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ON THE DEMAZURE CHARACTER FORMULA

BY A. JOSEPH

1. Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, $R \subset \mathfrak{h}^*$ the set of non-zero roots, $P(R) \subset \mathfrak{h}^*$ the lattice of integral weights, $P(R)^+ \subset P(R)$ the subset of dominant weights relative to R^+, with $B \subset R^+$ the set of simple roots and ρ the half sum of the positive roots. Set $P(R)^{++} = P(R)^+ + \rho$. For each $\alpha \in R$, let $s_\alpha \in \operatorname{Aut} \mathfrak{h}^*$ denote the reflection, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ the co-root and X_α the element of a Chevalley basis defined by the root α. Let W denote the subgroup of $\operatorname{Aut} \mathfrak{h}^*$ generated by the $s_\alpha : \alpha \in R^+$ and $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) the subalgebra of $\mathfrak{g}$ with basis $X_\alpha : \alpha \in R^+$ (resp. $X_{-\alpha} : \alpha \in R^+$). Set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. For each $\nu \in P(R)^+$ let $E(\nu)$ denote the simple finite dimensional $\mathfrak{g}$ module with highest weight ν. The extreme weights of $E(\nu)$ take the form $w\nu : w \in W$ and occur with multiplicity one. For each $w \in W$, let $e_{w\nu} \in E(\nu)$ denote a non-zero vector of weight $w\nu$.

1.2. For each Lie algebra $\mathfrak{a}$, let $U(\mathfrak{a})$ denote its enveloping algebra. For each $w \in W$, one has that $U(\mathfrak{n}) e_{w\nu}$ is an $\mathfrak{h}$ submodule of $E(\nu)$ and so admits a formal character. Demazure ([2], Thm. 2) gave a beautiful formula for the characters, thereby, generalizing the Weyl character formula. In attempting to generalize this formula to infinite root systems, V. Kac noticed that Demazure's analysis had a serious error the faulty step being ([1], Prop. 11) which is false (*see* Sect. 4). Here we introduce a family of functors $\mathscr{D}_\alpha : \alpha \in B$ defined purely algebraically which applied to the $\mathfrak{b}$ submodule $\mathbb{C}_\nu := \mathbb{C} e_\nu$ of $E(\nu)$ produce the $U(\mathfrak{n}) e_{w\nu}$. These functors are undoubtedly analogous to geometrically defined functors occuring in Demazure's paper [1]; yet we are able to develop the properties of these functors sufficiently so as to obtain Demazure's formula generically without the need for ([1], Prop. 11). A basic property needed is that they satisfy the so-called braid relations (2.17). We also need to prove some deep results on the homology spaces $H_i(\mathfrak{n}, U(\mathfrak{n}) e_{w\nu})$ generalising in a somewhat intricate fashion the classical results of Kostant [10]. We also show how to define the derived functions of products of the $\mathscr{D}_\alpha$ and there by, obtain analogues of Bott's Theorem and of [1], 5.5.

1.3. We show that Demazure's character formula implies the truth of ([1], Prop. 11) for *extreme weight* vectors. This in turn is shown to lead to an explicit formula for the annihilator of $e_{w_\vee}$ in $U(\mathfrak{n}^-)$, which is in fact the obvious generalization of the well-known formula when $w=1$. This is actually a rather remarkable result and any purely combinational proof would be extremely difficult. Through it we describe $H_1(\mathfrak{n}, U(\mathfrak{n})e_{w_\vee})$.

1.4. For each $\lambda\in\mathfrak{h}^*$ we let $M(\lambda)$ denote the Verma module with highest weight λ. This is a departure from our usual convention and furthermore, we use $w.\lambda$ to denote $w(\lambda+\rho)-\rho$. Unless otherwise specified all $U(\mathfrak{h})$ modules will be assumed to admit a decomposition into finite dimensional $\mathfrak{h}$ weight spaces of *integral weight*. In particular this applies to the so-called $\mathcal{O}$ category of finitely generated $U(\mathfrak{g})$ modules which are $U(\mathfrak{b})$ locally finite. We let $\mathbb{Z}[P]$ denote the group ring (over $\mathbb{Z}$) of formal characters generated by the $e^\lambda:\lambda\in P(R)$. For each $\alpha\in R$, we define (following Demazure) $\Delta_\alpha\in \operatorname{End}_{\mathbb{Z}}\mathbb{Z}[P]$ through

$$\Delta_\alpha u=\frac{u-s_\alpha.u}{1-e^{-\alpha}},$$

where $s_\alpha.e^\lambda:=e^{s_\alpha.\lambda}$.

2. Functors, braid relations and character formulae

2.1. (Recall 1.4). Let K denote the category of finite dimensional $U(\mathfrak{b})$ modules. For each $\alpha\in B$, let $\mathfrak{p}_\alpha$ denote the parabolic subalgebra $\mathbb{C}X_{-\alpha}\oplus\mathfrak{b}$ of $\mathfrak{g}$ and let K_α denote the category of finite dimensional $U(\mathfrak{p}_\alpha)$ modules. We define a functor $\mathscr{D}_\alpha$ from K to K_α as follows. Given $F\in \operatorname{Ob} K$ we define $\mathscr{D}_\alpha F$ to be the largest finite dimensional $U(\mathfrak{p}_\alpha)$ quotient of the induced module $U(\mathfrak{p}_\alpha)\otimes_{U(\mathfrak{b})}F$. Let us write $M=U(\mathfrak{p}_\alpha)\otimes_{U(\mathfrak{b})}F$, then say in the notation of [5], 2.1, we have $\mathscr{D}_\alpha F=M/D_\alpha^+M$. We remark that M has finite length (as in [3], 7.6.1) and so $\mathscr{D}_\alpha$ is well-defined.

2.2. It is clear that $\mathscr{D}_\alpha$ is covariant, right exact and that the canonical embedding $f\mapsto 1\otimes f$ of F into M defines a canonical $U(\mathfrak{b})$ module map $F\to\mathscr{D}_\alpha F$, whose image generates $\mathscr{D}_\alpha F$ over $U(\mathfrak{p}_\alpha)$. The reader may care to view $\mathscr{D}_\alpha$ as an adjoint to a Zuckerman functor for the pair $\mathfrak{p}_\alpha$, $\mathfrak{b}$ or as an algebraic analogue of the sheaf theoretic functor occuring in [1], 2.10.

LEMMA. — *Let* $F_1\to F_2\to F_3\to 0$ *be an exact sequence of finite dimensional* $U(\mathfrak{b})$ *modules. Then we have a commutative diagram*

$$\begin{array}{ccccccc} F_1 & \to & F_2 & \to & F_3 & \to & 0\\ \downarrow & & \downarrow & & \downarrow & & \\ \mathscr{D}_\alpha F_1 & \to & \mathscr{D}_\alpha F_2 & \to & \mathscr{D}_\alpha F_3 & \to & 0\end{array}$$

where the vertical maps are the canonical homomorphisms and the bottom row is an exact sequence of finite dimensional $U(\mathfrak{p}_\alpha)$ *modules.*

We have already noted the right exactness of $\mathscr{D}_\alpha$. The commutativity of the diagram obtains from the corresponding commutativity of the related diagram with $\mathscr{D}_\alpha F_i$ replaced by $U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} F_i$.

2.3. Set $X = X_\alpha$, $Y = X_{-\alpha}$, $H = [X, Y]$ and let $\mathfrak{s}_\alpha$ (or simply, $\mathfrak{s}$) denote the $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{p}_\alpha$ generated by X, Y. Set $\mathfrak{b}_\alpha = \mathbb{C}X \oplus \mathbb{C}H$. In computing $\mathscr{D}_\alpha F$ as an $\mathfrak{s}_\alpha$ module it is enough to know F as $\mathfrak{b}_\alpha$ module. This is because $\mathscr{D}_\alpha F$ is also the largest locally Y finite quotient of M. The technical advantage can be described as follows. Call a finite dimensional $\mathfrak{b}_\alpha$ module F a *string module* if it is cyclic as a $\mathbb{C}[X]$ module. Such a module is obviously H semi-simple and determined up to isomorphism by its highest weight μ and lowest weight ν. We denote this isomorphism class by $F(\mu, \nu)$. It is a submodule of the finite dimensional simple $\mathfrak{s}$ module $E(\mu)$ of highest weight μ if and only if $(\mu+\nu)/2 \in P(R)^+$. Again $F(\mu, \nu) \cong E[(\mu-\nu)/2] \otimes \mathbb{C}_{(\mu+\nu)/2}$, where $\mathbb{C}_\lambda$ denotes the one dimensional $\mathfrak{b}$ module of weight λ.

LEMMA. — *Every finite dimensional* H *semi-simple* $\mathfrak{b}$ *module* F *is a direct sum of string modules.*

Let l denote the index of nilpotence of $\mathrm{ad}_F X$. Choose $f \in F$ such that $(\mathrm{ad}\, X)^{l-1} f \neq 0$. We may assume that f is a H weight vector without loss of generality. Then the action of ad X on f generates a string submodule F_1 of F. It remains to show that F_1 is complemented in F as a $\mathfrak{b}_\alpha$ module. This obtains by the usual argument used in establishing Jordan normal form for nilpotent endomorphisms.

2.4. The computation of $\mathscr{D}_\alpha F$ on string modules is quite easy and can determined from the structure of $U(\mathfrak{s}_\alpha)$ modules in the $\mathscr{O}$ category for $\mathfrak{sl}(2)$. Here it is convenient to let the notations of 1.1 refer to $\mathfrak{s}_\alpha$ and for each $\lambda \in P(R)^+ - \rho$, let $\mathscr{O}_\lambda$ denote the $\mathscr{O}$ subcategory of $U(\mathfrak{s}_\alpha)$ modules whose generalized central character coincides with that the Verma module $M_\alpha(\lambda)$ (defined relative to $\mathfrak{s}_\alpha$). For $\lambda \in P(R)^+$ this category has just five indecomposable objects (*see* [6], 4.3 for example). The simple Verma module $V(\lambda) := M_\alpha(s_\alpha \lambda - 2\rho)$, the projective Verma module $P(\lambda) := M_\alpha(\lambda)$ the extension $T_\alpha(\lambda)$ [or simply, $T(\lambda)$] of $P(\lambda)$ by $V(\lambda)$ [we set $T(\lambda) = 0$ for $-\lambda \in P(R)^+$]. The quotient $E(\lambda)$ of $P(\lambda)$ by $V(\lambda)$ and the $\mathscr{O}$ dual $I(\lambda)$ of $P(\lambda)$. When $\lambda = -\rho$, this category has just one indecomposable object namely $V(-\rho)$, which we may also denote $T(-\rho)$.

