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Supercritical self-avoiding walks are space-filling

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Abstract. In this article, we consider the following model of self-avoiding walk: the probability of a self-avoiding trajectory γ between two points on the boundary of a finite subdomain of $\mathbb{Z}^d$ is proportional to $\mu^{-\mathrm{length}(\gamma)}$. When μ is supercritical (i.e. $\mu < \mu_c$ where μ_c is the connective constant of the lattice), we show that the random trajectory becomes space-filling when taking the scaling limit.

Résumé. Dans cet article, nous considérons le modèle suivant de marches auto-évitantes : la probabilité d'une trajectoire auto-évitante γ entre deux points fixés d'un sous-domaine fini de $\mathbb{Z}^d$ est proportionnelle à $\mu^{-\mathrm{length}(\gamma)}$. Lorsque le paramètre μ est supercritique (i.e. $\mu < \mu_c$ ou μ_c est la constante de connectivité du réseau), nous prouvons que la trajectoire aléatoire remplit l'espace lorsque l'on considère la limite d'échelle du modèle.

MSC: 60K35; 60C05; 82C41

Keywords: Self avoiding walk; Connective constant

1. Introduction

In 1953, Paul Flory [8] proposed considering self-avoiding walks (i.e. visiting every vertex at most once) on a lattice as a model for polymer chains. Self-avoiding walks have turned out to be a very interesting object, leading to rich mathematical theories and challenging questions; see [1,16].

Denote by c_n the number of n-step self-avoiding walks on the hypercubic lattice ($\mathbb{Z}^d$ with edges between nearest neighbors) started from some fixed vertex, e.g. the origin. Elementary bounds on c_n (for instance $d^n \le c_n \le 2d(2d-1)^{n-1}$) guarantee that c_n grows exponentially fast. Since an $(n+m)$-step self-avoiding walk can be uniquely cut into an n-step self-avoiding walk and a parallel translation of an m-step self-avoiding walk, we infer that $c_{n+m} \le c_n c_m$, from which it follows that there exists $\mu = \mu(d) \in (0, +\infty)$ such that $\mu := \lim_{n\to\infty} c_n^{1/n}$. The positive real number μ is called the *connective constant* of the lattice. The connective constant can be approximated in a number of ways, yet no closed formula exists in general. In the case of the hexagonal lattice, it was recently proved to be equal to $\sqrt{2+\sqrt{2}}$ in [7].

As it stands, the model has a strong combinatorial flavor. A more geometric variation was suggested by Lawler, Schramm and Werner [15]. Let us describe their construction now – they were interested in the two-dimensional case, but here we will not make this restriction. Let Ω be a simply connected domain in $\mathbb{R}^d$ with two points a, b on the boundary. For $\delta > 0$, let Ω_δ be the largest connected component of $\Omega \cap \delta\mathbb{Z}^d$ and let a_δ, b_δ be the two sites of Ω_δ closest to a and b respectively. We think of $(\Omega_\delta, a_\delta, b_\delta)$ as being an approximation of (Ω, a, b). See Fig. 1.

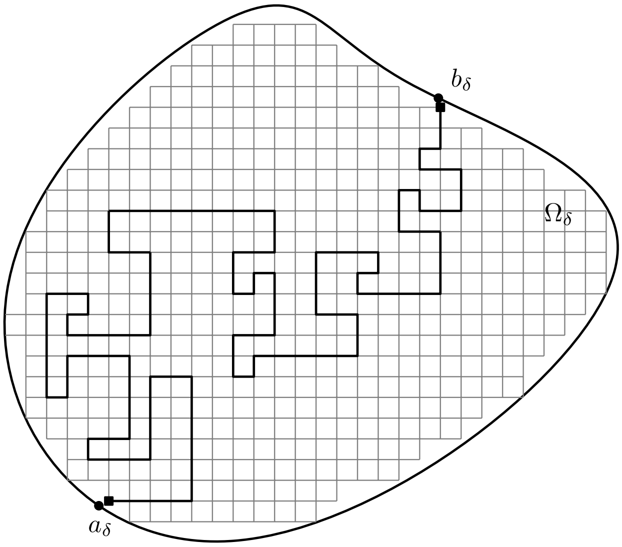

Fig. 1. A domain Ω with two points a and b on the boundary (circles) and the graph Ω_δ. The points a_δ and b_δ are depicted by squares. An example of a possible walk from a_δ to b_δ is presented. Note that there is a finite number of them.

Let $x > 0$. On $(\Omega_\delta, a_\delta, b_\delta)$, define a probability measure on the finite set of self-avoiding walks in Ω_δ from a_δ to b_δ by the formula

$$\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}(\gamma) = \frac{x^{|\gamma|}}{Z_{(\Omega_\delta,a_\delta,b_\delta)}(x)}, \tag{1}$$

where $|\gamma|$ is the length of γ (i.e. the number of edges), and $Z_{(\Omega_\delta,a_\delta,b_\delta)}(x)$ is a normalizing factor. A random curve γ_δ with law $\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}$ is called the *self-avoiding walk* with parameter x in $(\Omega_\delta, a_\delta, b_\delta)$. The sum $Z_{(\Omega_\delta,a_\delta,b_\delta)}(x) = \sum_\gamma x^{|\gamma|}$ (with the sum taken over all self-avoiding walks in Ω_δ from a_δ to b_δ) is sometimes called the *partition function* (or *generating function*) of self-avoiding walks from a_δ to b_δ in the domain Ω_δ.

