

# Kolmogorov's law of the iterated logarithm for noncommutative martingales

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**Abstract.** We prove Kolmogorov's law of the iterated logarithm for noncommutative martingales. The commutative case was due to Stout. The key ingredient is an exponential inequality proved recently by Junge and the author.

**Résumé.** Nous prouvons la loi de Kolmogorov du logarithme itéré pour des martingales non-commutatives. Le cas commutatif a été établi par Stout. L'ingrédient clé est une inégalité exponentielle prouvée récemment par Junge et l'auteur.

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## 1. Introduction

In probability theory, law of the iterated logarithm (LIL) is among the most important limit theorems and has been studied extensively in different contexts. The early contributions in this direction for independent increments were made by Khintchine, Kolmogorov, Hartman–Wintner, etc.; see [1] for more history of this subject. Stout generalized Kolmogorov and Hartman–Wintner's results to the martingale setting in [15,16]. The extension of LIL for independent sums in Banach spaces were due to Kuelbs, Ledoux, Talagrand, Pisier, etc.; see [12] and the references therein for more details in this direction. In the last decade, there has been new development for LIL results of dependent random variables; see [19,20] and the references therein for more details. However, it seems that the LIL in noncommutative (= quantum) probability theory has only been proved recently by Konwerska [10,11] for Hartman–Wintner's version. Even the Kolmogorov's LIL for independent sums in the noncommutative setting is not known. The goal of this paper is to prove Kolmogorov's version of LIL for noncommutative martingales.

Let us first recall Kolmogorov's LIL. Let  $(Y_n)_{n \in \mathbb{N}}$  be an independent sequence of square-integrable, centered, real random variables. Put  $S_n = \sum_{i=1}^n Y_i$  and  $s_n^2 = \operatorname{Var}(S_n) = \sum_{i=1}^n \mathbb{E}(Y_i^2)$ . Here and in the following  $\mathbb{E}$  denotes the expectation and Var denotes the variance. For any x > 0, we define the notation  $L(x) = \max\{1, \ln \ln x\}$ . In 1929, Kolmogorov proved that if  $s_n^2 \to \infty$  and

$$|Y_n| \le \alpha_n \frac{s_n}{\sqrt{L(s_n^2)}} \quad \text{a.s.}$$

for some positive sequence  $(\alpha_n)$  such that  $\lim_{n\to\infty} \alpha_n = 0$ , then

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{s_n^2 L(s_n^2)}} = \sqrt{2} \quad \text{a.s.}$$
<sup>(2)</sup>

Later on, Hartman–Wintner [4] proved that if  $(X_n)$  is an i.i.d. sequence of real, centered square-integrable random variables with variance  $Var(X_i) = \sigma^2$ , then

$$\limsup_{n \to \infty} \frac{S_n}{\sqrt{nL(n)}} = \sqrt{2}\sigma \quad \text{a.s.}$$

de Acosta [2] simplified the proof of Hartman–Wintner. To compare the two results, if the sequence  $(Y_n)$  are i.i.d. and uniformly bounded, then the two results coincide. Apparently, Hartman–Wintner's LIL does not contain Kolmogorov's version as a special case. However, Kolmogorov's LIL can be used in a truncation procedure to prove other LIL results; see, e.g., [16].

Kolmogorov's LIL was generalized to martingales by Stout [15]. Let  $(X_n, \mathcal{F}_n)_{n\geq 1}$  be a martingale with  $\mathbb{E}(X_n) = 0$ . Let  $Y_n = X_n - X_{n-1}$  for  $n \geq 1$ ,  $X_0 = 0$  be the associated martingale differences. Put  $s_n^2 = \sum_{i=1}^n \mathbb{E}[Y_i^2|\mathcal{F}_{i-1}]$ . Then Stout proved that if  $s_n^2 \to \infty$  and (1) holds, then  $\limsup_{n\to\infty} X_n/\sqrt{s_n^2 L(s_n^2)} = \sqrt{2}$  a.s.