Now suppose F is the string module $F(\mu, \nu)$. When $\mu + \nu \in P(R)^+$ then from say the embedding $F(\mu, \nu) \to E(\mu)$ and say ([6], 4.3) one easily checks that $M = U(\mathfrak{s}_\alpha) \otimes_{U(\mathfrak{b}_\alpha)} F$ is a direct sum of $P(\mu)$, $P(\mu-\alpha), \ldots, P(-\nu)$ and $T(-\nu-\alpha)$, $T(-\nu-2\alpha), \ldots$, if $\nu \leqq 0$, or of $P(\mu)$, $P(\mu-\alpha), \ldots, P(\nu)$ if $\nu \geqq 0$. Furthermore

$$(*) \qquad \mathscr{D}_\alpha F = \bigoplus_{k=0}^{t} E(\mu - k\alpha), \qquad t = \begin{cases} 1/2(\mu+\nu, \alpha^\vee) & \text{if } \nu \leqq 0, \\ 1/2(\mu-\nu, \alpha^\vee) & \text{if } \nu \geqq 0, \end{cases}$$

in this case and moveover, the canonical map $F \to \mathscr{D}_\alpha F$ is injective. On the other hand if $\mu + \nu \in -P(R)^{++}$, then a similar computation gives that $\mathscr{D}_\alpha F = 0$.

For each $\mathfrak{h}$ module F let $\Omega(F)$ denote the set of weights of F. Given any subset $\Omega \subset P(R)$ we let $\langle \Omega \rangle_\alpha$ denote the subset of $P(R)$ generated from Ω by taking for each $\mu \in \Omega$ the α-string joining μ to $s_\alpha \mu$.

LEMMA. — $\Omega(\mathscr{D}_\alpha F) \subset \langle \Omega(F) \rangle_\alpha$.

This is an immediate consequence of the above description of $\mathscr{D}_\alpha F$.

2.5. Let E be a finite dimensional $U(\mathfrak{p}_\alpha)$ module. Often E is a finite dimensional $U(\mathfrak{g})$ module viewed as a $U(\mathfrak{p}_\alpha)$ module by restriction. Take $M = U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} F$ as before. From the canonical isomorphism $U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} (E \otimes F) \xrightarrow{\sim} E \otimes M$ we obtain a surjection of $\mathscr{D}_\alpha(E \otimes F)$ onto $E \otimes \mathscr{D}_\alpha F$ of $U(\mathfrak{p}_\alpha)$ modules.

LEMMA. — $\mathscr{D}_\alpha(E \otimes F) \xrightarrow{\sim} E \otimes \mathscr{D}_\alpha F$.

It is enough to show that both sides are isomorphic as $\mathfrak{s}_\alpha$ modules. This derives from 2.4 by breaking F, considered as a $\mathfrak{b}_\alpha$ module, into string modules and computing term by term.

2.6. We may now describe the role of the operator Δ_α (notation 1.4) introduced by Demazure. Clearly each $F \in \operatorname{Ob} K$ admits a formal character. One has the:

LEMMA. — $\operatorname{ch} \mathscr{D}_\alpha F = \Delta_\alpha(\operatorname{ch}(\operatorname{Im}(F \to \mathscr{D}_\alpha F)))$.

It is enough to consider F as an $\mathfrak{s}_\alpha + \mathfrak{h}$ module. Then as in 2.3 we may decompose F into string modules $F(\mu, \nu)$. If $\mu + \nu = -P(R)^{++}$, then $\mathscr{D}_\alpha F = 0$. Otherwise the canonical map $F \to \mathscr{D}_\alpha F$ is injective and our formula is immediately verified using (*), or by the isomorphism $F \cong E[(\mu - \nu)/2] \otimes \mathbb{C}_{(\mu+\nu)/2}$ and 2.5.

2.7. Let E be a finite dimensional $U(\mathfrak{p}_\alpha)$ module. Breaking E into string modules it follows easily from 2.4, that the canonical map $E \to \mathscr{D}_\alpha E$ is bijective. In particular it follows that $\mathscr{D}_\alpha$ is an idempotent functor, that is $\mathscr{D}_\alpha F \xrightarrow{\sim} \mathscr{D}_\alpha(\mathscr{D}_\alpha F)$ for any $F \in \operatorname{Ob} K$. [Here we use the convention of forgetting the $U(\mathfrak{p}_\alpha)$ module structure of F.] Again if $F \in \operatorname{Ob} K$, then F can admit at most one $U(\mathfrak{p}_\alpha)$ module structure namely that given by the isomorphism $F \xrightarrow{\sim} \mathscr{D}_\alpha F$ and such an isomorphism exists if and only if each string submodule of F admits a $U(\mathfrak{s}_\alpha)$ module structure.

2.8. LEMMA. — *Let* F *be a* $U(\mathfrak{b})$ *submodule of a finite dimensional* $U(\mathfrak{p}_\alpha)$ *module* E. *Then:*

(i) *The canonical map* $F \to \mathscr{D}_\alpha F$ *is injective.*

(ii) *The map* $\mathscr{D}_\alpha F \to \mathscr{D}_\alpha E$ *has image* $U(\mathfrak{p}_\alpha)(\operatorname{Im}(F \to E))$.

(iii) *The map* $\mathscr{D}_\alpha F \to \mathscr{D}_\alpha E$ *is bijective if and only if* $\operatorname{Coker}(F \to E)$ *takes the form* $E' \otimes \mathbb{C}_{-\rho}$ *for some* $E' \in \operatorname{Ob} K_\alpha$.

(iv) *Let* $0 \to F_1 \to F_2 \to F_3 \to 0$ *be an exact sequence of objects of* K *with* $F_3 \cong F \otimes \mathbb{C}_\lambda$ *and* $(\lambda, \alpha^\vee) \in \mathbb{N}$.

Then $0 \to \mathscr{D}_\alpha F_1 \to \mathscr{D}_\alpha F_2 \to \mathscr{D}_\alpha F_3 \to 0$ *is exact.*

(i) It is enough to prove injectivity of $\mathfrak{s}_\alpha$ modules. This obtains from 2.4 by decomposition into string modules.

(ii) Since $\mathscr{D}_\alpha F$ is generated over $U(\mathfrak{p}_\alpha)$ by the image of F in $\mathscr{D}_\alpha F$, it follows that

$\mathrm{Im}(\mathscr{D}_\alpha \mathrm{F} \to \mathscr{D}_\alpha \mathrm{E})$ is generated by $\mathrm{Im}(\mathrm{F} \to \mathscr{D}_\alpha \mathrm{E})$ over $\mathrm{U}(\mathfrak{p}_\alpha)$. By 2.7 we have $\mathrm{E} \xrightarrow{\sim} \mathscr{D}_\alpha \mathrm{E}$ and we may identify $\mathrm{Im}(\mathrm{F} \to \mathscr{D}_\alpha \mathrm{E})$ with $\mathrm{Im}(\mathrm{F} \to \mathrm{E})$.

(iii) Set $\mathrm{Q} = \mathrm{Coker}(\mathrm{F} \to \mathrm{E})$ and consider the exact sequence $0 \to \mathrm{F} \to \mathrm{E} \to \mathrm{Q} \to 0$ of $\mathrm{U}(\mathfrak{b})$ modules. By the right exactness of $\mathscr{D}_\alpha$ we obtain $\mathscr{D}_\alpha \mathrm{F} \twoheadrightarrow \mathscr{D}_\alpha \mathrm{E}$ if and only if $\mathscr{D}_\alpha \mathrm{Q} = 0$. Assume this holds and break Q into string modules $\mathrm{F}(\mu, \nu)$. By 2.4 we must have $\mu + \nu \in -\mathrm{P}(\mathrm{R})^{++}$ for each string module. Now assume $\mathscr{D}_\alpha \mathrm{F} \xrightarrow{\sim} \mathrm{E}$. By 2.6 and (i) we obtain $\mathrm{ch}\,\mathrm{E} = \Delta_\alpha \mathrm{ch}\,\mathrm{F}$. Consequently $\mathrm{ch}\,\mathrm{Q} = \Delta_\alpha \mathrm{ch}\,\mathrm{F} - \mathrm{ch}\,\mathrm{F}$ and so $s_\alpha . \mathrm{ch}\,\mathrm{Q} = 0$. Recalling the definition of $\mathrm{F}(\mu, \nu)$, this implies that $(\mu + \nu, \alpha^\vee) = -2$ for each string module which hence takes the form $\mathrm{E}(\mu + \rho) \otimes \mathbb{C}_{-\rho}$. Consequently, $\mathrm{E}' := \mathrm{Q} \otimes \mathbb{C}_\rho$ admits a $\mathrm{U}(\mathfrak{p}_\alpha)$ structure by 2.7 and we obtain $\mathrm{Q} \cong \mathrm{E}' \otimes \mathbb{C}_{-\rho}$ as required. Conversely suppose $\mathrm{Q} = \mathrm{E}' \otimes \mathbb{C}_{-\rho}$ for some $\mathrm{U}(\mathfrak{p}_\alpha)$ submodule E. Then $\mathscr{D}_\alpha \mathrm{Q} \cong \mathrm{E}' \otimes \mathscr{D}_\alpha \mathbb{C}_{-\rho}$ by 2.5, whereas $\mathscr{D}_\alpha \mathbb{C}_{-\rho} = 0$ by 2.4 (*). Hence $\mathscr{D}_\alpha \mathrm{F} \to \mathscr{D}_\alpha \mathrm{E}$ is surjective. Now for Q of the given form one easily checks that $s_\alpha . \mathrm{ch}\,\mathrm{Q} = 0$ and so $\mathrm{ch}\,\mathrm{E} = \Delta_\alpha \mathrm{ch}\,\mathrm{F}$. By 2.6, 2.7 and (i) we obtain $\mathrm{ch}\,\mathscr{D}_\alpha \mathrm{F} = \Delta_\alpha \mathrm{ch}\,\mathrm{F} = \mathrm{ch}\,\mathrm{E} = \mathrm{ch}\,\mathscr{D}_\alpha \mathrm{E}$ and so $\mathscr{D}_\alpha \mathrm{F} \to \mathscr{D}_\alpha \mathrm{E}$ is bijective.

