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## ERGODICITY OF TORAL LINKED TWIST MAPPINGS

By Feliks PRZYTYCKI <sup>(1)</sup>

### 0. Introduction, notations

Let  $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$  be the standard torus and let  $P, Q$  be closed annuli in  $T^2$  defined by:

$$P = \{(x, y) \in T^2 : y_0 \leq y \leq y_1, x \text{ arbitrary}\}, \quad |y_1 - y_0| \leq 1,$$
$$Q = \{(x, y) \in T^2 : x_0 \leq x \leq x_1, y \text{ arbitrary}\}, \quad |x_1 - x_0| \leq 1\}.$$

Let  $f : [y_0, y_1] \rightarrow \mathbb{R}$ ,  $g : [x_0, x_1] \rightarrow \mathbb{R}$  be  $C^2$ -functions such that  $f(y_0) = g(x_0) = 0, f(y_1) = k,$   
 $g(x_1) = l$ , for some integers  $k$  and  $l$ .

Define  $F = F_f : P \cup Q \rightarrow P \cup Q$  and  $G = G_g : P \cup Q \rightarrow P \cup Q$  by:

$$F_f(x, y) = (x + f(y), y), \quad G_g(x, y) = (x, y + g(x)),$$

on  $P$  and  $Q$  respectively and  $F_f|_{Q \setminus P} = \text{id}, G_g|_{P \setminus Q} = \text{id}$ .

Define  $H = H_{f, g} = G_g \circ F_f$ .

Observe that  $F, G$  and  $H$  preserve the Lebesgue measure  $\mu$  on  $P \cup Q$ .

In the paper we shall consider  $f$  and  $g$  such that:

$$\frac{df}{dy}(y) \neq 0 \quad \text{and} \quad \frac{dg}{dx}(x) \neq 0$$

for every  $y \in [y_0, y_1]$  and  $x \in [x_0, x_1]$ .

If  $k > 0$  ( $k < 0$ ), what implies  $df/dy > 0$  ( $< 0$ ), define the slope of  $F$  by  $\alpha = \inf \{ df/dy(y) : y \in [y_0, y_1] \}$  (sup). Analogously define the slope  $\beta$  of  $G$ .

We call then  $F|_P$  a  $(k, \alpha)$ -twist,  $G|_Q$  an  $(l, \beta)$ -twist and  $H$  a toral linked twist mapping, t. l. t. m.

In the presented paper we prove the following.

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PROPOSITION. — *If a t. l. t. m.  $H$  is composed from  $(k, \alpha)$ - and  $(l, \beta)$ -twists, where  $k$  and  $l$  have opposite signs,  $|k|, |l| \geq 2$  and  $|\alpha \cdot \beta| \geq C_0 \approx 17,244\,45$ , then  $H$  and all its powers are ergodic. In fact  $H$  satisfies Bernoulli property.*

Together with facts known before we obtain:

THEOREM. — *Let  $H$  be a t. l. t. m. composed from  $(k, \alpha)$ - and  $(l, \beta)$ -twists.*

*If  $\alpha\beta > 0$  (i. e.  $kl > 0$ ), then  $H$  is Bernoulli.*

*If  $\alpha\beta < -4$ , then  $H$  is almost hyperbolic.*

*If  $\alpha\beta < -C_0 \approx 17,244\,45$  and  $|k|, |l| \geq 2$ , then  $H$  is Bernoulli.*

We add to the paper Appendix where we explain what we mean by “almost hyperbolic” and gather some facts from Pesin theory for mappings with singularities, which make background for this paper.

T. l. t. m. were introduced by Easton [4]. It seems however that the basic phenomena were observed earlier by Oseledec, *see* [5], Chap. 3.8.

The case  $kl > 0$  has been done by Burton and Easton in [2] and by Wojtkowski in [7].

Almost hyperbolicity when  $\alpha\beta < -4$  follows from Wojtkowski paper [7].

If  $kl > 0$ , global stable and unstable manifolds intersect each other since they are very long and go, roughly speaking, in different directions. This gives ergodicity.

