ANNALES SCIENTIFIQUES DE L'É.N.S.

M. FLATO
J. SIMON
H. SNELLMAN
D. STERNHEIMER

Simple facts about analytic vectors and integrability

Annales scientifiques de l'É.N.S. 4e série, tome 5, nº 3 (1972), p. 423-434
<http://www.numdam.org/item?id=ASENS_1972_4_5_3_423_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 5, 1972, p. 423 à 434.

SIMPLE FACTS ABOUT ANALYTIC VECTORS AND INTEGRABILITY

BY M. FLATO, J. SIMON, H. SNELLMAN AND D. STERNHEIMER

ABSTRACT. — A theorem which completes previous studies (mainly due to E. Nelson) concerning integrability criteria of representations of finite dimensional real Lie algebras is presented. This theorem states that if there exists a dense domain of analytic vectors for a representation by skew symmetric operators of a given basis of a finite dimensional real Lie algebra in a complex Hilbert space, and if in addition, this domain is invariant under the basis, then the Lie algebra representation generated by the basis is integrable to a unitary representation of a Lie group.

Introduction

The question of analytic vectors in representation theory of finite dimensional real Lie groups has two aspects : a global one, essentially the question of their density in the representation space, and an infinitesimal one, for which the most interesting problem is that of exponentiability from a Lie algebra representation to a Lie group representation.

Given a continuous representation $\mathfrak{T}$ of a real Lie group G in a quasi-complete locally convex topological vector space H [i. e. $\mathfrak{T}$ is a group representation $\mathfrak{T} : G \to \mathcal{L}(H)$ such that the map $(g, \varphi) \to \mathfrak{T}(g)\varphi$ is continuous on $G \times H$], a vector $\varphi \in H$ is said analytic (resp. differentiable) if the function $g \to \mathfrak{T}(g)\varphi$ is so. It has been shown by L. Gårding [3] and more generally by F. Bruhat [1] that the space of differentiable vectors is dense in H. When H is a Banach space, it has been shown with increasing generality and elegance by Harish-Chandra [6], P. Cartier and J. Dixmier [2], E. Nelson [11] and finally by L. Gårding [4], that the analytic vectors form a dense subspace. In contradistinction with the differentiable case this result needs no longer be true for more general spaces than Banach spaces, as simple examples can show.

The infinitesimal aspect is not so well studied, and it seems that some confusion exists in the literature. Following Nelson [11], we shall say that a vector $\varphi \in \mathrm{H}$ is analytic for an operator X acting on some domain in H if for some $t > 0$, the series $\sum_{n=0}^{\infty} \frac{t^n}{n!} \mathrm{X}^n \varphi$ is defined and absolutely convergent.

When dealing with a Lie algebra of operators having a common invariant dense domain, some confusion may arise between the following three notions :

(*a*) A vector $\varphi \in \mathrm{H}$ is said to be analytic for the *whole* Lie algebra if for some $t > 0$ and some basis $\{ \mathrm{X}_1, \ldots, \mathrm{X}_r \}$ of the Lie algebra, the series $\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{1 \leq i_1, \ldots, i_n \leq r} t^n \mathrm{X}_{i_1} \ldots \mathrm{X}_{i_n} \varphi$ is absolutely convergent. In the Banach case, this condition is equivalent to $\| \mathrm{X}_{i_1} \ldots \mathrm{X}_{i_n} \varphi \| \leq \mathrm{A}^n n!$ for some constant $\mathrm{A} > 0$.

(*b*) A vector $\varphi \in \mathrm{H}$ is analytic for *every* element X in the Lie algebra.

(*c*) A vector $\varphi \in \mathrm{H}$ is analytic for every element X_i in a given basis of the Lie algebra.

We say that a representation T of a real Lie algebra $\mathcal{G}$ is *integrable* if there exists a representation $\mathcal{T}$ of the connected and simply connected Lie group G, corresponding to $\mathcal{G}$, such that T is contained in the differential of $\mathcal{T}$ [i. e. every $\mathrm{X} \in \mathrm{T}(\mathcal{G})$, which we can write $\mathrm{X} = \mathrm{T}(x)$, $x \in \mathcal{G}$, coincides on its domain of definition with the infinitesimal generator of the one-parameter group $\mathcal{T}(\exp(tx))$, $t \in \mathbf{R}$].