When the domain (Ω, a, b) is fixed, we are interested in the scaling limit of the family (γ_δ), i.e. its geometric behavior when δ goes to 0. The qualitative behavior is expected to differ drastically depending on the value of x. A phase transition occurs at the value $x_c = 1/\mu$, where μ is the connective constant:

When $x < 1/\mu$: γ_δ converges to a deterministic curve corresponding to the geodesic between a and b in Ω (assuming it is unique – otherwise some adaptations need to be done). When rescaled, γ_δ should have Gaussian fluctuation of order $\sqrt{\delta}$ around the geodesic. The strong results of Ioffe [14] on the unrestricted self-avoiding walk would be a central tool for proving such a statement, though we are not aware of a reference for the details.

When $x = 1/\mu$: γ_δ should converge to a random simple curve. In dimensions four and higher, the limit is believed to be a Brownian excursion from a to b in the domain Ω. This is heuristically related to a number of rigorously proved results: in dimensions five and above to the work of Brydges and Spencer [5] and Hara and Slade [11,12] who showed that unrestricted self-avoiding walk converges to Brownian motion (see also the book [16]). Dimension four, the so-called *upper critical dimension*, is much harder, but recently some impressive results have been achieved using a supersymmetric renormalization group approach. These results are limited to continuous time weakly self-avoiding walk, see [2–4] and references within.

In dimension two, the scaling limit is conjectured to be the Schramm–Löwner Evolution of parameter 8/3, and in fact it was pointed out that this is true *if* the scaling limit exists as a continuous curve and is conformally invariant [15].

Finally, dimension three remains a mystery, and there is no clear candidate for the scaling limit of self-avoiding walk.

When $x > 1/\mu$: γ_δ is expected to become space-filling in the following sense: for any open set $U \subset \Omega$,

$$\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}[\gamma_\delta \cap U = \varnothing] \to 0$$

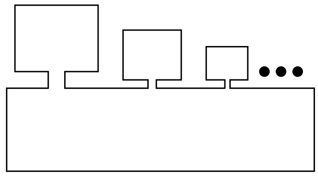

Fig. 2. A domain with mushrooms in infinitely many scales.

when δ goes to 0. On the one hand, let us mention that it is not clear in which sense (if any) (γ_δ) has a scaling limit when $d \geq 3$. On the other hand, the scaling limit is predicted [17], Conjecture 3, to exist in dimension two (in the case of the hexagonal lattice at least). It should be the Schramm–Löwner Evolution of parameter 8, which is conformally invariant.

One cannot hope that γ_δ would be space-filling in the strictest possible sense, namely that every vertex is visited. Nevertheless, one can quantify the size of the biggest hole not visited by the walk. The subject of this paper is the proof of a result which quantifies how γ_δ becomes space filling. Here is a precise formulation.

Theorem 1. *Let $\mathbb{D}$ be the unit disk and let a and b be two points on its boundary. For every $x > 1/\mu$, there exist $\xi = \xi(x) > 0$ and $c = c(x) > 0$ such that*

$$\mathbb{P}_{(\mathbb{D}_\delta, a_\delta, b_\delta, x)}\big[\text{there exists a component of } \mathbb{D}_\delta \setminus \Gamma_\delta^\xi \text{ with cardinality } > c\log(1/\delta)\big] \to 0$$

when $\delta \to 0$, where Γ_δ^ξ is the set of sites in $\mathbb{D}_\delta$ at graph distance less than ξ from γ_δ.

The theorem is stated for the unit disk to avoid various connectivity problems. Indeed, assume that at some given scale δ our domain Ω has a part which is connected by a "bridge" of width δ. Then the graph Ω_δ will have a large part connected by a single edge, which does not leave the self-avoiding walk enough space to enter and exit. Thus an analog of Theorem 1 will not hold for this Ω. It is not difficult to construct a single domain Ω with such "mushrooms" in many scales. See Fig. 2. In order to solve this issue, one can start from an arbitrary domain and expand it microscopically. This gives rise to the following formulation:

Theorem 2. *Let (Ω, a, b) be a bounded domain with two points on the boundary. For every $x > 1/\mu$, there exist $\xi = \xi(x) > 0$ and $c = c(x) > 0$ such that*

$$\mathbb{P}_{((\Omega+B(\xi\delta))_\delta, a_\delta, b_\delta, x)}\big[\exists \text{ a component of } \big(\Omega + B(\xi\delta)\big)_\delta \setminus \Gamma_\delta^\xi \text{ larger than } c\log(1/\delta)\big] \to 0$$

when $\delta \to 0$, where Γ_δ^ξ is the set of sites in $(\Omega + B(\xi\delta))_\delta$ at graph distance less than ξ from γ_δ.

Here $\Omega + B(\xi\delta)$ is Ω expanded by $\xi\delta$ i.e.

$$\Omega + B(\xi\delta) = \big\{z\colon \operatorname{dist}(z, \Omega) < \xi\delta\big\}.$$

Since ξ depends only on x, and $\delta \to 0$, this is a microscopic expansion.

The strategy of the proof is fairly natural. We first prove that in the supercritical phase, one can construct a lot (compared to their energy) of self-avoiding polygons in a prescribed box. Then, we show that the self-avoiding walk cannot leave holes that are too large, since adding polygons in the big holes to the self-avoiding walk would increase the entropy drastically while decreasing the energy in a reasonable way. In particular, an energy/entropy comparison shows that self-avoiding walks leaving big holes are unlikely. We present the proof *only in the case $d = 2$*, even though the reasoning carries over to all dimensions without difficulty (see Remark 8). One can also extend the result to other lattices with sufficient symmetry in a straightforward way (for instance to the hexagonal lattice).