If  $\alpha\beta < -4$ , global stable and unstable manifolds, although internally very long could have a very small diameter in  $P \cup Q$ . On Figure 1 we show what could happen with subsequent images of a local unstable manifold  $\gamma$  under iterations by  $H$ .

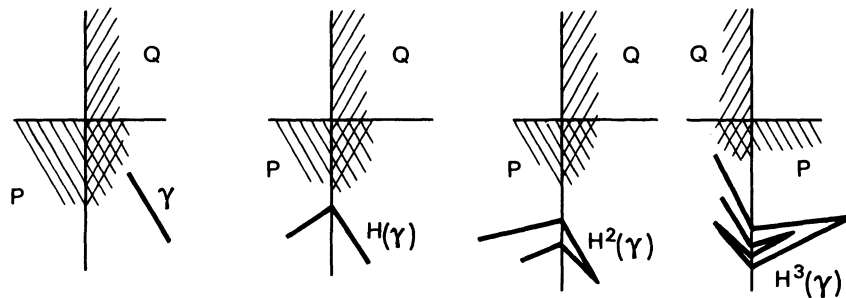


Fig. 1.

We prove however that if  $\alpha\beta < -C_0$ , this is not so, that piecewise differentiable global stable and unstable manifolds contain pieces winding along the whole annulus  $P(Q)$ . (A similar phenomenon appears in an example studied by Wojtkowski in [8].), § 1, will be devoted to this aim.

In section 2 we briefly list some generalizations of Theorem for larger classes of linked twist mapping. The reader can find details in [11].

1. Proof of proposition

To simplify notation assume that the functions  $f$  and  $g$  are linear.

We may assume  $|\alpha| = |\beta|$ . Otherwise we may change the coordinates on  $T^2$ , taking the coordinates  $(x, \sqrt{|\alpha/\beta|} \cdot y)$ . Then we consider the torus  $\mathbb{R}^2/\mathbb{Z} \times \sqrt{|\alpha/\beta|} \cdot \mathbb{Z}$ . We may assume that  $\alpha > 0$ . Let us repeat according to [2] and [14] the proof of almost hyperbolicity of  $H$ . The matrix:

$$DG \circ DF = \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ -\alpha & -\alpha^2 + 1 \end{pmatrix} \quad \text{for } \alpha > 2,$$

is hyperbolic. Its eigenvalues are:

$$\lambda_{\pm} = \frac{-\alpha^2 + 2 \pm \sqrt{\alpha^4 - 4\alpha^2}}{2}$$

and the expanding eigenvector  $(\xi_1, \xi_2)$  satisfies:

$$\xi_1/\xi_2 = -\left(\frac{\alpha}{2}\right) + \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1}.$$

Let us denote this number by  $L$ .

In  $\mathbb{R}^2$  let us take the cone  $C = \{(x, y) : L \leq x/y \leq 0\}$ .

It is easy to see that for every positive integers  $s, r$ ,  $DG^s \circ DF^r(C) \subset C$  and for every  $v \in C$ ,  $\|D(G^s \circ F^r)(v)\| \geq \lambda \|v\|$  where  $\lambda = |\lambda_-| > 1$ .

Denote  $P \cap Q = S$  and denote the first return mapping, to  $S$ , under  $F, G, H$ , by  $F_s, G_s, H_s$  respectively. It is easy to see that  $H_s = G_s \circ F_s$ .

For almost every  $x \in S$  there exists  $D(H_s)$  and  $D(H_s^{-1})$  and:

$$\int_s \log^+ \|DH_s^{-1}(x)\| d\mu(x) \leq \int_{P \cup Q} \log^+ \|DH^{-1}(x)\| d\mu(x) < +\infty.$$

Hence for all vectors tangent to  $s$  at  $x$  there exist Lyapunov exponents for  $DH_s$ . One of them, corresponding to the vectors from  $C$  is positive (not less than  $\log \lambda$ ), the other one is negative.