To state our main results, let us set up the noncommutative framework. Throughout this paper, we consider a noncommutative probability space  $(\mathcal{N}, \tau)$ . Here  $\mathcal{N}$  is a finite von Neumann algebra and  $\tau$  a normal faithful tracial state, i.e.,  $\tau(xy) = \tau(yx)$  for  $x, y \in \mathcal{N}$ . For  $1 \leq p < \infty$ , define  $||x||_p = [\tau(|x|^p)]^{1/p}$  and  $||x||_{\infty} = ||x||$  for  $x \in \mathcal{N}$ . In this paper  $|| \cdot ||$  will always denote the operator norm. The noncommutative  $L_p$  space  $L_p(\mathcal{N}, \tau)$  (or  $L_p(\mathcal{N})$  for short) is the completion of  $\mathcal{N}$  with respect to  $|| \cdot ||_p$ .  $\tau$ -measurable operators affiliated to  $(\mathcal{N}, \tau)$  are also called noncommutative random variables; see [3,17] for more details on the measurability and noncommutative  $L_p$  spaces. Let  $(\mathcal{N}_k)_{k=1,2,...} \subset \mathcal{N}$  be a filtration of von Neumann subalgebras with conditional expectation  $E_k : \mathcal{N} \to \mathcal{N}_k$ . Then  $E_k(1) = 1$  and  $E_k(axb) = aE_k(x)b$  for  $a, b \in \mathcal{N}_k$  and  $x \in \mathcal{N}$ . It is well known that  $E_k$  extends to contractions on  $L_p(\mathcal{N}, \tau)$  for  $p \ge 1$ ; see [7].

Following [10], a sequence  $(x_n)$  of  $\tau$ -measurable operators is said to be almost uniformly bounded by a constant  $K \ge 0$ , denoted by  $\limsup_{n \to \infty} x_n \le K$ , if for any  $\varepsilon > 0$  and any  $\delta > 0$ , there exists a projection e with  $\tau(1 - e) < \varepsilon$  such that

$$\limsup_{n \to \infty} \|x_n e\| \le K + \delta; \tag{3}$$

and  $(x_n)$  is said to be bilaterally almost uniformly bounded by a constant  $K \ge 0$ , denoted by  $\limsup_{n \to \infty} x_n \le K$ , if (3) is replaced by

$$\limsup_{n\to\infty}\|ex_ne\|\leq K+\delta.$$

Clearly,  $\limsup_{n \to \infty} x_n \leq K$  implies  $\limsup_{n \to \infty} x_n \leq K$ .

For a  $\tau$ -measurable operator x and t > 0, the generalized singular numbers [3] are defined by

$$\mu_t(x) = \inf \left\{ s > 0: \ \tau \left( \mathbb{1}_{(s,\infty)} \left( |x| \right) \right) \le t \right\}.$$

In this paper, we use  $1_A(a)$  to denote the spectral projection of an operator *a* on the Borel set *A*. According to [10], a sequence of operators  $(x_i)$  is said to be uniformly bounded in distribution by an operator *y* if there exists K > 0 such that  $\sup_i \mu_t(x_i) \le K \mu_{t/K}(y)$  for all t > 0. Let  $(x_n)$  be a sequence of mean zero self-adjoint independent random variables. Konwerska [11] proved that if  $(x_n)$  is uniformly bounded in distribution by a random variable *y* such that  $\tau(|y|^2) = \sigma^2 < \infty$ , then

$$\limsup_{n\to\infty}\frac{1}{\sqrt{nL(n)}}\sum_{i=1}^n x_i \leq C\sigma.$$

Note that if the sequence  $(x_n)$  is i.i.d., which is the case in the original version of Hartman–Wintner's LIL, then  $(x_n)$  is uniformly bounded in distribution by  $x_1$ . Essentially, the condition of uniform boundedness in distribution requires the sequence to be almost identically distributed.