2. Self-avoiding polygons in a square

In this section, we think of a walk as being indexed by (discrete) time t from 0 to n. For $m > 0$, let P_m be the set of self-avoiding polygons in $[0, 2m+1]^2$ that touch the middle of every face of the square: more formally, such that the edges $[(m,0),(m+1,0)]$, $[(2m+1,m),(2m+1,m+1)]$, $[(m,2m+1),(m+1,2m+1)]$ and $[(0,m),(0,m+1)]$ belong to the polygon, see Fig. 3. For $x > 0$, let $Z_m(x)$ be the partition function (with parameter x) of P_m, i.e.

$$Z_m(x) = \sum_{\gamma \in P_m} x^{|\gamma|}.$$

Proposition 3. *For $x > 1/\mu$, we have $\limsup_{m\to\infty} Z_m(x) = \infty$.*

It is classical that the number of self-avoiding walks with certain constraints grows at the same exponential rate as the number of self-avoiding walks without constraints (we will show it in our context in the proof of Lemma 5 below). For instance, let $x(v)$ and $y(v)$ be the first and the second coordinates of the vertex v. The number b_n of *self-avoiding bridges* of length n, meaning self-avoiding walks γ of length n such that $y(\gamma_0) = \min_{t\in[0,n]} y(\gamma_t)$ and $y(\gamma_n) = \max_{t\in[0,n]} y(\gamma_t)$, satisfies

$$e^{-c\sqrt{n}}\mu^n \le b_n \le \mu^n \tag{2}$$

for every n [10] (see also [16] for a modern exposition). This result harnesses the following theorem on integer partitions which dates back to 1917.

Theorem 4 (Hardy and Ramanujan [13]). *For an integer $A \ge 1$, let $P_D(A)$ denote the number of ways of writing $A = A_1 + \cdots + A_k$ with $A_1 > \cdots > A_k \ge 1$, for any $k \ge 1$. Then*

$$\log P_D(A) \sim \pi\sqrt{\frac{A}{3}}$$

as $A \to \infty$.

In the following, we need a class of walks with even more restrictive constraints. A *squared walk (of span k)* is a self-avoiding walk such that $\gamma_0 = (0,0)$, $\gamma_n = (k,k)$ and $\gamma \subset [0,k]^2$.

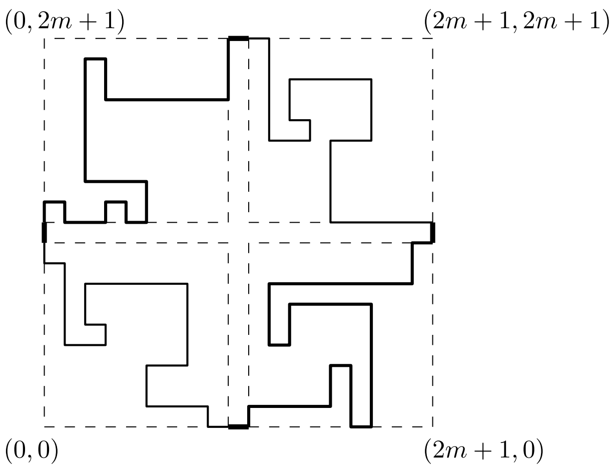

Fig. 3. By concatenating four walks in squares of size m (plus four edges), one obtains an element of P_m, i.e. a loop in the square of size $2m+1$ going through the middle of the sides.

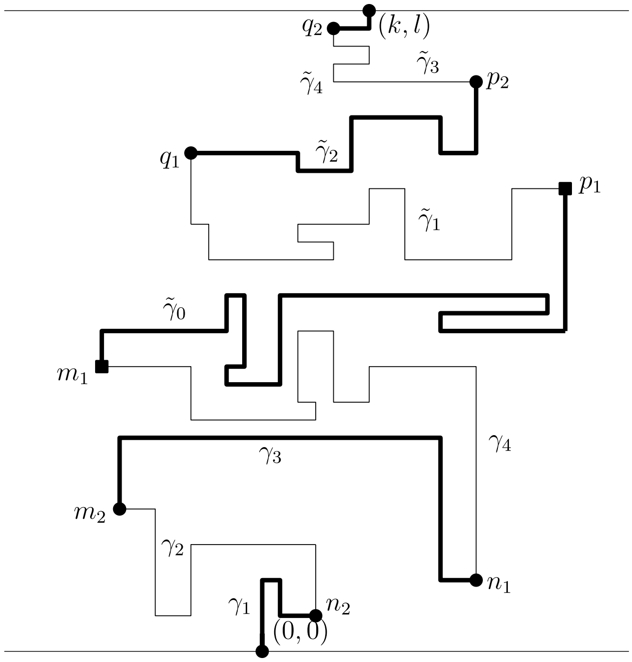

Fig. 4. The decomposition of a bridge into walks. One can construct a squared walk in a rectangle by reflecting non-bold walks and then concatenating all the walks together.