For almost every  $x \in S$  its  $H$ -orbit returns to  $S$  with positive frequency (this is a corollary from Birkhoff Ergodic Theorem, see [2], Lemma 4.4). Moreover  $\mu\left((P \cup Q) \setminus \left(\bigcup_{n=-\infty}^{+\infty} H^n(S)\right)\right) = 0$ . So Lyapunov exponents for  $H$  itself are nonzero almost everywhere and in view of Pesin Theory (see Appendix) for almost every  $x \in P \cup Q$  there exist local stable and unstable manifolds  $\gamma^s(x), \gamma^u(x)$  [Under the linearity assumption  $\gamma^{s(u)}(x)$ , are linear segments.]

To prove, ergodicity of  $H$  and all its powers it is enough to show that for almost every  $x, y \in S$ ,  $H^m(\gamma^u(x))$  intersects  $H^{-n}(\gamma^s(y))$  for all  $m, n$  large enough.

For any segment  $\gamma$  we denote by  $l_h(\gamma)$  and  $l_v(\gamma)$  the lengths of the orthogonal projections of  $\gamma$  to the horizontal, respect. vertical axes. We shall prove that for any linear segment  $\gamma \subset H_s^r(\gamma^u(x))$ , the image  $F_s(\gamma)$ , which is a union of linear segments, contains a segment  $\gamma'$  such that either  $l_h(\gamma') > \delta \cdot l_v(\gamma)$ , for a constant  $\delta > 1$  independent of  $\gamma$ , or  $\gamma'$  joins the left and right sides of  $S$ . In the latter case, due to  $|l| \geq 2$ ,  $G(\gamma')$  contains a segment joining the upper and lower sides of  $S$  (Fig. 2). (We shall call any segment in  $S$  joining the upper and lower sides of  $S$  a  $v$ -segment, and joining the left and right sides of  $S$  and  $h$ -segment.)

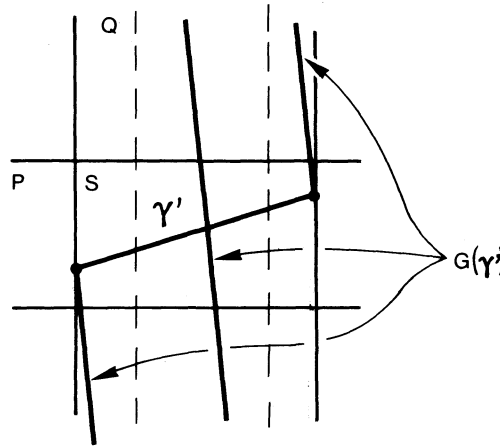


Fig. 2.

In the first case we act on  $\gamma'$  with  $G_s$  and find a segment  $\gamma'' \subset G_s(\gamma')$  such that either  $l_v(\gamma'') \geq \delta \cdot l_h(\gamma')$  or  $\gamma''$  is a  $v$ -segment. In the first case we continue the process.

We get a sequence of segments of exponentially growing length. So it must finish with a  $v$ -segment or an  $h$ -segment. Then, if we continue iterating with  $F$  and  $G$  alternately we find an  $h$ -segment or  $v$ -segment alternately at each step (since  $|k|, |l| \geq 2$ ). The same happens for all sufficiently high iterations of  $H^{-1}$  on  $\gamma^s(y)$ . Concluding:  $H^m(\gamma^u(x))$  contain  $v$ -segments and  $H^{-n}$  contain  $h$ -segments for all  $m, n$  sufficiently large. But the  $h$ -segments intersect  $v$ -segments.

So let us fix a segment  $\gamma \subset H_s^r(\gamma^u(x))$ . Let  $m_1 > 0$  be the first time when  $F^{m_1}(\gamma)$  intersects  $S$ . Then we have four possibilities:

- 1)  $F^{m_1}(\gamma)$  contains an  $h$ -segment. This case has just been discussed.
- 2) The right side  $F^{m_1}(\gamma)$  intersects  $S$  (Fig. 3).
- 3) The left side of  $F^{m_1}(\gamma)$  intersects  $S$  [this case is fully analogous to the case 2)].
- 4) Both sides of  $F^{m_1}(\gamma)$  intersect  $S$  (Fig. 4).

