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## GEOMETRIC VERSIONS OF WHITNEY REGULARITY FOR SMOOTH STRATIFICATIONS

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In 1965 Whitney [18] introduced two useful conditions on pairs of adjacent strata, known as  $(a)$ - and  $(b)$ -regularity. C. T. C. Wall [16] conjectured that these are equivalent to the conditions which we call  $(a_s)$  and  $(b_s)$  [12], which have “more obvious geometric content” [16]. Thom [9] showed that  $(a_s)$  and  $(b_s)$  are necessary, so it remained to prove sufficiency. This we had previously done for semianalytic strata ([11], [12]); in this paper we give the proof in the general case.

The plan of the proof is as follows.  $(a_s)$  says that the fibres of each  $C^1$  retraction onto the base stratum are transverse to the attaching stratum. We rephrase the question of whether  $(a_s)$  implies  $(a)$  to read, “Do transverse  $C^1$  foliations detect all  $(a)$ -faults?” We show that they do so in Theorem A by perturbing a foliation whose leaves are hyperplanes (transverse to the base stratum) by an infinite sequence of “ripples”, so as to detect a given  $(a)$ -fault. An example constructed with Anne Kambouchner [4] shows that this result is sharp, because there exist  $(a)$ -faults which are not detectable by transverse  $C^2$  foliations.

In paragraph 3 we prove that  $(b)$  follows from  $(b_s)$ , which says that for every  $C^1$  tubular neighbourhood of the base stratum; associated to which are a retraction  $\pi$  and a distance function  $\rho$ , the fibres of  $(\pi \times \rho)$ , which are embedded spheres, are transverse to the attaching stratum. The proof uses the corresponding result for  $(a)$ -regularity (Theorem A), and the method of proof is similar, if more complicated: we use the ripples constructed in paragraph 2 to perturb a foliation by spheres of the complement of the base stratum so as to detect a given  $(b)$ -fault. The example of [4] mentioned above provides a  $(b)$ -fault which cannot be detected by  $C^2$  tubular neighbourhoods.

These results form part of the author’s thesis [13] and were announced during the *Journées singulières* of Dijon in June 1978 [14].

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### 1. Definitions

THE WHITNEY CONDITIONS. — For completeness, we recall the definitions of (a)- and (b)-regularity.

Let  $X, Y$  be disjoint  $C^1$  submanifolds of  $\mathbb{R}^n$ , and let  $y$  be a point in  $Y \cap \overline{X}$ .  $X$  is (a)-regular over  $Y$  at  $y$  if:

(a) Given a sequence of points  $\{x_i\}$  in  $X$  tending to  $y$  such that  $T_{x_i} X$  tends to  $\tau$ , then  $T_y Y \subset \tau$ .

$X$  is (b)-regular over  $Y$  at  $y$  if:

(b) Given sequences  $\{x_i\}$  in  $X$ ,  $\{y_i\}$  in  $Y$ , both tending to  $y$ , such that  $T_{x_i} X$  tends to  $\tau$ , and the unit vector in the direction of  $\overrightarrow{x_i y_i}$  tends to  $\lambda$ , then  $\lambda \in \tau$ .

These conditions were first defined by Whitney in [18]. Accounts of them have been given by Thom in [9], by Mather in [6] and [7], by Wall in [16] and [17], by Gibson and Wirthmüller in [2], and by the author in [13] and [14].

THE GEOMETRIC VERSIONS. — Let  $(U, \varphi)$  be a  $C^1$  chart for  $Y$  at  $y$ ,

$$\varphi : (U, U \cap Y, y) \rightarrow (\mathbb{R}^n, \mathbb{R}^m \times O^{n-m}, O).$$

We have a  $C^1$  retraction

$$\pi_\varphi = \varphi^{-1} \circ \pi_m \circ \varphi : U \rightarrow U \cap Y,$$

and a  $C^1$  tubular function

$$\rho_\varphi = \rho_m \circ \varphi : U \rightarrow \mathbb{R}^+,$$

where

$$\pi_m(x_1, \dots, x_n) = (x_1, \dots, x_m, 0, \dots, 0) \quad \text{and} \quad \rho_m(x_1, \dots, x_n) = \sum_{i=m+1}^n x_i^2.$$

We refer to the tubular neighbourhood  $T_\varphi$  of  $U \cap Y$  associated to  $(U, \varphi)$ .

The following seems to be the clearest description of the conditions considered:

(a<sub>s</sub>) for every  $C^1$  chart  $(U, \varphi)$  for  $Y$  at  $y$ , there exists a neighbourhood  $V$  of  $y$ ,  $V \subset U$ , such that  $\pi_\varphi|_{V \cap X}$  is a submersion;

(b<sub>s</sub>) for every  $C^1$  chart  $(U, \varphi)$  for  $Y$  at  $y$ , there exists a neighbourhood  $V$  of  $y$ ,  $V \subset U$ , such that  $(\pi_\varphi, \rho_\varphi)|_{V \cap X}$  is a submersion.