(iv) We have only to show that the map $\mathscr{D}_\alpha \mathrm{F}_1 \to \mathscr{D}_\alpha \mathrm{F}_2$ is injective and so it is enough to show that $\mathrm{ch}\,\mathscr{D}_\alpha \mathrm{F}_2 = \mathrm{ch}\,\mathscr{D}_\alpha \mathrm{F}_1 + \mathrm{ch}\,\mathscr{D}_\alpha \mathrm{F}_3$. Let $\mathfrak{m}$ denote the nilradical of $\mathfrak{p}_\alpha$. As noted in 2.3 to compute the $\mathscr{D}_\alpha \mathrm{F}_i$ as $\mathfrak{s}_\alpha \oplus \mathfrak{h}$ modules we may forget their $\mathfrak{m}$ module structure. Thus consider $0 \to \mathrm{F}_1 \to \mathrm{F}_2 \to \mathrm{F}_3 \to 0$ as an exact sequence of $\mathfrak{c} := \mathfrak{b}_\alpha + \mathfrak{h}$ modules. It is enough to show that $0 \to \mathscr{D}_\alpha \mathrm{F}_1 \to \mathscr{D}_\alpha \mathrm{F}_2 \to \mathscr{D}_\alpha \mathrm{F}_3 \to 0$ is an exact sequence of $\mathrm{U}(\mathfrak{s}_\alpha + \mathfrak{h})$ modules as this will imply the required character formula. Set $\mathfrak{r} = \mathfrak{s}_\alpha + \mathfrak{h}$. Now we have an exact sequence

$$0 \to \mathrm{U}(\mathfrak{r}) \otimes_{\mathrm{U}(\mathfrak{c})} \mathrm{F}_1 \to \mathrm{U}(\mathfrak{r}) \otimes_{\mathrm{U}(\mathfrak{c})} \mathrm{F}_2 \to \mathrm{U}(\mathfrak{r}) \otimes_{\mathrm{U}(\mathfrak{c})} \mathrm{F}_3 \to 0,$$

of $\mathrm{U}(\mathfrak{r})$ modules which all lie in the corresponding $\mathscr{O}$ category. Under the hypothesis that F is a submodule of an $\mathfrak{r}$ module E it follows easily that $\mathrm{U}(\mathfrak{r}) \otimes_{\mathrm{U}(\mathfrak{c})} \mathrm{F}_3$ is projective in $\mathscr{O}$. (Break into string modules and recall the discussion in 2.4.) Thus the sequence splits and the right exactness of the functor $\mathrm{M} \to \mathrm{M}/\mathrm{D}_\alpha^+ \mathrm{M}$ implies the required assertion.

Remark. — One should regard 2.8 (iii) as an algebraic analogue of [1], Prop 10. Take $w \in \mathrm{W}$ and $\alpha \in \mathrm{B}$ such that $l(s_\alpha w) > l(w)$ [where $l(.)$ denotes the length function on W]. Take $\mathrm{F} = \mathrm{U}(\mathfrak{n}) e_{w\lambda}$, $\mathrm{E} = \mathrm{U}(\mathfrak{n}) e_{s_\alpha w \lambda}$. If we admit ([1], Prop. 11) then we have $\mathrm{E}/\mathrm{F} = \mathrm{E}' \otimes \mathbb{C}_{-\rho}$ for some $\mathrm{U}(\mathfrak{p}_\alpha)$ module E' and so by 2.6, 2.8 (i), 2.8 (iii) we obtain $\mathrm{ch}\,\mathrm{U}(\mathfrak{n}) e_{s_\alpha w \lambda} = \Delta_\alpha \mathrm{ch}\,\mathrm{U}(\mathfrak{n}) e_{w\lambda}$. This should be viewed as Demazure's proof of his character formula. Unfortunately, this method fails because of the failure of [1], Prop. 11.

2.9. A rather deeper property of the $\mathscr{D}_\alpha : \alpha \in \mathrm{B}$ emerges when we compose the functors [forgetting of course the $\mathrm{U}(\mathfrak{p}_\alpha)$ module structure at each step]. Let w_0 denote the unique longest element of W. With $s_i = s_{\alpha_i}$ let $w_0 = s, s_2 \ldots s_n$ be a reduced decomposition for w. Set $w_n = 1$ and for $i < n$, set $w_i = s_{i+1} s_{i+2} \ldots s_n$, $\mathscr{D}_i = \mathscr{D}_{\alpha_i}$ and $\mathscr{D}_{w_i} = \mathscr{D}_{i+1} \mathscr{D}_{i+2} \ldots \mathscr{D}_n$. Fix $\lambda \in \mathrm{P}(\mathrm{R})^+$. We should like to show that $\mathscr{D}_{w_0} \mathbb{C}_\lambda$ is an image of $\mathrm{M}(\lambda)$ [considered as a $\mathrm{U}(\mathfrak{b})$ module]. Set $\mathfrak{p}_i = \mathfrak{p}_{\alpha_i}$. There is an obvious $\mathrm{U}(\mathfrak{b})$ map of $\mathrm{F}(\lambda) := \mathrm{U}(\mathfrak{p}_1) \otimes_{\mathrm{U}(\mathfrak{b})} (\mathrm{U}(\mathfrak{p}_2) \otimes \ldots \otimes \mathrm{U}(\mathfrak{b}) \mathbb{C}_\lambda)$ to $\mathrm{M}(\lambda)$ and by definition of $\mathscr{D}_{w_0}$ there is a surjection of $\mathrm{F}(\lambda)$ onto $\mathscr{D}_{w_0} \mathbb{C}_\lambda$ which in turn surjects onto $\mathrm{E}(\lambda)$ [by 2.7 (iii) and right exactness]. Analyzing the kernel of $M(\lambda) \to E(\lambda)$ as the $(\lambda, \alpha): \alpha \in B$ become large one easily shows that the map $F(\lambda) \to M(\lambda)$ is surjective. Unfortunately, this map has a non-zero kernel in general and it is not obvious if this kernel is contained in the kernel of the map $F(\lambda) \to \mathcal{D}_{w_0} \mathbb{C}_\lambda$. We shall in fact show this but by a slightly roundabout method.

2.10. Preserve the notation of 2.9 and set $\beta_i = w_i^{-1} \alpha_i$. Recall the well-known fact that the β_i are pairwise distinct positive roots and satisfy $\{\beta_i\}_{i=m+l}^n = \{\gamma \in R^+ \mid w_m \gamma \in R^-\}$. In particular $\{\beta_i\}_{i=1}^n = R^+$. Fix $\lambda \in P(R)$ and set $\mathcal{Q}_i = \mathcal{D}_{i+1} \mathcal{D}_{i+2} \ldots \mathcal{D}_n \mathbb{C}_\lambda$, $\forall i$. Set $\mathfrak{s}_{\alpha_i} = \mathfrak{s}_i$. Since $\mathcal{Q}_i$ is a finite dimensional $\mathfrak{p}_{i+1}$ module it is also a finite dimensional $\mathfrak{s}_{i+1}$ module and so $s_{i+1} \mathcal{Q}_i = \mathcal{Q}_i$. Now set $\mathfrak{p}'_i = \mathfrak{p}_{\beta_i} := w_i^{-1}(\mathfrak{p}_{\alpha_i})$. It is a parabolic subalgebra of $\mathfrak{g}$ containing the Borel subalgebra $\mathfrak{b}'_i = w_i^{-1}(\mathfrak{p})$ of $\mathfrak{g}$.

Suppose we are given a finite dimensional $U(\mathfrak{b}'_i)$ module F. Then we can form $U(\mathfrak{p}'_i) \otimes_{U(\mathfrak{b}'_i)} F$ and we define $\mathcal{D}'_i F$ to be its largest finite dimensional $U(\mathfrak{p}'_i)$ quotient. Set $\mathcal{Q}'_i = \mathcal{D}'_{i+1} \mathcal{D}'_{i+2} \ldots \mathcal{D}'_n \mathbb{C}_\lambda$ which is well-defined since

$$\mathfrak{b}'_j = w_j^{-1}(\mathfrak{b}) \subset w_j^{-1} \mathfrak{p}_{j+1} = w_{j+1}^{-1} \mathfrak{p}_{j+1} = \mathfrak{p}'_{j+1}.$$

LEMMA. — *For each $i \in \mathbb{N}$, there is a linear isomorphism of $\mathcal{Q}_i$ onto $\mathcal{Q}'_i$ taking the $U(\mathfrak{p}_{i+1})$ module structure of $\mathcal{Q}_i$ to the $U(\mathfrak{p}'_{i+1})$ module structure of $\mathcal{Q}'_i$.*

The proof is by decreasing induction. For $i=n$, the assertion is trivial. Suppose we have established the assertion for $i+1$. We may denote the isomorphism conveniently by $\mathcal{Q}'_{i+1} = w_{i+1}^{-1} \mathcal{Q}_{i+1}$. Consider $\mathcal{Q}'_{i+1}$ (resp. $\mathcal{Q}_{i+1}$) as a $U(\mathfrak{b}'_{i+1})$ [resp. $U(\mathfrak{b})$] module. It is clear that w_{i+1}^{-1} extends to a linear isomorphism of $U(\mathfrak{p}_{i+1}) \otimes_{U(\mathfrak{b})} \mathcal{Q}_{i+1}$ onto $U(\mathfrak{p}'_{i+1}) \otimes_{U(\mathfrak{b}'_{i+1})} \mathcal{Q}'_{i+1}$ carrying the $U(\mathfrak{p}_{i+1})$ module structure of the former to the $U(\mathfrak{p}'_{i+1})$ module structure of the latter and hence defines the appropriate linear isomorphism of $\mathcal{Q}_i = \mathcal{D}_{i+1} \mathcal{Q}_{i+1}$ onto $\mathcal{Q}'_i = \mathcal{D}'_{i+1} \mathcal{Q}'_{i+1}$ which we can write as $w_{i+1}^{-1} \mathcal{Q}_i = \mathcal{Q}'_i$. Yet $s_{i+1} \mathcal{Q}_i = \mathcal{Q}_i$ and so we have the required linear isomorphism of $\mathcal{Q}_i$ onto $\mathcal{Q}'_i = w_i^{-1} \mathcal{Q}_i$.