Our main result is an extension of Stout's result to the noncommutative setting. Let  $(x_n)_{n\geq 0}$  be a noncommutative self-adjoint martingale with  $x_0 = 0$  and  $d_i = x_i - x_{i-1}$  the associated martingale differences. Define  $s_n^2 = \|\sum_{i=1}^n E_{i-1}(d_i^2)\|_{\infty}$  and  $u_n = [L(s_n^2)]^{1/2}$ .

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**Theorem 1.** Let  $0 = x_0, x_1, x_2, ...$  be a self-adjoint martingale in  $(\mathcal{N}, \tau)$ . Suppose  $s_n^2 \to \infty$  and  $||d_n||_{\infty} \le \alpha_n s_n/u_n$  for some sequence  $(\alpha_i)$  of positive numbers such that  $\alpha_n \to 0$  as  $n \to \infty$ . Then

$$\limsup_{n\to\infty}\frac{x_n}{s_nu_n}\leq 2.$$

So far as we know, this is the first result on the LIL for noncommutative martingales. A natural question is to ask for the lower bound of LIL. As observed in [10], however, one can only expect an upper bound for LIL in the general noncommutative setting. Indeed, consider a free sequence of semicircular random variables  $(x_n)$  (the so-called free Gaussian random variables [18]) such that the law of  $x_n$  is  $\gamma_{0,2}$  (in notation,  $x_n \sim \gamma_{0,2}$ ) for all n. Here  $\gamma_{0,2}$  has density function  $p(x) = \frac{1}{2\pi}\sqrt{4-x^2}$  for  $-2 \le x \le 2$ . Then it is well known in free probability theory that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i \sim \gamma_{0,2}$$

It follows that  $\lim_{n\to\infty} \sum_{i=1}^{n} x_i / \sqrt{nL(n)} = 0$  in the norm topology since a random variable with law  $\gamma_{0,2}$  is bounded. Therefore there is no reasonable notion of the positive LIL lower bound for the free semicircular sequence. Comparing our LIL results with classical ones, we lose a constant of  $\sqrt{2}$ . However, since there is no hope to obtain an LIL lower bound in the general noncommutative theory, we are more interested in the order of the fluctuation for general noncommutative martingales. It is also commonly acknowledged that going from the commutative theory to the noncommutative setting usually requires considerably more technologies [14]. Due to these reasons, it seems fair to have the constant 2 in the noncommutative martingale setting.

We will recall some preliminary facts in Section 2. The main result will be proved in Section 3. We will discuss some further questions in Section 4.

## 2. Preliminaries

In this section, we give some basic definitions and collect some preliminary facts. Let us recall the vector valued noncommutative  $L_p$  spaces for  $1 \le p \le \infty$  introduced by Pisier [13] and Junge [6]. Let  $(x_n)$  be a sequence in  $L_p(\mathcal{N})$  and define

$$\|(x_n)\|_{L_p(\ell_\infty)} = \inf\{\|a\|_{2p} \|b\|_{2p}: x_n = ay_n b, \|y_n\|_{\infty} \le 1\}.$$

Then  $L_p(\ell_{\infty})$  is defined to be the closure of all sequences with  $||(x_n)||_{L_p(\ell_{\infty})} < \infty$ . It was shown in [8] that if every  $x_n$  is self-adjoint, then

$$\|(x_n)\|_{L_p(\ell_{\infty})} = \inf\{\|a\|_p \colon a \in L_p(\mathcal{N}), a \ge 0, -a \le x_n \le a \text{ for all } n \in \mathbb{N}\}.$$

Similarly, Junge and Xu introduced in [8] the space  $L_p(\ell_{\infty}^c)$  with norm

$$\begin{aligned} \|(x_i)_{i \in I}\|_{L_p(\ell_{\infty}^c)} \\ &= \inf\{\|a\|_p \colon a \in L_p(\mathcal{N}), a \ge 0, -a \le x_i^* x_i \le a \text{ for all } i \in I\} \\ &= \inf\{\|b\|_p \colon x_i = y_i b, \|y_i\|_{\infty} \le 1 \text{ for all } i \in I\}. \end{aligned}$$

The following result is the noncommutative asymmetric version of Doob's maximal inequality proved by Junge [6]. We add a short proof to elaborate on the constant which is implicit in the original paper.