(b')-regularity. — As usual it is helpful to split (b) into two conditions, namely (a) and what Thom calls (b') in [10].  $X$  is (b')-regular over  $Y$  at  $y$  if for some  $C^1$  chart  $(U, \varphi)$  for  $Y$  at  $y$ :

(b') given a sequence  $\{x_i\}$  in  $X$  tending to  $y$ , such that  $T_{x_i} X$  tends to  $\tau$ , and the unit vector in the direction of  $\overrightarrow{x_i \pi_\varphi(x_i)}$  tends to  $\lambda$ , then  $\lambda \in \tau$ ;

(b) clearly implies (b') for any  $(U, \varphi)$ . Also (b) implies (a), since given any vector  $v$  in  $T_y Y$  and any sequence  $\{x_i\}$  in  $X$  we can choose  $\{y_i\}$  in  $Y$  coming in to  $y$  in the direction of  $v$  so

slowly that  $\frac{\overrightarrow{x_i y_i}}{|\overrightarrow{x_i y_i}|}$  tends to  $v$  (see Mather [6]). Conversely, if (a) holds and (b') holds for some  $(U, \varphi)$ , we arrive at (b) by decomposing the vector  $\lambda$  into the sum of two vectors, one in  $T_y Y$  and the other in  $T_y(\pi_\varphi^{-1}(y))$ . Thus we have:

LEMMA 1. —  $(b') + (a) \Leftrightarrow (b)$ .

SOME TERMINOLOGY. — The basic local situation when studying stratifications is as follows: the strata  $X$  and  $Y$  are  $C^1$  submanifolds of  $\mathbb{R}^n$  with  $Y \subset \overline{X} - X$ . We call  $Y$  the *base* stratum, and  $X$  the *attaching* stratum. When  $X$  is  $(b)$ -regular over  $Y$  at  $y$  in  $Y$ , we will say that the pair  $(X, Y)$  is  $(b)$ -regular at  $y$ , or that  $(X, Y)_y$  is  $(b)$ -regular. When  $(X, Y)_y$  is not  $(b)$ -regular, we say that  $(X, Y)_y$  is a  $(b)$ -*fault*: we justify this term below.

FAULTS AND DETECTORS. — When some equisingularity condition  $E$  is not satisfied at a point of a stratification, it is natural to call the point an  $E$ -*fault* (so retaining the geological terminology). Many proofs showing that one equisingularity condition implies another are by *reductio ad absurdum*: we suppose that the second condition fails, and then we show that the first condition necessarily fails as well. When we can do this we say we have *detected the fault* (the point where the second condition fails). In the same way counterexamples to implications between equisingularity conditions tend to be faults which are not detectable in some given way. Most of the results given in [13] consist of taking an equisingularity condition  $E$  and deciding whether possible detectors are effective or ineffective in detecting every  $E$ -fault.

## 2. (a)-regularity and transverse foliations

We first give a helpful reformulation of  $(a_s)$  suggested by Dennis Sullivan.

$(\mathcal{F}^k)$  Given a  $C^k$  foliation  $\mathcal{F}$  transverse to  $Y$  at  $y$ , there is some neighbourhood of  $y$  in  $\mathbb{R}^n$  in which  $\mathcal{F}$  is transverse to  $X$ .

It is easy to see that  $(a_s)$  is equivalent to  $(\mathcal{F}^1)$ . Given  $(\mathcal{F}^1)$ ,  $(a_s)$  follows since the fibres of each retraction  $\pi_\varphi$  define a foliation transverse to  $Y$  of codimension the dimension of  $Y$ . Given  $(a_s)$ ,  $(\mathcal{F}^1)$  follows by choosing  $\varphi$  such that the fibres of  $\pi_\varphi$  are contained in the leaves of the foliation.

So the question of whether  $(a_s)$  implies (a) can be formulated as: *do transverse  $C^1$  foliations detect (a)-faults?*

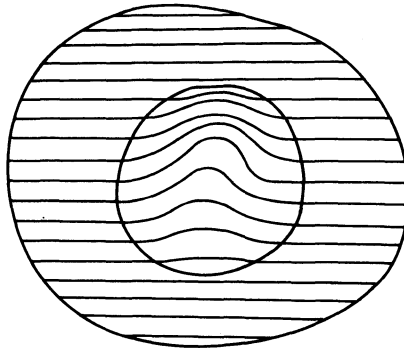
THEOREM A [*Transverse  $C^1$  foliations detect (a)-faults*]. — *Let  $X, Y$  be  $C^1$  submanifolds of  $\mathbb{R}^n$ , and let  $0 \in Y \subset \overline{X} - X$ . Then  $X$  is (a)-regular over  $Y$  at 0 if and only if  $X$  is  $(\mathcal{F}^1)$ -regular over  $Y$  at 0.*

*Proof.* — Thom (page 10 of [9]) shows how (a) implies  $(a_s)$ , and hence also  $(\mathcal{F}^1)$ . It remains to show that  $(\mathcal{F}^1)$  implies (a). We suppose that there is an (a)-fault at 0 given by a sequence  $\{x_i\} \in X$  tending to 0, with  $\tau = \lim T_{x_i} X$ , and  $T_0 Y \not\subset \tau$ .

