Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 4, 1457–1464
DOI: 10.1214/14-AIHP625

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Random two-component spanning forests

Adrien Kassel[a,1], Richard Kenyon[b,2] and Wei Wu[c]

[a] *Département de Mathématiques et Applications, École Normale Supérieure, 45 rue d'Ulm, 75230 Paris Cedex 05, France*
[b] *Mathematics Department, Brown University, 151 Thayer St., Providence, RI 02912, USA*
[c] *Applied Mathematics Department, Brown University, 182 George St., Providence, RI 02912, USA*

Received 26 September 2013; revised 21 April 2014; accepted 27 April 2014

Abstract. We study random two-component spanning forests (2SF) of finite graphs, giving formulas for the first and second moments of the sizes of the components, vertex-inclusion probabilities for one or two vertices, and the probability that an edge separates the components. We compute the limit of these quantities when the graph tends to an infinite periodic graph in $\mathbb{R}^d$.

Résumé. Nous étudions la mesure uniforme sur les forêts couvrantes à deux composantes connexes d'un graphe fini et donnons des formules pour les deux premiers moments de la taille des composantes, les probabilités d'inclusion d'un ou deux sommets dans la même composante, et la probabilité qu'une arête sépare les composantes. Nous calculons la limite des ces quantités lorsque l'on considère une suite de graphes finis qui tend vers un graphe infini périodique dans $\mathbb{R}^d$.

MSC: 82B20; 05C81

Keywords: Two-component spanning forests; Mean resistance; Torsional rigidity

1. Introduction

For $\mathcal{G}$ a finite connected graph with vertex set V, a *spanning tree* is a subgraph (V, A), where A is a set of edges, which contains no cycles and is connected. A *two-component spanning forest* (2SF) is a subgraph (V, B), where B is a set of edges, which contains no cycles and has exactly two connected components. A spanning tree of an n-vertex graph has $n-1$ edges; a 2SF has $n-2$ edges.

The matrix-tree theorem [8] (Theorem 2 below) equates the number of spanning trees $\kappa = \kappa(\mathcal{G})$ with the determinant of the reduced Laplacian. This result has led to an extensive study of the random spanning tree on many different families of graphs, see e.g. [1,2,13].

Let $\kappa_2 = \kappa_2(\mathcal{G})$ be the number of 2SFs. The ratio $\kappa_2(\mathcal{G})/\kappa(\mathcal{G})$ has an explicit expression in terms of the potential kernel which follows from [11] as explained in [6]. It reads

$$\frac{\kappa_2(\mathcal{G})}{\kappa(\mathcal{G})} = \sum_{uv\in E} A_{u,v}A_{v,u} + (A_{u,v} - A_{v,u})^2, \tag{1}$$

where $A_{u,v} = G^r_{u,u} - G^r_{u,v}$ is the potential kernel and G^r is the Green's function with Dirichlet boundary conditions at some vertex r.

This implies the following theorem (which was previously obtained by other means in [7,10]).

[1] Supported in part by Fondation Sciences Mathématiques de Paris.
[2] Supported by the NSF Grant DMS-1208191.

Theorem 1 ([6,7,10]). *For the $n \times n$ grid $\mathcal{G}$ we have*

$$\kappa_2(\mathcal{G}) = \kappa(\mathcal{G}) \frac{n^2}{8} \big(1 + o(1)\big), \quad \text{as } n \to \infty.$$

In this paper, we compute various properties of the random 2SF on the grid and other graphs. In particular we give exact expressions for the first two moments of the volume of the components, as well as vertex-inclusion probabilities.

A first application concerns the Abelian sandpile model. By [4], waves of topplings of avalanches of sandpiles started at a vertex v are in bijection with 2SFs where v is disconnected from the sink. Theorem 5 below hence yields the first moment of the volume covered by each such wave (see Section 2.4), where the sink is the boundary vertex.