**Theorem 2.** Let  $4 \le p \le \infty$ . Then, for any  $x \in L_p(\mathcal{N})$ , there exists  $b \in L_p(\mathcal{N})$  and a sequence of contractions  $(y_n) \subset \mathcal{N}$  such that

$$||b||_p \le 2^{2/p} ||x||_p$$
 and  $E_n x = y_n b$ , for all  $n \ge 0$ .

**Proof.** This follows from [6], Corollary 4.6. Indeed, setting  $r = p \ge 4$  and  $q = \infty$ , we find  $E_n x = az_n b$  for  $a, z_n \in \mathcal{N}$  and  $b \in L_p(\mathcal{N})$ . Let  $y_n = az_n/||az_n|| \in \mathcal{N}$  and  $b' = ||az_n||b \in L_p(\mathcal{N})$ . Then  $(y_n)$  is a sequence of contractions,  $E_n x = y_n b'$ , and

$$\|b'\|_p \le \|a\|_{\infty} \|b\|_p \sup_n \|z_n\|_{\infty} \le c(p,q,r) \|x\|_p,$$

where  $c(p,q,r) \le c_{q/(q-2)}^{1/2} c_{r/(r-2)}^{1/2} = c_1^{1/2} c_{p/(p-2)}^{1/2}$  and  $c_p$  is the constant in the dual Doob's inequality. Note that  $1 \le p/(p-2) \le 2$ . By Lemma 3.1 and Lemma 3.2 of [6], we find that  $c_p \le 2^{2(p-1)/p}$  for  $1 \le p \le 2$ . It follows that  $c(p,q,r) \le 2^{2/p}$ .

Suppose  $(x_i)_{m \le i \le n}$  is a martingale in  $L_p(\mathcal{N})$ . According to Theorem 2, there exist  $b \in L_p(\mathcal{N})$  and contractions  $(y_i)_{m \le i \le n} \subset \mathcal{N}$  such that  $x_i = y_i b$  for  $m \le i \le n$  and  $||b||_p \le 2^{2/p} ||x_n||_p$  for  $p \ge 4$ . It follows that

$$\|(x_i)_{m\leq i\leq n}\|_{L_p(\ell_{\infty}^c)}\leq 2^{2/p}\|x_n\|_p.$$

Doob's inequality will be used in this form in the proof of our main result.

Our proof of LIL for martingales relies on the following exponential inequality proved in [9]. Its proof was based on Oliveira's approach to the matrix martingales [5].

**Lemma 3.** Let  $(x_k)$  be a self-adjoint martingale with respect to the filtration  $(\mathcal{N}_k, E_k)$  and  $d_k = x_k - x_{k-1}$  be the associated martingale differences such that

(i)  $\tau(x_k) = x_0 = 0$ ; (ii)  $||d_k|| \le M$ ; (iii)  $\sum_{k=1}^n E_{k-1}(d_k^2) \le D^2 1$ . Then

$$\tau\left(\mathrm{e}^{\lambda x_n}\right) \leq \exp\left[(1+\varepsilon)\lambda^2 D^2\right]$$

for all  $\varepsilon \in (0, 1]$  and all  $\lambda \in [0, \sqrt{\varepsilon}/(M + M\varepsilon)]$ .