We shall adjust a codimension 1 foliation by hyperplanes parallel to a hyperplane containing  $\tau$  so as to be nontransverse to  $X$  at infinitely many  $x_i$ .

CONSTRUCTION 2.1 (RIPPLES). — Given a hyperplane  $H \in G_{n-1}(\mathbb{R})$ , a real number  $s \in [0, 1/2]$ , and a real number  $r > 0$ , we construct a  $C^1$  foliation  $\mathcal{F}_H^s$  of codimension 1 of the ball  $B_r^n$  of radius  $r$  with centre 0 in  $\mathbb{R}^n$  such that

- (1) for all  $x \in B_r^n - B_{(1/2)r}^n$ ,  $T_x \mathcal{F}_H^s = H$ ;
- (2) for all  $x \in B_{(1/2)r}^n$ ,  $d(H, T_x \mathcal{F}_H^s) \leq s$ ;
- (3) for all  $K \in G_{n-1}(\mathbb{R})$  such that  $d(K, H) = s$ , there is a unique  $x_K \in B_{(1/2)r}^n$  such that  $T_{x_K} \mathcal{F}_H^s = K$ ;
- (4) there is a  $C^1$  diffeomorphism  $\varphi_H^s : B_r^n \rightarrow B_r^n$  such that  $\varphi_H^s(\mathcal{F}_H^s)$  is the trivial foliation  $\mathcal{F}_H^0$  by hyperplanes parallel to  $H$ , and such that  $\varphi_H^s|_{B_r^n - B_{(1/2)r}^n} = \text{id}|_{B_r^n - B_{(1/2)r}^n}$ , and  $d\varphi_H^s$  tends to the identity uniformly as  $s$  tends to 0, i. e.  $\forall \varepsilon > 0, \exists s_\varepsilon > 0$  such that  $s < s_\varepsilon$  implies  $|d\varphi_H^s(x) - I| < \varepsilon$  for all  $x \in B_r^n$ .



Foliation with a ripple

We shall postpone the verification of Construction 2.1 until after the proof of Theorem A. The reader may in any case prefer to admit the verification as geometrically evident.

Choose a one-dimensional subspace  $V \subset T_0 Y$  such that  $V \not\subset \tau$ . Define a hyperplane  $H$  by  $\tau \oplus (\tau \oplus V)^\perp$ , where  $(\ )^\perp$  denotes orthogonal complement in  $T_0 \mathbb{R}^n$ .

Since  $T_{x_i} X$  tends to  $\tau$  as  $i$  tends to  $\infty$ , there is some  $i_0$  such that  $i \geq i_0$  implies  $V \not\subset T_{x_i} X$ . Then for all  $i \geq i_0$  define a hyperplane  $H_i$  by  $T_{x_i} X \oplus (T_{x_i} X \oplus V)^\perp \subset T_{x_i} \mathbb{R}^n$ . Then  $H_i$  tends to  $H$  as  $i$  tends to  $\infty$ . Pick  $i_1 \geq i_0$  such that  $|H_i - H| < 1/2$  for  $i \geq i_1$ .

Now pick an infinite sequence of pairwise disjoint balls  $B_{r_i}(x_i)$  with radius  $r_i$  and centre  $x_i$ . This is possible since 0 is the only accumulation point of  $\{x_i\}_{i=1}^\infty$ . Then for all  $i$ ,  $0 \notin B_{r_i}(x_i)$ .

For all  $i \geq i_1$ , place inside  $B_{r_i}(x_i)$  a "ripple": a foliated ball  $B_i = B_{(1/2)r_i}(y_i)$  with radius  $(1/2)r_i$ , centre  $y_i$ , and the foliation  $\mathcal{F}_i = \mathcal{F}_H^{|H_i - H|}$  given by Construction 2.1 such that  $x_i = x_{H_i}$ , i. e.  $T_{x_i} \mathcal{F}_i = H_i$ .

Define a foliation  $\mathcal{F}$  on  $\mathbb{R}^n$  by the trivial foliation  $\mathcal{F}_H$  by hyperplanes parallel to  $H$  on  $\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)$ , together with  $\mathcal{F}_i$  on  $B_i$  for all  $i \geq i_1$ ;  $\mathcal{F}$  will be a  $C^1$  foliation if we can define a  $C^1$

diffeomorphism  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  taking  $\mathcal{F}$  onto  $\mathcal{F}_H$ . Let  $\varphi|_{\mathbb{R}^n - (\bigcup_{i \geq i_1} B_i)} = \text{identity}$ , and  $\varphi|_{B_i} = \varphi_H^{|H_i - H|}$  as defined in Construction 2.1.

Since  $\varphi$  is continuous, to check that  $\varphi$  is a  $C^1$  diffeomorphism it is enough to check that  $d\varphi(x)$  can be extended continuously at 0.

Given  $\varepsilon > 0$ , (4) of Construction 2.1 gives us an  $s_\varepsilon > 0$ . Pick  $i_2 \geq i_1$  such that  $|H_i - H| < s_\varepsilon$  for all  $i \geq i_2$ . Let  $\delta = \min_{\substack{x \in B_i \\ i_1 \leq i \leq i_2}} \{|x|\}$ . Then  $\delta$  is well-defined and nonzero since  $0 \notin \bigcup_{i=i_1}^{i_2-1} B_i(x_i)$ .