Moreover, in the case of unitary representations in Hilbert spaces, if we have (or can construct) in H a dense invariant domain of analytic vectors for the *whole* Lie algebra $\mathrm{T}(\mathcal{G})$ [in the sense (*a*) above], then, following an argument due to Nelson [11] and developed in a more explicit form, using the Campbell-Hausdorff-Dynkin formula, by Goodman [5], the representation T is integrable.

It was also shown by Nelson [11] that analytic vectors for the " Laplacian " $\Delta = \mathrm{X}_1^2 + \ldots + \mathrm{X}_r^2$ relative to some basis are analytic for the representation. Therefrom followed the only important known integrability criterion : if the operators of a local representation T of a Lie algebra $\mathcal{G}$ in a Hilbert space are skew-symmetric on a common invariant dense domain on which the " Laplacian " Δ relative to some basis is essentially self-adjoint, then T is integrable to a unitary group representation.

The aim of this article is to present a new integrability condition, more general than the first Nelson criterion, which can be quite practical for applications. This new criterion states that in order to ensure integrability it is enough that the skew-symmetric operators of a given basis have a common dense invariant domain of analytic vectors [namely analytic in the sense (*c*) above]. We shall also extend this result to more general spaces at the cost of supposing the integrability to the one-parameter groups for a basis.

Such a result shows that analyticity on a dense domain in separate real coordinate directions of a real Lie group implies the existence of a dense domain of joint real analyticity, and therefore underlines the constraints inherent to integrable Lie algebra representations. We shall indeed have to rely more heavily on the abstract group structure, rather than merely use the Campbell-Hausdorff formula : We prove our main theorem upon utilizing some structural preliminaries, formula (6) which is based on duality, and the vector theory of differential equations.

I. — Structural preliminaries

Let $\mathcal{G}$ be a real Lie algebra, and denote by G the corresponding connected and simply connected real Lie group. The aim of this section is to derive a couple of identities in the Lie algebra $\mathcal{G}$ involving elements, their transforms under inner automorphisms defined by elements of G, and derivatives of local coordinates of the second kind relatively to some parameter.

As is well known, the exponential e^x, for x in some open neighbourhood V of the origin in $\mathcal{G}$, will realize some open neighbourhood of the identity in G. If $x_1, \ldots, x_r$ is a basis of $\mathcal{G}$, V can be chosen small enough so that for any $x \in \mathrm{V}$ we shall have $e^x = e^{t_1x_1} \ldots e^{t_rx_r}$, the coordinates of second kind $e^x \to (t_1, \ldots, t_r)$ being a local chart in G over some neighbourhood W of the identity, contained in e^{V}. Furthermore, we can suppose W convex, namely that if e^y and $e^x\, e^y$ belong to W, then $e^{tx}\, e^y \in \mathrm{W}$ when $0 \leqq t \leqq 1$ (and even when $-\varepsilon < t < 1 + \varepsilon$, for some $\varepsilon > 0$); this follows from the fact that (*cf.* Helgason [7], p. 34 and 92-94) the translates $t \to e^{tx}\, e^y$ of the one-parameter groups are the geodesics of the Cartan-Schouten connection.

Thus if $e^x \in \mathrm{W}$ and $0 \leqq t \leqq 1$, we shall have $e^{tx} = e^{t_1x_1} \ldots e^{t_rx_r}$ where the coordinates t_i are analytic in t.