We study the case 2) and make the *assumption* ( $\star$ ) that  $F_s(\gamma)$  does not contain any  $h$ -segment. We divide  $F^{m_1}(\gamma) \setminus S$  into three intervals  $I_1, I_2, I_3$ . Let us denote  $F^{m_1}(\gamma) \cap S = I_4$ .

Along  $I_2$  the rotation number  $f(y)$  changes by  $\alpha \cdot l_v(I_2)$ . So, for any integer  $n > 0$  such that  $1/n < \alpha \cdot l_v(I_2)$  there exists a horizontal circle in  $P$ , intersecting  $I_2$  on which the rotation

number is  $m/n$  for an integer  $m$ . So there exists an  $F$ -periodic point  $p \in I_2$  with the period  $[(1/\alpha \cdot l_v(I_2)) + 1]$ . It divides  $I_2$  into  $I'_2$  and  $I''_2$  (Fig. 3). The distance  $d$  between the different points of the  $F$ -orbit of  $p$ ,  $\text{Orb}_F(p)$ , is at least:

$$\frac{1}{[(1/\alpha \cdot l_v(I_2)) + 1]}$$

Let us denote the last (to the right) point of  $\text{Orb}_F(p)$  in  $S$  by  $p_1$  and the next one (to the right) by  $p_2$  (Fig. 3). Let  $m_2 > 0$  be the first time when  $F^{m_2}(p)$  is between  $p$  and  $RS$  — the right side of  $S$ . [Including  $p$  and  $p_1$ . Observe however that  $F^{m_2}(p) \neq p$ . Otherwise  $p$  would have its  $F$ -orbit disjoint with  $S$ . But  $F^{-m_1}(p) \in \gamma \subset S$ , a contradiction].

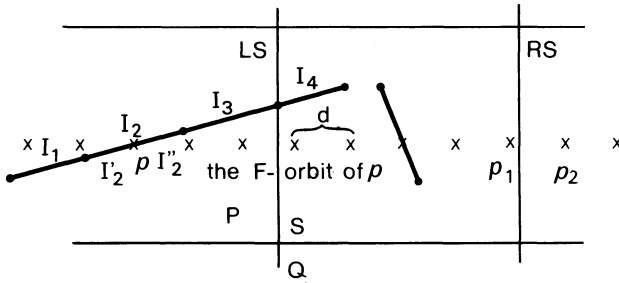


Fig. 3.

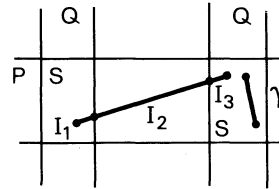


Fig. 4.

We denote  $\mathcal{J}_0 = I'_2 \cup I_3$  and  $\mathcal{J}_m = F(\mathcal{J}_{m-1} \setminus S)$  for  $m = 1, 2, \dots, m_2$ .

Then:

$$l_h(\mathcal{J}_m) \geq \min(d + l_h(I'_2 \cup I_3), l_h(I'_2 \cup I_3) + \alpha \cdot l_v(I'_2 \cup I_3))$$

for  $m = 1, 2, \dots, m_2$ .

If  $F^{m_2}(p)$  is between  $p$  and  $LS$  (the left side of  $S$ ), then  $l_h(\mathcal{J}_{m_2} \cap S) \geq d$ . If  $F^{m_2}(p) \in S \setminus \{p_1\}$ , then:

$$(2) \quad l_h(\mathcal{J}_{m_2} \cap S) \geq \min(d, l_h(I'_2 \cup I_3) + \alpha \cdot l_v(I'_2 \cup I_3)).$$

Assume  $F^{m_2}(p) = p_1$ . Let  $\text{dist}(p_1, RS) = \tau \cdot d$ .

Then  $\text{dist}(RS, p_2) = (1 - \tau) \cdot d$ . We have either (2) satisfied or  $l_h(\mathcal{J}_{m_2} \cap S) = \tau \cdot d$  and  $\mathcal{J}_{m_2} \cap S$  touches  $RS$  with its right end.