Then  $|x| < \delta$  implies  $x \notin \bigcup_{i=i_1}^{i_2-1} B_i$ , so

$$|d\varphi(x) - I| \leq \max_{\substack{x' \in B_i \\ i \geq i_2}} \{|d\varphi_H^{|H_i - H|}(x') - I|\}$$

$< \varepsilon$  by (4) of Construction 2.1, and the choices of  $s_\varepsilon$  and  $i_2$ .

Thus  $d\varphi(x)$  is continuous near 0, and  $d\varphi(0) = I$  (the identity matrix). Hence  $\mathcal{F}$  is a  $C^1$  foliation and  $T_0 \mathcal{F} = H$ , so that  $\mathcal{F}$  is transverse to  $Y$  at 0 ( $V \notin H$  by definition of  $H$ ). But for all  $i \geq i_1$ ,  $T_{x_i} \mathcal{F} = T_{x_i} \mathcal{F}_i = H_i$  and  $T_{x_i} X \subseteq H_i$  so that  $\mathcal{F}$  is nontransverse to  $X$  at  $x_i$ . This shows that  $X$  is not  $(\mathcal{F}^1)$ -regular over  $Y$  at 0, proving Theorem A.

VERIFICATION OF CONSTRUCTION 2.1. — It suffices to take  $H = \mathbb{R}^{n-1} \times 0 \subset \mathbb{R}^n$  and  $n = 2$ . For  $n > 2$  the calculations are similar.

Consider,

$$\begin{cases} y = \lambda + (1 - \lambda^2)^2 (x^2 - a^2)^2, & \lambda^2 \leq 1, \quad x^2 \leq a^2, \\ y = \lambda, & \lambda^2 \leq 1, \quad a^2 \leq x^2 \leq 1, \end{cases}$$

with the constant  $a$  in  $[0, 1]$  to be chosen shortly.

We shall prove that this defines a  $C^1$  foliation of  $[-1, 1]^2$  of codimension 1, with the leaves corresponding to fixed values of  $\lambda$ . [If  $n > 2$ , take  $x_n = \lambda + (1 - \lambda^2)^2 \left( \sum_{i=1}^{n-1} x_i^2 - a^2 \right)^2$ , etc.]

Multiplying by  $r/4$  gives a foliation of  $[-r/4, r/4]^2$  which fits into the ball  $B_{(1/2)r}(0)$  and extends trivially to a foliation  $\mathcal{F}_a$  of  $B_r(0)$  which satisfies (1). The leaf with normal vector furthest from  $(0 : 1)$  is clearly given by  $\lambda = 0$ , and this normal is  $(1 : \mp(8a^3)/(3\sqrt{3}))$  at the points  $((4/9)a^4, \pm a/\sqrt{3})$  (compare Construction 1 of [4]).

Write  $v_a = (8a^3)/(3\sqrt{3})$ . Then

$$|(1 : v_a) - (1 : 0)| = (v_a)/(1 + v_a^2)^{1/2}.$$

So, given  $s$ , choose  $a$  such that

$$\frac{v_a^2}{1 + v_a^2} = s^2,$$

i. e.

$$v_a^2 = \frac{s^2}{1-s^2}.$$

Then

$$a^6 = \frac{27s^2}{64(1-s^2)}.$$

With this choice of  $a$ , (2) and (3) of 2.1 are satisfied. Note that

$$(\star) \quad a^6 \leq 9/64 \quad \text{for } s \in \left[0, \frac{1}{2}\right].$$

Define  $\varphi_a : [-1, 1]^2 \rightarrow [-1, 1]^2$  by

$$\varphi_a(x, y) = \begin{cases} (x, y), & a^2 \leq x^2 \leq 1; \\ (x, y + (1-y^2)^2(x^2 - a^2)^2), & x^2 \leq a^2. \end{cases}$$

$\varphi_a$  is then a  $C^1$  map. Elementary calculation using  $(\star)$  shows that  $\varphi_a$  is injective. Now

$$d\varphi_a(x, y) = \begin{pmatrix} 1 & 0 \\ 4x(x^2 - a^2)(1-y^2)^2 & 1 - 4y(1-y^2)(x^2 - a^2)^2 \end{pmatrix} \quad \text{if } x^2 \leq a^2,$$

and  $d\varphi_a(x, y)$  is the identity matrix if  $a^2 \leq x^2 \leq 1$ .

Calculation using  $(\star)$  shows that  $d\varphi_a(x, y)$  is always nonsingular. Thus  $\varphi_a$  is a  $C^1$  diffeomorphism of  $[-1, 1]^2$ , which after scalar multiplication by  $r/4$  as described above may be extended by the identity to a  $C^1$  diffeomorphism of  $B_r(0)$  since  $d\varphi_a(x, \pm 1)$  is the identity matrix. It defines the foliation.