To simplify the notations and make the calculations more clear, we shall suppose (using e. g. Ado theorem) that the Lie algebra $\mathcal{G}$ is realized faithfully as a matrix algebra. Thus, some neighbourhood of the identity in G, containing of course W (and all the products of elements of W needed below), will also be realized as a matrix group neighbourhood. Therefore, from the identities

$$\frac{d}{dt}e^{tx} = x\,e^{tx} = e^{tx}\,x = \left(\frac{dt_1}{dt}x_1 + \ldots + \frac{dt_r}{dt}e^{t_1x_1}\ldots e^{t_{r-1}x_{r-1}}\,x_r\,e^{-t_{r-1}x_{r-1}}\ldots e^{-t_1x_1}\right)e^{tx}$$
$$= e^{tx}\left(e^{-t_rx_r}\ldots e^{-t_2x_2}\,x_1\frac{dt_1}{dt}e^{t_2x_2}\ldots e^{t_rx_r} + \ldots + x_r\frac{dt_r}{dt}\right),$$

we find

$$(1)\qquad \begin{cases} x = \dfrac{dt_1}{dt}x_1 + \ldots + \dfrac{dt_r}{dt}\,\mathrm{Int}\,(t_1\,x_1)\ldots\mathrm{Int}\,(t_{r-1}\,x_{r-1})\,x_r, \\ x = \mathrm{Int}\,(-\,t_r\,x_r)\ldots\mathrm{Int}\,(-\,t_2\,x_2)\,x_1\dfrac{dt_1}{dt} + \ldots + x_r\dfrac{dt_r}{dt}, \end{cases}$$

where Int (tx) denotes the inner automorphism Ad (e^{tx}) of $\mathcal{G}$ defined by $y \to e^{tx}\,y\,e^{-tx}$ in any realization.

Moreover if x, $y \in \mathcal{G}$ and e^y, $e^x e^y \in$ W, then, as we have seen, for $0 \leq t \leq 1$, $e^{tx}e^y \in$ W. We can then write :

$$e^{tx}\,e^y = e^{\alpha_1x_1}\ldots e^{\alpha_rx_r}, \qquad e^y = e^{\beta_1x_1}\ldots e^{\beta_rx_r}$$

and, from the identity

$$\frac{d}{dt}((e^{tx}\,e^y)\,e^{-y}) = \frac{d}{dt}(e^{\alpha_1x_1}\ldots e^{\alpha_rx_r}\,e^{-\beta_rx_r}\ldots e^{-\beta_1x_1})$$
$$= \left(x_1\frac{d\alpha_1}{dt} + \ldots + e^{\alpha_1x_1}\ldots e^{\alpha_{r-1}x_{r-1}}\,x_r\frac{d\alpha_r}{dt}e^{-\alpha_{r-1}x_{r-1}}\ldots e^{-\alpha_1x_1}\right)e^{tx}$$

we derive the relation

$$(2)\qquad x = \frac{d\alpha_1}{dt}x_1 + \ldots + \frac{d\alpha_r}{dt}\,\mathrm{Int}\,(\alpha_1\,x_1)\ldots\mathrm{Int}\,(\alpha_{r-1}\,x_{r-1})\,x_r.$$

In addition, as is well known, we have for all x, $y \in \mathcal{G}$ and $t \in \mathbf{R}$,

$$(3)\qquad e^{tx}\,y\,e^{-tx} = \mathrm{Int}\,(tx)\,y = \sum_{n=0}^{\infty}\frac{1}{n\,!}(\mathrm{ad}\,(tx))^n\,y.$$

II. — A basic formula

Let T be a representation of a Lie algebra $\mathcal{G}$ on a complex Hilbert space H by skew-symmetric operators defined over a common dense invariant domain D.

Such a representation is obviously strongly continuous, namely that if we have $y_n \to y$ in $\mathcal{G}$ (endowed with the usual Euclidean topology) then for every $\varphi \in D$, $T(y_n)\varphi \to T(y)\varphi$ in H. Indeed, writing

$$y_n = \sum_{k=1}^{r} \lambda_{n,k} x_k \quad \text{and} \quad y = \sum_{k=1}^{r} \lambda_k x_k$$

for a given basis $x_1, \ldots x_r$ of $\mathcal{G}$, we have $\lambda_{n,k} \to \lambda_k$, hence

$$T(y_n)\varphi - T(y)\varphi = \sum_{k=1}^{r} (\lambda_{n,k} - \lambda_k) T(x_k)\varphi \to 0 \quad \text{in H}.$$

In what follows, we shall write $X = T(x)$, $Y = T(y)$, ... The formula (3) can be transported to the representation in the following sense :