2.11. Retain the notation of 2.9, 2.10 and set $\mathfrak{b}^- = \mathfrak{n}^- \oplus \mathfrak{h}$. The point of the above construction is that there exists an obvious $U(\mathfrak{b}^-)$ module surjection ψ of $M(\lambda)$ onto $\mathcal{Q}'_0$. This is defined by identifying $M(\lambda)$ as a $U(\mathfrak{b}^-)$ module with $U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$ and then one easily sees that there exists a unique bijection of $U(\mathfrak{b}^-)$ modules of $M(\lambda)$ onto $U(\mathfrak{p}'_1) \otimes_{U(\mathfrak{b}'_1)} \otimes \ldots \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ and hence a unique $U(\mathfrak{b}^-)$ module surjection ψ of $M(\lambda)$ onto $\mathcal{Q}'_0$. The crucial point here is that $\mathfrak{p}'_i = \mathfrak{b}'_i \oplus \mathbb{C} X_{-\beta_i}$ and $\{\beta_i\}_{i=1}^n = R^+$. Now combined with 2.10 we obtain a linear surjection φ of $M(\lambda)$ onto $w_0 \mathcal{Q}'_0 = \mathcal{Q}_0$. Let v_λ denote the canonical generator of $M(\lambda)$ of weight λ.

LEMMA. — *For all $x \in U(\mathfrak{b})$, $m \in M(\lambda)$ one has $x\varphi(m) = \varphi(w_0 x w_0^{-1} m)$. In particular $\varphi(v_\lambda)$ generates $\mathcal{D}_{w_0} \mathbb{C}_\lambda$ as a $U(\mathfrak{b})$ module and is an $\mathfrak{h}$ weight vector of weight $w_0 \lambda$.*

Indeed

$$x\varphi(m) = x w_0^{-1} \psi(m) = w_0^{-1} w_0 x w_0^{-1} \psi(m) = w_0^{-1} \psi(w_0 x w_0^{-1} m) = \varphi(w_0 x w_0^{-1} m).$$

2.12. Retain the above notation.

LEMMA. — *Take* $\nu\in\Omega(\mathscr{D}_{w_i}\mathbb{C}_\lambda)$. *Then* $(\nu, \nu)\leqq(\lambda, \lambda)$ *and equality implies that* $\nu=y\lambda$ *for some* $y\leqq w_i$ *(Bruhat order) and this weight occurs with multiplicity one.*

The proof is by decreasing induction on i. It is trivial for $i=n$. Choose $\nu\in\Omega(\mathscr{D}_{w_{i-1}}\mathbb{C}_\lambda)$ setting $\alpha=\alpha_i$. By 2.4 there exists $\mu\in\Omega(\mathscr{D}_{w_i}\mathbb{C}_\lambda)$ such that (assuming for convenience that $k:=(\alpha^\vee, \mu)\geqq 0$) we have $\nu=\mu-l\alpha: 0\leqq l\leqq k$. Then

$$(\nu, \nu)=(\mu, \mu)-2l(\alpha, \mu)+l^2(\alpha, \alpha)=(\mu, \mu)+(l-k)(\alpha, \alpha)\leqq(\mu, \mu).$$

This gives the inequality $(\nu, \nu)\leqq(\mu, \mu)\leqq(\lambda, \lambda)$ and we see that equality implies that either $\nu=\mu$ or $\nu=s_\alpha\mu$. By definition of the Bruhat order, this gives the required assertion.

2.13. Choose $\lambda\in P(R)^+$, and let w_0 denote the largest element of W.

PROPOSITION. — *There exists a bijection* $\mathscr{D}_{w_0}\mathbb{C}_\lambda\xrightarrow{\sim}E(\lambda)$ *of* $U(\mathfrak{b})$ *modules.*

In the notation of 2.9 we have by 2.7 a surjection of $\mathscr{D}_{\alpha_i}U(\mathfrak{n})e_{w_i\lambda}$ onto $U(\mathfrak{p}_i)U(\mathfrak{n})e_{w_{i-1}\lambda}$. Then by the definition of $\mathscr{D}_{w_0}$, induction and the right exactness of the $\mathscr{D}_{\alpha_i}$ we obtain a surjection $\mathscr{D}_{w_0}\mathbb{C}_\lambda\twoheadrightarrow E$ of $U(\mathfrak{b})$ modules. By 2.11, $\mathscr{D}_{w_0}\mathbb{C}_\lambda$ is a cyclic $U(\mathfrak{b})$ module with cyclic vector $f_{w_0\lambda}$ of weight $w_0\lambda$. Set $k_\alpha=(\alpha^\vee, \lambda)$, $\forall\alpha\in B$. Consider $X_\alpha^{k_\alpha+1}f_{w_0\lambda}$. It is a vector of weight $\nu=w_0\lambda+(k_\alpha+1)\alpha$. One has

$$(\nu, \nu)=(\lambda, \lambda)+(k_\alpha+1)^2(\alpha, \alpha)+2(k_\alpha+1)(\alpha, w_0\lambda)=(\lambda, \lambda)+(k_\alpha+1)(\alpha, \alpha)>(\lambda, \lambda).$$

Hence $X_\alpha^{k_\alpha+1}f_{w_0}\lambda=0$ by 2.12. Hence $\mathscr{D}_{w_0}\mathbb{C}_\lambda$ is a $U(\mathfrak{n})$ quotient of the $U(\mathfrak{n})$ module $U(\mathfrak{n})/\bigoplus_{\alpha\in B}U(\mathfrak{n})X_\alpha^{k_\alpha+1}$. If we recall that $E(\lambda)$ as a $U(\mathfrak{n}^-)$ module is isomorphic to

$$M(\lambda)/\sum_{\alpha\in B}M(s_\alpha.\lambda)\cong U(\mathfrak{n}^-)/\sum_{\alpha\in B}U(\mathfrak{n}^-)X_\alpha^{k_\alpha+1}$$

it follows easily that $\mathscr{D}_{w_0}\mathbb{C}_\lambda$ is a quotient of $E(\lambda)$. Hence the proposition.

Remark. — Similarly $\mathscr{D}_w\mathbb{C}_\lambda\xrightarrow{\sim}U(\mathfrak{n})e_{w\lambda}$ whenever $w\in W$ is the longest element of any subgroup generated by a subset of the simple refections.

2.14. Take $w\in W$, $\lambda\in P(R)^+$. It follows from 2.13 that $\mathscr{D}_{w_0}\mathbb{C}$ is independent of the reduced decomposition of w_0 used to define $\mathscr{D}_{w_0}$. An obvious question is whether or not the functors $\mathscr{D}_w$ are also independent of the reduced decomposition. To show this we need the following:

LEMMA. — *Let* F *be a finite dimensional* $U(\mathfrak{b})$ *module (with integral weights). Then for* $\lambda\in P(R)^+$ *sufficiently large* [i.e. *the* $(\lambda, \alpha^\vee):\alpha\in B$ *are all sufficiently large*] *there exists a finite dimensional* $U(\mathfrak{g})$ *module* E *and a surjection*

$$E\otimes\mathbb{C}_\lambda\to F\to 0,$$

of $U(\mathfrak{b})$ *modules.*

Since all modules are finite dimensional it is enough to exhibit an injection $F^* \to E^* \otimes \mathbb{C}_{-\lambda}$ of dual modules [given a $U(\mathfrak{b})$ structure through the principle antiautomorphism].

Set $G = F^* \otimes \mathbb{C}_\lambda$, with λ chosen sufficiently large to ensure that $\Omega(G) \subset P(R)^+$ and that $\Omega(G) \cap (s_\alpha(\Omega(G)) - \mathbb{N}R^+) = \varnothing$. The former condition implies [*via* the action the centre using the Harish-Chandra isomorphism ([3], 7.4)] that

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} G \xrightarrow{\sim} \bigoplus_{\mu \in \Omega(G)} M(\mu),$$

whereas the second condition implies that each surjection $M(\mu) \twoheadrightarrow E(\mu)$ is injective on the weight subspaces of weight $\nu \in \Omega(G)$. Consequently, the composed map

$$G \xrightarrow{g \to 1 \otimes g} U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} G \to \bigoplus_{\mu \in \Omega(G)} E(\mu)$$

is injective. Tensoring by $\mathbb{C}_{-\lambda}$ then proves the Lemma.

2.15. Let $\mathscr{D}$, $\mathscr{D}'$ be the functors $\mathscr{D}_w$ corresponding to two possibly different reduced decompositions of $w \in W$.

PROPOSITION. — $\mathscr{D} \xrightarrow{\sim} \mathscr{D}'$.

We must show that for each $F \in \operatorname{Ob} K$ one has a natural isomorphism $\mathscr{D} F \xrightarrow{\sim} \mathscr{D}' F$. Recalling 2.8 it is enough to establish the braid relations (for example that $\mathscr{D}_\alpha \mathscr{D}_\beta \mathscr{D}_\alpha F \xrightarrow{\sim} \mathscr{D}_\beta \mathscr{D}_\alpha \mathscr{D}_\beta F$ when $\{\alpha, \beta\}$ generate a system of type A_2). It then amounts to constructing natural isomorphisms $\mathscr{D} F \xrightarrow{\sim} \mathscr{D}\mathscr{D}_\beta F$, $\mathscr{D}' F \xrightarrow{\sim} \mathscr{D}_\alpha \mathscr{D}' F$, where for example $\mathscr{D} = \mathscr{D}_\alpha \mathscr{D}_\beta \mathscr{D}_\alpha, \mathscr{D}' = \mathscr{D}_\beta \mathscr{D}_\alpha \mathscr{D}_\beta$.