A second application, in the planar case, concerns cycle-rooted spanning trees. On a planar graph the dual of a 2SF is a *cycle-rooted spanning tree (CRST)*, that is, a set of $n = |V|$ edges connecting all vertices (and thus containing a unique cycle). By [10], the expected length of the cycle in the scaling limit is related to the so-called "looping constant" of loop-erased random walk (LERW), the density of sand in recurrent Abelian sandpiles, and derivatives of the Tutte polynomial at $(1,1)$ (see also [6,7]). Our results can be interpreted as computing the first two moments of the area of the unique cycle, as well as the probabilities that the cycle encloses a given face or pair of faces, and the probability that an edge is in the unique cycle.

2. Finite graphs

2.1. *Spanning trees and potential theory*

Let $\mathcal{G} = (V, E)$ be an undirected finite connected graph, endowed with a function $c : E \to \mathbb{R}_{>0}$ (which we call *conductance* or *weight*), and b a marked vertex. The *Laplacian operator* $\Delta : \mathbb{R}^V \to \mathbb{R}^V$ is defined by

$$\Delta f(v) = \sum_{v' \sim v} c_{vv'} \big(f(v) - f(v')\big),$$

where the sum is over the neighbors of v. The *Dirichlet Laplacian* Δ_D is defined on $\mathbb{R}^{V \setminus \{b\}}$ by the same formula (in which the sum, however, ranges over all of V, not just $V \setminus \{b\}$); in the natural basis indexed by V, Δ_D is the submatrix of Δ obtained by removing b's row and column.

The operator Δ_D is invertible, see Theorem 2 below. Let G denote the *Green's function with Dirichlet boundary conditions at b*; it is the inverse of Δ_D.

Entries of G have both probabilistic and potential-theoretic interpretations: $G_{u,v}$ is the expected number of visits to v of a conductance-biased random walk from u to b. It is also the voltage at v when one unit of current flows from u to b. $G_{u,u}$ in particular is the resistance between u and b [3]. See also (3) below.

Given two directed edges $e = u_1 v_1$, $e' = u_2 v_2$ we define the *transfer current* to be

$$T\big(e, e'\big) = c\big(e'\big)\big(G(u_1, u_2) - G(u_1, v_2) - G(u_2, v_1) + G(v_1, v_2)\big).$$

The transfer current is used to compute edge inclusion probabilities for random spanning trees [2], for example

$$\Pr(\text{edge } e \text{ is in the tree}) = T(e, e). \tag{2}$$

The quantity $T(e, e')$ is also the amount of current crossing edge e' when one unit of current flows in at u_1 and out at v_1.

We define

$$\kappa = \kappa(\mathcal{G}) = \sum_{\text{trees } T} \prod_{e \in T} c(e)$$

to be the weighted sum of the collection of spanning trees.

Theorem 2 (The matrix-tree theorem [8]). $\kappa(\mathcal{G}) = \det \Delta_D$.

For general graphs a useful identity is

$$G_{u,u} = \frac{\kappa(\mathcal{G}_{u\sim b})}{\kappa(\mathcal{G})}, \tag{3}$$

where $\mathcal{G}_{u\sim b}$ is the graph $\mathcal{G}$ with u and b identified. This can be proved by comparing the Laplacian of $\mathcal{G}$ and $\mathcal{G}_{u\sim b}$ which differ in only a single entry; see e.g. [8].

2.2. *Vertex-inclusion probabilities*

For any 2SF of $\mathcal{G}$, we define the *floating component* to be the component not containing b.

In the following, let $\mathbb{P}$ denote the probability measure on 2SFs which assigns to each 2SF a probability proportional to its weight. Let $\kappa_2 = \kappa_2(\mathcal{G})$ be the weighted sum of 2SFs.

