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REAL ALGEBRAIC CURVES

BY BENEDICT H. GROSS AND JOE HARRIS

In this paper we investigate the geometry and topology of real algebraic curves. After some introductory material on real abelian varieties in paragraph 1, we take up the study of a real curve X and its Picard scheme Pic X in paragraphs 2 and 3. In paragraph 4 we show how the topological invariants of a real curve X are determined by the action of complex conjugation on the group $H_1(X(\mathbb{C}), \mathbb{Z}/2)$ and in paragraph 5 we show how the topological invariants of X determine the number of real theta-characteristics of each parity in Pic X. We illustrate the general theory with a discussion of real hyper-elliptic curves in paragraph 6, real plane curves in paragraph 7, and real trigonal curves in paragraph 8, and end with some remarks on real moduli.

Acknowledgments

Much of the material in paragraphs 2 and 3 is classical—see, for example, the papers of Klein ([7], [8]); Weichold [16], Hurwitz [5] and Witt [19]. The *existence* of real theta-characteristics on any real curve X was first proved by Atiyah and Serre [2]; real abelian varieties and real moduli were discussed by Shimura in [14]. We also wish to thank Bill Thurston, who showed us a nice approach to spin structures, and Alan Landman, for several helpful discussions.

1. Real abelian varieties

Let A be an abalian variety of dimension g over $\mathbb{R}$. Let $A(\mathbb{R})^0$ denote the connected component of the identity in the group $A(\mathbb{R})$ of real points.

PROPOSITION 1.1. — (1) $A(\mathbb{R})^0$ *is a real torus of dimension* g.

(2) $A(\mathbb{R})/A(\mathbb{R})^0$ *is an elementary abelian* 2-*group*.

(3) $A(\mathbb{R}) \simeq (\mathbb{R}/\mathbb{Z})^g \times (\mathbb{Z}/2)^d$ *with* $0 \leq d \leq g$.

Proof. — (1) Since $A(\mathbb{R})^0$ is a connected, compact, abelian real Lie group of dimension g, it must be isomorphic to the torus $(\mathbb{R}/\mathbb{Z})^g$.

(2) Consider the map $\mathbb{N} : A(\mathbb{C}) \to A(\mathbb{R})$ defined by $\mathbb{N}(P) = P + \overline{P}$. Since $\mathbb{N}$ is a continuous homomorphism and $A(\mathbb{C})$ is compact and connected, the image $\mathbb{N}\, A(\mathbb{C})$ is a closed connected subgroup of $A(\mathbb{R})$. Since it contains $2\,A(\mathbb{R})$ it has finite index and is also open. Consequently $\mathbb{N}\, A(\mathbb{C}) = A(\mathbb{R})^0$ and the quotient $A(\mathbb{R})/A(\mathbb{R})^0$ is killed by 2.

(3) Since $A(\mathbb{R})^0$ is a divisible group, the exact sequence:

$$0 \to A(\mathbb{R})^0 \to A(\mathbb{R}) \to A(\mathbb{R})/A(\mathbb{R})^0 \to 0 \quad \text{splits.}$$

Hence $A(\mathbb{R}) \simeq (\mathbb{R}/\mathbb{Z})^g \times (\mathbb{Z}/2)^d$. The bound on d follows from a count of the 2-division points: $(\mathbb{Z}/2)^{g+d} \simeq A(\mathbb{R})_2 \subseteq A(\mathbb{C})_2 \simeq (\mathbb{Z}/2)^{2g}$.

Let $\check{A}$ denote the dual variety over $\mathbb{R}$ and let $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Let $n(A) = \mathrm{Card}(A(\mathbb{R})/A(\mathbb{R})^0)$.

Proposition 1.2. — $n(A) = n(\check{A})$.

Proof. — The Weil pairing $A(\mathbb{C})_2 \times \check{A}(\mathbb{C})_2 \to \mu_2$ is non-degenerate and G-equivariant. Hence:

$$\mathrm{Card}(A(\mathbb{R})_2) = \mathrm{Card}(\check{A}(\mathbb{R})_2).$$

But $n(A) = 2^d$ if and only if $A(\mathbb{R})_2 \simeq (\mathbb{Z}/2)^{g+d}$.

Proposition 1.3. — *If* $n(A) = 2^d$, *then* $\hat{H}^i(G, A(\mathbb{C})) \simeq (\mathbb{Z}/2)^d$ *for all* $i \in \mathbb{Z}$ (*where* $\hat{H}^i$ *is Tate cohomology*).

Proof. — Since G is cyclic, its Tate cohomology is periodic with period 2. We have already seen that:

$$\hat{H}^0(G, A(\mathbb{C})) = A(\mathbb{R})/\mathbb{N}\, A(\mathbb{C}) = A(\mathbb{R})/A(\mathbb{R})^0 \simeq (\mathbb{Z}/2)^d.$$

Tate has shown that $\hat{H}^1(G, A(\mathbb{C}))$ is isomorphic to the dual of $\hat{H}^0(G, \check{A}(\mathbb{C}))$ [15]; so by Proposition 1.2, $\hat{H}^1(G, A(\mathbb{C})) \simeq (\mathbb{Z}/2)^d$.

Corollary 1.4. — (1) *If* $n(A) = 1$ *then every principal homogeneous space for* A *is trivial.*

(2) *If* $0 \to A \to B \to C \to 0$ *is exact, then:*

(*a*) $n(A) = 1 \Rightarrow n(B) = n(C)$;

(*b*) $n(B) = 1 \Rightarrow n(A) = n(C)$;

(*c*) $n(C) = 1 \Rightarrow n(A) = n(B)$.

Proof. — (1) The principal homogeneous spaces for A correspond to elements of $\hat{H}^1(G, A(\mathbb{C}))$ [9], which is trivial by 1.3.

(2) This follows from the exact cohomology sequence:

$$\hat{H}^{-1}(G, C) \to \hat{H}^0(G, A) \to \hat{H}^0(G, B) \to \hat{H}^0(G, C) \to \hat{H}^1(G, A) \to \dots$$

and 1.3.

2. Real curves: geometry

Let X be a complete, non-singular, geometrically connected curve of genus g over $\mathbb{R}$. Let Pic be the Picard scheme of X over $\mathbb{R}$, and Pic^d the subscheme representing divisor classes of degree d. Pic^d is a principal homogenous space for the Jacobian $\mathrm{J}=\mathrm{Pic}^0$ of X, which is an abelian variety of dimension g over $\mathbb{R}$ [13].

The real points of the Picard scheme consist of those complex divisor classes which are invariant under the action of $\mathrm{G}=\mathrm{Gal}(\mathbb{C}/\mathbb{R})$:

(2.1) $$\mathrm{Pic}(\mathbb{R})=\mathrm{Pic}(\mathbb{C})^{\mathrm{G}}.$$

This group contains the subgroup $\mathrm{Pic}(\mathbb{R})^+$ of those classes represented by a G-invariant divisor $\mathbf{a}$. Elements of $\mathrm{Pic}(\mathbb{R})$ correspond to complex line bundles on X which are isomorphic to their complex conjugates; elements of $\mathrm{Pic}(\mathbb{R})^+$ correspond to algebraic line bundles which may be defined over $\mathbb{R}$.

For $d \geqq 0$ let $\mathrm{S}^d\mathrm{X}$ denote the symmetric d-fold product of X. The usual map:

$$\varphi:\ \mathrm{S}^d\mathrm{X} \to \mathrm{Pic}^d,$$
$$(p_1, \ldots, p_d) \to \mathbf{a} = \sum_{i=1}^{d} (p_i),$$

is defined over $\mathbb{R}$. Clearly φ maps $\mathrm{S}^d\mathrm{X}(\mathbb{R})$ into $\mathrm{Pic}^d(\mathbb{R})^+$.

PROPOSITION 2.1. — *If* $d \geqq g$ *the map* $\varphi : \mathrm{S}^d\mathrm{X}(\mathbb{R}) \to \mathrm{Pic}^d(\mathbb{R})^+$ *is surjective.*

Proof. — For any complex divisor class $\mathbf{a}$ let $\mathrm{L}(\mathbf{a})$ denote the associated complex line bundle and:

$$h^i(\mathbf{a}) = \dim_{\mathbb{C}} \mathrm{H}^i(\mathrm{X}, \mathrm{L}(\mathbf{a})).$$

If $\mathbf{a}$ is in $\mathrm{Pic}(\mathbb{R})^+$ let $\mathrm{L}_{\mathbb{R}}(\mathbf{a})$ denote the associated real bundle; then $\mathrm{H}^i(\mathrm{X}, \mathrm{L}_{\mathbb{R}}(\mathbf{a}))$ is a real vector space of dimension $h^i(\mathbf{a})$.

Now assume $\mathbf{a} \in \mathrm{Pic}(\mathbb{R})^+$ has degree $d \geqq g$. Then by the Theorem of Riemann-Roch, $h^0(\mathbf{a}) \geqq 1$. Hence there is a function $f \in \mathbb{R}(\mathrm{X})^*$ such that the divisor $\mathbf{b}=\mathbf{a}+\mathrm{div}(f)$ is *effective*. This divisor gives a point P in $\mathrm{S}^d\mathrm{X}(\mathbb{R})$ with $\varphi(\mathrm{P})=\mathbf{a}$.

PROPOSITION 2.2:

(1) *If* $\mathrm{X}(\mathbb{R}) \neq \emptyset$ *then* $\mathrm{Pic}(\mathbb{R})^+ = \mathrm{Pic}(\mathbb{R})$.

(2) *If* $\mathrm{X}(\mathbb{R}) = \emptyset$ *then* $\mathrm{Pic}(\mathbb{R})^+$ *has index* 2 *in* $\mathrm{Pic}(\mathbb{R})$. *If* a *is a class generating the quotient then* $h^0(\mathbf{a}) = h^1(\mathbf{a}) = 0 \pmod 2$ *and* $\deg \mathbf{a} \equiv g-1 \pmod 2$.

Proof. — Let D denote the group of complex divisors on X and P the subgroup of principal divisors. The exact sequence of $\mathrm{G}=\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ modules:

$$0 \to \mathrm{P} \to \mathrm{D} \to \mathrm{Pic}(\mathbb{C}) \to 0,$$

gives the exact cohomology sequence :

$$0 \to \mathrm{Pic}(\mathbb{R})^{+} \to \mathrm{Pic}(\mathbb{R}) \xrightarrow{\delta} \mathrm{H}^{1}(\mathrm{G}, \mathrm{P}) \to 0,$$

as $\mathrm{H}^1(\mathrm{G}, \mathrm{D})=0$ (D is an induced $\mathbb{Z}[\mathrm{G}]$-module). Since G is cyclic, the transition map δ may be identified with the homomorphism:

$$\mathrm{Pic}(\mathbb{R}) \to \hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{P}),$$

$$\mathbf{a} \to \mathrm{div}(f)=\mathbf{a}-\overline{\mathbf{a}}.$$

To calculate $\hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{P})$ consider the exact sequence:

$$0 \to \mathbb{C}^* \to \mathbb{C}(\mathrm{X})^* \underset{\mathrm{div}}{\to} \mathrm{P} \to 0.$$

This gives the cohomology sequence:

$$\begin{array}{ccccc} 0 \to \hat{\mathrm{H}}^{-1}(\mathrm{G}, \mathrm{P}) & \underset{\alpha}{\to} & \hat{\mathrm{H}}^{0}(\mathrm{G}, \mathbb{C}^*) & \underset{\beta}{\to} & \hat{\mathrm{H}}^{0}(\mathrm{G}, \mathbb{C}(\mathrm{X})^*), \\ & & \wr\| & & \wr\| \\ & & \mathbb{R}^*/\mathbb{R}^*_{+} & & \mathbb{R}(\mathrm{X})^*/\mathbb{N}\mathbb{C}(\mathrm{X})^*. \end{array}$$

The map α takes the divisor $\mathrm{div}(f)$ into $\mathbb{N} f (\mathrm{mod}\ \mathbb{R}^*_{+})$. Since α is an injection, the group $\mathrm{Pic}(\mathbb{R})/\mathrm{Pic}(\mathbb{R})^{+}$ has order at most 2. It has order 2 precisely when there is a function $f \in \mathbb{C}(\mathrm{X})^*$ with $\mathbb{N} f=-1$. Writing $f=u+iv$ with $u, v \in \mathbb{R}(\mathrm{X})^*$ we see that X maps to the real quadric $\mathrm{N}: \{u^2+v^2+1=0\}$. This is impossible when X has a real point.