Lemma 1. — *For any $x, y \in \mathcal{G}$ and $\varphi \in D$, the series*

$$A(tX, Y)\varphi = \sum_{n=0}^{\infty} \frac{t^n}{n!} ((\text{ad } X)^n Y)\varphi$$

[*where* $(\text{ad } X) Y \varphi = (XY - YX)\varphi, \ldots$] *is convergent for all* $t \in \mathbf{R}$ *and we have*

(4) $$T(\text{Int}(x) y)\varphi = A(X, Y)\varphi.$$

Indeed, (4) follows from (3) and from the above-mentioned strong continuity. Changing x into tx, we obtain the convergence of the series $A(tX, Y)\varphi$ for all $t \in \mathbf{R}$.

Lemma 2. — *For any $x, y \in \mathcal{G}$ and $\varphi \in D$ we have*

(5) $$YX^m \varphi = \sum_{p=0}^{m} \binom{m}{p} X^p ((-\text{ad } X)^{m-p} Y)\varphi.$$

The verification of this (known) formula is straightforward.

Proposition 1. — *Let* T *and* T′ *be representations of the real Lie algebra $\mathcal{G}$ by skew-symmetric operators over common invariant domains* D *and* D′ *(respectively), dense in* H, *with* $D \subset D'$, *and such that for any $y \in \mathcal{G}$, $Y = T(y)$ is the restriction to* D *of* $Y' = T'(y)$. *Then if* D *is a domain of analytic vectors for some* $X = T(x) \in T(\mathcal{G})$ *we have, denoting by* $\langle\, , \rangle$ *the scalar product in* H, *for any* $t \in \mathbf{R}$, $\varphi \in D$ *and* $\psi \in D'$:

(6) $$\langle -e^{tX} Y\varphi, \psi \rangle = \langle e^{tX}\varphi, A(tX', Y')\psi \rangle.$$

From a result of Nelson [11], the closure $\overline{X}$ of X (and of X′) is skew-adjoint and therefore generates a unique one-parameter unitary group which we denote by $e^{t\overline{X}}$. For all $\varphi \in D$, the functions $t \to e^{t\overline{X}} \varphi$ and $t \to e^{t\overline{X}} Y \varphi$ are analytic in **R**. By lemma 1, the function $t \to A(t X', Y') \psi$, with $\psi \in D'$, is also analytic in **R**. The functions $a(t) = \langle - e^{t\overline{X}} Y \varphi, \psi \rangle$ and $b(t) = \langle e^{t\overline{X}} \varphi, A(t X', Y') \psi \rangle$ are therefore also analytic for all real t. Now we have

$$\frac{d^n a}{dt^n}(0) = \langle - X^n Y \varphi, \psi \rangle,$$

$$\frac{d^n b}{dt^n}(0) = \sum_{p=0}^{n} \binom{n}{p} \langle X^p \varphi, (\operatorname{ad} X')^{n-p} Y' \psi \rangle.$$

Since $Y \subset Y' \subset -Y'^*$ (and the same for X), we obtain from (5) that $\frac{d^n a}{dt^n}(0) = \frac{d^n b}{dt^n}(0)$, hence $a(t) = b(t)$ for all $t \in \mathbf{R}$.

Corollary 1. — *Under the conditions of proposition* 1, $e^{-t\overline{X}} \psi$ *belongs to the domain* D (Y*) *of the adjoint* Y* *of* Y *for all* $\psi \in D'$ *and* $t \in \mathbf{R}$, *and*

$$T' (\operatorname{Int}(tx)\, y) \psi = - e^{t\overline{X}} Y^* e^{-t\overline{X}} \psi. \tag{7}$$

Indeed, due to the continuity in φ of the right-hand side of (6), $e^{-t\overline{X}} \psi \in D(Y^*)$, whence the formula (since D is dense in H).