Theorem 3. *Let Σ be the floating component of a $\mathbb{P}$-random 2SF on $\mathcal{G}$. The probability that vertex u is in Σ is*

$$\mathbb{P}(u) = \frac{\kappa}{\kappa_2} G_{u,u}. \tag{4}$$

The probability that two vertices u and v are in Σ is

$$\mathbb{P}(u, v) = \frac{\kappa}{\kappa_2} G_{u,v}. \tag{5}$$

Proof. Let $\mathcal{G}_u$ be the graph $\mathcal{G}$ with an additional edge e_u connecting the wired boundary b to u. The event $\{u \in \Sigma\}$ has an interpretation in terms of spanning trees of $\mathcal{G}_u$: it is the event that e_u is contained in a spanning tree of $\mathcal{G}_u$. Thus, letting $T_{\mathcal{G}_u}$ denote the transfer current on $\mathcal{G}_u$, we have

$$\mathbb{P}(u) = \frac{1}{\kappa_2(\mathcal{G})} \sum_{\substack{\text{spanning trees } T \\ \text{containing } e_u}} w(T) = \frac{\kappa(\mathcal{G}_u)}{\kappa_2(\mathcal{G})} T_{\mathcal{G}_u}(e_u, e_u) = \frac{\kappa(\mathcal{G})}{\kappa_2(\mathcal{G})} G_{u,u}, \tag{6}$$

where the second equality follows from (2) and the third one from (3).

We now condition on the event that $u \in \Sigma$. Wire u and b together and construct a spanning tree of $\mathcal{G}_{u\sim b}$ using Wilson's algorithm [13] starting at v. The conditional probability $\mathbb{P}(v \in \Sigma | u \in \Sigma)$, as a function of v, is the harmonic function with boundary values 1 at u and 0 at b, hence we have

$$\mathbb{P}(v \in \Sigma | u \in \Sigma) = \frac{G_{u,v}}{G_{u,u}}. \tag{7}$$

The result follows. □

Theorem 4. *The probability that edge e connects Σ to Σ^c is*

$$\mathbb{P}(e \in \partial\Sigma) = \frac{\kappa T(e, e)}{c(e)\kappa_2}. \tag{8}$$

Proof. By [2], see (2), $\kappa T(e, e)$ is the weighted sum of spanning trees containing edge e. □

2.3. *First and second moments of the size of Σ*

Let $\partial\Sigma$ denote the boundary of Σ, that is, the set of edges with exactly one endpoint in Σ. Let $|\partial\Sigma|$ denote the sum of weights of edges in $\partial\Sigma$.

Lemma 1.

$$\mathbb{E}\big(|\partial\Sigma|\big) = \frac{\sum_{e\in E}\kappa(\mathcal{G})\mathbb{P}(e\in T)}{\kappa_2(\mathcal{G})} = \frac{\kappa(\mathcal{G})(|V|-1)}{\kappa_2(\mathcal{G})}. \tag{9}$$

Proof. The first equality follows from (8), upon multiplying both sides of (8) by $c(e)$ and then summing over all edges. The second equality follows from the fact that every spanning tree has exactly $|V|-1$ edges. □

Let $\ell^* = \mathbb{E}(|\partial\Sigma|)$ be the quantity in (9). Summing (4) and (5) over all vertices, we obtain the following volume moments.

Theorem 5. *We have*

$$\mathbb{E}\big(|\Sigma|\big) = \ell^*\frac{|V|}{|V|-1}R \quad \textit{and} \quad \mathbb{E}\big(|\Sigma|^2\big) = \ell^*\frac{|V|}{|V|-1}\mathbb{E}(\tau_b),$$

where $R = \sum_{v\in V} G_{v,v}/|V|$ is the mean resistance between v and b for a uniform random v, and

$$\mathbb{E}(\tau_b) = \frac{1}{|V|}\sum_{u,v\in V} G_{u,v}$$

is the expected hitting time to b for the conductance-biased random walk started at a uniform starting vertex.

2.4. *Pinned bush*

For any vertex $z_0 \neq b$, let $\mathbb{P}_{z_0}$ be the probability distribution of the random 2SF conditioned so that its floating component Σ contains z_0.