$\varphi_H^s$  will be the inverse of the resulting diffeomorphism. It only remains to verify (4) of Construction 2.1, i. e. to show that  $d(\varphi_a^{-1})$  tends uniformly to the identity matrix as  $a$  tends to 0; but this follows from the same result for  $d\varphi_a$ , and this in turn follows from the expression above.

**COROLLARY 2.2.** — *(a)-regularity is a  $C^1$  diffeomorphism invariant.*

*Proof.* —  $(\mathcal{F}^1)$  is clearly a  $C^1$  invariant.

Having shown that transverse  $C^1$  foliations detect  $(a)$ -faults, we refer to [4] for an example of an  $(a)$ -fault which is not detectable by transverse  $C^2$  foliations, showing that Theorem A is sharp.

### 3. $(b)$ -regularity and tubular neighbourhoods

Let  $X, Y$  be disjoint  $C^1$  submanifolds of  $\mathbb{R}^n$ . We say that  $X$  is  $(b_s)$ -regular over  $Y$  if (using the notation of Mather [7]) for all  $C^1$  tubular neighbourhoods  $T$  of  $Y$ , there is a neighbourhood  $N$  of  $Y$  in  $|T|$  such that  $(\pi_T, \rho_T)|_{X \cap N}$  is a submersion. We have already defined  $(b_s)$ -regularity at a point  $y$  of  $Y \cap \overline{X}$ . The following lemma justifies our use of the term  $(b_s)$ -regularity in both the local and global cases.

LEMMA 3.1. — *X is  $(b_s)$ -regular over Y if and only if X is  $(b_s)$ -regular over Y at y, for all  $y \in Y$ .*

*Proof.* — *If.* Given a sequence of points on X tending to Y, at which  $(\pi_T, \rho_T)_X$  is not submersive, there must be some convergent subsequence with a limit  $y_0$  in Y. The implication follows.

*Only if.* Given a point  $y_0$  of Y and a  $C^1$  tubular neighbourhood  $T_\phi$  of a neighbourhood  $U \cap Y$  of  $y_0$  in Y defined by a  $C^1$  chart  $(U, \phi)$  for Y at  $y_0$ , it will suffice to find a  $C^1$  tubular neighbourhood T of Y and a neighbourhood  $U'$  of  $y_0$ ,  $U' \subset U$ , such that  $T|_{U' \cap Y} = T_\phi|_{U' \cap Y}$ . This follows from the Tubular Neighbourhood Theorem of [7], which is proved in [6].

For a simpler proof, let  $\psi$  be a  $C^1$  diffeomorphism of  $\mathbb{R}^n$  which is the identity outside some neighbourhood of  $y_0$ , and such that there is a smaller neighbourhood W of  $y_0$ ,  $W \subset U$ , such that the fibres of the retraction  $\psi \circ \pi_\phi \circ \psi^{-1}$  intersect  $\psi(W)$  in a  $C^1$  field of planes transverse to  $\psi(Y)$ , and such that  $\rho_\phi \circ \psi^{-1}$  is the square of the function measuring distance from  $\psi(Y)$  in  $\mathbb{R}^n$ . Extend this local  $C^1$  field to a globally defined [over  $\psi(Y)$ ]  $C^1$  field of planes (whose dimension is the codimension of Y) transverse to  $\psi(Y)$ . In Theorem 4.5.1 of [3] Hirsch shows how to obtain a tubular neighbourhood of  $\psi(Y)$ , so that the transverse planes contain the fibres of the associated retraction. (There is also a very careful proof of this fact by Munkres on page 51 of [8].) Pulling back by  $\psi^{-1}$  we have a tubular neighbourhood T of Y with the required properties. This completes the proof of Lemma 3.1.

In [16] C.T.C. Wall conjectured that  $(b_s)$ -regularity is a necessary and sufficient condition for  $(b)$ -regularity. Applying Lemma 3.1, together with the convention that X is  $(b)$ -regular over Y when X is  $(b)$ -regular over Y at y for all y in Y, we see that the local and global versions of the conjecture are equivalent. We now prove the local version.

THEOREM B. — *Let X, Y be disjoint  $C^1$  submanifolds of  $\mathbb{R}^n$ , and let  $0 \in Y$ . Then X is  $(b)$ -regular over Y at 0 if and only if X is  $(b_s)$ -regular over Y at 0.*

*Proof.* — “Only if” was proved by Mather as Lemma 7.3 in [6], and in fact in 1964 by Thom on page 10 of [9]. For another published proof see Lemma 2.3 of [19].

It is left to prove “if”.

Suppose X is  $(b_s)$ -regular over Y at 0. It follows at once that X is  $(a_s)$ -regular over Y at 0 (see § 1), so that we can apply Theorem A to show that (a) holds. Suppose (b) fails: we shall derive a contradiction. By Lemma 1, (b') must fail for every  $C^1$  retraction onto Y.