Now assume $\mathrm{X}(\mathbb{R})=\emptyset$; we must show $\mathrm{Pic}(\mathbb{R})^{+} \neq \mathrm{Pic}(\mathbb{R})$. If equality held, φ would map $\mathrm{S}^d\mathrm{X}(\mathbb{R})$ surjectively onto $\mathrm{Pic}^d(\mathbb{R})$ for $d \geqq g$ (Proposition 2.1). Since $\mathrm{S}^d\mathrm{X}(\mathbb{R})$ is connected when d is even, $\mathrm{Pic}^d\mathrm{X}(\mathbb{R}) \simeq \mathrm{J}(\mathbb{R})$ would be connected. Hence every principal homogeneous space for J would be trivial (Corollary 1.4) and, in particular, $\mathrm{Pic}^d(\mathbb{R})$ would be connected for all $d \in \mathbb{Z}$. This contradicts the fact that φ is surjective for all $d \geqq g$; when d is *odd*, $\mathrm{S}^d\mathrm{X}(\mathbb{R})$ is empty.

Hence $\mathrm{Pic}(\mathbb{R})^{+}$ has index 2 in $\mathrm{Pic}(\mathbb{R})$ and there is a function f in $\mathbb{C}(\mathrm{X})^*$ with $\mathbb{N} f=-1$ (Compare, Witt [19]). Writing $(f)=\mathbf{a}-\overline{\mathbf{a}}$ we see that $\mathbf{a}$ generates the quotient $\mathrm{Pic}(\mathbb{R})/\mathrm{Pic}(\mathbb{R})^{+}$. "Multiplication by f" gives an isomorphism:

$$\mathrm{H}^0(\mathrm{X}, \overline{\mathrm{L}(\mathbf{a})}) \underset{f}{\simeq} \mathrm{H}^0(\mathrm{X}, \mathrm{L}(\mathbf{a})).$$

Complex conjugation gives a $\mathbb{C}$-anti-linear isomorphism:

$$\mathrm{H}^0(\mathrm{X}, \mathrm{L}(\mathbf{a})) \underset{\tau}{\simeq} \mathrm{H}^0(\mathrm{X}, \overline{\mathrm{L}(\mathbf{a})}).$$

The composition $j=f \circ \tau$ is a complex anti-linear automorphism of $\mathrm{H}^0(\mathrm{X}, \mathrm{L}(\mathbf{a}))$ of order 4; it gives this complex vector space a quaternionic structure. Hence $h^0(\mathbf{a}) \equiv 0 \pmod 2$. By the Theorem of Riemann-Roch:

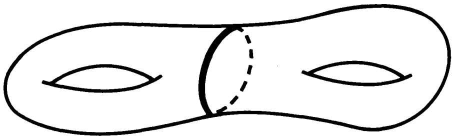
$$(\star) \qquad h^0(\mathbf{a})-h^1(\mathbf{a})=1-g+\deg(\mathbf{a}),$$

so to prove $h^1(\mathbf{a})=0 \pmod 2$ it suffices to show that $\deg(\mathbf{a}) \equiv g-1 \pmod 2$. To see this, take a generator $\mathbf{a}$ for the quotient with $\deg(\mathbf{a}) \geqq 2g-1$. Then $h^1(\mathbf{a})=0$ and the desired congruence follows from ($\star$).

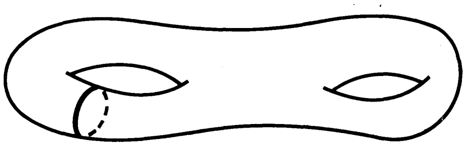
3. Real curves: topology

For any variety X over $\mathbb{R}$, let $n(X)$ denote the number of non-trivial connected components of $X(\mathbb{R})$.

The real locus $X(\mathbb{R})$ of a curve X of genus g consists of $n(X)$ disjoint circles. The complement of the real locus in the complex locus has either one or two connected components [7]. Put $a(X)=0$ if $X(\mathbb{R})$ divides the complex locus, and $a(X)=1$ if $X(\mathbb{C})-X(\mathbb{R})$ remains connected.



$n(X)=1$
$a(X)=0$



$n(X)=1$
$a(X)=1$

The quotient of $X(\mathbb{C})$ by complex conjugation is a connected 2-manifold M with $n(X)$ boundary components. M has Euler characteristic $(1-g)$ and is orientable if and only if $a(X)=0$. From the classification of 2-manifolds we obtain the following restrictions on the topological invariants $n(X)$ and $a(X)$.

PROPOSITION 3.1. — (1) $0 \leqq n(X) \leqq g+1$.

(2) *If* $n(X)=0$ *then* $a(X)=1$. *If* $n(X)=g+1$ *then* $a(X)=0$.

(3) *If* $a(X)=0$ *then* $n(X) \equiv g+1 \pmod 2$.

Klein proved that all pairs $(n(X), a(X))$ which are permitted by Proposition 3.1 actually occur for some real curve X of genus g ([7], [8]).

Now recall the varieties $S^d X$ and $\mathrm{Pic}^d X$ defined in paragraph 2. Put $W^d = \mathrm{Image}(\varphi : S^d X \to \mathrm{Pic}^d X)$.

PROPOSITION 3.2. — (1):

$$n(S^d X) = \sum_{s=0}^{[d/2]} \binom{n(X)}{d-2s}, \qquad d \geqq 0.$$

(2) *If* $n(X)>0$ *then:*

$$n(W^d) = n(S^d X), \qquad d \geqq 0,$$

$$n(J) = n(\mathrm{Pic}^d X) = 2^{n(X)-1}.$$

We thank Shimura for showing us the following simple argument.

Proof. — (1) Let $C_1, \ldots, C_{n(X)}$ be the distinct components of $X(\mathbb{R})$ and put $W = \{(p, \bar{p}) \in X(\mathbb{C}) \times X(\mathbb{C})\}$. The connected set:

$$U(i_1, \ldots, i_r) = C_{i_1} \times C_{i_2} \times \ldots \times C_{i_r} \times W^s,$$

maps to $S^d X(\mathbb{R})$ if $d = r + 2s$. By the definition of $S^d X$, the totality of the sets $U(i_1, \ldots, i_r)$ cover $S^d X(\mathbb{R})$. There are some obvious intersections; for example the sets:

$$C_1 \times C_2 \times C_2 \times W^s$$

and:

$$C_1 \times W \times W^s,$$

have the subset $C_1 \times p_2 \times p_2 \times W^s$ in common. Taking this into account, we find that $n(S^d X)$ is the number of combinations $(i_1, \ldots, i_r)$ of *distinct* indices from $\{1, 2, \ldots, d\}$ with $0 \leqq r \leqq d$ and $d - r = 2s$ even.

(2) If $n(X) > 0$, then by Proposition 2.2, $\mathrm{Pic}(\mathbb{R})^+ = \mathrm{Pic}(\mathbb{R})$. I claim that $W^d(\mathbb{R}) = \varphi(S^d X(\mathbb{R}))$. Indeed if $p \in W^d(\mathbb{R})$ we may choose a divisor $\mathbf{a}$ representing p with $\mathbf{a} = \bar{\mathbf{a}}$. Since $h^0(\mathbf{a}) \geqq 1$ we may find $f \in \mathbb{R}(X)^*$ such that $\mathbf{a} + (\operatorname{div} f)$ is effective and fixed by complex conjugation. This gives a point in $S^d X(\mathbb{R})$ mapping to p.

Since the map $\varphi^d : S^d X(\mathbb{R}) \to W^d(\mathbb{R})$ is a surjection, $n(W^d) \leqq n(S^d X)$. To prove equality it suffices to show that the images of two distinct components of $S^d X(\mathbb{R})$ cannot overlap. But if $\mathbf{a}$ and $\mathbf{a}'$ are two effective real divisors of degree d with $\varphi^d(\mathbf{a}) = \varphi^d(\mathbf{a}')$, then there is a function $f \in \mathbb{R}(X)^*$ of degree $e \leqq d$ with $\operatorname{div}(f) = \mathbf{a} - \mathbf{a}'$. The function $f_{\mathbb{R}} : X(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ induces a continuous map:

$$F: \quad \mathbb{P}^1(\mathbb{R}) \to S^e X(\mathbb{R}), \qquad y \to f^{-1}(y).$$

Since $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ is connected, the divisors $\mathbf{b} = f^{-1}(0)$ and $\mathbf{b}' = f^{-1}(\infty)$ lie in the same connected component of $S^e X(\mathbb{R})$. Hence $\mathbf{a} = \mathbf{b} + \mathbf{c}$ and $\mathbf{a}' = \mathbf{b}' + \mathbf{c}$ lie in the same component of $S^d X(\mathbb{R})$. This completes the proof that $n(W^d) = n(S^d X)$.

Since $n(X) \leqq g + 1$ by Proposition 3.1, we have:

$$n(S^d X) = n(W^d) = 2^{n(X)-1} \qquad \text{for} \quad d \geqq g.$$

Since $W^d = \mathrm{Pic}^d$ for $d \geqq g$, this counts the number of components of Pic^d when d is large. Since Pic^d is a (trivial) principal homogeneous space for the Jacobian, we have $n(J) = n(\mathrm{Pic}^d) = 2^{n(X)-1}$ for all d.

When $n(X) = 0$ it is more difficult to compute $n(W^d)$, as $\mathrm{Pic}(\mathbb{R})^+ \neq \mathrm{Pic}(\mathbb{R})$. However, the number of components of Pic^d depends only on the parity of d and g.

PROPOSITION 3.3. — *Assume* $n(X) = 0$.

(1) *If* $g \equiv 0 \pmod 2$ *then:*

$$n(\mathrm{Pic}^d) = 1, \qquad d \in \mathbb{Z}.$$

(2) *If* $g \equiv 1 \pmod 2$ *then:*

$$\left.\begin{array}{r} n(\mathrm{Pic}^{2d+1})=0 \\ n(\mathrm{J})=n(\mathrm{Pic}^{2d})=2 \end{array}\right\} d\in\mathbb{Z}.$$

Proof. — By Part (1) of Proposition 3.2:

$$n(\mathrm{S}^{2d}\mathrm{X})=1,$$
$$n(\mathrm{S}^{2d+1}\mathrm{X})=0.$$

If g is even, then $\mathrm{Pic}^{2d}(\mathbb{R})^+=\mathrm{Pic}^{2d}(\mathbb{R})$ by Proposition 2.2. Consequently, $n(\mathrm{W}^{2d})=n(\mathrm{S}^{2d}\mathrm{X})=1$ for all $d\geqq 0$. But $\mathrm{W}^{2d}=\mathrm{Pic}^{2d}$ for $2d\geqq g$, so Pic^{2d} has one component for d large. Since it is a (trivial) principal homogeneous space for the Jacobian, $n(\mathrm{J})=1$. Corollary 1.4 now implies that $n(\mathrm{Pic}^d)=1$ for all d, as *any* principal homogeneous space for J is trivial.