Lemma 5. *For c sufficiently large and n even, the number a_n of squared walks of length n satisfies*

$$a_n \geq \mu^n \mathrm{e}^{-c\sqrt{n}}.$$

Proof. *Step 1: Rectangles.* Let Λ_n be the set of self-avoiding bridges of length n starting at the origin. Let Σ_n be the set of n-step self-avoiding walks for which there exists (k, l) such that $\gamma_0 = (0, 0)$, $\gamma_n = (k, l)$ and $\gamma \subset [0, k] \times [0, l]$. We construct a map from Λ_n to Σ_n.

Fix $\gamma \in \Lambda_n$ and denote by m_1 the first time at which $x(\gamma_{m_1}) = \min_{t\in[0,n]} x(\gamma_t)$, see Fig. 4. Then, define n_1 to be the first time at which $x(\gamma_{n_1}) = \max_{t\in[0,m_1]} x(\gamma_t)$. One can then define recursively m_k, n_k, by the formulæ

$$m_k = \min\Big\{r \leq n_{k-1}\colon\ x(\gamma_r) = \min_{t\in[0,n_{k-1}]} x(\gamma_t)\Big\},$$

$$n_k = \min\Big\{r \leq m_k\colon\ x(\gamma_r) = \max_{t\in[0,m_k]} x(\gamma_t)\Big\}.$$

We stop the recursion the first time m_k or n_k equals 0. For convenience, if the first time is n_k, we add a further step $m_{k+1} = 0$. We are then in possession of a sequence of integers $m_1 > n_1 > m_2 > \cdots > m_r \geq n_r \geq 0$ and a sequence of walks $\gamma_{2r-1} = \gamma[n_1, m_1]$, $\gamma_{2r-2} = \gamma[m_2, n_1], \ldots, \gamma_1 = \gamma[0, m_r]$. Note that the width of the walks γ_i is strictly increasing (see Fig. 4 again).

Similarly, let p_1 be the last time at which $x(\gamma_{p_1}) = \max_{t\in[m_1,n]} x(\gamma_t)$ and q_1 the last time at which $x(\gamma_{q_1}) = \min_{t\in[p_1,n]} x(\gamma_t)$. Then define recursively p_k and q_k by the following formula

$$p_k = \max\Big\{r \geq q_{k-1}\colon\ x(\gamma_r) = \max_{t\in[q_{k-1},n]} x(\gamma_t)\Big\},$$

$$q_k = \max\Big\{r \leq p_k\colon\ x(\gamma_r) = \min_{t\in[p_k,n]} x(\gamma_t)\Big\}.$$

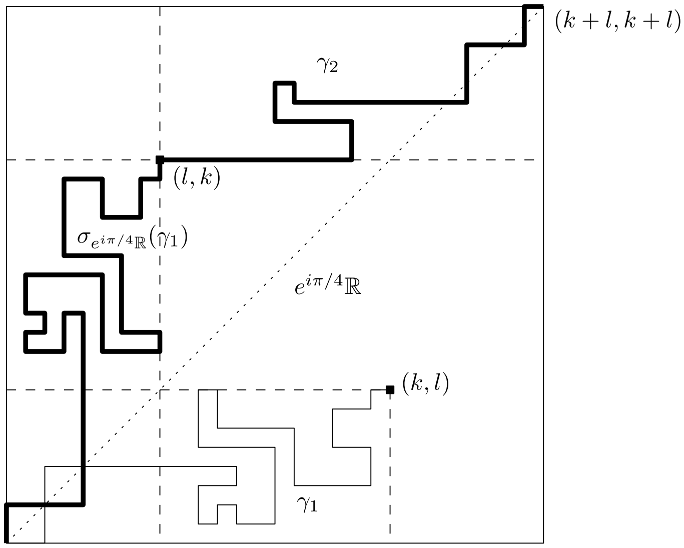

Fig. 5. This figure depicts the passage of two walks in the rectangle $[0,k]\times[0,l]$ to a walk in the square $[0,k+l]^2$.

This procedure stops eventually and we obtain another sequence of walks $\tilde{\gamma}_0=\gamma[m_1,p_1]$, $\tilde{\gamma}_1=\gamma[p_1,q_1]$, etc. This time, the width of the walks is strictly decreasing, see Fig. 4 one more time.

For a walk ω, we set $\sigma(\omega)$ to be its reflexion with respect to the vertical line passing through its starting point. Let $f(\gamma)$ be the concatenation of $\gamma_1,\sigma(\gamma_2),\gamma_3,\dots,\sigma(\gamma_r),\tilde{\gamma}_0,\sigma(\tilde{\gamma}_1),\tilde{\gamma}_2$ and so on. This walk is contained in the rectangle with corners being its endpoints so that f maps Λ_n on Σ_n.

In order to estimate the cardinality of Σ_n, we remark that each element of Σ_n has a limited number of possible preimages under f. More precisely, the map which gives $f(\gamma)$ and the widths of the walks (γ_i) and $(\tilde{\gamma}_i)$ is one-to-one (the reverse procedure is easy to identify). The number of possible widths for γ_i and $\tilde{\gamma}_i$ is the number of pairs of *decreasing* sequences partitioning an integer $l\le n$. This number is bounded by $\mathrm{e}^{c\sqrt{n}}$ (Theorem 4). Therefore, the number of possible preimages under f is bounded by $\mathrm{e}^{c\sqrt{n}}$. Using (2), the cardinality of Σ_n is thus larger than $\mathrm{e}^{-c\sqrt{n}}b_n\ge \mathrm{e}^{-2c\sqrt{n}}\mu^n$.

So far n was not restricted to be even.