Let  $\pi_1$  (resp.  $\pi_2$ ) be the local linear retraction defined near 0 of  $\mathbb{R}^n$  onto Y (resp.  $T_0 Y$ ) orthogonal to  $T_0 Y$ . Then (b') fails for  $\pi_1$ , and there is a sequence  $\{x_i\}$  in X tending to 0 such that

$$\lambda_i = \frac{\overrightarrow{x_i \pi_1(x_i)}}{|\overrightarrow{x_i \pi_1(x_i)}|}$$

tends to a limit  $\lambda$ , and  $T_{x_i} X$  tends to a limit  $\tau$ , and  $\lambda \notin \tau$ .

The  $C^1$  diffeomorphism defined near 0,

$$\begin{aligned} \alpha: \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\ p &\mapsto p + (\pi_2(p) - \pi_1(p)), \end{aligned}$$



preserves  $\{\lambda_i\}$ ,  $\lambda$  and  $\tau$ , and sends  $Y$  onto  $T_0 Y$ , hence we may identify  $Y$  with  $\mathbb{R}^m \times \mathbb{O}^{n-m}$  in  $\mathbb{R}^n$ . Write  $\pi: \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{O}^{n-m}$  for the projection mapping  $(y_1, \dots, y_n)$  to  $(y_1, \dots, y_m, 0, \dots, 0)$ . Then, continuing to write  $\{x_i\}$  and  $X$  for their images by  $\alpha$ , we have that

$$\lambda_i = \frac{\overrightarrow{x_i \pi(x_i)}}{|\overrightarrow{x_i \pi(x_i)}|}$$

tends to  $\lambda$ , which is not contained in  $\tau = \lim T_{x_i} X$ .

Now let  $A$  be a linear automorphism of  $\mathbb{O}^m \times \mathbb{R}^{n-m}$  such that  $A(\lambda)$  and  $A(\tau \cap \mathbb{R}^{n-m})$  are orthogonal. By applying the linear change of coordinates  $(I_m, A): \mathbb{R}^m \times \mathbb{R}^{n-m} \leftrightarrow$  we may suppose that  $\lambda$  and  $\tau$  are orthogonal. The function measuring distance from  $Y$  is  $\rho: \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , taking  $(y_1, \dots, y_n)$  to  $\sum_{i=m+1}^n y_i^2$ . We shall construct a  $C^1$  diffeomorphism  $\varphi$  of  $\mathbb{R}^n$  with  $\varphi|_{\mathbb{R}^m \times \mathbb{O}^{n-m}} = \text{identity}$ , such that the tangent space to  $X$  is contained in the tangent space to the fibre of  $\rho_\varphi = \rho \circ \varphi$  on an infinite subsequence of the sequence  $\{x_i\}$ , so that  $(b_s)$  fails for  $(X, Y)$  at  $0$ .

As in the proof of Theorem A, pick an infinite sequence of pairwise disjoint balls  $B_{r_i}(x_i) = B_i$  with centre  $x_i$  and radius  $r_i$  such that  $Y \cap B_i = \emptyset$ . Then  $0 \in B_i$  for all  $i$ . We shall obtain  $\varphi$  by perturbing the foliation of  $\mathbb{R}^n - (\mathbb{R}^m \times \mathbb{O}^{n-m})$  by the level hypersurfaces of  $\rho$ , within each  $B_i$ .

Let  $H = \lambda^\perp \in G_{n-1}(\mathbb{R})$ , and note that  $H = \tau \oplus (\tau \oplus \lambda)^\perp$  because  $\tau$  and  $\lambda$  have been assumed orthogonal. Since  $T_{x_i} X$  tends to  $\tau$ , and  $\lambda_i$  tends to  $\lambda$ , as  $i$  tends to  $\infty$ , there is some  $i_0$  such that  $i \geq i_0$  implies  $\lambda_i \notin T_{x_i} X$ . Then for all  $i \geq i_0$  we define a hyperplane

$$H_i = T_{x_i} X \oplus (T_{x_i} X \oplus \lambda_i)^\perp \subset T_{x_i} \mathbb{R}^n.$$

$H_i$  tends to  $H$  as  $i$  tends to  $\infty$ . Pick  $i_1 \geq i_0$  such that  $|H_i - H| < 1/4$  for  $i \geq i_1$ .

Let  $\delta_i > 0$ . Then it is clear that we can find a  $C^1$  diffeomorphism  $\psi_i: (B_i, x_i) \rightarrow$ , equal to the identity near  $\partial B_i$ , such that  $d\psi_i(x_i) = I_n$  (the identity matrix),