If g is odd, then $\mathrm{Pic}^{2d+1}(\mathbb{R})^+=\mathrm{Pic}^{2d+1}(\mathbb{R})$ by Proposition 2.2. Hence $\mathrm{Pic}^{2d+1}(\mathbb{R})$ is trivial for all d. On the other hand, $n(\mathrm{S}^{2d}\mathrm{X})=1$, so $\mathrm{Pic}^{2d}(\mathbb{R})^+$ is connected when $2d>g$. Hence $\mathrm{J}(\mathbb{R})^+$ is connected; since it has index 2 in $\mathrm{J}(\mathbb{R})$ we must have $n(\mathrm{J})\leqq 2$. But the Jacobian cannot be connected, as it has a non-trivial principal homogeneous space Pic^1. Hence $n(\mathrm{Pic}^{2d})=2$ for all d.

4. Packaging the topological data

We will show how Proposition 3.2 gives information on the 2-divisibility of an element in $\mathrm{Pic}(\mathbb{R})$, and how the topological invariants $n(\mathrm{X})$ and $a(\mathrm{X})$ are stored in the $\mathbb{Z}/2$-homology of $\mathrm{X}(\mathbb{C})$.

LEMMA 4.1. — *If* $f\in\mathbb{R}(\mathrm{X})^*$, *then* $\mathrm{div}(f)=\mathbf{a}$ *has an* even *number of points on each component of* $\mathrm{X}(\mathbb{R})$.

Proof. — Restricting f to the component C_i gives a continuous map $f_i:\mathrm{C}_i\to\mathbb{P}^1(\mathbb{R})$. Going around the loop C_i, we see $f_i(t)$ changes sign precisely when we cross a point of $\mathrm{div}(f)\cap\mathrm{C}_i$ which occurs with odd multiplicity. Since the total number of sign changes is even, this gives the Lemma.

For any divisor $\mathbf{a}$ fixed by G let:

$$c_i(\mathbf{a})=\mathrm{Card}\{\mathrm{C}_i\cap\mathbf{a}\}\qquad(\mathrm{mod}\ 2).$$

By Lemma 4.1 this is independent of the choice of representative $\mathbf{a}$, and we obtain a surjective homomorphism:

$$c:\quad \mathrm{Pic}(\mathbb{R})^+\to(\mathbb{Z}/2)^{n(\mathrm{X})},$$
$$\mathbf{a}\to(\ldots,c_i(\mathbf{a}),\ldots).$$

PROPOSITION 4.2. — (1) *If* $n(X)>0$, $\ker c=2\,\mathrm{Pic}(\mathbb{R})$. [*A class divisible by* 2 *in* Pic (ℝ) *iff it has an even number of points on each component of the real locus.*]

(2) *If* $n(X)=0$:

$$2\,\mathrm{Pic}(\mathbb{R})=\begin{cases}\mathrm{Pic}(\mathbb{R})^+, & g=0 \pmod 2,\\ 2\,\mathrm{Pic}(\mathbb{R})^+, & g\equiv 1 \pmod 2.\end{cases}$$

Proof. — If $n(X)=0$ then the determination of 2 Pic (ℝ) follows from Proposition 3.3.

Now assume $n(X)>0$, so $\mathrm{Pic}(\mathbb{R})^+=\mathrm{Pic}(\mathbb{R})$. Clearly $\ker c\subseteq 2\,\mathrm{Pic}(\mathbb{R})$. Suppose $c(\mathbf{a})\equiv 0$; then $\mathbf{a}$ has even degree $2d$. Let $p\in X(\mathbb{R})$ and $\mathbf{b}=\mathbf{a}-2\,d(p)$; then $\mathbf{b}\in J(\mathbb{R})$ and $c(\mathbf{b})\equiv 0$. To show $\mathbf{a}$ is divisible by 2 it suffices to show $\mathbf{b}\in J(\mathbb{R})^0$; this follows from the proof of Proposition 3.2.

COROLLARY 4.3 (compare [2]). — *The canonical class* $\mathbf{k}$ *is divisible by* 2 *in* Pic (ℝ).

Proof. — First we must check that $\mathbf{k}\in\mathrm{Pic}(\mathbb{R})^+$. Let f be a non-constant function in $\mathbb{R}(X)^*$ and let $\omega=df$. Then $\mathbf{k}=\mathrm{div}(\omega)$ clearly lies in $\mathrm{Pic}(\mathbb{R})^+$. When $g\equiv 1 \pmod 2$ and $n(X)=0$ $\mathbf{k}$ lies in $2\,\mathrm{Pic}(\mathbb{R})^+$ as it has degree $2g-2\equiv 0 \pmod 4$.

Now view ω as a complex (1, 0) form on $X(\mathbb{C})$: as such it gives rise to a meromorphic section of the line bundle $\mathrm{Hom}_{\mathbb{C}}(\tau,\mathbb{C})$, where τ is the complex tangent bundle of $X(\mathbb{C})$. Let Ω denote the dual section of τ; then Ω is a meromorphic vector field on $X(\mathbb{C})$, and by Proposition 4.2, it suffices to show Ω has even index on each component C_i of $X(\mathbb{R})$.

Since ω is defined over ℝ, the foliation determined by Ω remains fixed under the action of complex conjugation on $X(\mathbb{C})$. In particular, for $t\in C_i$ the vector Ω_t is a *real* multiple α_t of the tangent vector to C_i at t. Passing a point of index m changes the sign of α_t by $(-1)^m$. Since the total number of sign changes must be even, Ω has even index on C_i.

We now show how the $\mathbb{Z}/2$-homology of a real curve X neatly packages the topological invariants $n(X)$ and $a(X)$. The vector space:

$$V=H_1(X(\mathbb{C}),\mathbb{Z}/2)=J(\mathbb{C})_2,$$

of dimension $2g$ carries the additional structure of a symplectic space with involution. The symplectic form e on V is given by the Weil pairing [which in this case is the reduction of the intersection pairing on $H_1(X(\mathbb{C}),\mathbb{Z})$ modulo 2]. The involution $\tau\in\mathrm{Sp}(V)$ is induced from the action of complex conjugation on $X(\mathbb{C})$.

Clearly the isomorphism class of V (as a symplectic space with involution) is determined by the conjugacy class of τ in Sp(V). We can always find a symplectic basis for V with respect to which the matrix of τ takes the form:

$$\tau=\begin{pmatrix} I_g & H\\ 0 & I_g\end{pmatrix},\qquad H^t=H.$$

The isomorphism class of V is then determined by the class of $H=((h_{ij}))$ as a symmetric bilinear form over $\mathbb{Z}/2$. The class of H is known to be determined by the two invariants [6]:

$$\mathrm{rank}\,(H),$$

$$\mathrm{diag}(H)=\begin{cases}0 & \text{if } h_{ii}=0 \text{ all } i,\\ 1 & \text{otherwise.}\end{cases}$$

The following proposition shows that the invariants $n(X)$ and $a(X)$ completely determine the structure of V, and that conversely, the isomorphism class of V determines $n(X)$ and $a(X)$ (if X has a real point).

PROPOSITION 4.4:

(1) *If* $n(X)>0$ *then:*

$$\operatorname{rank}(H)+n(X)=g+1,$$
$$\operatorname{diag}(H)=a(X).$$

(2) *If* $n(X)=0$ *then:*

$$\operatorname{rank}(H)=2[g/2],$$
$$\operatorname{diag}(H)=0.$$

Proof. — Since $J(\mathbb{R})_2=V^{\langle\tau\rangle}\simeq(\mathbb{Z}/2)^{2g-\operatorname{rank}(H)}$, we have $n(J)=2^{g-\operatorname{rank}(H)}$ by Proposition 1.1. On the other hand:

$$n(J)=\begin{cases}2^{n(X)-1}, & n(X)>0,\\ 1, & n(X)=0, \quad g\equiv 0 \pmod 2,\\ 2, & n(X)=0, \quad g\equiv 1 \pmod 2.\end{cases}$$

by Proposition 3.2 and 3.3. This gives the desired formula for rank (H). Now consider the linear form:

$$f:\quad V\to\mathbb{Z}/2,$$
$$v\to e(v, v^{\tau}).$$

Clearly $\operatorname{diag}(H)=0$ if and only if f is identically zero. Let c_X denote the class of the real locus $X(\mathbb{R})$ in $V=H_1(X(\mathbb{C}), \mathbb{Z}/2)$; I claim we have the formula:

$$(\star)\qquad f(v)=e(c_x, v)\quad \text{for all } v\in V.$$

This will complete the proof, as e is non-degenerate and $c_X=0$ iff $X(\mathbb{R})=\emptyset$ or $a(X)=0$. [$X(\mathbb{R})$ divides $X(\mathbb{C})$ into 2 components.] To prove ($\star$), represent v and c_X by cycles A and X so that A, X, and $A\tau$ meet transversally. Let E denote the intersection pairing on $H_1(X(\mathbb{C}), \mathbb{Z})$; then e is obtained by reducing E (mod 2). Consequently:

$$\begin{aligned} e(v, v^{\tau}) &\equiv E(A, A^{\tau}) \pmod 2\\ &=\operatorname{Card}\{p\in A\cap A^{\tau}\}\\ &=\operatorname{Card}\{p\in A\cap A^{\tau}, p\in X(\mathbb{R})\}+\operatorname{Card}\{p\in A\cap A^{\tau}, p\notin X(\mathbb{R})\}\\ &\equiv\operatorname{Card}\{p\in A\cap A^{\tau}, p\in X(\mathbb{R})\} \pmod 2\\ &=\operatorname{Card}\{p\in X(\mathbb{R})\cap A\}\\ &=E(X, A)\equiv e(c_X, v) \pmod 2. \end{aligned}$$

Note. — The non-singular bilinear form of rank (H) determined by V is the intersection form of the compact surface $\tilde{M}$ obtained from $M=X(\mathbb{C})/\langle\tau\rangle$ by glueing $n(X)$ discs to the boundary.

5. Real theta-characteristics

We begin with a brief review of the complex theory ([2], [11]). Let X be a complex curve of genus g and S the set of its theta-characteristics:

$$S=\{\mathbf{a}\in \mathrm{Pic}^{g-1}(\mathbb{C}) : 2\,\mathbf{a}=\varkappa\}.$$

The set S is a principal homogeneous space for $V=J(\mathbb{C})_2$ and has cardinality 2^{2g}.

Each $\mathbf{a}\in S$ gives rise to a map:

$$q_{\mathbf{a}} : V \to \mathbb{Z}/2,$$
$$v\to h^0(\mathbf{a}+v)-h^0(\mathbf{a}) \pmod 2,$$

which is a quadratic refinement of the Weil pairing $e : V\times V\to \mu_2$. More precisely, after identifying μ_2 with $\mathbb{Z}/2$ we have the formula:

$$e(v, w)=q_{\mathbf{a}}(v+w)-q_{\mathbf{a}}(v)-q_{\mathbf{a}}(w).$$

If $\langle x_1, \ldots, x_g; y_1, \ldots, y_g\rangle$ is a symplectic basis for V and q is *any* quadratic refinement of e with values in $\mathbb{Z}/2$, define the invariant:

$$\mathrm{Arf}(q)=\sum_{i=1}^{g} q(x_i)\,q(y_i).$$

This invariant is independent of the basis chosen; for $q=q_{\mathbf{a}}$ we have the formula:

$$\mathrm{Arf}(q_{\mathbf{a}})\equiv h^0(\mathbf{a}) \pmod 2.$$

We say $\mathbf{a}$ is even or odd if $h^0(\mathbf{a})$ is even or odd respectively.