Define  $\tilde{\mathcal{J}}_0 = I_1 \cup I'_2$  and  $\tilde{\mathcal{J}}_m = F(\tilde{\mathcal{J}}_{m-1} \setminus S)$  for  $m = 1, \dots, m_2$ : We have either:

$$l_h(\tilde{\mathcal{J}}_{m_2} \cap S) \geq \min((1 - \tau) d, \alpha \cdot l_v(I_1 \cup I'_2) + l_h(I_1 \cup I'_2))$$

or  $\tilde{\mathcal{J}}_{m_2} \cap S$  touches  $LS$  with its left end [the number  $(1 - \tau) d$  appears because there can exist  $m : 0 < m < m_2$  for which  $F^m(p) = p_2$ ]. So due to assumption ( $\star$ ):

$$l_h(\mathcal{J}_{m_2} \cup \tilde{\mathcal{J}}_{m_2}) \cap S \geq \min(d, \alpha \cdot l_v(I_3) + l_h(I_3)), \\ \alpha \cdot l_v(I_1) + l_h(I_1)).$$

Thus in order to have  $l_h((\mathcal{C}_{m_2} \cup \mathcal{F}_{m_2}) \cap S) \geq \delta \cdot l_v(\gamma)$  it is enough that the following inequalities hold:

$$(3) \quad d \geq \delta \cdot l_v(\gamma),$$

$$(4) \quad \alpha \cdot l_v(I_3) + l_h(I_3) \geq \delta \cdot l_v(\gamma),$$

$$(5) \quad \alpha \cdot l_v(I_1) + l_h(I_1) \geq \delta \cdot l_v(\gamma).$$

For (3) it is enough that:

$$\frac{\alpha \cdot l_v(I_2)}{1 + \alpha \cdot l_v(I_2)} \geq \delta \cdot l_v(\gamma)$$

or:

$$(6) \quad l_v(I_2) \geq \frac{\delta \cdot l_v(\gamma)}{\alpha(1 - \delta \cdot l_v(\gamma))}.$$

We can assume here  $1 - \delta \cdot l_v(\gamma) > 0$  since the assumption  $(\star)$  implies:

$$(7) \quad l_v(\gamma) \cdot (L + \alpha) < 2$$

and  $L + \alpha \geq -1 + \sqrt{17} > 3$ . We can replace (4), (5) by

(8) [and (9)]

$$l_v(I_{3(1)}) \geq \frac{\delta \cdot l_v(\gamma)}{L + 2\alpha}.$$

(due to the fact that  $\gamma \in C$  — the cone  $L \leq x/y \leq 0$ ). The work would be also done if:

$$l_h(I_4) \geq \delta \cdot l_v(\gamma)$$

which follows from:

$$(10) \quad l_v(I_4) \geq \frac{\delta \cdot l_v(\gamma)}{L + \alpha}.$$

There exists  $\delta > 1$  and a partition of  $F^{m_1}(\gamma) \setminus S$  into  $I_1, I_2, I_3$  satisfying (6), (8), (9), or the inequality (10) is satisfied if:

$$(11) \quad l_v(\gamma) = \sum_{i=1}^4 l_4(I_i) > l_v(\gamma) \left( \frac{1}{\alpha(1 - l_v(\gamma))} + \frac{2}{2\alpha + L} + \frac{1}{\alpha + L} \right).$$

We divide both sides by  $l_v(\gamma)$  and due to (7) we obtain the condition:

$$(12) \quad 1 > \frac{1}{\alpha(1 - (2/(L + \alpha)))} + \frac{2}{2\alpha + L} + \frac{1}{\alpha + L}$$

[recall that  $L = -(\alpha/2) + \sqrt{(\alpha/2)^2 - 1}$ ].