Using the expression of $\mathbb{P}(u\in\Sigma|z_0\in\Sigma)$ in (7), and summing over u, we obtain that the conditional expected size of Σ satisfies

$$\mathbb{E}\big(|\Sigma|:\ z_0\in\Sigma\big) = \sum_v \frac{G_{v,z_0}}{G_{z_0,z_0}} = \frac{\mathbb{E}(\tau_D^{z_0})}{G_{z_0,z_0}},$$

where $\tau_D^{z_0}$ is the exit time of a conductance-biased random walk started at z_0.

3. Infinite graphs

Let $\mathcal{G} = \mathbb{Z}^d$ with constant conductances 1. Let $\mathcal{G}_n = \mathcal{G}\cap[-n,n]^d$. We wire all vertices of $\mathcal{G}\setminus\mathcal{G}_n$ into a single vertex which plays the role of the boundary of $\mathcal{G}_n$.

In this setting the potential kernel on $\mathcal{G}_n$ converges to the potential kernel on $\mathcal{G}$. On $\mathcal{G}_n$ we have $A_{u,v} = 1/2d + \mathrm{o}(1)$ for any edge uv not within $\mathrm{O}(1)$ of the boundary. Since there are $dn^d(1+\mathrm{o}(1))$ edges in $\mathcal{G}_n$, formula (9) gives

$$\kappa_2(\mathcal{G}_n) = \kappa(\mathcal{G}_n)n^d/(4d)\big(1+\mathrm{o}(1)\big).$$

By (1), the expected boundary size of the floating component is then

$$\mathbb{E}\big(|\partial\Sigma|\big) = 4d + \mathrm{o}(1). \tag{10}$$

More generally, let $\mathcal{G}$ be a graph in $\mathbb{R}^d$, periodic under translations in $\mathbb{Z}^d$. Again let $\mathcal{G}_n = \mathcal{G}\cap[-n,n]^d$. In this setting again the potential kernel on $\mathcal{G}_n$ converges to the potential kernel on $\mathcal{G}$.

Combining (1) and (9), we find (to leading order)

$$\mathbb{E}\big(|\partial\Sigma|\big) = n_0\bigg[\sum_{e\in \mathrm{f.d.}} A_{u,v}A_{v,u} + (A_{u,v}-A_{v,u})^2\bigg]^{-1},$$

where the sum is over edges in a single fundamental domain $[0,1)^d$, and n_0 is the number of vertices per fundamental domain.

4. Euclidean domains

Here we consider scaling limits. Although our results apply in greater generality (see the last but one paragraph of this section), for simplicity we consider only the case of subgraphs of $\mathbb{Z}^d$.

Let D be a domain in $\mathbb{R}^d$ (for some $d \geq 2$) with boundary a smooth hypersurface. Let $\mathcal{G}_n$ be the nearest-neighbor graph of $\frac{1}{n}\mathbb{Z}^d$ with all vertices outside of D wired to a single vertex called b. All edges have conductance 1.

By (10) above, $\mathbb{E}_n(|\partial\Sigma|) \to 4d$ as $n \to \infty$. For general graphs $\mathcal{G}$ we denote this limit $\ell^* = \ell^*(\mathcal{G})$ when it exists.

Denote R_n to be the mean resistance (from a uniformly chosen vertex to the wired boundary) associated with $\mathcal{G}_n$. We obtain the following.

Theorem 6. *Suppose $d \geq 2$. Let $z \neq z' \in D$, and z_n, z'_n be points on $\mathcal{G}_n$ within distance $\mathrm{O}(1/n)$ of z, z', respectively. As $n \to \infty$, we have*

$$\mathbb{P}_n(z_n, z'_n \in \Sigma) = \frac{4d}{|D|n^{2d-2}} g_D^0(z, z') + \mathrm{o}(n^{-d}).$$

As $n \to \infty$, we have (for $d \geq 3$)

$$\mathbb{E}_n(|\Sigma|) = 4dR^* + \mathrm{o}(1),$$
$$\mathbb{E}_n(|\Sigma|^2) = 4dC(D)|D|n^2 + \mathrm{o}(n^2),$$

where $R^* = \lim_{n\to\infty} R_n$, and $C(D) = \lim_{n\to\infty} \frac{\mathbb{E}(\tau_b)}{|D|n^2}$ is the expected exit time from D for Brownian motion started at a uniform point in D, divided by $|D|$.