Remark. — The analyticity of the vectors in D is not completely used here (and in the following). We only use the invariance of D under $T(\mathcal{G})$ and the quasianalytic property of the function $a - b$, namely that if $\frac{d^n}{dt^n}(a - b)(0) = 0$, then $a(t) - b(t) = 0$ for all $t \in \mathbf{R}$. It would therefore be enough to suppose that the vectors of D belong to some " quasianalytic class ", such that all the functions $a - b$ belong to a quasianalytic class of functions on **R** [defined e. g. by some *a priori* inequalities on the derivatives $\frac{d^n}{dt^n}(a - b)(t)$].

III. — Main hypothesis and first consequences

Hypothesis (C). — T *is a representation of a Lie algebra* $\mathcal{G}$ *on a dense invariant domain* D *of vectors that are analytic for all skew-symmetric representatives* $X_i = T(x_i)$ *of a basis* $x_1, \ldots, x_r$ of $\mathcal{G}$.

Lemma 3. — *Hypothesis* (C) *being satisfied, define* H_∞ *as the intersection of the domains of all monomials* $\overline{X}_{i_1} \ldots \overline{X}_{i_n}$, *for all* $1 \leq i_1, \ldots, i_n \leq r$,

$n \in \mathbf{N}$. *Let* X'_i *be the restriction of* $\overline{X}_i$ *to* H_∞ *and define, for all*

$$y = \sum_{i=1}^{r} \lambda_i x_i \in \mathcal{G}, \qquad Y' \equiv T'(y) \equiv \sum_{i=1}^{r} \lambda_i X'_i$$

(with invariant domain H_∞*). Then* T' *is a representation of* $\mathcal{G}$ *by skew-symmetric operators (on* H_∞*) and we have, for any two elements* x_i *and* x_j *in the basis and* $\psi \in H_\infty$ *:*

(8)
$$A(t X'_i, X'_j)\psi = e^{t\overline{X}_i} \overline{X}_j e^{-t\overline{X}_i} \psi.$$

By definition, H_∞ contains D and is invariant under all

$$\overline{X}_i = \overline{X}'_i = -X_i^* = -X_i'^*,$$

hence under all Y' which, due to the hypothesis made, are skew-symmetric. By definition also, T' is linear. Now, if

$$[x_i, x_j] = \sum_{k=1}^{r} c_{ijk} x_k$$

we have, for all $\psi \in H_\infty$ and $\varphi \in D$:

$$\langle (X'_i X'_j - X'_j X'_i)\psi, \varphi \rangle = \langle \psi, (X_j X_i - X_i X_j)\varphi \rangle$$
$$= \left\langle \psi, -\sum_k c_{ijk} X_k \varphi \right\rangle = \left\langle \sum_k c_{ijk} X'_k \psi, \varphi \right\rangle.$$

Therefore T' is a representation and we can apply formula (7) with $D' = H_\infty$ to $y = x_j$, whence (8).

LEMMA 4. — *Under hypothesis* (C), *the above-defined domain* H_∞ *is invariant under* $T'(\mathcal{G})$ *and under all one-parameter groups* $e^{t\overline{X}_i}$, *and if* $t_1, \ldots, t_r$ *are differentiable functions of some parameter* t, *then for all* $\varphi \in H_\infty$ *the vector-valued function*

$$t \to e^{t_1 \overline{X}_1} \ldots e^{t_r \overline{X}_r} \varphi$$

has a first derivative in t.

From (8) and the invariance of H_∞ under $T'(\mathcal{G})$ we obtain

$$A(t X'_i, X'_{j_1}) \ldots A(t X'_i, X'_{j_n})\psi = e^{t\overline{X}_i} \overline{X}_{j_1} \ldots \overline{X}_{j_n} e^{-t\overline{X}_i} \psi$$

for all base elements $x_i, x_{j_1}, \ldots, x_{j_n}$ and all $\psi \in H_\infty$, and $e^{-t\overline{X}_i}\psi$ belongs to the domain of all operators $\overline{X}_{j_1} \ldots \overline{X}_{j_n}$, whence the invariance of H_∞ under the $e^{-t\overline{X}_i}$.

The differentiability property follows by induction from the differentiability of the vector-valued function $t \to U(t)\varphi(t)$ where $t \to \varphi(t) \in H_\infty$

is strongly differentiable and $t \to U(t)$ is a unitary operator-valued function strongly differentiable on H_∞ [such that the map $(t, \varphi) \to U(t)\varphi$ is continuous $\mathbf{R} \times H \to H$].