By 2.14, there exist λ, $\mu \in P(R)^+$, finite dimensional $U(\mathfrak{g})$ modules E, E′ and an exact sequence

$$(*) \qquad E' \otimes \mathbb{C}_\mu \to E \otimes \mathbb{C}_\lambda \to F \to 0$$

of $U(\mathfrak{b})$ modules. From 2.2 and 2.5 we obtain the commutative diagram:

$$\begin{array}{ccccccc} E' \otimes \mathbb{C}_\mu & \to & E \otimes \mathbb{C}_\lambda & \to & F & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ E' \otimes \mathscr{D}_\beta \mathbb{C}_\mu & \to & E \otimes \mathscr{D}_\beta \mathbb{C}_\lambda & \to & \mathscr{D}_\beta F & \to & 0 \end{array}$$

with exact rows, where the vertical maps are the canonical ones. Functorially and 2.5 gives

$$\begin{array}{ccccccc} E' \otimes \mathscr{D} \mathbb{C}_\mu & \to & E \otimes \mathscr{D} \mathbb{C}_\lambda & \to & \mathscr{D} F & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ E' \otimes \mathscr{D}\mathscr{D}_\beta \mathbb{C}_\mu & \to & E \otimes \mathscr{D}\mathscr{D}_\beta \mathbb{C}_\lambda & \to & \mathscr{D}\mathscr{D}_\beta F & \to & 0 \end{array}$$

By 2.13 the first two vertical maps are isomorphisms and hence so is the last. Again from (∗) we obtain

$$E' \otimes \mathscr{D}' \mathbb{C}_\mu \to E \otimes \mathscr{D}' \mathbb{C}_\lambda \to \mathscr{D}' F \to 0,$$

which by 2.2 gives the commutative diagram.

$$\begin{array}{ccccc} E' \otimes \mathscr{D}' \mathbb{C}_\mu & \to E \otimes \mathscr{D}' \mathbb{C}_\lambda & \to \mathscr{D}' F \to 0 \\ \downarrow & \downarrow & \downarrow \\ E' \otimes \mathscr{D}_\alpha \mathscr{D}' \mathbb{C}_\mu & \to E \otimes \mathscr{D}_\alpha \mathscr{D}' \mathbb{C}_\lambda & \to \mathscr{D}_\alpha \mathscr{D}' F \to 0 \end{array}$$

which by 2.13 gives $\mathscr{D}' F \xrightarrow{\sim} \mathscr{D}_\alpha \mathscr{D}' F$ as required. For functorality use (D) in 5.2.

2.16. For each $B' \subset B$, let $\mathfrak{p}_{B'}$ denote the subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{p}_\alpha : \alpha \in B'$.

LEMMA. — *Fix* $F \in \mathrm{Ob}\, K$. *Choose* $B' \subset B$ *such that for all* $\alpha \in B$, *the canonical map* $F \xrightarrow{\sim} \mathscr{D}_\alpha F$ *is an isomorphism for all* $\alpha \in B'$. *Then the* $\mathfrak{b}$ *module structure on* F *extends uniquely to a* $\mathfrak{p}_{B'}$ *module structure.*

Uniqueness follows from 2.7 applied to each $\alpha \in B'$. For existence recall 2.14 and choose finite dimentional $U(\mathfrak{g})$ modules E, E′ and $\lambda, \lambda' \in P(R)^+$ such that

$$(*) \qquad E' \otimes \mathbb{C}_{\lambda'} \to E \otimes \mathbb{C}_\lambda \to F \to 0,$$

is an exact sequence of $U(\mathfrak{b})$ modules. Let $E_{B'}(\lambda')$, $E_{B'}(\lambda)$ denote the finite dimensional simple $U(\mathfrak{p}_{B'})$ modules with highest weights λ, λ'. Applying successively the $\mathscr{D}_\alpha : \alpha \in B'$ to (∗) we obtain by right exactness, 2.13 and the hypothesis of the lemma, an exact sequence

$$E' \otimes E_{B'}(\lambda') \xrightarrow{\varphi} E \otimes E_{B'}(\lambda) \to F \to 0,$$

of $U(\mathfrak{b})$ modules. Yet by 2.2 and 2.8 we see that φ is also a $U(\mathfrak{p}_\alpha)$ module map for each $\alpha \in B'$. Hence φ is a $U(\mathfrak{p}_{B'})$ module map and $F = \mathrm{coker}\, \varphi$ is a $U(\mathfrak{p}_{B'})$ module.

2.17. Fix $\lambda \in P(R)^+$ and take $w \in W$. Set $|\lambda| = (\lambda, \lambda)$. It follows from 2.10 that $\mathscr{D}_w \mathbb{C}_\lambda$ is a cyclic $U(\mathfrak{n})$ module generated by a weight vector $f_{w\lambda}$ of weight $w\lambda$. Furthermore, if $\alpha \in B$ satisfies $s_\alpha w > w$, then $|\alpha + w\lambda| > |\lambda|$ and so by 2.12 we have $X_\alpha f_{w\lambda} = 0$. We remark that one may also use this last remark to prove the first assertion inductively using at each step that $\mathscr{D}_{s_\alpha w} \mathbb{C}_\lambda$ is s_α stable. (This is anyway the reasoning behind 2.10.) Actually one can prove the following more precise result. Fix $\alpha \in B$ and let $\mathfrak{m}$ denote the nilradical of $\mathfrak{p}_\alpha$.

LEMMA. — *Suppose* $F \in \mathrm{Ob}\, K$ *is generated by a weight vector* f *of weight* ν *satisfying* $X_\alpha f = 0$. *If* $(\nu, \alpha) < 0$, *then* $\mathscr{D}_\alpha F = 0$. *Otherwise set* $n = (\nu, \alpha^\vee)$ *which is a non-negative integer. Set* $I = \mathrm{Ann}_{U(\mathfrak{m})} f$, *set*

$$J = U(\mathfrak{p}_\alpha) X_\alpha + U(\mathfrak{p}_\alpha) I + \sum_{H \in \mathfrak{h}} U(\mathfrak{p}_\alpha)(H - (\nu, H)),$$

and set $L = J + U(\mathfrak{p}_\alpha) X_{-\alpha}^{n+1}$. *Then* $\mathscr{D}_\alpha F \cong U(\mathfrak{p}_\alpha)/L$.

The first assertion is obvious. For the second set $M = U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} F$. Since $F \cong U(\mathfrak{n})/U(\mathfrak{n}) X_\alpha + U(\mathfrak{n}) I$ [because the sum $U(\mathfrak{b}) X_\alpha + U(\mathfrak{m})$ is direct] we have $M \cong U(\mathfrak{p}_\alpha)/J$. Now L is both ad $\mathfrak{h}$ and ad X_α stable in $U(\mathfrak{m})$ and so by a Lemma of Verma ([1], Lemma 6) it follows that $U(\mathfrak{p}_\alpha)/L$ is a finite dimensional $U(\mathfrak{p}_\alpha)$ quotient of M. Conversely by $\mathfrak{sl}(2)$ module theory the image $\bar{f}$ of $1 \otimes f$ in $\mathscr{D}_\alpha F$ which generates $\mathscr{D}_\alpha F$ over $U(\mathfrak{p}_\alpha)$ and satisfies $J\bar{f} = 0$ must also satisfy $X_{-\alpha}^{n+1} \bar{f} = 0$. Hence the assertion of the Lemma.

Remarks. — We also see that the non-zero vector $X_{-\alpha}^n \bar{f}$ generates $\mathscr{D}_\alpha F$ over $U(\mathfrak{p}_\alpha)$ and since $X_{-\alpha}^{n+1} \bar{f} = 0$, it also generates $\mathscr{D}_\alpha F$ over $U(\mathfrak{b})$. It is a vector of weight $s_\alpha \nu$. If $F \in \mathrm{Ob}\, K$ is a submodule of a finite dimensional $U(\mathfrak{g})$ module E, then the canonical map $F \to \mathscr{D}_\alpha F$ is an embedding. It is generally false that $\mathscr{D}_\alpha F$ embeds in E (a counterexample obtains from Section 4). Yet under the hypothesis of the Lemma it could still be true that $\mathscr{D}_\alpha F$ is a finite dimensional $U(\mathfrak{p}_\alpha)$ submodule of the largest finite dimensional $U(\mathfrak{g})$ quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\alpha)} \mathscr{D}_\alpha F$. (This fails without the hypothesis of the Lemma.) Combined with 2.13 it would imply that the surjection $\mathscr{D}_w \mathbb{C}_\lambda \to U(\mathfrak{n}) e_{w\lambda}$ is bijective.

2.18. Fix $\lambda \in P(R)^+$. For each $w \in W$, we set $F(w\lambda) := U(\mathfrak{n}) e_{w\lambda}$. We should like to show that our surjection $\varphi_w : \mathscr{D}_w \mathbb{C}_\lambda \to F(w\lambda)$ is bijective using backward induction. For this we recall that the homology spaces $H_*(\mathfrak{n}, F(w\lambda))$ admit an $\mathfrak{h}$ module structure and we let a subscript $\mu \in P(R)$ designate the μ weight space.