In the case 4) the situation is simpler.  $F^{m_1}(\gamma)$  divides into  $I_1, I_2, I_3$  as on Figure 4. We need either  $l_h(I_1) \geq \delta \cdot l_v(\gamma)$ , or  $l_h(I_3) \geq \delta \cdot l_v(\gamma)$ , or  $l_h F(I_2) - l_h(I_2) \geq \delta \cdot l_v(\gamma)$ . [The sufficiency

of the last inequality follows from the following: Lift everything to  $\mathbb{R}^2$ , denote two consecutive components of the lift of  $Q$  by  $Q_1$  and  $Q_2$ . Then, assumed  $(\star)$ ,  $F(I_2)$  has a component  $\tilde{I}$  of its lift between left sides of  $Q_1$  and  $Q_2$  or between right sides of  $Q_1$  and  $Q_2$ . Otherwise  $\tilde{I}$  would intersect a component of the lift of  $I_2$ , which would imply the existence of an  $F$ -fixed point  $q \in S$ . But  $F^{-m_1}(q) \in \gamma \subset S$  — a contradiction].

For this it suffices that:

$$(L + \alpha) \cdot l_v(I_1) \geq \delta \cdot l_v(\gamma),$$

or:

$$(L + \alpha) \cdot l_v(I_3) \geq \delta \cdot l_v(\gamma),$$

or:

$$\alpha \cdot l_v(I_2) \geq \delta \cdot l_v(\gamma).$$

For that it is enough if:

$$l_v(\gamma) = \sum_{i=1}^3 l_v(I_i) \geq l_v(\gamma) \left( \frac{2}{L + \alpha} + \frac{1}{\alpha} \right)$$

i. e. :

$$(13) \quad 1 > \frac{2}{L + \alpha} + \frac{1}{\alpha}$$

$$(12) \quad \text{is satisfied for } \alpha > \alpha_0 \approx 4,152\,643;$$

$$(13) \quad \text{is satisfied for } \alpha > \alpha_1 \approx 3,239.$$

This gives the constant  $C_0 = \alpha_0^2 \approx 17,244\,45$  in the statement of Proposition.

*Remark.* — If  $l_v(\gamma)$  is small, then in (13) we can write  $(L + m_1 \alpha)$  instead of  $L + \alpha$ , where  $m_1$  is large. So (13) can be replaced by:

$$1 > \frac{1}{\alpha}.$$

Also (12) can be replaced by:

$$(14) \quad 1 > \frac{1}{\alpha} + \frac{2}{2\alpha + L} + \frac{1}{\alpha + L},$$

since we can omit  $l_v(\gamma)$  in the denominator of the ratio  $1/(1 - l_v(\gamma))$  of (11):

$$(14) \quad \text{holds for } \alpha > \alpha_2 \approx 3,183\,590.$$

In this case the  $H^m$ -images of any unstable segment  $\gamma^u(x)$ , for  $m$  sufficiently large, contain segments of length larger than a constant. The same concerns  $H^{-n}$ -images of  $\gamma^s(y)$ , for  $n$  large. Does it imply that there exists a decomposition into *finite* number of Bernoulli components?



## 2. Generalizations

Denote by (  $\natural$  ) the places where we are not exact in statements.

1.  $P$  and  $Q$  can be replaced by two finite families of annuli embedded (  $\natural$  ) into a surface,  $\{P_i\}_{i=1}^p$  and  $\{Q_i\}_{i=1}^q$ , such that  $P_i \cap P_j = Q_i \cap Q_j = \emptyset$  for  $i \neq j$  and  $P_i$  intersect  $Q_j$  transversally (  $\natural$  ). Assume that  $M = \bigcup_{i=1}^p P_i \cup \bigcup_{j=1}^q Q_j$  is connected. Consider  $H$  which is a composition of two mappings:  $(\alpha_i, k_i)$ -twists on  $\bigcup_{i=1}^p P_i$  with  $(\beta_j, l_j)$ -twists on  $\bigcup_{j=1}^q Q_j$ .

If  $H$  preserves a probability measure on  $M$  which restriction to each  $P_i$  and  $Q_j$  is equivalent to Lebesgue measure with bounded density, if  $|\alpha_i|, |\beta_j|$  are large enough and  $|k_i|, |l_j| \geq 2$ , then  $H$  is Bernoulli.