The *parity* of $h^0(\mathbf{a})$ is a *topological* invariant which remains unchanged under analytic deformations of X [11]. Thurston has shown us a simple topological description of the map $q_{\mathbf{a}}$. Let ω be a meromorphic differential on X with div $(\omega)=2.\mathbf{a}$ and let Ω be the dual vector field, defined as in the proof of Corollary 4.3. Represent $v\in V=H_1(X(\mathbb{C}), \mathbb{Z}/2)$ by a simple closed curve C which does not pass through any of the singularities of Ω. The turning number $W_C(\Omega)$ of Ω around C is then well-defined (up to sign) and:

$$(\star)\qquad q_{\mathbf{a}}(v)\equiv W_C(\Omega)+1 \pmod 2.$$

To prove this formula, one first checks that the right-hand side is independent of the curve C representing v. Indeed, if $C-C'=\partial M$ where M is an oriented 2-manifold with boundary, then:

$$\chi(M)=\mathrm{Index}_{\mathrm{Int}\,M}\,\Omega+W_C(\Omega)-W_{C'}(\Omega).$$

Since both the Euler characteristic of M and index of Ω are *even*, the right-hand side of $(\star)$ depends only on v.

More generally, if C is *any* curve on $X(\mathbb{C})$ representing the class v, we may define:

$$p_a(v) \equiv W_C(\Omega) + C.C + n(C) \qquad (\text{mod } 2),$$

where $n(C)$ is the number of components and C.C is the self-intersection number of C. Again the right-hand side is independent of the curve chosen. Clearly p_a gives a quadratic refinement of the intersection pairing (mod 2), as $W_C(\Omega)$ and $n(C)$ are both additive. To prove (★) one checks that $p_a \equiv q_a$ when X is hyper-elliptic; this is a purely combinatorial question (*see* § 6). Next one observes that both p_a and q_a are invariant under analytic deformation of X; since the moduli space of complex curves of genus g is *connected*, this gives the equality $p_a = q_a$ for any X. Similarly, using hyperelliptic curves, one can obtain the count:

$$\text{Card } S^{\text{even}} = 2^{g-1}(2^g+1),$$
$$\text{Card } S^{\text{odd}} = 2^{g-1}(2^g-1).$$

Now assume that the curve X is defined over $\mathbb{R}$ and put:

$$S(\mathbb{R}) = S \cap \text{Pic}^{g-1}(\mathbb{R}).$$

We have already seen that $S(\mathbb{R})$ is non-empty (Corollary 4.3); since it is a principal homogeneous space for $J(\mathbb{R})_2$ we find:

$$\text{Card } S(\mathbb{R}) = \text{Card } J(\mathbb{R})_2 = 2^{g+d} \qquad \text{where} \quad n(J) = 2^d.$$

This number depends only on g and $n(X)$. The number of real characteristics of each parity depends on the further invariant $a(X)$.

Proposition 5.1. — *Assume* $n(J) = 2^d$ *and let* c_X *denote the class of* $X(\mathbb{R})$ *in* $V = H_1(X(\mathbb{C}), \mathbb{Z}/2)$:

$$\textit{If} \quad c_X \equiv 0 \quad \textit{then} \quad \begin{cases} \text{Card } S(\mathbb{R})^{\text{even}} = 2^{g-1}(2^d+1), \\ \text{Card } S(\mathbb{R})^{\text{odd}} = 2^{g-1}(2^d-1). \end{cases}$$

$$\textit{If} \quad c_X \not\equiv 0 \quad \textit{then} \quad \begin{cases} \text{Card } S(\mathbb{R})^{\text{even}} = 2^{g-1}(2^d), \\ \text{Card } S(\mathbb{R})^{\text{odd}} = 2^{g-1}(2^d). \end{cases}$$

Note. — (1) If $n(X) > 0$ then $d = n(X) - 1$ and $a(X) = 0$ iff $c_X \equiv 0$. Proposition 5.1 may be restated as follows:

$$\text{Card } S(\mathbb{R})^{\text{even}} = 2^{g-1}(2^{n(X)-1} + 1 - a(X));$$
$$\text{Card } S(\mathbb{R})^{\text{odd}} = 2^{g-1}(2^{n(X)-1} - 1 + a(X)).$$

(2) If $n(X) = 0$ then $c_X \equiv 0$ and Proposition 5.1 may be restated as follows:

$$\text{Card } S(\mathbb{R})^{\text{even}} = \begin{cases} 3.2^{g-1}, & g \text{ odd}, \\ 2^g, & g \text{ even}. \end{cases}$$

$$\text{Card } S(\mathbb{R})^{\text{odd}} = \begin{cases} 2^{g-1}, & g \text{ odd}, \\ 0, & g \text{ even}. \end{cases}$$

Put $S(\mathbb{R})^+ = S(\mathbb{R}) \cap \mathrm{Pic}^{g-1}(\mathbb{R}^+$; this is a principal homogeneous space for $J(\mathbb{R})^+ \simeq (\mathbb{Z}/2)^g$ which is trivial iff g is odd [2]. Consequently:

$$\text{Card } S(\mathbb{R})^+ = \begin{cases} 2^g, & g \text{ odd}, \\ 0, & g \text{ even}. \end{cases}$$

$$\text{Card } (S(\mathbb{R}) - S(\mathbb{R})^+) = 2^g.$$

By Proposition 2.2, every $\mathbf{a} \in S(\mathbb{R}) - S(\mathbb{R})^+$ is *even*; also notice that the quotient $\mathrm{Pic}(\mathbb{R})/\mathrm{Pic}(\mathbb{R})^+$ is always generated by an even theta-characteristic !

Proof of 5.1. — Consider the 2^g orbits of $J(\mathbb{R})_2^0$ on the set S. These orbits are indexed by the linear forms $f : J(\mathbb{R})_2^0 \to \mathbb{Z}/2$. Indeed, the subspace $J(\mathbb{R})_2^0$ is maximal isotropic for e, so for any $\mathbf{a} \in S$ the map $f = q_{\mathbf{a}}$ $J(\mathbb{R})_2^0 \to \mathbb{Z}/2$ is *linear*. Furthermore, the form f depends only on the orbit of $\mathbf{a}$ under $J(\mathbb{R})_2^0$; call this orbit S_f.

From the formula $h^0(\mathbf{a}+v) = h^0(\mathbf{a}) + q_{\mathbf{a}}(v)$ we can deduce that Card S_f^{even} = Card $S_f^{\text{odd}} = 2^{g-1}$ when $f \neq 0$. Similarly, when $f = 0$ all the elements of S_0 have the same parity; in fact they are all *even* as S contains more even theta-characteristics than odd. Since $S(\mathbb{R})$ is the union of 2^d such orbits, we have:

$$\text{Card } S(\mathbb{R})^{\text{even}} = \begin{cases} 2^{g-1}(2^d+1) & \text{if } S_0 \subset S(\mathbb{R}), \\ 2^{h-1}(2^d) & \text{if } S_0 \not\subset S(\mathbb{R}). \end{cases}$$

To complete the proof we must show S_0 is real if and only if $c_X \equiv 0$.

Take $\mathbf{a} \in S_0$ and let $v_0 = \mathbf{a}^\tau - \mathbf{a}$ in V. Then:

$$\begin{aligned} e(v_0, v) &= q_{\mathbf{a}}(v_0+v) - q_{\mathbf{a}}(v_0) - q_{\mathbf{a}}(v) \\ &\equiv h^0(\mathbf{a}+v_0+v) + h^0(\mathbf{a}+v_0) + h^0(\mathbf{a}+v) + h^0(\mathbf{a}) \\ &\equiv h^0(\mathbf{a}^\tau+v) + h^0(\mathbf{a}^\tau) + h^0(\mathbf{a}+v) + h^0(\mathbf{a}). \end{aligned}$$

Since $h^0(\mathbf{b}) = h^0(\mathbf{b}^\tau)$ for any class $\mathbf{b}$:

$$e(v_0, v) = h^0(\mathbf{a}+v^\tau) + h^0(\mathbf{a}+v) = q_{\mathbf{a}} v^\tau) + q_{\mathbf{a}}(v).$$

By the proof of 1.1, $v + v^\tau \in J(\mathbb{R})_2^0$. Since $\mathbf{a} \in S_0$ we must have $q_{\mathbf{a}}(v+v^\tau) = 0$. Hence:

$$e(v_0, v) = q_{\mathbf{a}}(v+v^\tau) - q_{\mathbf{a}}(v) - q_{\mathbf{a}}(v^\tau) = e(v, v^\tau) = e(c_X, v) \quad \text{by Proposition 4.4.}$$

Since e is non-degenerate, $v_0 \equiv c_X$ in V. Consequently $\mathbf{a} = \mathbf{a}^\tau$ if and only if $c_X \equiv 0$.

Finally, we wish to discuss the components in the real locus of $\theta = W^{g-1}$ for a curve with $n(X) > 0$. Proposition 3.2 gives the formulas:

$$n(\theta) = n(\mathrm{Pic}^{g-1}) \qquad \text{if } 0 < n(X) < g+1,$$

$$n(\theta) = n(\mathrm{Pic}^{g-1}) - 1 \qquad \text{if } n(X) = g+1.$$

More precisely, the components of $\mathrm{Pic}^{g-1}(\mathbb{R})$ correspond to the fibres of the map:

$$c: \quad \mathrm{Pic}^{g-1}(\mathbb{R}) \to (\mathbb{Z}/2)^{n(X)},$$

$$\mathbf{a} \to (\ldots, c_i(\mathbf{a}), \ldots)$$

defined in paragraph 4. This map is not surjective: its image consists of all $n(X)$-tuples with $\sum_{i=1}^{n(X)} c_i \equiv g-1 \pmod 2$. When $0<n(X)<g+1$ each component of $\mathrm{Pic}^{g-1}(\mathbb{R})$ contains exactly one component of $\theta(\mathbb{R})$; when $n(X)=g+1$ each component of $\mathrm{Pic}^{g-1}(\mathbb{R})$ contains one component of $\theta(\mathbb{R})$ *except* for the component $c^{-1}(1, 1, \ldots, 1)$, which contains no effective divisors. In all cases, each component of $\mathrm{Pic}^{g-1}(\mathbb{R})$ contains exactly 2^g real theta-characteristics.

Now suppose $n(X)<g+1$ but $X(\mathbb{R})$ divides $X(\mathbb{C})$ into 2 components. By Proposition 5.1 there is exactly *one* component U of $\theta(\mathbb{R})$ with no odd theta-characteristics. We can identify U using the following.

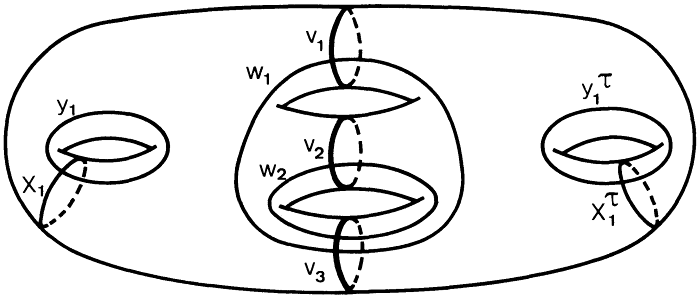
PROPOSITION 5.2. — $c(U)=(1, 1, \ldots, 1)$.