Lemma. — *Suppose that φ_z is injective for all $z \in W$ satisfying $l(z) > l(w)$. Then if φ_w is not injective, there exists $\mu \in -P(R)^+$ such that $H_1(\mathfrak{n}, F(w\lambda))_\mu \neq 0$ and $(\mu, \mu) < (\lambda, \lambda)$.*

Set $L = \ker \varphi_w$. Since $\mathscr{D}_w \mathbb{C}_\lambda$ is a cyclic $U(\mathfrak{n})$ module generated by a cyclic vector of weight $w\lambda$ (*see* 2.17) the natural map $H_1(\mathfrak{n}, F(w\lambda))_\mu \to (L/\mathfrak{n}L)_\mu$ is surjective. By finite dimensionality and weight space decomposition we have $L \neq 0$ implies $(L/\mathfrak{n}L)_\mu \neq 0$ for some $\mu \in P(R)$. To show that $(\mu, \alpha) \leqq 0 : \alpha \in B$ it is enough to show that $(L/X_\alpha L)_\mu \neq 0$ implies $(\mu, \alpha) \leqq 0$. If $s_\alpha w < w$ then $\mathscr{D}_\alpha \mathscr{D}_w = \mathscr{D}_w$ by 2.7 and 2.15, so in this case $\mathscr{D}_w \mathbb{C}_\lambda$ [and of course also $F(w\lambda)$] is a $U(\mathfrak{p}_\alpha)$ module. Then L is a $U(\mathfrak{p}_\alpha)$ module and the assertion follows from $\mathfrak{sl}(2)$ module theory. If $s_\alpha w > w$, then from the exact sequence

$$0 \to L \to \mathscr{D}_w \mathbb{C}_\lambda \to F(w\lambda) \to 0,$$

we obtain from 2.8 (iv) the exact sequence

$$0 \to \mathscr{D}_\alpha L \to \mathscr{D}_{s_\alpha w} \mathbb{C}_\lambda \to \mathscr{D}_\alpha F(w\lambda) \to 0.$$

Now $\mathscr{D}_{s_\alpha w} \mathbb{C}_\lambda \xrightarrow{\sim} F(s_\alpha w\lambda)$ by the induction hypothesis. Yet $\mathscr{D}_\alpha F(w\lambda) \twoheadrightarrow F(s_\alpha w\lambda)$ and so $\mathscr{D}_\alpha L = 0$. Breaking L up into string modules and using 2.4 proves the assertion in this case.

2.19. We need to get some information on $H_*(\mathfrak{n}, F(w\lambda))$. This is done inductively using 2.7 (iii) and the following lemma:

Lemma. — *Choose $F \in \mathrm{Ob}\, K_\alpha$, $\nu \in P(R)$ such that $(\alpha^\vee, \nu) \in \mathbb{N}$. Then there is a natural isomorphism $H_i(\mathfrak{n}, F)_{s_\alpha \nu} \xrightarrow{\sim} H_{i+1}(\mathfrak{n}, F)_{\nu+\alpha}$ and both spaces vanish if $s_\alpha \nu = \nu + \alpha$. Again if*

$F \in \mathrm{Ob}\,K$ *takes the form* $E \otimes \mathbb{C}_{-\rho} : E \in \mathrm{Ob}\,K_\alpha$, *then* $H_i(\mathfrak{n}, F)_{s_\alpha \nu} \to H_{i+1}(\mathfrak{n}, F)_\nu$ *and both spaces vanish if* $s_\alpha \nu = \nu$.

Both parts are equivalent. Setting $\mathfrak{m}_\alpha = \mathfrak{m}$ which is of codimension one in $\mathfrak{n}$ we have by Hochschild-Serre or direct calculation the exact sequences

$$(*) \qquad 0 \to H_0(\mathfrak{n}/\mathfrak{m}, H_i(\mathfrak{m}, F)) \to H_i(\mathfrak{n}, F) \to H_1(\mathfrak{n}/\mathfrak{m}, H_{i-1}(\mathfrak{m}, F)) \to 0.$$

The result then obtains as an easy consequence of the $\mathfrak{s}_\alpha$ module structure of $H_i(\mathfrak{m}, F)$.

Remark. — This result easily implies the famous Bott-Kostant Theorem. My original proof was a little longer and used a special case of the lemma below which I think is of independent interest. The use of (*) was suggested by the referee and although I had considered this before, I had not previously dared to think that such an easy proof was possible in which not even the centre of $U(\mathfrak{g})$ is used.

2.20. Fix $\nu \in P(R)^+$ and choose $w \in W$, $\gamma \in R^+$ such that $l(s_\gamma w) = l(w) + 1$, that is $w \xrightarrow{\gamma} s_\gamma w$ in the notation of [3], 7.7.3, *see* also 3.6. Recall (as noted in 3.6) that we can write $s_\gamma w = w_1 s_\alpha w_2$ with $\alpha \in B$, $w_1, w_2 \in W$ such that $w = w_1 w_2$ and $l(s_\gamma w) = l(w_1) + l(w_2) + 1$. Set $E = E(\nu)$. Recall that for each $\beta \in B$ we have an exact self-adjoint functor (coherent continuation across the β-wall) θ_β on $\mathcal{O}$ satisfying $\theta_\beta E = 0$ (*see* [10], 3.4, 3.6 for example). Let λ be the unique element of $W.\nu$ in $-P(R)^+$.

LEMMA. — *In the* $\mathcal{O}$ *category for* $\mathfrak{g}$ *one has:*

(i) $\dim \mathrm{Ext}^1(M(w.\lambda), M(s_\gamma w.\lambda)) = 1$.

(ii) *Let* T *denote the unique extension of* $M(s_\gamma w.\lambda)$ *by* $M(w.\lambda)$ *defined through the conclusion of* (i). *Then* $\mathrm{Ext}^i(T, E) = 0$, $\forall i$.

(i) Let $L(\mu)$ denote the unique simple quotient of $M(\mu)$. Recall the Casselman-Schmidt vanishing theorem which asserts that $\mathrm{Ext}^i(M(w.\lambda), M(y.\lambda)) = 0$ unless $y \geqq w$ and $l(y) - l(w) \geqq i$. [This is proved first for $\mathrm{Ext}^i(M(w.\lambda), L(y.\lambda))$ by an easy induction on length.]. In particular $\mathrm{Ext}^2(M(w.\lambda), M(s_\gamma w.\lambda)) = 0$. Substitute this result in the middle term of the short exact sequence of [10], 1.9.5 (iv) (noting a few differences of notation and convention) for $j = 2$. The first term there must vanish and resubstitution in this exact sequence for $j = 1$ and use of [10], 1.9.8, gives the required conclusion. We remark (*see* [10], 3.14) that in this proof T obtains in a rather natural way from the embedding $M(w.\lambda) \subsetneq M(s_\gamma w.\lambda)$.

(ii) The proof is by induction on $l(w_2)$. Let $w_2 = s_1 s_2 \ldots s_k : s_k : s_i = s_{\alpha_i}$ be a reduced decomposition and let T_l denote the unique non-trivial extension of $M(w_1 s_\alpha s_1 \ldots s_l.\lambda)$ by $M(w_1 s_1 \ldots s_l.\lambda)$. By say ([10], 3.6 (ii)) we have $T_0 = \theta_\alpha M(w_1.\lambda)$ and then

$$\mathrm{Ext}^i(T_0, E) \cong \mathrm{Ext}^i(\theta_\alpha M(w_1.\lambda), E) \cong \mathrm{Ext}^i(M(w_1.\lambda), \theta_\alpha E) = 0.$$

Set $\theta_l = \theta_{\alpha_l}$ and consider $\theta_{l+1} T_l$. Since T_l has $L := L(w_1 s_1 s_2 \ldots s_l.\lambda)$ as its unique quotient and $\theta_{l+1} L \neq 0$ ([10], 3.6 (iv), (v)) we obtain from [10], 3.12 (ii), a surjective map $I'' : \theta_{l+1} T \to T_l$. By [10], 3.6, we obtain $[\ker I''] = [M(s_\gamma w'.\lambda)] + [M(w'.\lambda)]$ in the Grothendieck group where $w' = w_1 s_1 s_2 \ldots s_{l+1}$. By Casselman-Schmidt,

$\mathrm{Ext}^1(M(s_\gamma w'.\lambda), M(w'.\lambda))=0$ and so $S:=\ker I''$ is an extension of $M(s_\gamma w'.\lambda)$ by $M(w'.\gamma\lambda)$. We have $\mathrm{Ext}^i(\theta_{l+1} T_l, E)=\mathrm{Ext}^i(T, \theta_{l+1} E)=0$ and $\mathrm{Ext}^i(T_l, E)=0$ by the induction hypothesis. Hence $\mathrm{Ext}^i(S, E)=0$, $\forall i$. By [10], 3.12 (i), we have an embedding $I': T_l \to S$. It easily follows that S cannot be a trivial extension, and hence by (i) and the above, coincides with T_{l+1}. This proves (ii).

2.21. We can now prove the main result of this section. We shall call $\mu\in P(R)^+$ *sufficiently large* if the $(\lambda, \alpha^\vee):\alpha\in B$ are sufficiently large and positive. We call $\mu\in P(R)$ *sufficiently large* if $\lambda:=w\mu\in P(R)^+$ is sufficiently large. Any sufficiently large $\mu\in P(R)$ is regular, i.e. admits a unique $w\in W$ such that $w\mu\in -P(R)^+$.

THEOREM. — *Assume $\lambda\in P(R)^+$ sufficiently large. Then for each $w\in W$ one has a bijection $\mathscr{D}_w \mathbb{C}_\lambda \tilde{\to} F(w\lambda)$. Furthermore, if $H_i(\mathfrak{n}, F(w\lambda))_\mu\neq 0$, then μ is sufficiently large and if $y\mu\in -P(R)^+$ then $l(y)\geqq i$.*

Both assertions are established (simultaneously) by backward induction on $l(w)$. We have already seem (2.18) that the second and the induction hypothesis implies the first assertion. The first result for $w=w_0$ is just 2.13. The second result for $w=w_0$ holds because $F(w_0\lambda)=E(\lambda)$ is a finite dimensional $U(\mathfrak{g})$ module and then the assertion follows from a classical theorem of Kostant. (It also follows from 2.19 as below.)

Now assume that we have proved both assertions for all z with $l(z)>l(w)$. Set $B'=\{\alpha\in B\mid s_\alpha w<w\}$ and

$$\rho_w=\sum_{\alpha\in B'}\omega_\alpha,$$

where ω_α denotes the fundamental weight corresponding to $\alpha\in B$. Observe that

$$s_\alpha(\mu-\rho_w)+\rho_w=\begin{cases} s_\alpha\mu+\alpha:\alpha\in B',\\ s_\alpha\mu:\alpha\notin B'.\end{cases}$$

We assume w fixed and set $y_*\mu=y(\mu-\rho_w)+\rho_w$, $\forall y\in W$.