$$|j^1(\psi_i)(p) - j^1(\text{id}_{\mathbb{R}^n})(p)| < \delta_i$$

and

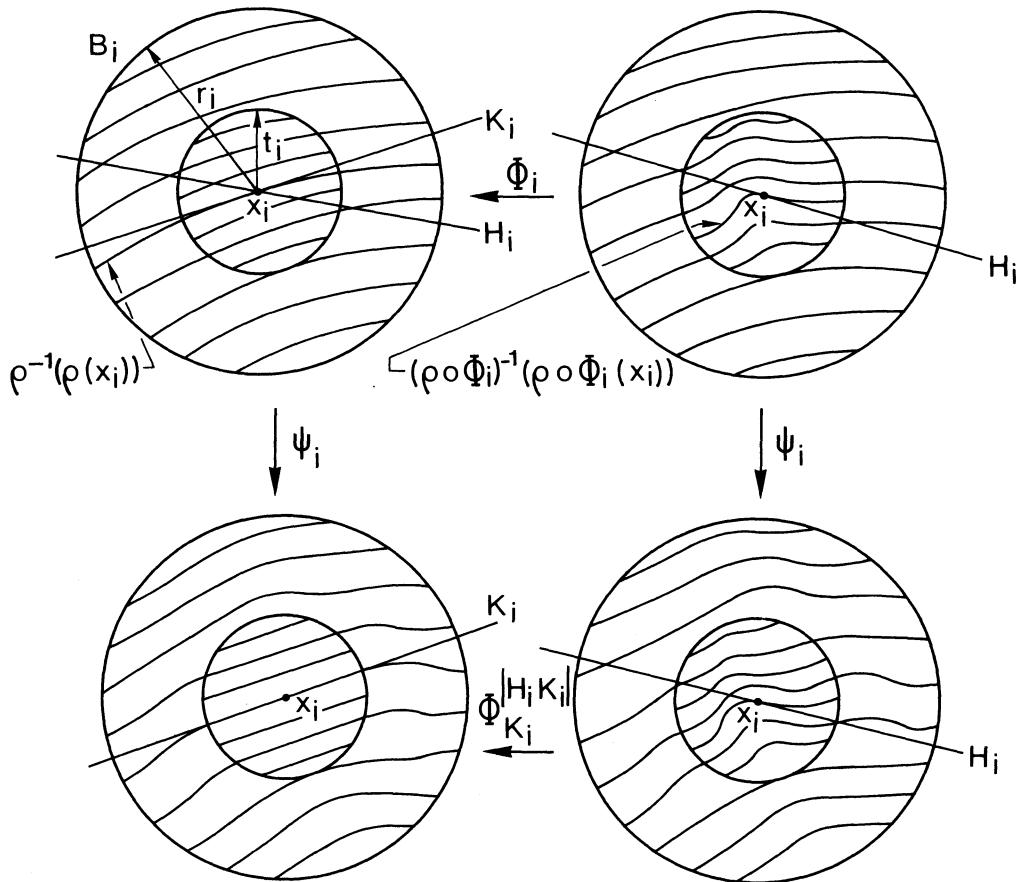
$$|j^1(\psi_i^{-1})(p) - j^1(\text{id}_{\mathbb{R}^n})(p)| < \delta_i$$

for all  $p \in B_i$ , and such that for some  $t_i, 0 < t_i < r_i$ , the image by  $\psi_i$  of the foliation of  $B_i(x_i)$  by the level hypersurfaces of  $\rho$  is the trivial foliation by hyperplanes parallel with  $K_i = T_{x_i}(\rho^{-1}(\rho(x_i)))$ . Now  $K_i = \lambda_i^\perp$ , by definition of  $\lambda_i$ , and so  $K_i$  tends to  $H = \lambda^\perp = (\lim \lambda_i)^\perp$  as  $i$  tends to  $\infty$ . Pick  $i_2 \geq i_1$  such that  $|K_i - H| < 1/4$  for all  $i \geq i_2$ . Then  $|K_i - H_i| \leq 1/2$  for  $i \geq i_2$ , by our choice of  $i_1$  and  $i_2$ .

For all  $i \geq i_2$  we now perturb the trivial foliation of  $B_i(x_i)$  by planes parallel with  $K_i$  by placing inside  $B_i(x_i)$  a "ripple": a foliated ball  $B_{(1/2)t_i}(y_i)$  of radius  $(1/2)t_i$ , centre  $y_i$ , with the

foliation  $\mathcal{F}_{K_i}^{|H_i-K_i|}$  given by Construction 2.1, such that  $x_i = x_{H_i}$  (the tangent at  $x_i$  to the leaf of the foliation passing through  $x_i$  is  $H_i$ ). In the notation of 2.1,  $\varphi_{K_i}^{|H_i-K_i|}$  is the  $C^1$  diffeomorphism defining the resulting foliation of  $B_i(x_i)$ , and we may extend  $\varphi_{K_i}^{|H_i-K_i|}$  by the identity to the rest of  $B_i$ .

Set  $\varphi_i = \psi_i \circ \varphi_{K_i}^{|H_i-K_i|} \circ \psi_i^{-1}: B_i \rightarrow B_i$ . Then  $\varphi_i$  is a  $C^1$  diffeomorphism, and the tangent space at  $x_i$  to  $(\rho \circ \varphi_i)^{-1}(\rho(\varphi_i(x_i)))$  is  $H_i$  which contains  $T_{x_i} X$  by definition (we have used here for the second time that  $d\psi_i(x_i) = I_n$ ). Compare the Figure 3.2.



We have yet to fix  $\delta_i$ . It is easy to verify that  $\sup_{p \in B_i} |d\varphi_i(p) - I_n|$  may be set as near as we please to  $\sup_{p \in B_i} |d\varphi_{K_i}^{|H_i-K_i|}(p) - I_n|$ , by choosing  $\delta_i$  small.