IV. — Main theorem

THEOREM 1. — *Let* T *be a Lie algebra representation in a complex Hilbert space satisfying hypothesis* (C). *Then* T *is integrable to a unique unitary group representation.*

Let x, y be any elements of $\mathcal{G}$ close enough to 0 so that e^x, e^y, $e^x e^y$, and therefore $e^{tx} e^y$ for $0 \leq t \leq 1$, belong to the neighbourhood W of the identity of G introduced in section I. We shall write

$$(9) \qquad \begin{cases} e^{tx} e^y = e^{\alpha_1 x_1} \dots e^{\alpha_r x_r}, \\ e^{tx} = e^{t_1 x_1} \dots e^{t_r x_r}, \\ e^y = e^{\beta_1 x_1} \dots e^{\beta_r x_r}. \end{cases}$$

For any $z \in \mathcal{G}$ such that $e^z \in W$, we write (in a unique way, once the basis is chosen) $e^z = e^{z_1 x_1} \dots e^{z_r x_r}$ and define

$$(10) \qquad \mathcal{T}(e^z) = e^{z_1 \overline{X}_1} \dots e^{z_r \overline{X}_r}.$$

Since G is generated by finite products of elements of W, we only have to show that the group law holds in W, i. e. that for any e^x, $e^y \in W$ such that $e^x e^y \in W$, we have $\mathcal{T}(e^x e^y) = \mathcal{T}(e^x)\,\mathcal{T}(e^y)$, and that $T(\mathcal{G})$ is on D the differential of $\mathcal{T}(G)$. The unicity is obvious since relation (10) is a necessary condition.

From lemma 4, for $\varphi \in H_\infty$, $\mathcal{T}(e^{tx})\varphi$ and $\mathcal{T}(e^{tx} e^y)\,\mathcal{T}(e^y)^{-1}\varphi$ are differentiable functions of t. Since

$$\frac{d}{dt_i} e^{t_i \overline{X}_i} \varphi = \overline{X}_i e^{t_i \overline{X}_i} \varphi = e^{t_i \overline{X}_i} \overline{X}_i \varphi,$$

we have by direct computation

$$\frac{d}{dt}\mathcal{T}(e^{tx})\varphi = \left(\frac{dt_1}{dt}\overline{X}_1 + \dots + \frac{dt_r}{dt} e^{t_1 \overline{X}_1} \dots e^{t_{r-1}\overline{X}_{r-1}} \overline{X}_r e^{-t_{r-1}\overline{X}_{r-1}} \dots e^{-t_1 \overline{X}_1}\right).\mathcal{T}(e^{tx})\varphi$$

and similarly

$$\frac{d}{dt}\mathcal{T}(e^{tx})\varphi = \mathcal{T}(e^{tx})\left(\frac{dt_1}{dt} e^{-t_r \overline{X}_r} \dots e^{-t_2 \overline{X}_2} \overline{X}_1 e^{t_2 \overline{X}_2} \dots e^{t_r \overline{X}_r} + \dots + \frac{dt_r}{dt}\overline{X}_r\right)\varphi.$$

From relations (1), (4), (8), (9), and (10) we then get :

$$(11) \qquad \frac{d}{dt}\mathcal{T}(e^{tx})\varphi = X'\,\mathcal{T}(e^{tx})\varphi = \mathcal{T}(e^{tx})\,X'\varphi.$$

On the other hand we have by direct computation, for all $\varphi \in H_\infty$

$$\frac{d}{dt} \mathfrak{T}(e^{tx} e^{y}) \mathfrak{T}(e^{y})^{-1} \varphi$$
$$= \left(\frac{d\alpha_1}{dt} \overline{X}_1 + \ldots + \frac{d\alpha_r}{dt} e^{\alpha_1 \overline{X}_1} \ldots e^{\alpha_{r-1} \overline{X}_{r-1}} \overline{X}_r e^{-\alpha_{r-1} \overline{X}_{r-1}} \ldots e^{-\alpha_1 \overline{X}_1} \right) \mathfrak{T}(e^{tx} e^{y}) \mathfrak{T}(e^{y})^{-1} \varphi.$$