Assume $H_i(\mathfrak{n}, F(w\lambda))_\mu\neq 0$. Fix $\alpha\in B$ and assume $(\alpha, \mu-\rho_w)\leqq 0$.

If $\alpha\in B'$, then by 2.19 we have a strict inequality and

(1) $$H_i(\mathfrak{n}, F(w\lambda))_\mu \tilde{\to} H_{i+1}(\mathfrak{n}, F(w\lambda))_{s_{\alpha *}\mu}.$$

If $\alpha\notin B'$ we consider the exact sequence

$$0\to F(w\lambda)\to F(s_\alpha w\lambda)\to Q\to 0.$$

We remark that $F(s_\alpha w\lambda)$ is a $U(\mathfrak{p}_\alpha)$ module and furthermore, $\mathscr{D}_\alpha F(w\lambda)\tilde{\to} F(s_\alpha w\lambda)$ by our previous induction hypothesis. We conclude from 2.8 (iii) that $Q\cong E\otimes\mathbb{C}_{-\rho}$ for some $E\in \mathrm{Ob}\, K_\alpha$. This gives the following three possibilities

(i) $$H_i(\mathfrak{n}, F(s_\alpha w\lambda))_\mu\neq 0,$$

or $H_{i+1}(\mathfrak{n}, Q)_\mu \neq 0$ and then by 2.19 we have $(\alpha, \mu-\rho_w)<0$ and either

(ii) $$H_{i+1}(\mathfrak{n}, F(w\lambda))_{s_{\alpha *}\mu} \neq 0,$$

or

(iii) $$H_{i+2}(\mathfrak{n}, F(s_\alpha w\lambda))_{s_{\alpha *}\mu} \neq 0.$$

Now assume that $H_j(\mathfrak{n}, F(z\lambda))_{y_* \mu}=0$ for all $j \in \mathbb{N}$, $z=s_\alpha w>w : \alpha \in B \setminus B'$, and all $y \in W$. It follows from the above analysis that the $y_* \mu : y \in W$ are pairwise distinct.

Recalling that $H_i(\mathfrak{n}, F(w\lambda))$ must vanish unless $i \in \{0, 1, 2, \ldots, \text{Card } R^+\}$ and that $H_n(\mathfrak{n}, F(w\lambda)) \cong \mathbb{C}_{\lambda+2\rho}$ (considered as an $\mathfrak{h}$ module) we conclude that $\mu = w_*(\lambda+2\rho)$ for some $w \in W$ satisfying $l(w)=n-i$. Moreover, since $H_0(\mathfrak{n}, F(w\lambda)) \cong \mathbb{C}_{w\lambda}$ we obtain $w\lambda = w_{0*}(\lambda+2\rho) = w_0\lambda + 2(\rho_w - \rho)$ and consequently $w=w_0$. This is just the usual argument to describe $H_*(\mathfrak{n}, E(\lambda))$.

If $w \neq w_0$ we conclude that $H_j(\mathfrak{n}, F(z\lambda))_{y_* \mu} \neq 0$ for some $j \in \mathbb{N}$, $z=s_\alpha w>w : \alpha \in B \setminus B'$ and some $y \in W$. Moreover, let us assign to the pair $s_{\alpha *}\mu$, μ the number $\nu(s_{\alpha *}\mu, \mu)$ being 1 if $(\alpha, \mu-\rho_w)<0$ and -1 if $(\alpha, \mu-\rho_w)>0$ and define

$$\nu(y_* \mu, \mu) = \sum_{i=1}^{n-1} \nu(y_{i*}\mu, y_{i+1*}\mu),$$

where $y = s_{\alpha_1} \ldots s_{\alpha_n}$ is a reduced decomposition and $y_i = s_{\alpha_i} \ldots s_{\alpha_n}$. Then we can assume without loss of generality that $j = i + \nu(y_* \mu, \mu)$ or $j = i+1+\nu(y_* \mu, \mu)$ depending on whether case (i) or case (iii) holds.

The difficulty in completing the proof is that we must now replace ρ_w by ρ_z in the definition of $(*)$. Yet if λ is sufficiently large it follows by our previous conclusion and backward induction that μ is sufficiently large (i.e. it is close to some $w\lambda$) and hence regular. More precisely for any definition of $(*)$ the $y_* \mu : y \in W$ are pairwise distinct and $y_* \mu \in -P(R)^+$ if and only if $y\mu \in -P(R)^+$. It is then clear that the above formula relating i, j give the second assertion and hence the Theorem.

2.22. Fix a reduced decomposition $w_0 = s_1 s_2 \ldots s_n : s_i = s_{\alpha_i} : \alpha_i \in B$ of w_0 and set $w_i = s_{i+1} s_{i+2} \ldots s_n$. Set $\Delta_{\alpha_i} = \Delta_i$.

COROLLARY (*The Demazure character formula*). — *Assume* $\lambda \in P(R)^+$ *sufficiently large. Then*

$$\text{ch } F(w_i \lambda) = \Delta_{i+1} \Delta_{i+2} \ldots \Delta_n e^\lambda.$$

By 2.20, $\mathscr{D}_{w_i} \mathbb{C}_\lambda \xrightarrow{\sim} F(w_i\lambda)$ the latter being a $U(\mathfrak{b})$ submodule of $E(\lambda)$. Hence by 2.6 and 2.8 (i) we obtain

$$\text{ch } F(w_{i-1}\lambda) = \text{ch } \mathscr{D}_{w_{i-1}} \mathbb{C}_\lambda = \Delta_i (\text{ch } \mathscr{D}_{w_i} \mathbb{C}_\lambda) = \text{ch } F(w_i \lambda).$$

Since $\text{ch } F(\lambda) = e^\lambda$ the result follows by induction.

Remark. — In particular $\Delta_w := \Delta_{\alpha_i}\Delta_{\alpha_2}\dots\Delta_{\alpha_k}$ is independent of the reduced decomposition $w=s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_k}$ of W. This was shown by Demazure purely combinatorially [2].

3. Annihilators

3.1. Fix $\nu\in P(R)^+$. Recall that the weights of $E(\nu)$ of the form $w\nu : w\in W$ occur with multiplicity one. The corresponding vectors in $E(\nu)$ which we denote by $e_{w\nu}$ are called extreme weight vectors. Here we show that ([1], Prop. 1) which fails in general, still holds for extreme weight vectors if we admit the Demazure character formula. This we show implies in turn that the obvious formula for $\mathrm{Ann}_{U(\mathfrak{n}^-)}e_{w\nu}$ is in fact the correct one. One may attempt to verify this formula directly; but it involves some rather difficult and delicate combinatorics.

3.2. Let t denote the Chevalley antiautomorphism of $\mathfrak{g}$ defined by ${}^tX_\alpha=X_{-\alpha}:\forall\alpha\in R$. Fix $\alpha\in B$ and set $\mathfrak{p}={}^t\mathfrak{p}_\alpha$, $\mathfrak{b}^-={}^t\mathfrak{b}$. Let $\mathfrak{m}^-$ denote the nilradical of $\mathfrak{p}^-$. Set $X=X_\alpha$, $Y=X_{-\alpha}$, $H=[X, Y]$, $\mathfrak{s}=\mathbb{C}X\oplus\mathbb{C}H\oplus\mathbb{C}Y$.

Let E be a finite dimensional cyclic $U(\mathfrak{p}^-)$ module with a cyclic weight vector e satisfying $Xe=0$. Obviously $E=U(\mathfrak{n}^-)e$. Choose $n\in\mathbb{N}$ such that $Y^{n+1}e=0$, $Y^n e\neq 0$. Set $f=Y^n e$, $F=U(\mathfrak{n}^-)f$ which is a $U(\mathfrak{b}^-)$ module.

PROPOSITION. — *The following two assertions are equivalent.*

(i) $\mathrm{Ann}_{U(\mathfrak{n}^-)}e=U(\mathfrak{n}^-)\,\mathrm{Ann}_{U(\mathfrak{m}^-)}e+U(\mathfrak{n}^-)Y^{n+1}$.

(ii) $\mathrm{ch}\,E=\Delta_\alpha\,\mathrm{ch}\,F$.

(i) $\Rightarrow$ (ii). This is due to Demazure. We give his proof for completion. Set $I=\mathrm{Ann}_{U(\mathfrak{m}^-)}e$. Since $Xe=0$ it follows that I is ad X (and ad H) stable. Then by a Lemma of Verma ([1], Lemma 6) it follows that $M:=U(\mathfrak{n}^-)/U(\mathfrak{n}^-)I+U(\mathfrak{n}^-)Y^{n+1}$ admits a unique $U(\mathfrak{p}^-)$ module structure with a cyclic $U(\mathfrak{n}^-)$ vector $\bar{1}$ satisfying $X\bar{1}=0$, $H\bar{1}=n\bar{1}$. The $U(\mathfrak{b}^-)$ submodule N generated over $U(\mathfrak{n}^-)$ by $Y^n\bar{1}$ identifies with $U(\mathfrak{n}^-)Y^n/U(\mathfrak{n}^-)I+U(\mathfrak{n}^-)Y^{n+1})\cap U(\mathfrak{n}^-)Y^n$ which is in turn isomorphic to $U(\mathfrak{n}^-)Y^n+U(\mathfrak{n}^-)I/U(\mathfrak{n}^-)I+U(\mathfrak{n}^-)Y^{n+1}$. Consequently M/N is isomorphic to $U(\mathfrak{n}^-)/U(\mathfrak{n}^-)I+U(\mathfrak{n}^-)Y^n$ which by Verma's Lemma admits a similar $U(\mathfrak{p}^-)$ module structure to M except that its cyclic vector $\bar{1}$ now satisfies $H\bar{1}=(n-1)\bar{1}$. We deduce that there exists a finite dimensional $U(\mathfrak{p}^-)$ module E′ and an exact sequence

$$0\to N\to M\to E'\otimes\mathbb{C}_{-\rho}\to 0$$

of $U(\mathfrak{b}^-)$ modules. Hence $\mathrm{ch}\,M=\Delta_\alpha\,\mathrm{ch}\,N$ by 2.8 (iii). Now E is a quotient of M and so isomorphic to M if and only if

$$\mathrm{Ann}_{U(\mathfrak{n}^-)}e=U(n^-)\,\mathrm{Ann}_{U(\mathfrak{m}^-)}e+U(n^-)Y^{n+1}.$$

Furthermore, in this case $F:=U(\mathfrak{n}^-)Y^n e$ and so $F\cong N$. Hence (i)$\Rightarrow$(ii).