Proof. — Assume $\mathbf{a} \in S(\mathbb{R})$ has odd multiplicity on each component T_i of $X(\mathbb{R})$. We must show $h^0(\mathbf{a}) \equiv \mathrm{Arf}(q_{\mathbf{a}})=0$.

Let v_i denote the class of C_i in $V=H_1(X(\mathbb{C}), \mathbb{Z}/2)$. Then $q_{\mathbf{a}}(v_i)=c_i(\mathbf{a})+1$. Indeed, by Thurston's formula for $q_{\mathbf{a}}$ we must show that $c_i(\mathbf{a})=w_{C_i}(\Omega) \pmod 2$, where Ω is the vector field dual to the differential ω with divisor $2\mathbf{a}$. This is clear, as $\Omega\, 1_{C_i}$ is always a real multiple of the tangent field.

Now assume $X(\mathbb{R})$ divides $X(\mathbb{C})$ and let $k=(g+1-n(X))/2$. *Choose a symplectic basis for V of the form:*

$$\langle v_1, \ldots, v_{n(X)-1}, x_1, \ldots, x_k, x_1^\tau, \ldots, x_k^\tau; w_1, \ldots, w_{n(X)-1}, y_1, \ldots, y_k, y_1^\tau, \ldots, y_k^\tau \rangle.$$



Then

$$\mathrm{Arf}(q_{\mathbf{a}})=\sum_{i=1}^{n(X)-1} q_{\mathbf{a}}(v_i)\, q_{\mathbf{a}}(w_i)+\sum_{i=1}^{k} q_{\mathbf{a}}(x_i)\, q_{\mathbf{a}}(y_i)+\sum_{i=1}^{k} q_{\mathbf{a}}(x_i^\tau)\, q_{\mathbf{a}}(y_i^\tau).$$

Since:

$$q_{\mathbf{a}}(x_i)=q_{\mathbf{a}}(x_i^\tau) \quad \text{and} \quad q_{\mathbf{a}}(y_i)=q_{\mathbf{a}}(y_i^\tau),$$

$$\mathrm{Arf}(q_{\mathbf{a}})=\sum_{i=1}^{n(X)-1} q_{\mathbf{a}}(v_i)\, q_{\mathbf{a}}(w_i).$$

But when $c_i(\mathbf{a})=1$ we must have $q_{\mathbf{a}}(v_i)=0$. Hence $\mathrm{Arf}(q_{\mathbf{a}})=0$.

COROLLARY 5.3. — $X(\mathbb{R})$ *divides* $X(\mathbb{C})$ *if and only if* $n(X)>0$, $n(X) \equiv g+1 \pmod 2$, *and the component* $c^{-1}(1, 1, 1, \ldots, 1)$ *in* $\mathrm{Pic}^{g-1}(\mathbb{R})$ *contains no odd theta-characteristics.*

6. Real hyper-elliptic curves

A curve X over $\mathbb{C}$ of genus $g \geqq 2$ is hyper-elliptic if the canonical map $X \underset{\pi}{\rightarrow} \mathbb{P}(H^0(X, L(\mathbf{k}))) = \mathbb{P}^{g-1}$ is 2-to-1 onto a rational normal curve Y of degree $g-1$ [3]. If p_Y is any point on $Y(\mathbb{C})$, the divisor $\pi^{-1}(p_Y)$ gives a representative for a class $\mathbf{d}$ in $\mathrm{Pic}^2(\mathbb{C})$ with $h^0(\mathbf{d}) = 2$. The properties $\deg(\mathbf{d}) = h^0(\mathbf{d}) = 2$ completely characterize $\mathbf{d}$, and the map $X \rightarrow \mathbb{P}(H^0(X, L(\mathbf{d})))$ is a 2-sheeted cover of $\mathbb{P}^1$ ramified at $2g+2$ points. Without loss of generality, we may assume the branch points $\{p_1, \ldots, p_{2g+2}\}$ lie in the finite plane, and X is given by the equation:

$$y^2 = \prod_{i=1}^{2g+2} (x - p_i).$$

The theta-characteristics on X are easy to describe [11]. Any semi-canonical class may be represented by a divisor of the form:

$$\mathbf{a} = (m-1)\mathbf{d} + \mathbf{e},$$

where $\mathbf{d}$ is described above, $0 \leqq m \leqq [(g+1)/2]$ and $\mathbf{e}$ is a formal sum of elements in a subset E of the $(2g+2)$ branch points. Clearly $\mathrm{Card}(E) = g+1-2m$; this representation of $\mathbf{a}$ is unique except when $m=0$, in which case the subset E may be replaced by its complement. In any case we have the formula:

$$h^0(\mathbf{a}) = m,$$

the linear series $|\mathbf{a}|$ having (when $m > 0$) the divisor $\mathbf{e}$ as fixed part.

Using this description of the characteristics, and the fact that the differential:

$$\omega = \frac{dx}{y} \prod_{p_i \in E} (x - p_i),$$

has divisor $2\mathbf{a}$, it is an easy exercise to verify Thurston's formula $q_{\mathbf{a}}(v) \equiv w_C(\Omega) + 1 \pmod 2$ on complex hyper-elliptic curves.

Now suppose X is a hyper-elliptic curve which is defined over $\mathbb{R}$.

PROPOSITION (6.1. — (1) $\mathbf{d} \in \mathrm{Pic}^2_X(\mathbb{R})$.

(2) *If* $X(\mathbb{R}) \neq \emptyset$ *or g is even,* X *may be represented as a 2-sheeted cover of* $\mathbb{P}^1$ *over* $\mathbb{R}$.

Proof. — Since $h^0(\mathbf{d}^\tau) = h^0(\mathbf{d})$ and $\deg(\mathbf{d}^\tau) = \deg(\mathbf{d})$ we must have $\mathbf{d}^\tau = \mathbf{d}$. Hence $\mathbf{d} \in \mathrm{Pic}^2(\mathbb{R})$. If $X(\mathbb{R}) \neq \emptyset$ or $g \equiv 0 \pmod 2$, then $\mathrm{Pic}^2(\mathbb{R}) = \mathrm{Pic}^2(\mathbb{R})^+$ and $\mathbf{d}$ corresponds to a line-bundle L on X which is defined over $\mathbb{R}$. The map $X \rightarrow \mathbb{P}(H^0(X, L)) = \mathbb{P}^1$ is the desired cover. Alternatively, the canonical map is 2-to-1 onto a rational curve Y of degree $g-1$; if $Y(\mathbb{R}) \neq \emptyset$ then $Y \simeq \mathbb{P}^1$ over $\mathbb{R}$.

When $g \geqq 3$ is odd and $X(\mathbb{R}) = \emptyset$, the curve Y may be isomorphic to either $\mathbb{P}^1$ or the conic $N = \{u^2 + v^2 = -1\}$. Since $\mathbf{d} \sim \pi^{-1}(p_Y)$ we find.

PROPOSITION 6.2. — *If g is odd and $X(\mathbb{R})=\emptyset$ then one of the following occurs:*

(1) $\mathbf{d}\in \mathrm{Pic}^2(\mathbb{R})^+$ *and* X *may be represented as a 2-sheeted cover of* $\mathbb{P}^1$ *over* $\mathbb{R}$.

(2) $\mathbf{d}\notin \mathrm{Pic}^2(\mathbb{R})^+$ *and* X *may be represented as a 2-sheeted cover of* N *over* $\mathbb{R}$.

In all cases but (2) of 6.2, the curve X is given by the real equation:

$$y^2=f(x), \qquad \deg f=2g+2.$$

In case (2), X is given by the pair of equations:

$$u^2+v^2=-1,$$
$$y^2=f(u, v), \qquad \deg f=g+1.$$

The fact that the curve X is hyper-elliptic puts surprisingly strong restrictions on the pair $(n(X), a(X))$ of its topological invariants. Recall that $a(X)=0$ iff $X(\mathbb{R})$ divides $X(\mathbb{C})$ into 2 components.

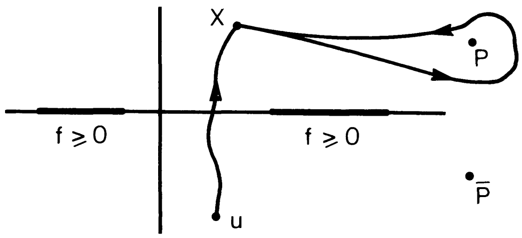
PROPOSITION 6.3. — *Let* X *be a real hyper-elliptic curve with* $a(X)=0$. *Then one of the following occurs.*

(1) $n(X)=g+1$.

(2) $n(X)=\begin{cases} 1 \text{ if } g\equiv 0 \pmod 2, \\ 2 \text{ if } g\equiv 1 \pmod 2. \end{cases}$

Note. — This result shows that *entire components* of the real moduli contain no hyper-elliptic curves once $g\geqq 4$ (*see* § 9).

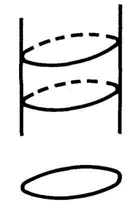
Proof. — Assume $a(X)=0$ and X is given by the equation $y^2=f(x)$. If $f(x)$ has a non-real root p, we can connect the points (x, y) and $(x, -y)$ on $X(\mathbb{C})-X(\mathbb{R})$ by lifting a loop on $\mathbb{P}^1(\mathbb{C})-\mathbb{P}^1(\mathbb{R})$ based at x and winding once around either p or $\bar{p}$ (but around no other branch point). If $f(x)$ has a real root, we can connect the point (u, v) to either (x, y) or $(x, -y)$ on $X(\mathbb{C})-X(\mathbb{R})$ by lifting a path from u to x on $\mathbb{P}^1(\mathbb{C})-\{z\in\mathbb{P}^1(\mathbb{R}): f(z)\geqq 0\}$.



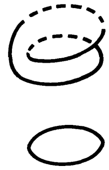
The assumption that $a(X)=0$ therefore implies that either *all* or *none* of the roots of $f(x)$ are real.

If all the roots of $f(x)$ are real, then $n(X)=g+1$ and the two branches of y disconnect $X(\mathbb{C})-X(\mathbb{R})$. If none of the roots of $f(x)$ are real, the two components of $X(\mathbb{C})-X(\mathbb{R})$ are given by $\operatorname{Im} x>0$ and $\operatorname{Im} x<0$. In the latter case, $X(\mathbb{R})$ is an unramified double cover of

$\mathbb{P}^1(\mathbb{R}) \simeq S^1$ which may be either trivial (disconnected) or non-trivial (connected). The two cases are distinguished by the parity of g: $X(\mathbb{R})$ is always a submanifold of the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(g+1)(\mathbb{R})$. This is a cylinder when g is odd and a Moebius strip where g is even.



$g \equiv 1 \pmod 2$



$g \equiv 0 \pmod 2$

One can also attack 6.3 by counting real theta characteristics on X and using Proposition 5.1. For example, assume X is given by the equation $y^2 = f(x)$ where f has degree $2g+2$ with $2n > 0$ real roots. Then $n(X) = n$ and the theta-characteristic $\mathbf{a} = (m-1)\mathbf{d} + \mathbf{e}$ is real if and only if $E^\tau = E$. Hence E will consist of k pairs of conjugate branch points and $g+1-2m-2k$ real branch points. Of the:

$$\frac{1}{2}\sum_{k=0}^{g+1-n}\binom{g+1-n}{k}\binom{2g+2-2n}{g+1-2k} + \sum_{m=1}^{[(g+1)/2]}\left(\sum_{k=0}^{g+1-n}\binom{g+1-n}{k}\binom{2g+2-2n}{g+1-2m-2k}\right) = 2^{g+n-1},$$

real theta-characteristics on X, exactly 2^{g+n-2} will be even when $n < g+1$, and $2^{g-1}(2^g+1)$ will be even when $n = g+1$. This shows that $X(\mathbb{R})$ can only disconnect $X(\mathbb{C})$ in the latter case.