Hence, from relations (2), (4), (8), (9) and (10) we obtain that for all $\varphi \in H_\infty$, $\mathfrak{T}(e^{tx} e^{y}) \mathfrak{T}(e^{y})^{-1} \varphi$, which belongs to H_∞, is also a differentiable solution of the vector-valued differential equation (with values in H_∞ and derivation in the H-topology)

$$\frac{d}{dt} u(t) = X' u(t), \qquad u(t) \in H_\infty. \tag{12}$$

Such an equation has a unique solution (*cf.* e. g. Kato [9], p. 481). Indeed one checks easily that for any solution $u(s) \in H_\infty$, and $0 \leq s \leq t \leq 1$, $\mathfrak{T}(e^{(t-s)x}) u(s)$ is differentiable in s and that

$$\frac{d}{ds} (\mathfrak{T}(e^{(t-s)x}) u(s)) = - \mathfrak{T}(e^{(t-s)x}) X' u(s) + \mathfrak{T}(e^{(t-s)x}) X' u(s) = 0.$$

Therefore $\mathfrak{T}(e^{(t-s)x}) u(s)$ does not depend on s. Equalling its values for $s = 0$ and $s = t$ we obtain $u(t) = \mathfrak{T}(e^{tx}) u(0)$, whence the unicity of the solution of (12) in H_∞ and the group law (which we can extend from H_∞ to H by continuity)

$$\mathfrak{T}(e^{tx} e^{y}) = \mathfrak{T}(e^{tx}) \mathfrak{T}(e^{y}).$$

Moreover, relation (11) shows that $T(\mathcal{G})$ is the restriction to D of the differential of $\mathfrak{T}(G)$.

V. — Complements

1. Hypothesis (B). — T *is a representation of a Lie algebra* $\mathcal{G}$ *in a Hilbert space* H *by skew-symmetric operators* $X = T(x) \in \mathcal{G}$ *having a common dense invariant domain* D *of analytic vectors.*

This hypothesis, more restrictive than hypothesis (C), can be of independent interest when extended to more general spaces. The proof of integrability is simpler in that case : Using an argument similar to that of proposition 1 one shows that the smallest domain H_ω containing D and invariant under the operators $e^{\overline{X}}$, $x \in \mathcal{G}$ (obtained by acting upon D with all polynomials in such operators) is an invariant set of analytic vectors for all operators $\overline{X}$, $x \in \mathcal{G}$; this domain can replace H_∞ in the proof.

2. More general representations. — Let T be a representation of a Lie algebra $\mathcal{G}$ by operators defined over a dense domain D in a quasi-complete locally convex space H. Then lemmas 1 and 2 still hold. In order to have an equivalent of proposition 1, we have to suppose the existence of one-parameter groups. This existence was automatic in the case of skew-symmetric operators in Hilbert space having a dense domain of analytic vectors, but it is a difficult question in the more general case. We shall mention here only the Hille-Yosida necessary and sufficient condition for a closed operator to generate a continuous one-parameter group of operators on a Banach space in terms of the resolvent (*cf.* Hille-Phillips [8], p. 364).

Let us denote by H* the antidual of the quasi-complete locally convex space H, namely the space of semilinear continuous functionals $H \to \mathbf{C}$ endowed with the strong topology, the antiduality being realized by the sesquilinear form which we write $(\psi, \varphi) \to \langle \psi, \varphi \rangle$. We shall denote by Y* the adjoint of a densely defined operator Y in H. We recall that in the integrable case, the contragredient of a continuous Lie group representation in a non-semi-reflexive space H needs not be continuous on the whole dual (*cf.* e. g. Bruhat [1]) but will be continuous on a closed invariant subspace of H*. We have then :

Proposition 2. — *Suppose that there exist both a dense domain* D *in* H *and some domain* D* *in* H* *such that :*

(i) D *is invariant under the operators* $Y = T(y)$, $y \in \mathcal{G}$.
D* *is invariant under the operators* Y*.