For $d = 2$, the expression for $\mathbb{E}_n(|\Sigma|^2)$ is the same above, and we have

$$\mathbb{E}_n(|\Sigma|) = \frac{4\log n}{\pi} + \mathrm{o}(\log n).$$

Proof. The sequence of graphs $(\mathcal{G}_n)_{n\geq 1}$ has the following approximation property: the discrete Green's function converges under rescaling by n^{2-d} to the continuous Green's function g_D^0 on D with Dirichlet boundary condition [9]: $n^{d-2}G(z_n, z'_n) \to g_D^0(z, z')$.

The result follows by passing to the limit in Theorem 5 and using convergence of random walk to Brownian motion to get the convergence of the expected exit time (with the right Brownian time–space scaling). The convergence of the mean resistance to a limit on the infinite network follows from Rayleigh's principle [1].

In particular, $R^* < \infty$ exists for $d \geq 3$ by transience of the random walk [3]. For $d = 2$, $\lim_{n\to\infty} R_n/\log n = 1/(2\pi)$ by explicit asymptotics of the Green's kernel [9]. □

The mean resistance is the normalized trace of the Green's function, and so for the cubic grid of sidelength n can be computed by explicit diagonalization. We have

$$R_n = n^{-d} \sum_{k_1,\dots,k_d \overset{*}{=} 1}^{n} \left(4\sum_{i=1}^{d} \sin^2(\pi k_i/n)\right)^{-1}, \tag{11}$$

where the $\overset{*}{=}$ indicates that we leave off the term in which all $k_j = n$, and

$$R^* = \frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \frac{1}{2d - 2\cos\theta_1 - \cdots - 2\cos\theta_d} \,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_d.$$

Table 1
Constants for three planar regular lattices

	Square grid	Hexagonal grid	Triangular grid
ℓ^*	8	6	12
$\mathbb{E}_n(A)/(\log n/\pi)$	4	$3\sqrt{3}$	$2\sqrt{3}$
$\mathbb{E}_n(A^2)/(C(D)n^2)$	4	$3\sqrt{3}$	$2\sqrt{3}$

For fixed $|D|$, the constant $C(D)$ is maximized for a sphere [12]. In the case of a cuboid D in $\mathbb{Z}^d$ of side lengths $a_1, \dots, a_d$, it is equal (the proof uses the explicit expression for the eigenvalues of the Laplacian) to

$$C(D) = \frac{4^{2d}}{\pi^{2d+2}} \sum_{n_i \geq 1, \text{odd}} \prod_{i=1}^{d} \frac{1}{a_i n_i^2} \left(\sum_{i=1}^{d} \frac{n_i^2}{a_i^2} \right)^{-1}. \tag{12}$$

It is interesting to note that in dimension two, $C(D) = P(D)/|D|^2$, where $P(D)$ is what Pólya calls the *torsional rigidity* of the cross-section D, and which is, in mechanical terms, a measure of the resistance to torsion of a cylindrical beam with cross-section D, defined by $1/P(D) = \inf_f w(f)$ is the infimum, over all smooth functions f over D vanishing on the boundary, of $w(f) := \frac{\int_D |\nabla f|^2}{4(\int_D f)^2}$.

Theorem 6 is valid in greater generality: one can take any periodic graph in $\mathbb{R}^d$ for which random walk converges to Brownian motion, and replace the constants $4d$ in the statement with ℓ^*. Indeed, we need only the property that the discrete Green's function converges to the continuous one. Both ℓ^* and R^* can be computed, since the Green's function is an explicit integral of a rational function and ℓ^* and R^* are obtained from the Green's function. For the square, triangular, and hexagonal lattices ℓ^* is easily computed; we list the relevant quantities for these cases in Table 1.