Another important tool in our proof is a noncommutative version of Borel–Cantelli lemma. To state this result, we recall from [10] that for a self-adjoint sequence  $(x_i)_{i \in I}$  of random variables, the column version of tail probability is by definition

$$\operatorname{Prob}_{c}\left(\sup_{i \in I} \|x_{i}\| > t\right) = \inf\left\{s > 0: \exists \text{ a projection } e \text{ with } \tau(1-e) < s \text{ and } \|x_{i}e\|_{\infty} \le t \text{ for all } i \in I\right\}$$

for t > 0. It is immediate that

$$\operatorname{Prob}_{c}\left(\sup_{i\in I}\|x_{i}\| > t\right) \leq \operatorname{Prob}_{c}\left(\sup_{i\in I}\|x_{i}\| > r\right) \tag{4}$$

for  $t \ge r$  and that if  $a_i \ge 1$  for  $i \in I$ , then

$$\operatorname{Prob}_{c}\left(\sup_{i\in I}\|x_{i}\|>t\right) \leq \operatorname{Prob}_{c}\left(\sup_{i\in I}\|a_{i}x_{i}\|>t\right).$$
(5)

Using the notation  $\text{Prob}_c$ , we state two lemmas which are taken from [10].

**Lemma 4 (Noncommutative Borel–Cantelli lemma).** Let  $\bigcup_n I_n = \{n \in \mathbb{N}: n \ge n_0\}$  for some  $n_0 \in \mathbb{N}$  and  $(z_n)$  be a sequence of self-adjoint random variables. If for any  $\delta > 0$ ,

$$\sum_{n\geq n_0} \operatorname{Prob}_{\mathcal{C}}\left(\sup_{m\in I_n} \|z_m\| > \gamma + \delta\right) < \infty,$$

then

 $\limsup_{n\to\infty} z_n \leq \gamma.$ 

**Lemma 5** (Noncommutative Chebyshev inequality). Let  $(x_i)_{i \in I}$  be a self-adjoint sequence of random variables. For t > 0 and  $1 \le p < \infty$ ,

$$\operatorname{Prob}_{c}\left(\sup_{n}\|x_{n}\|>t\right)\leq t^{-p}\|x\|_{L_{p}(\ell_{\infty}^{c})}^{p}.$$

## 3. Law of the iterated logarithm

According to [1], the original proof of Kolmogorov's LIL is comparably expensive as that of Hartman–Wintner. However, our proof of Kolmogorov's LIL here seems to be relatively easier than (the upper bound of) Hartman–Wintner's version for the commutative case due to the exponential inequality (Lemma 3).

**Proof of Theorem 1.** Let  $\eta \in (1, 2)$  be a constant which we will determine later. To avoid annoying subscripts, we write  $s(k_i) = s_{k_i}$  in the following. Using the stopping rule in [15], we define  $k_0 = 0$  and for  $n \ge 1$ ,

$$k_n = \inf \left\{ j \in \mathbb{N} \colon s_{j+1}^2 \ge \eta^{2n} \right\}.$$

Then  $s_{k_n+1}^2 \ge \eta^{2n}$  and  $s_{k_n}^2 < \eta^{2n}$ . Note that given  $\varepsilon' > 0$  there exists  $N_1(\varepsilon') > 0$  such that for  $n > N_1(\varepsilon')$ ,

$$s_{k_n+1}^2 u_{k_n+1}^2 / \left( s(k_{n+1})^2 u(k_{n+1})^2 \right)$$
  

$$\geq \eta^{-2} \ln \ln \eta^{2n} / \ln \ln \eta^{2(n+1)} \geq \left( 1 - \varepsilon' \right)^2 \eta^{-2}.$$

Then  $s_m u_m \ge (1 - \varepsilon')\eta^{-1}s(k_{n+1})u(k_{n+1})$  for  $k_n < m \le k_{n+1}$ . For any  $\delta' > 0$ , we can find  $\delta, \varepsilon' > 0$  and  $\eta \in (1, 2)$  such that  $1 + \delta' > \eta(1 + \delta)(1 - \varepsilon')^{-1}$ . Fix  $\beta > 0$  which will be determined later. Using the notation  $\text{Prob}_c$  with order relations (4) and (5), we have for  $n > N_1(\varepsilon')$ 