7. Real plane curves

In this section we assume X is a smooth real plane curve of degree d. The genus of X is given by the formula $g = ((d-1)(d-2))/2$. Restricting the hyperplane bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ to X gives a divisor class $\mathbf{b} \in \mathrm{Pic}^d(\mathbb{R})^+$ with $h^0(\mathbf{b}) = 3$, the divisor $(d-3)\mathbf{b}$ is equal to the canonical class $\mathbf{k}_X$. Since the embedding $X \hookrightarrow \mathbb{P}^2$ is projectively normal, $h^0(n\mathbf{b}) = \binom{n+2}{2}$ for $n < d$ [3].

The real locus $X(\mathbb{R})$ consists of $n(X)$ disjoint circles; Harnack proved that all possible values $1 \leqq n(X) \leqq g+1$ are actually attained ([1], [7]). Call a circle $S^1 \to \mathbb{P}^2(\mathbb{R})$ an *oval* if it is homotopic to zero in $\mathbb{P}^2(\mathbb{R})$ and a *pseudo-line* if it represents the non-trivial class in $\pi_1(\mathbb{P}^2(\mathbb{R}))$. When d is even all of the components of $X(\mathbb{R})$ are ovals in $\mathbb{P}^2(\mathbb{R})$; when d is odd exactly one component is a pseudo-line. Indeed, the number of components not homotopic to zero must be congruent to $d \pmod 2$, and no two components can intersect. Each oval C_i in $X(\mathbb{R})$ disconnects $\mathbb{P}^2(\mathbb{R})$: the two components of the complement being homeomorphic to a disc and a moebius strip. We call the disc the *interior* of C_i and say that two components are *nested* if one lies in the interior of the other.

The nesting of ovals on real plane curves has been extensively studied ([1], [18]). Here we make some remarks relating the nesting of $X(\mathbb{R}) \subsetneq \mathbb{P}^2(\mathbb{R})$ to the topological type of $X(\mathbb{R}) \subsetneq X(\mathbb{C})$. First note that plane curves do not exhaust the possible topological types.

PROPOSITION 7.1. — *If* $d \equiv 5 \pmod 8$ *and* $n(X)=1$, *then:*

$$a(X)=1 \qquad [X(\mathbb{R}) \textit{ does } \text{not } \textit{divide } X(\mathbb{C})].$$

Proof. — If $d \equiv 5 \pmod 8$ then $g \equiv 0 \pmod 2$. A curve with $n(X)=1$ will have $a(X)=1$ if and only if it has a real odd theta-characteristic (Proposition 4.1). But $\mathbf{a}=((d-3)/2).\mathbf{b}$ is clearly a real theta-characteristic and $h^0(\mathbf{a})=(d^2-1)/2 \equiv 1 \pmod 2$.

We now consider plane curves of degree $d \leqq 5$ in more detail:

$\mathbf{d}=1$, X is a line and $g=0$. There is only one possibility:

$$n(X)=1, \qquad a(X)=0, \qquad X \simeq \mathbb{P}^1.$$

$\mathbf{d}=2$, X is a conic and $g=0$. There are two possiblities:

$$n(X)=0, \qquad a(X)=1, \qquad X \simeq \mathbb{N},$$
$$n(X)=1, \qquad a(X)=0, \qquad X \simeq \mathbb{P}^1.$$

$\mathbf{d}=3$, X is an elliptic curve and $g=1$. There are two possible configurations:

$$n(X)=1, \qquad a(X)=1, \qquad \Delta<0,$$
$$n(X)=2, \qquad a(X)=0, \qquad \Delta>0.$$

where $\Delta=g_2^3-27g_3^2$ is the discriminant of any Weierstrass model $y^2=4x^3-g_2x-g_3$ for X.

$\mathbf{d}=4$, X is a plane quartic, which is the canonical model of any non-elliptic curve of genus $g=3$.

An effective complex divisor $\mathbf{a}=(p_1)+(p_2)$ will be semi-canonical iff the line $\overline{p_1 p_2}$ is bitangent to X at p_1 and p_2. Since no divisor $\mathbf{a}$ of degree 2 has $h^0(\mathbf{a}) \geqq 2$, the odd theta-characteristics on X are in one-to-one correspondence with the 24 bitangent lines to $X(\mathbb{C})$ in $\mathbb{P}^2(\mathbb{C})$.

The *real* bitangent lines are related to the nesting of ovals on $\mathbb{P}^2(\mathbb{R})$ by the following:

LEMMA 7.2. — *If* C_1 *and* C_2 *are two components of* $X(\mathbb{R})$, *there is a line* $L \subset \mathbb{P}^2(\mathbb{R})$ *which is tangent to each if and only if* C_1 *and* C_2 *are* not *nested.*

Proof. — We can define an involution $g: C_1 \times C_2 \to C_1 \times C_2$ as follows. For every pair of points $p_1 \in C_1$ and $p_2 \in C_2$ the line $\overline{p_1 p_2}$ meets $X(\mathbb{R})$ in two additional points $q_1 \in C_1$ and $q_2 \in C_2$. Define $g((p_1, p_2))=(q_1, q_2)$. Then bitangent lines correspond to *fixed points* of the involution g, which we will compute using the Lefschetz fixed point formula.

Since g is the pull-back, *via* the natural embedding $C_1 \times C_2 \to \mathrm{Pic}_X^2(\mathbb{R})$, of the involution $\mathbf{a} \to \mathbf{k}_X - \mathbf{a}$, we see that g_* acts as -1 on the tangent space to $C_1 \times C_2$ at each fixed point. Hence g will have a fixed point iff its Lefschetz number $l(g)$ is non-zero. By the fixed point formula:

$$l(g)=\sum_{i=0}^{2}(-1)^i \operatorname{Tr}(g_* \mid H_i(C_1\times C_2, \mathbb{Z}))=\det(1-g_*\mid H_1(C_1\times C_2, \mathbb{Z})).$$

Hence $l(g)\neq 0$ iff g_* acts as the scalar -1 on $H_1(C_1\times C_2, \mathbb{Z})$. This will occur iff $\operatorname{Tr}(g_*\mid H_1(C_1\times C_2, \mathbb{Z}))=-2$ in which case $l(g)=4$.

To compute the trace on first homology, let η_i be the 1-cycle obtained by holding fixed a point $p\in C_j$ with $j\neq i$. The cycles η_i and η_2 form a basis of $H_1(C_1\times C_2, \mathbb{Z})$, and the coefficient of η_i in $g_*(\eta_i)$ is given by the degree of the involution on C_i obtained by interchanging points on C_i collinear with p. This degree is clearly -1 when p is exterior to C_i and $+1$ when p is interior. Hence the trace of g_* is 0 if the ovals are nested and -2 if they are mutually exterior. In the latter case $l(g)\neq 0$ and there are 4 bitangent lines between C_1 and C_2.

We can now enumerate the possible configurations of real plane quartics. Note that nesting of ovals can only occur when $n(X)=2$, by Bezout's Theorem. This is also the only case where $n(X)$ does not *a priori* determine $a(X)$. In fact, we have the following possible configurations:

$n(X)$	position of ovals	$a(X)$	# real bitangents
0.	–	1	4
1.	–	1	4
2.	nested	0	4
2.	non- nested	1	8
3.	–	1	16
4.	–	0	28

Indeed, when $n(X)=2$ we have $a(X)=0$ if only if the component $c^{-1}(1, 1)$ of $\operatorname{Pic}^2_X(\mathbb{R})$ contains no odd theta-characteristics (Proposition 5.3). This means there are no bitangent lines connecting C_1 and C_2, which by 7.2, occurs if and only if C_1 and C_2 are nested.

The fact that no component of $\operatorname{Pic}^2_X(\mathbb{R})$ can contain more than 4 odd theta-characteristics shows that the total number of lines bitangent to a single component of $X(\mathbb{R})$ is at most 4. This affords a simple proof of a classical Theorem of Zeuthen.

PROPOSITION 7.3. — *Of the* 24 *complex Weierstrass points on a real non-hyper-elliptic curve of genus* 3, *no more than* 8 *may be real.*

Proof. — The Weierstrass points on X are exactly the inflectionary points in the canonical model. If C is any component of $X(\mathbb{R})$, the number of flexes on C is exactly *twice* the number of lines bitangent to C. To see this, give C an orientation. At any point p of C other than a flex or point of contact with a bitangent line, the tangent line to C at p meets C residually in:

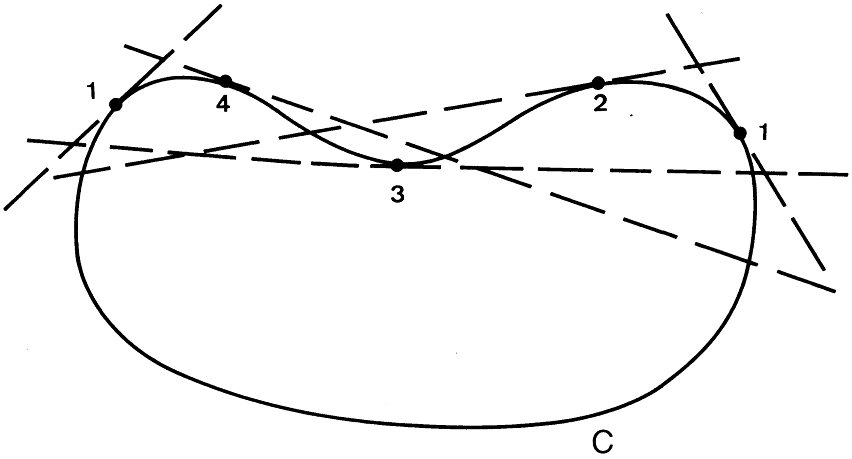
(1) no points;

(2) 2 points in the positive half of the tangent line;

(3) 2 points, one in each half of the tangent line;

(4) 2 points in the negative half of the tangent line.

As p moves around C, the disposition of this residual intersection changes from (1) to (2) at the first point of contact with a bitangent, from (2) to (3) or (3) to (4) at an ordinary flex, and from (4) to (1) at the second point of contact with a bitangent.



Hence there are exactly two flexes between each pair of bitangent points of contact. Since there are at most 4 pairs of bitangent points, this gives the Lemma. Note that a hyperflex occurs in the interior of an interval of type (1) and counts as two flexes and one pair of bitangents.

$\mathbf{d}=5$, X is a plane quintic and has genus $g=6$. The canonical series on X is cut out (completely) by the conics in the plane.

There is one distinguished odd theta-characteristic $\mathbf{b}$ with $h^0(\mathbf{b})=3$ which is cut out by lines in $\mathbb{P}^2$; the odd th ta-characteristics other than $\mathbf{b}$ correspond in a one-to-one manner to the quinti-tangent conics to X.

When $n(X)=1$ we must have $a(X)=1$ by 7.1. Similarly, $a(X)=1$ when $n(X)=2$, 4, 6 and $a(X)=0$ when $n(X)=7$. Only the cases $n(X)=3$, 5 are ambiguous.

