1$ with probability $p_x p_{x+1}(1-p_{x+2})$. The generator of this process is also identical to that of Example 5.4, with γ_x^n and ι_x^n of the form

$$\gamma_x^n=\gamma^n(p_y\colon x\leq y<x+n), \qquad \iota_x^n=\iota^n(p_y\colon x-n<y\leq x)$$

for some polynomial functions $\gamma^n, \iota^n : [0,1]^n \to [0,+\infty)$, where $n\in\{1,\dots,k\}$.

5.3. *K-exclusion process with speed change and traffic flow model*

Let $\mathcal{K} := \{-k,\dots,k\}\setminus\{0\}$, and $\alpha=((\upsilon(x),\beta_x^1)\colon x\in\mathbb{Z})$ be an ergodic $[0,+\infty)^{2k}\times(0,+\infty)$-valued field, where $\upsilon(x)=(\upsilon_z(x)\colon z\in\mathcal{K})$. We define the following dynamics. Set

$$\Theta(x,\eta) := \{y\in\mathbb{Z}\colon y-x\in\mathcal{K}, \eta(y)<K\},$$
$$Z(\alpha,x,\eta) := \sum_{z\in\Theta(x,\eta)}\upsilon_{z-x}(x).$$

In configuration η, if $Z(\alpha,x,\eta)>0$, a particle at x picks a site y at random in $\Theta(x,\eta)$ with probability $Z(\alpha,x,\eta)^{-1}\upsilon_{y-x}(x)$, and jumps to this site at rate β_x^1. If $Z(\alpha,x,\eta)=0$, nothing happens. For instance, if $\upsilon_z(x)\equiv 1$, the particle uniformly chooses a site with strictly less than K particles. The corresponding generator is given by (69), with

$$c_\alpha(x,y,\eta)=\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{Z(\alpha,x,\eta)>0\}}\mathbf{1}_{\Theta(x,\eta)}(y)Z(\alpha,x,\eta)^{-1}\upsilon_{y-x}(x).$$

Hence, the microscopic flux (25) writes

$$\widetilde{\jmath}(\alpha,\eta)=\beta_0^1\mathbf{1}_{\{\eta(0)>0\}}Z(\alpha,0,\eta)^{-1}\sum_{z\in\mathcal{K}}z\upsilon_z(0)\mathbf{1}_{\{\eta(z)<K\}}.$$

This process can be compared with a bond-disordered K-exclusion process in which a particle at x jumps to y with rate $\alpha(x,y)=\upsilon_{y-x}(x)$. The difference is that in the latter, the particle could pick a location occupied by K particles, in which case the jump is suppressed. In the former, the particle first eliminates sites occupied by K particles and picks a site occupied by strictly less than K particles whenever there is at least one. This results in a speed change K-exclusion process, that is the jump rate from x to y has the form $c_{x,y}(\eta)\mathbf{1}_{\{\eta(x)>0\}}\mathbf{1}_{\{\eta(y)<K\}}$. To illustrate this, consider a nearest-neighbor example: we take $K=1$, $k=1$, $\upsilon_1(x)=p(x)\in[0,1]$, $\upsilon_{-1}(x)=1-p(x)$. If sites $x-1$ and $x+1$ are free, in both processes the particle at x moves with rate β_x^1 to a site picked in $\{x-1,x+1\}$ with probabilities $p(x)$ and $1-p(x)$. Now assume $x+1$ is free and $x-1$ occupied. If $p(x)=0$, nothing happens in either process. If $p(x)>0$, at rate β_x^1, the particle at x moves to $x+1$ in the speed change process, while in the bond-disordered process it moves to $x+1$ with probability $p(x)$ and attempts in vain to jump to $x-1$ with probability $1-p(x)$.

Assume $K = 1$, and consider the totally asymmetric case, where $\upsilon_z(x) = 0$ for $z < 0$. Recalling that the totally asymmetric exclusion process is a classical simplified model of single-lane traffic flow (without overtaking) where particles represent cars, the above model can be viewed as a trafficflow model with maximum overtaking distance k. This is true also for Example 5.4 in Section 5.2, in the totally asymmetric setting $\iota_x^i = 0, 1 \le i \le k$. However in the latter model, an overtaking car has only one choice for its new position.

Though it is not clear from this formulation, we can rephrase this dynamics as a $2k$-step model, which is thus strongly attractive by Lemma 5.1. To this end we take a random field of the form $\beta_x = (\beta_x^1, \dots, \beta_x^1)$, and define $q_x := q(\upsilon(x))$, where $q(\upsilon_z\colon z \in \mathcal{K})$ is the distribution of a random self-avoiding path $(Z_1, \dots, Z_{2k})$ in $\mathcal{K}$ such that

$$\mathbf{P}(Z_1 = y) = \frac{\upsilon_y}{\sum_{z \in \mathcal{K}} \upsilon_z}, \tag{91}$$

$$\mathbf{P}(Z_i = y | Z_1, \dots, Z_{i-1}) = \frac{\upsilon_y}{\sum_{z \in \mathcal{K} \setminus \{Z_1, \dots, Z_{i-1}\}} \upsilon_z} \quad \text{for } 2 \le i \le 2k. \tag{92}$$

For this model, assumption (81) is always satisfied, while (80) reduces to the existence of a constant $c > 0$ and a probability distribution $p(\cdot)$ on $\mathbb{Z}$ satisfying assumption (A1), such that

$$\inf_{x \in \mathbb{Z}} \upsilon_{\cdot}(x) \ge c p(\cdot).$$

The link between the two models comes from:

Lemma 5.2. *Assume* $(Z_1, \dots, Z_{2k}) \sim q(\upsilon_z\colon z \in \mathcal{K})$. *Let* Θ *be a nonempty subset of* $\{z \in \mathcal{K}\colon \upsilon_z \ne 0\}$, $\tau := \inf\{i \in \{1, \dots, 2k\}\colon Z_i \in \Theta\}$, *and* $Y = Z_\tau$. *Then*

$$\mathbf{P}(Y = y) = \mathbf{1}_\Theta(y) \frac{\upsilon_y}{\sum_{y' \in \Theta} \upsilon_{y'}}.$$

Proof. For all $t \ge 2$, let Θ_{t-1} be the set of self-avoiding paths $(z_1, \dots, z_{t-1})$ of size $t - 1$ on $\mathcal{K} \setminus \Theta$. For $y \in \Theta$, by (91)–(92),

$$\begin{aligned}
\mathbf{P}(Y = y) &= \sum_{t=1}^{2k} \mathbf{P}(Z_t = y, \tau = t) \\
&= \mathbf{P}(Z_1 = y) + \sum_{t=2}^{2k} \sum_{(z_1, \dots, z_{t-1}) \in \Theta_{t-1}} \mathbf{P}(Z_1 = z_1, \dots, Z_{t-1} = z_{t-1}) \frac{\upsilon_y}{\sum_{z \in \mathcal{K} \setminus \{z_1, \dots, z_{t-1}\}} \upsilon_z}, \\
&= C \upsilon_y,
\end{aligned}$$

where C is independent of $y \in \Theta$, whence the result. □

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