5. Spanning unicycles on planar graphs

A *cycle-rooted spanning tree* (CRST), or *unicycle*, is a spanning subgraph which is connected and has a unique cycle, that is, is the union of a spanning tree and an additional edge. We let $\lambda(\mathcal{G})$ be the weighted sum of the collection of CRSTs.

On a planar graph the dual of a 2SF (take duals of all edges not in the 2SF) is a CRST. It is natural to assign conductances to the planar dual $\mathcal{G}^*$ which are the reciprocals of the conductances of $\mathcal{G}$; then duality gives a bijection from 2SFs to CRSTs which multiplies the weight by a constant (the reciprocal of the product of all conductances of $\mathcal{G}$).

In the planar case, we can use planar duality to translate the previous statements about the floating component of a 2SF into statements about the unique loop of a weighted spanning unicycle.

On a planar graph embedded in the plane, the dual of a spanning unicycle on $\mathcal{G}$ is a 2SF on the dual graph $\mathcal{G}^*$, for which we choose the marked vertex b^* to be the outer boundary face of $\mathcal{G}$. Let V^* be the vertex set of $\mathcal{G}^*$.

Theorem 4 in this setting shows that for a CRST on the dual of a planar graph $\mathcal{G}$, the probability that edge e^*, dual of edge e, is in the unique cycle is

$$\mathbb{P}\big(e^* \text{ in cycle}\big) = \frac{\kappa(\mathcal{G})}{c(e)\kappa_2(\mathcal{G})} T_{\mathcal{G}}(e,e) = \frac{c(e^*)\kappa(\mathcal{G}^*)}{\lambda(\mathcal{G}^*)} \big(1 - T_{\mathcal{G}^*}\big(e^*, e^*\big)\big).$$

Theorem 3 and Theorem 5 translate to the following result in this setting. Denote A to be the area (i.e. the number of faces enclosed) of the unique cycle of a random spanning unicycle.

Theorem 7. *Let f, f' be two faces of $\mathcal{G}$. Then*

$$\mathbb{P}(f \text{ enclosed}) = \frac{\kappa(\mathcal{G})}{\lambda(\mathcal{G})} G^*_{f,f}, \qquad \mathbb{P}\big(f, f' \text{ enclosed}\big) = \frac{\kappa(\mathcal{G})}{\lambda(\mathcal{G})} G^*_{f,f'}.$$

We also have

$$\mathbb{E}(A) = \frac{\kappa(\mathcal{G})}{\lambda(\mathcal{G})} \sum_{v^* \in V^*} G^*_{v^*,v^*} \quad \text{and} \quad \mathbb{E}\big(A^2\big) = \frac{\kappa(\mathcal{G})}{\lambda(\mathcal{G})} \big|V^*\big|^2 \mathbb{E}(\tau_{b^*}),$$

where $\mathbb{E}(\tau_{b^*})$ *is the expected hitting time of* b^* *for a conductance-biased random walk started at a uniformly chosen starting vertex of* $\mathcal{G}^*$.

Let $\mathcal{G}_n$ be the $n \times n$ grid with unit conductances. In [5] were computed up to constants the moments of the combinatorial area A of the uniform unicycle on $\mathcal{G}_n$ (whose probability distribution we denote by ν_n), or equivalently, the moments of the size of the floating component of the uniform 2SF:

Theorem 8 ([5]). *For all integer* $k \geq 2$, *there is a constant* $C_k > 0$, *such that* $\mathbb{E}(A_n^k) = C_k n^{2k-2}(1 + \mathrm{o}(1))$, *as* $n \to \infty$.

We give a sketch of the proof for completeness.