When $n(X)=3$ the 2 ovals of $X(\mathbb{R})$ may be nested or not. In the nested case $a(X)=0$. Indeed no quinti-tangent real conic can meet each component of $X(\mathbb{R})$ once, so the component $c^{-1}(1, 1, 1)$ of $\mathrm{Pic}_X^5(\mathbb{R})$ contains no odd theta-characteristics and we may apply 5.3. In the non-nested case, we suspect that $a(X)$ is always 1 but have no proof. One can construct examples with $a(X)=1$ as follows. Take a real quartic Y with 2 non-nested components plus a real line L with $L(\mathbb{R})$ disjoint from $Y(\mathbb{R})$. Since $Y(\mathbb{C})-Y(\mathbb{R})$ is connected and the 4 points of $Y \cap L$ come in complex conjugate pairs, the singular quintic Y.L is also not disconnected by its real locus. The same will hold for a small deformation X of Y.L (which is smooth) as the singular points are all complex.

When $n(X)=5$ there can be no nesting of ovals (by Bezout's Theorem). Although there are no apparent topological distinctions among real plane quintics with 5 components, we shall see that both cases $a(X)=0$ and $a(X)=1$ actually occur! As usual, we must check whether the component $c^{-1}(1, 1, 1, 1, 1)$ in $\mathrm{Pic}_X^5(\mathbb{R})$ contains odd theta-characteristics: i. e., whether there are quinti-tangent conics which meet each component of $X(\mathbb{R})$ once.

Consider the involution $\mathbf{g}: C_1 \times \ldots \times C_5 \to C_1 \times \ldots \times C_5$ which sends a divisor $\mathbf{a}=(p_1)+\ldots+(p_5)$ to the divisor $(q_1)+\ldots+(q_5) \sim \mathbf{k}_X - \mathbf{a}$, where the q_i are the residual intersection of a conic through $p_1, \ldots, p_5$ with $X(\mathbb{R})$. This conic is unique, as no four of the p_i are collinear. As in the quartic case, we find:

$$a(X)=1 \quad \Leftrightarrow \quad g \text{ has a fixed point} \quad \Leftrightarrow \quad l(g) \neq 0,$$

where $l(g)=\det(1-g_* \mid H_1(C_1 \times \ldots \times C_5, \mathbb{Z})$ is the Lefschetz number.

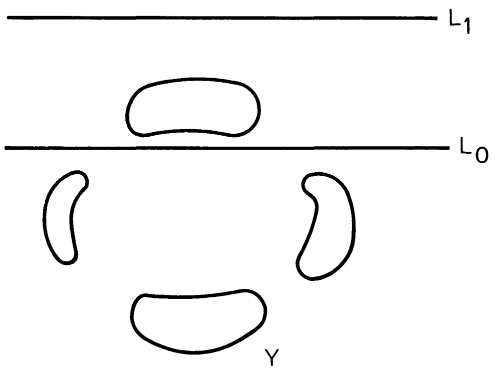
Let C_5 denote the pseudo-line in $X(\mathbb{R})$ and for $i=1, 2, \ldots, 5$ let η_i denote the class of $p_1 \times \ldots \times C_i \times \ldots \times p_5$ in $H_1(C_1 \times \ldots \times C_5, \mathbb{Z})$. By arguments similar to Lemma 7.2, the coefficient of η_i in $g_*(\eta_i)$ is -1 for $i=1, 2, 3, 4$. The key to the situation is the coefficient of η_5 in $g_*(\eta_5)$; this is precisely the degree of the involution f on C_5 interchanging the points lying on a conic through the four points $p_i \in C_i, i=1, 2, 3, 4$. The degree of f will be $+1$ and the Lefschetz number $l(g)$ will be 0 precisely when f has no fixed points: i. e., when no conic in the pencil through $p_1, \ldots, p_4$ is tangent to C_5. As the same topological considerations apply if we replace C_5 by a homologous line (which we call the line at infinity), we have established the criterion:

$$l(g)=0 \quad \Leftrightarrow \quad \text{the pencil of conics through } p_1, p_2, p_3, p_4 \text{ contains no parabolas.}$$

But a pencil of conics through 4 points in the finite real plane will consist entirely of hyperbolas if and only if cone of the four points lies inside the triangle with vertices at the remaining three. In general, the three lines $\overline{p_1 p_2}$, $\overline{p_1 p_3}$, and $\overline{p_2 p_3}$ divide $\mathbb{P}^2(\mathbb{R})$ into 4 regions, of which exactly 3 are not met by the pseudo-line C_5. We say C_4 is *surrounded* by C_1, C_2 and C_3 if C_4 lies in the remaining region of $\mathbb{P}^2(\mathbb{R})$. Since no line meets 3 of the components $C_1, \ldots, C_4$ this condition does *not* depend on the choice of $p_i \in C_i$, and we have proved:

LEMMA 7.4. — *If X is a real plane quintic with $n(X)=5$, then $a(X)=1$ if and only if none of the ovals $C_1, \ldots, C_4$ is surrounded by the other three.*

We point out that both $a(X)=0$ and $a(X)=1$ actually occur among quintics with 5 components. Start with a plane quartic Y with four components, which may be obtained by deforming slightly the sum E.F of two ellipses meeting in 4 points. Now add a line L with $L(\mathbb{R})$ disjoint from $Y(\mathbb{R})$; if we vary the coefficients of the singular quintic Y.L slightly we obtain a smooth curve X with 5 components, where C_5 corresponds to L. If L is positioned as L_0 in the picture, then C_1, C_2 and C_3 will surround C_4 and $a(X)=0$. If L is positioned as L_1 then no bounded component surrounds another and $a(X)=1$.



To summarize, the list of possible configurations for plane quintics is as follows:

$n(X)$	position of ovals	$a(X)$	card $S(\mathbb{R})_{odd}$
1.	–	1	32
2.	–	1	64
3.	nested	0	96
3.	non- nested	1	128
		(0)	(96)??
4.	–	1	256
5.	surrounded oval	0	480
5.	no surrounded oval	1	512
6.	–	1	1024
7.	–	0	2016

Note. – What are the restrictions on the pair $(n(X), a(X))$ when the curve X has a non-singular real plane model of degree d? For example, can $X(\mathbb{R})$ disconnect $X(\mathbb{C})$ when $n(X)<[(d+1)/2]$? We *can* construct examples which disconnect when $n(X)=[(d+1)/2]$: when d is even take a small deformation of $n(X)$ concentric circles in the finite plane, when d is odd take a deformation of $n(X)-1$ concentric circles and the line at infinity.

8. Real trigonal curves

A non-hyperelliptic complex curve X of genus $g \geqq 4$ is *trigonal* if there is a divisor class $\mathbf{a} \in \mathrm{Pic}^3_X(\mathbb{C})$ with $h^0(\mathbf{a})=2$. The map $X \underset{\pi}{\rightarrow} \mathbb{P}(H^0(X, L(\mathbf{a})))$ then exhibits X as a 3-sheeted cover of $\mathbb{P}^1$. When $g \geqq 5$ the two properties $\deg(\mathbf{a})=3$ and $h^0(\mathbf{a})=2$ uniquely determine the trigonal class; when $g=4$ a non-hyperelliptic curve will have either 1 or 2 trigonal classes.

The intersection of all quadrics in $\mathbb{P}^{g-1}$ containing the canonical model of a trigonal curve X is a rational normal scroll Y. Each line of Y meets X in exactly 3 points: the rulings cut out the trigonal series. As an abstract complex variety, Y is a $\mathbb{P}^1$-bundle $\mathbb{F}_k=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(k)) \rightarrow \mathbb{P}^1$. The integer k satisfies the congruence $k \equiv g \pmod 2$; for a generic trigonal curve $k \leqq 1$ [3].

Now assume that X is trigonal and defined over $\mathbb{R}$. When $g \geqq 5$ the class $\mathbf{a}$ is unique, so $\mathbf{a} \in \mathrm{Pic}^3(\mathbb{R})$. Similarly, the scroll Y is always real; when $g \geqq 5$ its unique ruling is also real. We have the following possiblities.

If $Y(\mathbb{R}) = \emptyset$ then X has no real points. Since $\deg(\mathbf{a}) \equiv 1 \pmod 2$ the class $\mathbf{a}$ cannot lie in $\mathrm{Pic}^3(\mathbb{R})^+$, so $g \equiv 0 \pmod 2$. Furthermore, $Y \simeq \mathbb{F}_k$ over $\mathbb{C}$ with $k \equiv g+2 \pmod 4$. We can construct pointless trigonal curves of any even genus $g \geqq 4$ as follows. If $g = 2(2m-1)$ take a curve of type $(3, 2m)$ on $Y = \mathbb{P}^1 \times N$, where $N = \{u^2 + v^2 = -1\}$. If $g = 4m$ take three times the section at ∞ in the $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O}_N \oplus \mathcal{O}_N(2)) \to N$ plus $(2m+4)$ fibres.

Now assume $Y(\mathbb{R}) \neq \emptyset$ and $g \geqq 5$. Since every real line of Y meets $X(\mathbb{C})$ in 3 points, at least one of which is real, X *also* has real points. Furthermore, Y is a *real* $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, so $Y(\mathbb{R})$ is either a torus (g even) or a Klein bottle (g odd). Hence we have established.

PROPOSITION 8.1. — *Let X be a real trigonal curve of genus $g \geqq 5$ which lies on the rational normal scroll $Y \subseteq \mathbb{P}^{g-1}$. Then either:*

(1) $Y(\mathbb{R}) = \emptyset$:

$$g \equiv 0 \pmod 2,$$
$$n(X) = 0,$$
$$Y(\mathbb{C}) \simeq \mathbb{F}_k(\mathbb{C}) \quad \text{with} \quad k \equiv g+2 \pmod 4.$$

(2) $Y(\mathbb{R}) \simeq S^1 \times S^1$:

$$g \equiv 0 \pmod 2,$$
$$n(X) > 0,$$
$$Y(\mathbb{C}) \simeq \mathbb{F}_k(\mathbb{C}) \quad \text{with} \quad k \equiv 0 \pmod 2.$$

(3) $Y(\mathbb{R}) \simeq$ Klein bottle :

$$g \equiv 1 \quad (mod\ 2),$$
$$n(X) > 0,$$
$$Y(\mathbb{C}) \simeq \mathbb{F}_k(\mathbb{C}) \quad \text{with} \quad k \equiv 1 \pmod 2.$$

When X has genus 4, its canonical model lies on a unique real quadric $Y \subset \mathbb{P}^3$. If Y is singular its ruling is unique and hence real. If Y is non-singular (which is the generic case) it has two rulings. If these rulings are both real, $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and $Y(\mathbb{R})$ is a hyperboloid; if they are switched by complex conjugation, $Y \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{P}^1)$ and $Y(\mathbb{R})$ is a sphere. By studying degenerations of curves on Y and on $\mathbb{P}^2$ (where X may be viewed as a plane quintic with 2 nodes), we can show that all topological pairs $(n(X), a(X))$ occur for trigonal curves X of genus 4 on the hyperboloid, and all pairs with the possible exception of $n(X) = 1$, $a(X) = 0$ occur for curves on the sphere. As in the case of plane curves, the invariant $a(X)$ is often determined by the configuration of the locus $X(\mathbb{R}) \subseteq Y(\mathbb{R})$. For example, if $n(X) = 3$ and $Y(\mathbb{R})$ is a sphere, we find:

$a(X) = 1 \Leftrightarrow$ every 2 components of $X(\mathbb{R})$ lie in the same connected component of the complement of the third in $Y(\mathbb{R})$.

$a(X) = 0 \Leftrightarrow$ one component of $X(\mathbb{R})$ separates the remaining two on $Y(\mathbb{R})$.

The proof uses (5.3) and the Lefschetz fixed-point formula.