Let  $\delta_i$  be chosen such that,

$$(\star) \quad \sup_{p \in B_i} |d\varphi_i(p) - I_n| \leq 2 \sup_{p \in B_i} |d\varphi_{K_i}^{|H_i-K_i|}(p) - I_n|.$$

Define  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by setting  $\varphi|_{\mathbb{R}^n - (\bigcup_{i \geq i_2} B_i)} = \text{identity}$ , and  $\varphi|_{B_i} = \varphi_i$  for  $i \geq i_2$ . Since  $\varphi$  is continuous, to verify that  $\varphi$  is a  $C^1$  diffeomorphism it is enough to check that  $p \mapsto d\varphi(p)$  extends continuously at 0.

Given  $\varepsilon > 0$ , (4) of Construction 2.1 gives an  $s(1/2)\varepsilon > 0$ . Pick  $i_3 \geq i_2$  such that  $|H_i - H|$  and  $|K_i - H|$  are each less than  $(1/2)s_{(1/2)\varepsilon}$  for all  $i \geq i_3$ . Then  $|H_i - K_i| < s_{(1/2)\varepsilon}$  for all  $i \geq i_3$ . Let  $\delta = \min_{\substack{p \in B_i \\ i_2 \leq i < i_3}} \{|p|\}$ .

Then  $\delta$  is well-defined and nonzero since  $0 \notin \bigcup_{i=i_2}^{i_3-1} B_i$ .

Let  $p \in \mathbb{R}^n$  be such that  $|p| < \delta$ . Then  $p \notin \bigcup_{i=i_2}^{i_3-1} B_i$ , and thus

$$\begin{aligned} |d\varphi(p) - I_n| &\leq \max_{\substack{p' \in B_i \\ i \geq i_3}} \{|d\varphi_i(p') - I_n|\} \\ &\leq 2 \max_{\substack{p' \in B_i \\ i \geq i_3}} \{|d\varphi_{K_i}^{|H_i - K_i|}(p') - I_n|\} \quad [\text{by } (\star)] \\ &\leq 2 \cdot (1/2)\varepsilon \quad (\text{by choice of } i_3 \text{ and } s(1/2)\varepsilon - \text{see 2.1}) = \varepsilon. \end{aligned}$$

Hence  $d\varphi(p)$  is continuous at 0, and  $d\varphi(0)$  is the identity matrix.

By construction, the fibre of  $\rho_\varphi = \rho \circ \varphi$  is not transverse to  $X$  at  $x_i$ , and hence neither is the fibre of  $(\pi_\varphi, \rho_\varphi) = (\pi \circ \varphi, \rho \circ \varphi)$ , so that  $(\pi_\varphi, \rho_\varphi)|_X$  is not a submersion near  $x_i$ . Hence we have shown that  $X$  fails to be  $(b_s)$ -regular over  $Y$  at 0, using the hypothesis that  $X$  is not  $(b)$ -regular over  $Y$  at 0.

This completes the proof of Theorem B.

**COROLLARY 3.3.** — *(b)-regularity is a  $C^1$  invariant.*

This contrasts with the (stronger) generic regularity conditions of Kuo [5] and Verdier [15] which are not  $C^1$  invariants, as shown by the examples in [13] and [1], although they are  $C^2$  invariants.

*Note 3.4.* — Theorem B is sharp, i.e.  $C^2$  tubular neighbourhoods do not detect all  $(b)$ -faults. Consider example 2 of [4]. There we have a  $(b)$ -fault, since it is an  $(a)$ -fault. However for all  $C^1$  distance functions  $\rho_\varphi$  [associated to a  $C^1$  chart  $(U, \varphi)$  for  $Y$  at 0], the fibres of  $\rho_\varphi$  are transverse to  $X$  near 0. For, all limiting tangent planes to  $X$  at 0 contain the  $z$ -axis, and near 0 all points  $(x, y, z)$  on  $X$  have  $x/z$  small, and at such points the normal to the fibre of  $\rho_\varphi$  will be close to  $(0 : 0 : 1)$ . To see that near 0, if  $(x, y, z)$  is on  $X$ , then  $x/z$  is small, notice that the  $x$ -coordinate of the points in each barrow  $B_n$  is bounded above by  $m_n r_n$ , while the  $z$ -coordinate is bounded below by  $m_n$  and  $r_n$  tends to 0 as  $n$  tends to  $\infty$  and we approach 0.

Since it is shown in [4] that all  $C^2$  retractions have their fibres transverse to  $X$  near 0, it follows that for all  $C^2$  tubular neighbourhoods  $T_\varphi$  of  $Y$ , the fibres of  $(\pi_\varphi, \rho_\varphi)$  are transverse to  $X$  near 0.

*Note 3.5. — A semianalytic version of Theorem B.*

We refer to [12] for a proof that  $(b_s)$  implies  $(b)$  when  $X$  and  $Y$  are semianalytic. A careful reading of the proof in [12] shows that semianalytic  $(b)$ -faults can be detected by  $C^1$  semianalytic tubular neighbourhoods, i. e. we can suppose that the detecting chart  $(U, \varphi)$  has a semianalytic graph.

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