Step 2: Squares. We have bounded from below the number of n-step self-avoiding walks 'contained in a rectangle'. We now extend this bound to the case of squares. There exist $k,l\le n$ such that the number of elements of Σ_n with (k,l) as an ending point is larger than $\mathrm{e}^{-2c\sqrt{n}}\mu^n/n^2$. By taking two arbitrary walks of Σ_n ending at (k,l), one can construct a $2n$-step self-avoiding walk with $\gamma_0=(0,0)$ and $\gamma_{2n}=(k+l,k+l)$ contained in $[0,k+l]^2$ by reflecting orthogonally to $\mathrm{e}^{\mathrm{i}\pi/4}\mathbb{R}$ the first walk, and then concatenating the two, see Fig. 5. We deduce that $a_{2n}\ge \mu^{2n}\mathrm{e}^{-4c\sqrt{n}}/n^4$. This shows the lemma for n sufficiently large, and one can increase c if necessary to handle all even n. □

Proof of Proposition 3. Squared walks with length n were defined as walks between corners of some $m\times m$ square, but m was not fixed. Fix now m to be such that the number of such walks is maximized (and then it is at least a_n/n where a_n is the total number of squared walks). It is interesting to remark that finding the maximal m as an explicit function of n, even asymptotically, seems difficult, probably no easier than the $\mathrm{SLE}_{8/3}$ conjecture. But we do not need to know its value. From any quadruplet $(\gamma_1,\gamma_2,\gamma_3,\gamma_4)$ of such squared self-avoiding walks, one can construct a self-avoiding polygon of P_m as follows (see Fig. 3):

- translate γ_1 and γ_3 by $(m+1,0)$ and $(0,m+1)$ respectively,
- rotate γ_2 and γ_4 by an angle $\pi/2$, and then translate them by $(m,0)$ and $(2m+1,m+1)$ respectively,
- add the four edges $[(m,0),(m+1,0)]$, $[(2m+1,m),(2m+1,m+1)]$, $[(m,2m+1),(m+1,2m+1)]$ and $[(0,m),(0,m+1)]$.

Since each walk is contained in a square, one can easily check that we obtain a $(4n+4)$-step polygon in P_m. Using Lemma 5, we obtain

$$Z_m(x) \geq x^{4n+4}\left(\frac{a_n}{n}\right)^4 \geq \left(\frac{x^{n+1}\mu^n \mathrm{e}^{-c\sqrt{n}}}{n}\right)^4.$$

When n goes to infinity, the right-hand side goes to infinity and the claim follows readily. □

3. Proof of the main results

The strategy is the following. We first show that for some hole (namely it will be a connected union of boxes of some size m), the probability that the self-avoiding walk gets close to it without intersecting it can be estimated in terms of $Z_m(x)$. This claim is the core of the argument, and is presented in Proposition 7. Next, we show that choosing m large enough (or equivalently $Z_m(x)$ large enough), the probability to avoid some connected union of k boxes decays exponentially fast in k, thus implying Theorem 1.

Let $m > 0$. A *cardinal edge* of a (square) box B of side length $2m+1$ is an edge of the lattice in the middle of one of the sides of B. For $m \in \mathbb{N}$, two boxes B and B' of side length $2m+1$ are said to be *adjacent* if they are disjoint and each has a cardinal edge, $[xy]$ and $[zt]$ respectively, such that $x \sim z$, $y \sim t$ (see Fig. 6). A family F of boxes is called *connected* if every two boxes can be connected by a path of adjacent boxes in F.

To simplify the picture, we will assume that all our boxes have their lower left corner in $(2m+2)\delta\mathbb{Z}^2$. When Ω_δ is fixed, such boxes included in Ω_δ are called *m-boxes* and the set of m-boxes is denoted by $\mathcal{F}(\Omega_\delta, m)$.

Let us return to the issue of domain regularity discussed after Theorem 1. With the definitions above we can now explain that, in fact, our only requirement from the domain is that the family of all boxes in $\mathcal{F}(\Omega_\delta, m)$ is connected. Let us state this formally. The set Γ_δ^ξ is as in Theorem 1.

Theorem 6. *For every $x > 1/\mu$, there exists $m = m(x)$ and $c(x) > 0$ such that for every domain Ω and every $\delta > 0$ such that $\mathcal{F}(\Omega_\delta, m)$ is connected, one has, for every a and b in the boundary of Ω, and every $\lambda > 0$,*

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}\big(\exists \text{ a component of } \Omega_\delta \setminus \Gamma_\delta^{6m} \text{ of size } > \lambda\big) \leq \frac{C(x,\Omega)}{\delta^2}\mathrm{e}^{-c(x)\lambda}.$$