In particular assume that changes of coordinates on  $M$  from  $P_i$  to  $Q_j$  are only translations composed with multiplication by  $-1$ . Then:

If for every pair  $i, j$  such that  $P_i \cap Q_j \neq \emptyset$ ,  $|\alpha_i \beta_j| \geq 4$ , then  $H$  is almost hyperbolic.

If  $|\alpha_i \beta_j| > X(i) \cdot Y(j)$  where  $X(i), Y(j)$  are respectively the largest solutions of the equations:

$$1 = \frac{2q(i)}{X} + \frac{q(i)}{X-3q(i)} + \frac{2}{X-1},$$

$$1 = \frac{2p(j)}{Y} + \frac{p(j)}{Y-3p(j)} + \frac{2}{Y-1},$$

where  $q(i)$ , respectively  $p(j)$ , is the number of components of  $P_i \cap \bigcup_{s=1}^q Q_s$ , respectively of  $Q_j \cap \bigcup_{s=1}^p P_s$ , then  $H$  is Bernoulli.

For the sufficient estimates on  $\alpha_j, \beta_j$ , to have almost hyperbolicity and Bernoulli property, in the general case see [11], § 2.

This generalization includes the Bowen example [1], Chap. 10, F. Devaney generalized t. l. t. m. [3] and Thurston pseudo-Anosov examples built from Dehn twists on two thickened simple closed curves (families of curves) "filling up" a surface, see [10], § 6.

2. We can consider a composition of a finite number of families of twists alternately on the annuli  $\{P_i\}$  and  $\{Q_j\}$  (rather than to compose two family only), so that every annulus is twisted at least once and all twists on it go in the same direction.

Then also as in the generalization 1, if slopes of twists are large enough,  $H$  is Bernoulli.

3.  $H$  is Bernoulli if the above annuli  $P_i, Q_i$  are sufficiently narrow collars for the circles  $A_i, B_j$  respectively, embedded into a compact orientable surface  $N$ , such that  $A_i$  intersect  $B_j$  transversally.

Indeed, then, since  $|k_i|, |l_j| \neq 0$  (as  $\geq 2$ ), slopes of the twists are large (if we consider linear twists, i. e. powers of Dehn twists).

Due to this we can find in any isotopy class of orientation preserving homeomorphism on  $N$  a piecewise linear homeomorphism which is Bernoulli (for a probabilistic measure which support is  $N$ ) with arbitrarily small measure entropy.

4. We can prove Bernoulli property in the generalization 1 with the assumptions  $|k_i|, |l_j| \geq 2$  weakened in various ways (see [11], the section "graphs of linkage").

For example this assumption can be completely dropped if for every  $i, j : p(i), q(j) \geq 3$ .

### APPENDIX

We refer in this paper to the so-called Pesin Theory [9] for maps with singularities, recently developed by Katok and Strelcyn [6]. For the comfort of the reader we list below the results.

#### KATOK-STRELCYN, (K-S)-CONDITIONS

Let  $X$  be a complete metric space with a metric  $\rho$ . Let  $N \subset X$  be an open subset which is a Riemannian manifold with a Riemannian metric inducing  $\rho|_N$ . Assume that there exists a number  $r > 0$  such that for each  $x \in N$ ,  $\exp_x$  restricted to the ball  $B(x) = B(x, \min(r, \text{dist}_\rho(x, X \setminus N)))$  is injective.

Let  $\mu$  be a probability measure on  $X$  and  $\Phi$  be a  $\mu$ -preserving,  $C^2$ -,  $1-1$  mapping defined on an open set  $V \subset N$ , into  $N$ . Denote  $\text{sing } \Phi = X \setminus V$ .