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{x_{m}}{s_{m}u_{m}} \right\| > \beta\left(1 + \delta'\right)\right)$$
$$\leq \operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1 + \delta)\right).$$
(6)

By Lemma 5 and Theorem 2, we have for  $p \ge 4$ ,

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta)\right)$$
$$\leq \left(\lambda\beta(1+\delta)\right)^{-p} \left\| \left( \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right)_{k_{n} < m \leq k_{n+1}} \right\|_{L_{p}(\ell_{\infty}^{c})}^{p}$$
$$\leq \left(\lambda\beta(1+\delta)\right)^{-p} \left(2^{2/p}\right)^{p} \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right\|_{p}^{p}.$$

Using the elementary inequality  $|u|^p \le p^p e^{-p} (e^u + e^{-u})$ , functional calculus and Lemma 3 with  $M = \alpha(k_{n+1}) \times s(k_{n+1})/u(k_{n+1})$ ,  $D^2 = s(k_{n+1})^2$ , we find

$$\begin{aligned} \left\| \frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})} \right\|_{p}^{p} \\ &\leq p^{p} \mathrm{e}^{-p} \tau \left( \exp\left(\frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})}\right) + \exp\left(-\frac{\lambda x(k_{n+1})}{s(k_{n+1})u(k_{n+1})}\right) \right) \\ &\leq 2 \left(\frac{p}{\mathrm{e}}\right)^{p} \exp\left(\frac{(1+\varepsilon)\lambda^{2}}{u(k_{n+1})^{2}}\right) \end{aligned}$$

provided  $0 \le \lambda \le \frac{\sqrt{\varepsilon}u(k_{n+1})^2}{(1+\varepsilon)\alpha(k_{n+1})}$  and  $0 < \varepsilon \le 1$ . Hence we obtain

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta)\right)$$
$$\leq 8\left(\frac{p}{\lambda\beta(1+\delta)e}\right)^{p} \exp\left(\frac{(1+\varepsilon)\lambda^{2}}{u(k_{n+1})^{2}}\right).$$

Now optimizing in p gives  $p = \lambda \beta (1 + \delta)$  and thus,

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta)\right)$$
$$\leq 8\exp\left(\frac{(1+\varepsilon)\lambda^{2}}{u(k_{n+1})^{2}} - \beta(1+\delta)\lambda\right).$$

Put  $\lambda = \beta (1 + \delta) u(k_{n+1})^2 / (2(1 + \varepsilon))$ . Since  $\alpha_n \to 0$ , for any  $\varepsilon > 0$ , there exists  $N_2 > 0$  such that for  $n > N_2$ ,  $0 < \alpha(k_{n+1}) \le \frac{2\sqrt{\varepsilon}}{\beta(1+\delta)}$ , which ensures that we can apply Lemma 3. This also implies  $p \ge 4$  for large *n*. It follows that

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta) \right) \leq \left(\ln s(k_{n+1})^{2}\right)^{-\beta^{2}(1+\delta)^{2}/(4(1+\varepsilon))}$$

Notice that  $s(k_{n+1})^2 \ge s(k_n+1)^2 \ge \eta^{2n}$ . Setting  $\beta = 2$  in the beginning of the proof, we have

$$\operatorname{Prob}_{c}\left(\sup_{k_{n} < m \leq k_{n+1}} \left\| \frac{\lambda x_{m}}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda\beta(1+\delta)\right) \leq \left[ (2\ln\eta)n \right]^{-(1+\delta)^{2}/(1+\varepsilon)}$$

By choosing  $\varepsilon$  small enough so that  $(1 + \delta)^2/(1 + \varepsilon) > 1$ , we find that for  $n_0 = \max\{N_1, N_2\}$ ,

$$\sum_{n\geq n_0} \operatorname{Prob}_c \left( \sup_{k_n < m \leq k_{n+1}} \left\| \frac{\lambda x_m}{s(k_{n+1})u(k_{n+1})} \right\| > \lambda \beta(1+\delta) \right) < \infty.$$

Then (6) and Lemma 4 give the desired result.