Proof of Theorem 8. Let $\mathcal{H}_n$ be the graph $\mathcal{G}_n$ scaled to fit in the square $D = [0, 1]^2$. Let μ_n be the measure on unicycles on $\mathcal{H}_n$, weighted by the square of the area of the cycle. In [5] it is shown that μ_n converges as $n \to \infty$ to a measure μ with the property that the probability of a cycle of positive area is positive. For a cycle of area A in $\mathcal{G}_n$, the Radon–Nikodym derivative between μ_n and ν_n is $\mathrm{d}\mu_n/\mathrm{d}\nu_n = A^2/\mathbb{E}_{\nu_n}(A^2)$.

We have

$$\frac{\mathbb{E}_{\nu_n}(A^k)}{\mathbb{E}_{\nu_n}(A^2)} = \mathbb{E}_{\mu_n}\big(A^{k-2}\big) = n^{2k-4}\mathbb{E}_\mu\big(\theta^{k-2}\big)\big(1 + \mathrm{o}(1)\big),$$

where θ is the scaled Euclidean area (in [0, 1]). By Theorem 6,

$$\mathbb{E}_{\nu_n}\big(A^2\big) = C(D)|D|n^2\big(1 + \mathrm{o}(1)\big)$$

thus we have

$$\mathbb{E}_{\nu_n}\big(A^k\big) = n^{2k-2}C(D)\mathbb{E}_\mu\big(\theta^{k-2}\big)\big(1 + \mathrm{o}(1)\big). \qquad \square$$

6. Questions

(1) Can one compute the constants C_k in the higher moments of Theorem 8?
(2) Can one compute the expected length of the cycle of the spanning unicycle in higher dimension, for example in $\mathbb{Z}^3$?
(3) The probability that three distinct vertices are in Σ seems to be a much harder quantity to compute. Can this probability be written in terms of the Green's function?

Acknowledgements

We thank Cédric Boutillier, Yuval Peres, and David Wilson for helpful discussions and feedback, as well as the anonymous referee for spotting an incorrection in the first version.

References

[1] I. Benjamini, R. Lyons, Y. Peres and O. Schramm. Uniform spanning forests. *Ann. Probab.* **29** (2001) 1–65. MR1825141
[2] R. Burton and R. Pemantle. Local characteristics, entropy, and limit theorems for spanning trees and domino tilings via transfer impedances. *Ann. Probab.* **21** (3) (1993) 1329–1371. MR1235419
[3] P. G. Doyle and J. L. Snell. *Random Walks and Electric Networks*. *Carus Mathematical Monographs* **22**. Mathematical Association of America, Washington, DC, 1984. MR0920811

[4] E. V. Ivashkevich, D. V. Ktitarev and V. B. Priezzhev. Waves of topplings in an Abelian sandpile. *Phys. A* **209** (1994) 347–360.
[5] A. Kassel and R. Kenyon. Random curves on surfaces induced by the Laplacian determinant. Preprint, 2012. Available at arXiv:1211.6974.
[6] A. Kassel and D. B. Wilson. The looping rate and sandpile density of planar graphs. Preprint, 2014. Available at arXiv:1402.4169.
[7] R. W. Kenyon and D. B. Wilson. Spanning trees of graphs on surfaces and the intensity of loop-erased random walk on $\mathbb{Z}^2$. Preprint, 2011. Available at arXiv:1107.3377.
[8] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Verteilung galvanischer Ströme geführt wird. *Ann. Phys. Chem.* **72** (1847) 497–508.
[9] G. F. Lawler and V. Limic. *Random Walk: A Modern Introduction*. *Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge, 2010. MR2677157
[10] L. Levine and Y. Peres. The looping constant of $\mathbb{Z}^d$. *Random Structures Algorithms* **45** (2014) 1–13. MR3231081
[11] W. Myrvold. Counting k-component forests of a graph. *Networks* **22** (7) (1992) 647–652. MR1189292
[12] G. Pólya. Torsional rigidity, principal frequency, electrostatic capacity, and symmetrization. *Quart. Appl. Math.* **6** (1948) 267–277. MR0026817
[13] D. B. Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-Eigth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)* 296–303. ACM, New York, 1996. MR1427525