(K-S, 1) There exist positive constants  $a, C_1$  such that for every  $\varepsilon > 0$ :

$$\mu(B(\text{sing } \Phi, \varepsilon)) \leq C_1 \varepsilon^a.$$

[ $B(\text{sing } \Phi, \varepsilon)$  means the neighbourhood of  $\text{sing } \Phi$  with radius  $\varepsilon$ .]:

$$(K-S, 2) \left\{ \begin{array}{l} \int \log^+ \|D\Phi(x)\| d\mu(x) < \infty, \\ \int \log^+ \|D\Phi^{-1}(x)\| d\mu(x) < \infty \end{array} \right.$$

(K-S, 3) There exist positive constants  $b, C_2$  such that for every  $x \in X \setminus \text{sing } \Phi$ :

$$\|D^2\Phi(x)\| \leq C_2 (\text{dist}(x, \text{sing } \Phi))^{-b}$$

[By  $\|D^2\Phi(x)\|$  we mean  $\sup \{ \|D^2(\exp_z^{-1} \circ \Phi \circ \exp_y)\| : x \in B(y), \Phi(x) \in B(z) \}$ .]

*Remark.* — If  $\Phi = \Phi_n \circ \dots \circ \Phi_1$  and  $\Phi$  preserves  $\mu$  we can replace the (K-S)-conditions for  $\Phi$  by the analogous conditions for each  $\Phi_i$  separately, with  $\text{Sing } \Phi_i$  and  $\mu_i = (\Phi_{i-1} \circ \dots \circ \Phi_1)_*(\mu)$  respectively.

**THEOREM.** — (a) *If  $\Phi$  satisfies the (K-S)-conditions then for almost every  $x \in X$ , for all vectors tangent at  $x$ , there exist Lyapunov exponents and there exist local stable and unstable*

manifolds  $\gamma^s(x), \gamma^u(x)$ , corresponding to negative and positive exponents respectively. Denote by  $\Lambda^{s(u)}(k)$  the set of points where the number of negative (positive) Lyapunov exponents computed with multiplicities is equal to  $k$  [i. e.  $\dim \gamma^{s(u)}(x) = k$ ]. Consider  $\Lambda^{s(u)}(k)$  if  $\mu(\Lambda^{s(u)}(k)) > 0$ . Then for a sequence of sets  $\Lambda^{s(u)}(k, m)$  increasing with  $m$  which exhaust almost all of  $\Lambda^{s(u)}(k)$ , the families  $\{\gamma^{s(u)}(x) : x \in \Lambda^{s(u)}(k, m)\}$  are absolutely continuous.

(b) If we assume additionally that the measure  $\mu$  is equivalent to the Riemannian measure on  $N$  and all Lyapunov exponents are almost everywhere different from 0, then  $X$  decomposes into a countable family of positive measure  $\mu$ ,  $\Phi$ -invariant pairwise disjoint sets  $X = \bigcup_{i=1}^{\infty \text{ (or } N)} \Lambda_i$  such that for every  $i$ ,  $\Phi|_{\Lambda_i}$  is ergodic and  $\Lambda_i = \bigcup_{j=1}^{j(i)} \Lambda_i^j$  where  $\Lambda_i^j \cap \Lambda_i^{j'} = \emptyset$  for  $j \neq j'$ ,  $\Phi|_{\Lambda_i}$  permutes  $\Lambda_i^j$  and for each  $j$   $\Phi^{j(i)}|_{\Lambda_i^j}$  is Bernoulli. (In the above situation we sometimes call the system almost hyperbolic and say that it decomposes into a countable family of Bernoulli components.)

(c) If additionally for almost every  $z, z' \in X$  there exist integers  $m, n$  such that  $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$  then in the decomposition of  $X$  we have only one set  $\Lambda_i = \Lambda_1$  (i. e.  $N=1$ ). In particular  $\Phi$  is ergodic.

(d) If additionally for almost all points  $z, z' \in X$  and every pair of integers  $m, n$  large enough  $\Phi^m(\gamma^u(z)) \cap \Phi^{-n}(\gamma^s(z')) \neq \emptyset$  then all powers of  $\Phi$  are ergodic. This implies  $j(1) = 1$  so  $\Phi$  is a Bernoulli system.

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