## 4. Further questions

Without the growing condition on martingale differences  $d_n$ , Stout proved Hartman–Wintner's LIL in [16] under the additional assumption that the martingale differences are stationary ergodic. At the time of this writing, it is still not clear to us whether a "genuine" version (i.e., it does not satisfy Kolmogorov's growing condition) of Hartman–Wintner's LIL is possible for noncommutative martingales. It would be interesting to see such a result in the future.

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## References

 H. Bauer. Probability Theory. de Gruyter Studies in Mathematics 23. Walter de Gruyter, Berlin, 1996. Translated from the fourth (1991) German edition by Robert B. Burckel and revised by the author. MR1385460

#### Q. Zeng

- [2] A. de Acosta. A new proof of the Hartman–Wintner law of the iterated logarithm. Ann. Probab. 11 (2) (1983) 270–276. MR0690128
- [3] T. Fack and H. Kosaki. Generalized *s*-numbers of τ-measurable operators. *Pacific J. Math.* **123** (2) (1986) 269–300. MR0840845
- [4] P. Hartman and A. Wintner. On the law of the iterated logarithm. Amer. J. Math. 63 (1941) 169–176. MR0003497
- [5] R. Imbuzeiro Oliveira. Concentration of the adjacency matrix and of the Laplacian in random graphs with independent edges. ArXiv e-prints, 2009.
- [6] M. Junge. Doob's inequality for non-commutative martingales. J. Reine Angew. Math. 549 (2002) 149–190. MR1916654
- [7] M. Junge and Q. Xu. Noncommutative Burkholder/Rosenthal inequalities. Ann. Probab. 31 (2) (2003) 948–995. MR1964955
- [8] M. Junge and Q. Xu. Noncommutative maximal ergodic theorems. J. Amer. Math. Soc. 20 (2) (2007) 385-439. MR2276775
- [9] M. Junge and Q. Zeng. Noncommutative martingale deviation and Poincaré type inequalities with applications. Preprint, 2012.
- [10] M. Konwerska. The Law of the Iterated Logarithm in Noncommutative Probability. ProQuest LLC, Ann Arbor, MI, 2008. Ph.D. Thesis, Univ. Illinois at Urbana–Champaign. MR2712889
- [11] M. Konwerska. The law of the iterated logarithm in noncommutative probability. Preprint, 2012.
- [12] M. Ledoux and M. Talagrand. Probability in Banach Spaces: Isoperimetry and Processes. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)] 23. Springer, Berlin, 1991. MR1102015
- [13] G. Pisier. Non-commutative vector valued L<sub>p</sub>-spaces and completely p-summing maps. Astérisque 247 (1998) vi+131. MR1648908
- [14] G. Pisier and Q. Xu. Non-commutative L<sup>p</sup>-spaces. In Handbook of the Geometry of Banach Spaces, Vol. 2 1459–1517. North-Holland, Amsterdam, 2003. MR1999201
- [15] W. F. Stout. A martingale analogue of Kolmogorov's law of the iterated logarithm. Z. Wahrsch. Verw. Gebiete 15 (1970) 279–290. MR0293701
- [16] W. F. Stout. The Hartman–Wintner law of the iterated logarithm for martingales. Ann. Math. Statist. 41 (1970) 2158–2160.
- [17] M. Terp. L<sup>p</sup>-spaces associated with von Neumann algebras. Notes, Copenhagen Univ., 1981.
- [18] D. V. Voiculescu, K. J. Dykema and A. Nica. Free Random Variables. CRM Monograph Series 1. Amer. Math. Soc., Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. MR1217253
- [19] W. B. Wu. Strong invariance principles for dependent random variables. Ann. Probab. 35 (6) (2007) 2294–2320. MR2353389
- [20] O. Zhao and M. Woodroofe. Law of the iterated logarithm for stationary processes. Ann. Probab. 1 (2008) 127–142. MR2370600