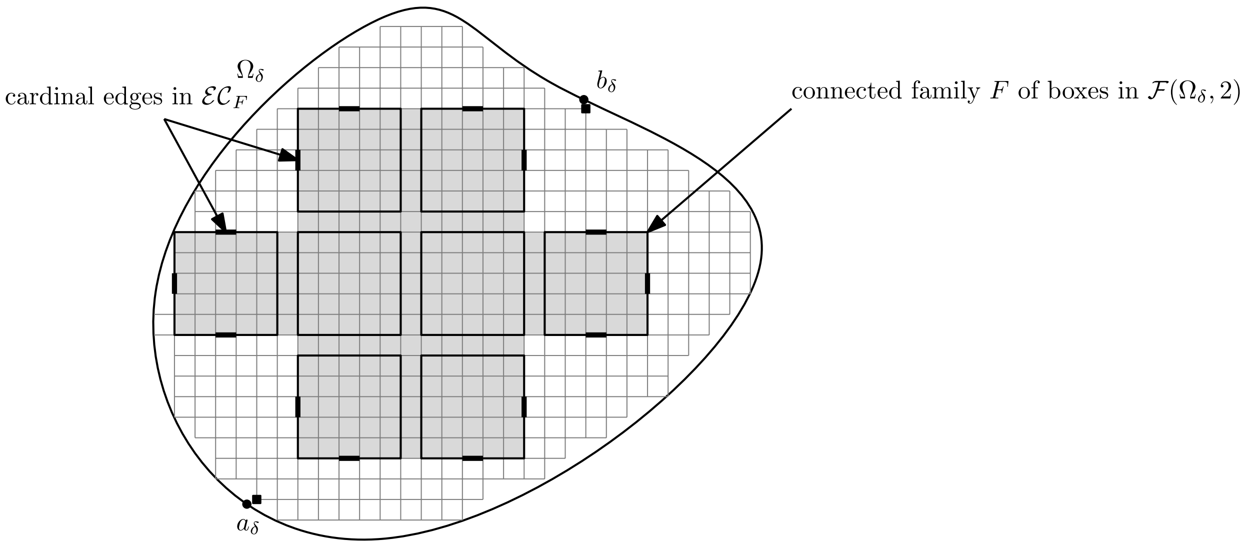

Fig. 6. A discrete domain with a connected component of adjacent boxes of size 5 ($m = 2$). Edges of $\mathcal{E}_F$ lie in the gray area.

Proof of Theorem 1 given Theorem 6. Here our domain is $\mathbb{D}$. Clearly, the family of all boxes in $\mathbb{D}_\delta$ is connected (it is an interval in every row and every column), hence Theorem 6 applies. Taking $\lambda = C_1 \log(1/\delta)$ for C_1 sufficiently large gives the result. □

Proof of Theorem 2 given Theorem 6. Again, all we have to show is that the family of boxes in $(\Omega + B(\xi\delta))_\delta$ is connected for ξ sufficiently large. Taking $\xi = 6m$ we see that every box in $\Omega + B(6m\delta)$ can be connected to a box in Ω, and any two boxes in Ω can be connected by taking a path γ in Ω between them (here is where we use that Ω is connected) and checking that $\gamma + B(6m\delta)$ contains a path of connected boxes. □

Hence we need to prove Theorem 6. Let $\delta > 0$. For $F \in \mathcal{F}(\Omega_\delta, m)$, let $\mathcal{V}_F$ be the set of vertices in boxes of F, and let $\mathcal{E}_F$ be the set of edges with both end-points in $\mathcal{V}_F$. For two subsets A and B of the vertices of Ω_δ define the *box distance* $\operatorname{boxdist}(A, B)$ between them as the size of the smallest set of connected boxes containing one box in A and one box in B, minus 1. The boxes do not have to be different, but then the distance is 0 – if no such connected set exists, then the distance is ∞.

Proposition 7. *Let (Ω, a, b) be a domain with two points on the boundary. Fix $\delta > 0$ and $m \in \mathbb{N}$ and assume $\mathcal{F}(\Omega_\delta, m)$ is connected. Then there exists $C(x, m) < \infty$ such that for every $F \in \mathcal{F}(\Omega_\delta, m)$,*

$$\mathbb{P}_{(\Omega_\delta, a_\delta, b_\delta, x)}\big(\operatorname{boxdist}(\gamma_\delta, \mathcal{V}_F) = 1\big) \le C(x, m) Z_m(x)^{-|F|}.$$

Proof. For $F \in \mathcal{F}(\Omega_\delta, m)$, let $\mathcal{EC}_F$ be the set of *external cardinal edges* of F i.e. all cardinal edges in boxes of F which have neighbors outside of F. Let S_F be the set of self-avoiding polygons included in $\mathcal{E}_F$ visiting all the edges in $\mathcal{EC}_F$. Let $Z_F(x)$ be the partition function of polygons in S_F. We have:

Claim. *For $F \in \mathcal{F}(\Omega_\delta, m)$, $Z_F(x) \ge Z_m(x)^{|F|}$.*

Proof. We prove the result by induction on the cardinality of $F \in \mathcal{F}(\Omega_\delta, \xi)$. If the cardinality of F is 1, $Z_F(x) = Z_m(x)$ by definition. Consider $F_0 \in \mathcal{F}(\Omega_\delta, \xi)$ and assume the statement true for every $F \in \mathcal{F}(\Omega_\delta, \xi)$ with $|F| < |F_0|$. There exists a box B in F_0 such that $F_0 \setminus \{B\}$ is still connected. Therefore, for every pair $(\gamma, \gamma') \in S_{\{B\}} \times S_{F_0 \setminus \{B\}}$, one can associate a polygon in S_F in a one-to-one fashion. Indeed, B is adjacent to a box $B' \in F_0 \setminus \{B\}$ so that one of the four cardinal edges (called $[ab]$) of B is adjacent to a cardinal edge $[cd]$ of B'. Note that $[cd]$ belongs to $\mathcal{EC}_{F_0 \setminus \{B\}}$. Then, by changing the edges $[cd]$ and $[ab]$ of γ and γ' into the edges $[ac]$ and $[bd]$, one obtains a polygon in S_F. Furthermore, the construction is one-to-one and we deduce

$$Z_{F_0}(x) \ge Z_{F_0 \setminus \{B\}}(x) Z_B(x) \ge Z_m^{|F_0 \setminus \{B\}|} Z_m(x) = Z_m(x)^{|F_0|}.$$ □

Consider the set Θ_F of walks not intersecting F yet reaching to a neighboring box. Let e be a cardinal edge of a box in F which neighbours a box visited by γ. For each $\gamma \in \Theta_F$, consider a self-avoiding polygon $\ell = \ell(\gamma)$ (ℓ standing for “link”) satisfying the following three properties:

- it contains e and is included in $(\Omega_\delta \setminus \mathcal{E}_F) \cup \{e\}$,
- it intersects γ either at just one edge, or at two adjacent edges only (to intersect means to intersect along edges),
- it has length smaller than $10m + 10$ (for simplicity, we will bound the length by $100m$).