We now consider trigonal curves X of genus $g \geqq 5$ with $n(X) > 0$ in more detail. Since the trigonal class **a** lies in $\mathrm{Pic}^3(\mathbb{R})^+$, it corresponds to a real cover $X \underset{\pi}{\rightarrow} \mathbb{P}^1$ of degree 3. Hence $X(\mathbb{R})$ has an *odd* number of components having *odd* degree over $\mathbb{P}^1(\mathbb{R})$. We can separate into the following four cases:

(1) $X(\mathbb{R})$ has 1 component of degree 3 over $\mathbb{P}^1(\mathbb{R})$.
(2) $X(\mathbb{R})$ has 3 components of degree 1 over $\mathbb{P}^1(\mathbb{R})$.
(3) $X(\mathbb{R})$ has 1 component of degree 1; 1 component of degree 2 over $\mathbb{P}^1(\mathbb{R})$.
(4) $X(\mathbb{R})$ has 1 component C_1 of degree 1; $n(X)-1$ components of degree 0 over $\mathbb{P}^1(\mathbb{R})$.

In the first three cases $X(\mathbb{R}) = \pi^{-1}(\mathbb{P}^1(\mathbb{R})) \subset X(\mathbb{C})$ and $a(X) = 0$. Hence g must be even in cases (1) and (2), and g must be odd in Case (3). By degenerating curves on Y, we can prove these cases indeed occur.

We know much less about the general Case (4). For example, are all values $1 \leqq n(X) \leqq g+1$ possible? Even for trigonal curves of a fixed topological type, there is a *further* discrete invariant to consider: the class of the distinguished component C_1 of $X(\mathbb{R})$ in the homology of the real scroll $Y(\mathbb{R})$. When g is even, we may find a basis of $H_1(Y(\mathbb{R}), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ by taking cycles corresponding to the ruling of Y and a hyperplane section. The class of C_1 then has type $(1, \alpha)$. To see that α may vary, start with a trigonal curve X' of genus $g-2$ on $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Add a fibre to X' and deform to a smooth trigonal curve X of genus g. Then $\alpha(X) = \alpha(Y) + 1$ or $\alpha(X) = \alpha(Y) - 1$, depending on the direction of deformation. Hence the moduli of trigonal curves with fixed g, $n(X)$, and $a(X)$ need *not* be connected (compare 9).

9 Real moduli

Let J be a principally polarized abelian variety of dimension g over $\mathbb{R}$. The polarization induces a unimodular alternating pairing on $T(J) = H_1(J(\mathbb{C}), \mathbb{Z})$:

$$E: \quad T(J) \times T(J) \rightarrow \mathbb{Z}(1),$$

which is equivariant for the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The fixed space $T(J)^+$ has rank g and is maximal isotropic for E; furthermore the pairing:

$$(9.1) \qquad \left\{ \begin{array}{l} T(J)^+ \otimes \mathbb{R} \times H^0(J, \Omega^1/\mathbb{R}) \rightarrow \mathbb{R}, \\ \qquad (m, \omega) \mapsto \int_m \omega \end{array} \right.$$

is non-degenerate.

Fix a basis $\langle m_1, \ldots, m_g \rangle$ for $T(J)^+$ and extend this to a symplectic basis $\langle m_1, \ldots, m_g; n_1, \ldots, n_g \rangle$ for $T(J)$. With respect to this basis, the matrix for complex conjugation on $T(J)$ has the form:

$$(9.2) \qquad \tau = \begin{pmatrix} I_g & H \\ 0 & -I_g \end{pmatrix} \qquad \text{where} \quad H^{\tau} = H.$$

If we change basis by a symplectic automorphism α which fixes $T(J)^+$ we find:

$$\alpha\tau\alpha^{-1}=\begin{pmatrix} I & H+2B \\ 0 & -I \end{pmatrix} \quad \text{if} \quad \alpha=\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}, \qquad B^\tau=B,$$

$$\alpha\tau\alpha^{-1}=\begin{pmatrix} I & AHA^t \\ 0 & -I \end{pmatrix} \quad \text{if} \quad \alpha=\begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}.$$

Hence, without loss of generality, we may assume that τ has the form (9.2) where H is a *fixed lift* to $\mathbb{Z}$ of one of the inequivalent symmetric bilinear forms over $\mathbb{Z}/2$ described on Page 14. The basis $\langle m_1, \ldots, m_g; n_1, \ldots, n_g \rangle$ is the then uniquely determined up to the action of the group:

$$\Gamma_H=\{A \in GL(g, \mathbb{Z}) : AHA^t=H\} \quad Sp(2g, \mathbb{Z}),$$

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix}.$$

Now let $\langle \omega_1, \ldots, \omega_g \rangle$ be the basis of $H^0(A, \Omega^1/\mathbb{R})$ which is dual to $\langle m_1, \ldots, m_g \rangle$ under the pairing (9.1). The complex $g \times g$ matrix:

$$Z=\left(\left(\int_{n_i} \omega_j\right)\right)=\frac{1}{2}H+iY,$$

is symmetric, with imaginary part Y *positive definite*. A different choice of basis for T(J) gives the matrix:

$$Z'=\frac{1}{2}H+iY' \qquad \text{where} \quad Y'=AYA^t, \qquad A \in \Gamma_H.$$

Hence J determines a point in the quotient of the cone $\mathscr{X}_H=\{(1/2)H+iY : Y \quad 0\}$ in Siegel upper half-space by the discrete subgroup Γ_H in $Sp(2g, \mathbb{Z})$. Conversely, using Weil's criterion [17], it is easy to show that any point in the cone $\mathscr{X}_H$ represents the period matrix of a principally polarized abelian variety *over* $\mathbb{R}$. Fixing lifts H_i of the inequivalent symmetric bilinear forms on a vector space of dimension g over $\mathbb{Z}/2$, we obtain the following (compare Shimura [14]).

PROPOSITION 9.3. — *The set of isomorphism classes of principally polarized abelian varieties of dimension g over $\mathbb{R}$ is in one-to-one correspondence with the points of the real-analytic space of dimension g over $\mathbb{R}$ is in one-to-one correspondence with the points of the real-analytic space* $\mathscr{A}_g=\bigcup_i \mathscr{X}_{H_i}/\Gamma_{H_i}$.

Note. — The number of components of $\mathscr{A}_g$ is $(3g+1)/2$ if g is odd and $(3g+2)/2$ if g is even; each component has real dimension $g(g+1)/2$, and two varieties J and J′ lie in the same component if $H_1(J(\mathbb{C}), \mathbb{Z}/2) \simeq H_1(J'(\mathbb{C}), \mathbb{Z}/2)$ as symplectic spaces with involutions. The individual components are themselves quite interesting. If $n(J)=2^g$ we may take $H=0$ and $\Gamma_H=GL(g, \mathbb{Z})$; the moduli of such varieties is the classical space of lattices:

$$\mathscr{X}_H/\Gamma_H=O(g)\backslash GL(g, \mathbb{R})/GL(g, \mathbb{Z}).$$

At the other extreme, if $n(A)=1$ and $H=I_g$ the group $\Gamma_H$nis finite of order $2^g(g!)$.

The space $\mathscr{A}_g$ has a natural involution α which may be described as follows. The real forms of a principally polarized abelian variety J over $\mathbb{R}$ are classified by elements of the set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}_{\mathbb{C}}(J))$ [17]. The assignment $\tau \mapsto -1_J$ is a one-cocycle on $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ with values in $\mathrm{Aut}_{\mathbb{R}}(J)$, we denote the corresponding real form of J by J^*. Then $J^*(\mathbb{R}) = \{p \in J(\mathbb{C}) : p^\tau = -p\}$, and J^* is $\mathbb{R}$-isomorphic to J iff $\varphi^\tau = -\varphi$ for some $\varphi \in \mathrm{Aut}_{\mathbb{C}}(J)$. Generically $\mathrm{Aut}_{\mathbb{C}}(J) = \langle \pm 1 \rangle$, so J and J^* represent the two distinct real forms of J.

The map $J \to J^*$ induces an involution α of $\mathscr{A}_g$, and even an involution of each component $\mathscr{X}_H/\Gamma_H$. To see that J and J^* lie in the same component, notice that $\tau^* = -\tau$, so both complex conjugations have the same action on $H_1(J(\mathbb{C}), \mathbb{Z}/2)$. For example, when $n(J) = 2^g$ we may take $H = 0$, and the involution α on $\mathscr{X}_H$ is given by $i Y \mapsto i Y^{-1}$.

We now turn to the moduli $\mathscr{M}_g$ of real curves of genus g. To a curve X we associate the abelian variety $J = \mathrm{Pic}^0 X$ which is principally polarized by a translate of the theta divisor $\theta \subset \mathrm{Pic}^{g-1} X$. This gives a map on the level of real moduli.

$$t: \quad \mathscr{M}_g \to \mathscr{A}_g$$
$$X \mapsto J.$$

The space $\mathscr{M}_0$ consists of 2 points which correspond to the curves $\mathbb{P}^1$ and $N = \{u^2 + v^2 = -1\}$. Both map to the single point of $\mathscr{A}_0$. The space $\mathscr{M}_1$ has 3 components, which correspond to curves with $n(X) = 0, 1, 2$. Each component is analytically isomorphic to an open interval, and the latter two make up the moduli $\mathscr{A}_1$ of elliptic curves. (For an analytic description of $\mathscr{A}_1$ using Jacobi's parametrization *see* [4].) The map t identifies the curves in the component where $n(X) = 0$ with those in the component where $n(X) = 2$.

When the genus of X is greater than 1, however, we have a real analog of the Torelli Theorem.

THEOREM 9.4. – *If $g \geqq 2$ the map $t: \mathscr{M}_g \to \mathscr{A}_g$ is one-to-one.*

Proof. – By Torelli's Theorem, J determines X over $\mathbb{C}$. For $g \geqq 2$ the natural map:

$$f: \quad \mathrm{Aut}_{\mathbb{C}}(X) \to \mathrm{Aut}_{\mathbb{C}}(J), \tag{9.5}$$

is an *injection*. The image of f is all of $\mathrm{Aut}_{\mathbb{C}}(J)$ when X is hyperelliptic; otherwise $\mathrm{Aut}_{\mathbb{C}}(J) = \langle \pm 1 \rangle \times f(\mathrm{Aut}_{\mathbb{C}}(X))$ [10]. In any case, the map:

$$f_*: \quad H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}_{\mathbb{C}}(X)) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), \mathrm{Aut}_{\mathbb{C}}(J)),$$

is *injective*. Hence two distinct real forms X and X′ of a fixed curve will have non-isomorphic Jacobians. This shows t is one-to-one.

What can we say about the image of t when $g \geqq 2$? Since any inclusion $X \to J$ induces a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ isomorphism $H_1(X(\mathbb{C}), \mathbb{Z}/2) \simeq H_1(J(\mathbb{C}), \mathbb{Z}/2)$, all curves of a fixed topological type $(n(X), a(X))$ go into a single component $\mathscr{X}_H/\Gamma_H$. Seppälä [12] (*see* also Klein [7]) has shown that the moduli of real curves of a fixed topological type form a *connected* real analytic space of dimension $3g-3$. Hence each component or $\mathscr{A}_g$ contains a single component of Jacobians, with the exception of the component where:

$$H=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

which contains the additional component corresponding to curves with $n(X)=0$.

Note that the image $t(\mathscr{M}_g)$ is *not* stable under the natural involution α of $\mathscr{A}_g$ once $g \geqq 3$. α does preserve the Jacobians of hyper-elliptic curves: if X is given by the equation $y^2=f(x)$ [or $y^2=f(u, v)$] then X^* is given by the equation $y^2=-f(x)$ [or $y^2=-f(u, v)$].

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