(ii) D* *is composed of analytic vectors for some* $X^* = T(x)^*$,

or (ii)′ D *is composed of analytic vectors for some* $X = T(x)$.

(iii) X* *coincides on* D* *with the weak* generator of a weakly continuous one-parameter group* $\mathcal{C}_x^*(t)$,

or (iii)′ X *coincides on* D *with the weak generator of a weakly continuous one-parameter group* $\mathcal{C}_x(t)$ *on* H.

Then we have, for all $\varphi \in D$, $\psi \in D^*$:

under hypotheses (i), (ii), (iii)

$$\langle \mathcal{C}_x^*(t)\, Y^* \psi, \varphi \rangle = \langle \mathcal{C}_x^*(t)\, \psi, A(-tX, Y)\, \varphi \rangle \tag{13}$$

and under hypotheses (i), (ii)′, (iii)′

$$\langle \psi, \mathcal{C}_x(t)\, Y \varphi \rangle = \langle A(-tX^*, Y^*)\, \psi, \mathcal{C}_x(t)\, \varphi \rangle . \tag{13'}$$

Indeed, denote the left-hand side by $a(t)$ and the right-hand side by $b(t)$. Then these real-analytic functions are equal since, because of (5) :

$$\frac{d^n a}{dt^n}(0) = \langle X^{*n} Y^* \psi, \varphi \rangle = \sum_{p=0}^{n} \binom{n}{p} \langle X^{*p} \psi, ((-\operatorname{ad} X)^{n-p} Y) \varphi \rangle = \frac{d^n b}{dt^n}(0)$$

in the first case, and similarly in the second case since $y \to -Y^*$ defines a representation of $\mathcal{G}$ on D^*.

Here also, as we noticed in the end of paragraph II, a quasianalyticity property of $a-b$ would be enough, a fact which might be useful when H is not a Banach space.

To proceed further we shall suppose that H is semi-reflexive, i. e. algebraically equal to its bidual H^{**}, and that D^* is dense in H^*. Then X is closable with closure $\overline{X} = X^{**}$ and we have, under hypotheses (i), (ii), (iii) of proposition 2 :

$$\text{(14)} \qquad T(\operatorname{Int}(tx)\, y)\, \varphi = \tilde{\mathcal{E}}_x(t)\, \tilde{Y}\, \tilde{\mathcal{E}}_x(-t)\, \varphi$$

for all $\varphi \in D$, where $\tilde{Y}$ is the adjoint of the restriction of Y^* to D^* and $\tilde{\mathcal{E}}_x(t)$ is the adjoint of $\mathcal{E}^*_x(t)$.

Hypothesis (C′). — T *is a representation of a Lie algebra* $\mathcal{G}$ *on a dense invariant domain* D *in a semi-reflexive locally convex space* H *such that :*

(C'_1) *There exists a dense invariant domain* D^* *in the antidual* H^* *of* H *for all* Y^*, *the operators* $Y = T(y)$, $y \in \mathcal{G}$, *being defined with domain* D.

(C'_2) D^* *is composed of analytic vectors for all* $X^*_i = T(x_i)^*$ *representatives of a given basis* $\{x_i\}$ $(i = 1, \ldots, r)$ *of* $\mathcal{G}$, *and* X^*_i *is the closure of its restriction to* D^*.

(C'_3) *The closures* $\overline{X}_i$ *of the* X_i *generate continuous one-parameter groups* $e^{t\overline{X}_i}$ *on* H (*i. e. such that the associated maps* $\mathbf{R} \times H \to H$ *are continuous*).

We then prove exactly in the same way as before :

Theorem 3. — *Under hypothesis* (C′), T *is integrable.*

Remark 1. — When H is barreled (hence isomorphic to its strong bidual H^{**}) one can show that, under hypothesis (C′), the contragredient representation $T^* : y \to -T(y)^*$ (with domain D^*) is also integrable [even if the analogue of (C'_2) for T^* is not true].

Remark 2. — After completing this article one of us (M. F.) was told by E. Nelson (who is cordially acknowledged) that R. T. Moore [10] also dealt with the problem of exponentiation of Lie algebras of operators.