One can easily check that such a polygon always exists, see Fig. 7.

Now, consider the map f that associates to $(\gamma_1, \gamma_2) \in \Theta_F \times S_F$ the symmetric difference $\gamma = f(\gamma_1, \gamma_2)$ of γ_1, $\ell(\gamma_1)$ and γ_2 (symmetric difference here meaning as sets of edges). Note that the object that we obtain is a walk from a_δ to b_δ in Ω_δ, which can be verified to be self-avoiding by noting that each vertex has degree 0 or 2 and that the set is connected. Further, its length is equal to $|\gamma_1| + |\ell(\gamma_1)| + |\gamma_2| - 4$ or $|\gamma_1| + |\ell(\gamma_1)| + |\gamma_2| - 6$ (this being due to the fact that γ_1 and $\ell(\gamma_1)$ intersect at one or two adjacent edges – each intersection reduces the length by 2 edges). Now, given a path γ there is a limited number of ways it may be written as $f(\gamma_1, \gamma_2)$. Indeed, e can be located by the only two paths that exit F, and that gives γ_2. Given e, ℓ has only a limited number of possibilities, say 4^{100m}, and once one

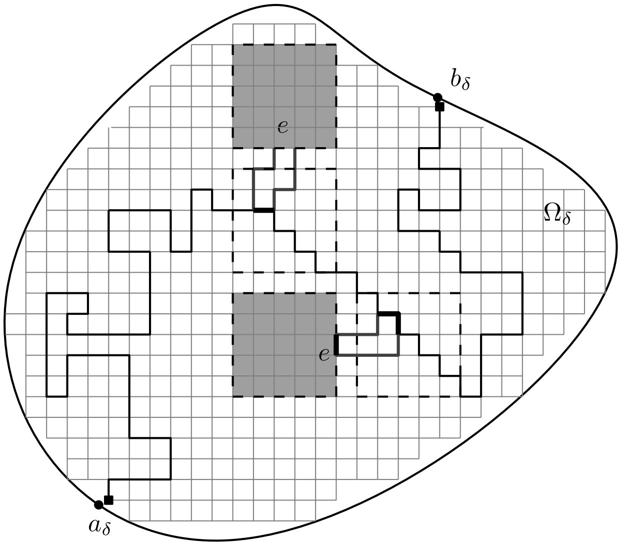

Fig. 7. Two examples of the polygon ℓ (the small N- and L-shaped loops). It overlaps the curve in one edge exactly, except in the second case, where we have no choice but overlapping the walk on two edges.

knows ℓ this gives γ_1. We can now write

$$\begin{aligned} Z_{\Theta_F}(x)\cdot Z_F(x) &= \Big(\sum_{\gamma_1\in\Theta_F} x^{|\gamma_1|}\Big)\Big(\sum_{\gamma_2\in S_F} x^{|\gamma_2|}\Big) \\ &\le \max\big(1,x^{-100m}\big)\sum_{\gamma_1\in\Theta_F,\gamma_2\in S_F} x^{|\gamma_1|+|\ell(\gamma_1)|+|\gamma_2|} \\ &\le \max\big(1,x^{-100m}\big)\max\big(x^4,x^6\big)\sum_{\gamma_1\in\Theta_F,\gamma_2\in S_F} x^{|f(\gamma_1,\gamma_2)|} \\ &\le 4^{100m}\max\big(x^6,x^{-100m+4}\big)\sum_{\gamma\in f(\Theta_F\times S_F)} x^{|\gamma|} \\ &\le 4^{100m}\max\big(x^6,x^{-100m+4}\big)Z_{(\Omega_\delta,a_\delta,b_\delta)}(x), \end{aligned}$$

where in the first inequality we used the fact that $\ell(\gamma_1)$ has length smaller than $100m$, in the second the fact that $|f(\gamma_1,\gamma_2)|$ equals $|\gamma_1|+|\ell(\gamma_1)|+|\gamma_2|-4$ or $|\gamma_1|+|\ell(\gamma_1)|+|\gamma_2|-6$, and in the third the fact that f is at most 4^{100m}-to-one. Using the claim, the previous inequality implies

$$\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}\big(\text{boxdist}(\gamma_\delta,\mathcal{V}_F)=1\big)=\frac{Z_{\Theta_F}(x)}{Z_{(\Omega_\delta,a_\delta,b_\delta)}(x)}\le\frac{C(x,m)}{Z_F(x)}\le\frac{C(x,m)}{Z_m(x)^{|F|}}. \qquad \square$$

Proof of Theorem 6 in dimension 2. Let $x>1/\mu$ and let (Ω,a,b) be a domain with two points on the boundary. Let A_n be the number of connected subsets of $\mathbb{Z}^2$ containing 0. It is well known that $\overline{\lim}\sqrt[n]{A_n}$ is finite (see e.g. Theorem 4.20 in [9]). Let therefore $\lambda=\lambda(2)$ satisfy $A_n\le\lambda^n$ for all n. We now apply Proposition 3 and get some $m=m(x,2)$ such that $Z_m(x)>2\lambda$.

Let $\delta>0$ and consider the event $\mathcal{A}(s)$ that there exists a connected set S of cardinality s at distance larger than $6m$ of γ_δ. Every box intersecting S must be disjoint from γ_δ, so there must exist a connected family of at least $s/(2m+1)^2$ boxes of size $2m+1$ covering S and not intersecting γ_δ. We may assume this family is maximal among families covering S and not intersecting γ_δ. Since the family of boxes is maximal, and because of the condition

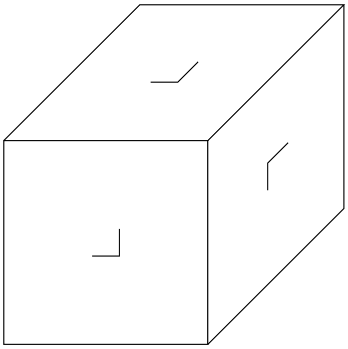

Fig. 8. Cardinal edges in three dimensions.

of the theorem that the family of all boxes is connected, the box-distance between the union of boxes and γ_δ is 1. Proposition 7 implies

$$\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}\big[\mathcal{A}(s)\big] \le \sum_{F\in\mathcal{F}(\Omega_\delta,\xi):|F|\ge s/(2m+1)^2} C(x,m)\big[Z_m(x)\big]^{-|F|}.$$

By the definition of λ, the number of families of connected boxes of size K in $\mathcal{F}(\Omega_\delta,\xi)$ is bounded by $(C(\Omega)/\delta^2)\lambda^K$ (since up to translation they are connected subsets of a normalized square lattice), where $C(\Omega)=C(\Omega,x,m)$ depends on the area of Ω, x and m. Therefore, for $c>0$,

$$\begin{aligned}\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}\big[\mathcal{A}(s)\big] &\le C(x,m)\frac{C(\Omega)}{\delta^2}\sum_{i\ge s/(2m+1)^2}\left(\frac{\lambda}{Z_m(x)}\right)^i\\ &\le \frac{C(x,m,\Omega)}{\delta^2}2^{-s/(2m+1)^2}\end{aligned}$$

and the theorem follows. □

Remark 8. *Let us briefly describe what needs to be changed in higher dimensions. The notion of cardinal edge must be extended: in the box* $[0,2m+1]^d$*, cardinal edges for the face* $[0,2m+1]^{d-1}\times\{0\}$ *are all the edges joining vertices in* $\{m,m+1\}^{d-1}\times\{0\}$ *of the form*

$$\big[(\underbrace{m+1,\dots,m+1}_{i-1\ terms},\underbrace{m,\dots,m}_{d-i\ terms},0),(\underbrace{m+1,\dots,m+1}_{i\ terms},\underbrace{m,\dots,m}_{d-i-1\ terms},0)\big]$$

for $1\le i\le d-1$*. See Fig. 8 for an example in 3 dimensions. We only consider part of the edges joining vertices in* $\{m,m+1\}^{d-1}\times\{0\}$ *because all these edges should belong to a self-avoiding polygon. Similarly, cardinal edges can be defined for every face. It can be shown that the number of polygons included in some box* $[0,2m+1]^d$ *and visiting all the cardinal edges grows exponentially at the same rate as the number of self-avoiding walks. The proofs then apply mutatis mutandis.*

4. Questions

The supercritical phase exhibits an interesting behavior. We know that the curve becomes space-filling, yet we have very little additional information. For instance, a natural question is to study the length of the curve. It is not difficult to show that the length is of order $1/\delta^2$, yet a sharper result would be interesting:

Problem 9. *For $x > 1/\mu$, show that there exists $\theta(x) > 0$ such that for every $\varepsilon > 0$ and every sufficiently regular domain (Ω, a, b),*

$$\mathbb{P}_{(\Omega_\delta,a_\delta,b_\delta,x)}\big(\big||\gamma_\delta| - \theta(x)\cdot|\Omega_\delta|\big| > \varepsilon|\Omega_\delta|\big) \longrightarrow 0 \quad \textit{when } \delta \to 0.$$

The quantity $\theta(x)$ would thus be an averaged density of the walk. Note that the existence of $\theta(x)$ seems natural since the space-filling curve should look fairly similar in different portions of the space.

Another challenge is to try to say something nontrivial about the critical phase. Recently, the uniformly chosen self-avoiding walk on $\mathbb{Z}^d$ was proved to be sub-ballistic [6]. A natural question would be to prove that it is *not* space-filling.

Problem 10. *When $x = 1/\mu$ and (Ω, a, b) is sufficiently regular, show that the sequence (γ_δ) does not become space-filling.*

Finally, we recall the conjecture made in [17] concerning the two-dimensional limit in the supercritical phase.

Conjecture 11 (Smirnov). *Let (Ω, a, b) be a simply connected domain of $\mathbb{C}$ and consider approximations by the hexagonal lattice. The law of (γ_δ) converges to the chordal Schramm–Löwner Evolution in (Ω, a, b)*

- *with parameter $8/3$ if $x = 1/\mu$,*
- *with parameter 8 if $x > 1/\mu$.*

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