(ii) $\Rightarrow$ (i). This follows from 2.17 and 5.4 (iv), (v).

3.3. Choose α, $\beta \in R^+$ such that $\alpha-\beta$ is not a root and fix k, $l \in \mathbb{N}$. Let J be a left ideal of $U(n^-)$ containing $L := U(n^-) X_{-\alpha}^{k+1} + U(n^-) X_{-\beta}^{l+1}$. Take $\nu \in P(R)^+$ such that $(\alpha^\vee, \nu) = k$, $(\beta^\vee, \nu) = l$. Set $r = ((s_\alpha \beta)^\vee, \nu)$.

LEMMA :

(i) *Choose* $s \in \mathbb{N}$ *such that* $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{s+1} \in J$ *for all* $t = 0, 1, 2, \ldots, k$. *Then* $(\operatorname{ad} X_\alpha)^t X_{-s_\alpha \beta}^{s+1} \in J$, $\forall t \in \mathbb{N}$.

(ii) *One may always choose* $s \leqq r$ *in* (i). *In particular* $(\operatorname{ad} X_\alpha)^t X_{-s_\alpha \beta}^{r+1} \in L$, $\forall t \in \mathbb{N}$.

(i) It is immediate that the hypothesis implies that $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{s+1} \in J$, $\forall t \in \mathbb{N}$. We may write $s_\alpha \beta = \beta + v\alpha : v \in \mathbb{N}$ and then $X_{-s_\alpha \beta}^{s+1} \in \mathbb{C} (\operatorname{ad} X_{-\alpha})^{v(s+1)} X_{-\beta}^{s+1} \subset J$. Furthermore, from the relation $[\operatorname{ad} X_\alpha, (\operatorname{ad} X_{-\alpha})^m] = m (\operatorname{ad} H_\alpha - (m-1)) (\operatorname{ad} X_{-\alpha})^{m-1}$ and the fact that $\beta - \alpha$ is not a root one easily shows that $(\operatorname{ad} X_\alpha)^t X_{-s_\alpha \beta}^{s+1} = (\operatorname{ad} X_{-\alpha})^{v(s+1)-t} X_{-\beta}^{s+1}$ up to a non-zero scalar. Hence (i).

(ii) We may assume without loss of generality that α, β are simple roots and let R′ denote the subroot system of R that they generate. Assume first that β is a long root. If $\gamma \in R' \cap R^+$ then either $\gamma = \beta$ or $\gamma = m\alpha + n\beta$ with $m \geqq n$ from which it follows that $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{s+1} \in U(n^-) X_{-\beta}^{s+1-t}$. Yet because β is long we have $(\beta, \beta) = -2(\alpha, \beta)$ and so $((s_\alpha \beta)^\vee, \nu) = (\beta^\vee, \nu) + (\alpha^\vee, \nu)$ and so $r = k + l$. We conclude that $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{r+1} \in L$, $\forall t \leqq k$ as required. Now assume that β is a short root and set $u = (\alpha, \alpha)/(\beta, \beta)$. If $\gamma \in R' \cap R^+$ then either $\gamma = \beta$, or $\gamma = m\alpha + n\beta$ with $m \geqq un$, from which it follows that $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{s+1} \in U(n^-) X_{-\beta}^{s+1-ut}$. Yet because β is short we have $r = ((s_\alpha \beta)^\vee, \nu) = (\beta^\vee, \nu) + u(\alpha^\vee, \nu) = l + uk$. We conclude that $(\operatorname{ad} X_{-\alpha})^t X_{-\beta}^{r+1} \in L$, $\forall t \leqq k$ as required.

Remark. — Of course (ii) also results from Verma module theory. Except for G_2 (and this will cause us some difficulties later on) it implies the well-known result that $X_{-\alpha}^{(\alpha^\vee, \nu)+1} \in \operatorname{Ann}_{U(n^-)} e_\nu$ where e_ν is the highest weight vector of the simple $U(\mathfrak{g})$ module of highest weight $\nu \in P(R)^+$. (The difficulty in G_2 is that not every root, viz., $2\alpha+\beta$ can be written as a product of simple reflections applied to a simple root such that after applying say s_α the resulting root lies on the end of an α-string). Nevertheless, we still get the fact that $M(\nu)$ has a finite dimensional quotient and the above purely combinatorial proof is relatively easy.

3.4. Fix $\nu \in P(R)^+$ and $w \in W$. For each $\alpha \in R^+$, set

$$k_\alpha(w) = \begin{cases} 0: & (\alpha^\vee, w\nu) \leqq 0. \\ (\alpha^\vee, w\nu): & \textit{otherwise}. \end{cases}$$

and

$$I_{w\nu} = \sum_{\alpha \in R^+} U(n^-) X_{-\alpha}^{k_\alpha(w)+1}.$$

It is well-known and easy to check that $I_{w\nu} \subset \operatorname{Ann}_{U(n^-)} e_{w\nu}$.

THEOREM. — *For each* $\nu \in P(R)^+$, $w \in W$ *one has* (*given the Demazure character formula*) *that*

$$I_{w\nu} = \operatorname{Ann}_{U(n^-)} e_{w\nu}.$$

The proof is by decreasing induction on $l(w)$. It is trivial for the unique longest element w_0 of W. Choose $\alpha\in B$ such that $l(s_\alpha w)<l(w)$ and show that the assertion for w implies that for $s_\alpha w$. Set $\mathfrak{p}^-={}^t\mathfrak{p}_\alpha$, with $\mathfrak{m}$ the nilradical of $\mathfrak{p}^-$. Set $X=X_\alpha$, $Y=X_{-\alpha}$. By hypothesis $(\alpha^\vee, w\nu)\leqq 0$ and so $Y e_{w\nu}=0$. From the decomposition $U(\mathfrak{n}^-)=U(\mathfrak{m}^-)\oplus U(\mathfrak{n}^-)Y$ and the induction hypothesis it follows [writing $k_\beta=k_\beta(w)$] that

$$\operatorname{Ann}_{U(\mathfrak{m}^-)} e_{w\nu}=\sum_{k\in\mathbb{N}}(\operatorname{ad} Y)^k\Big(\sum_{\beta\in R^+\setminus\{\alpha\}} U(\mathfrak{m}^-)X_{-\beta}^{k_\beta+1}\Big).$$

Since $\operatorname{Ann}_{U(\mathfrak{m}^-)} e_{s_\alpha w\nu}=s_\alpha(\operatorname{Ann}_{U(\mathfrak{m}^-)} e_{w\nu})$ (*see* proof of 3.2) we obtain from 3.2 and the hypothesis that the Demazure formula holds that

$$(*)\qquad \operatorname{Ann}_{U(\mathfrak{n}^-)} e_{s_\alpha w\nu}=U(\mathfrak{n}^-)Y^{k_\alpha+1}+U(\mathfrak{n}^-)\sum_{k\in\mathbb{N}}(\operatorname{ad} X)^k\Big(\sum_{\beta\in R^+\setminus\{\alpha\}} U(\mathfrak{m}^-)X_{-\beta}^{k_\beta+1}\Big).$$

(Here we have also used that $R^+\setminus\{\alpha\}$ is s_α stable.) Now although the bracketed term on the extreme right hand side of (*) is not ad X stable, yet we show that ad X can be dropped due to the term $U(\mathfrak{n}^-)Y^{k_\alpha+1}$. Then the right hand side of (*) just becomes $I_{s_\alpha w\nu}$ as required.

The above rather remarkable fact is proved by decomposing $R^+\setminus\{\alpha\}$ as a disjoint union of α-strings. Any such string S takes the form $\beta, \beta+\alpha, \ldots, \beta+k\alpha : k\in\mathbb{N}$ where $\beta-\alpha$, $\beta+(k+1)\alpha$ are not roots. It is clear that we only have to show that for each α-string S

$$\operatorname{ad} X\Big(\sum_{\gamma\in S} U(\mathfrak{m}^-)X_{-\gamma}^{k_\gamma+1}\Big)\subset U(\mathfrak{n}^-)Y^{k_\alpha+1}+\sum_{\gamma\in S} U(\mathfrak{n}^-)X_{-\gamma}^{k_\gamma+1}.$$

Decomposing $\mathfrak{m}^-$ into $\mathfrak{s}$ modules, this quickly reduces to the corresponding computation for each rank 2 subsystem separately.

When $\{\alpha, \beta\}$ form an $A_1\times A_1$ system the assertion is trivial. For the remaining cases we recall the well-known result that

$$(**)\qquad \operatorname{Ann}_{U(\mathfrak{n}^-)} e_\nu=\sum_{\alpha\in B} U(\mathfrak{n}^-)X_{-\alpha}^{l_\alpha+1}:\qquad l_\alpha=k_\alpha(1).$$

This follows from the isomorphism $M(\nu)/\sum_{\alpha\in B} M(s_\alpha.\nu)\xrightarrow{\sim} E(\nu)$, the fact that $M(\nu)$ can be identified with $U(\mathfrak{n}^-)$ as a $U(\mathfrak{n}^-)$ module and then each submodule $M(s_\alpha.\nu)$ identifies with $U(\mathfrak{n}^-)X_{-\alpha}^{l_\alpha+1}$.

Now suppose $\{\alpha, \beta\}$ form a system of type A_2. When $w=w_0=s_\alpha s_\beta s_\alpha$ the term in question, namely $U(\mathfrak{n}^-)X_{-(\alpha+\beta)}+U(\mathfrak{n}^-)X_{-\beta}$ is already ad X_α stable so there is nothing to prove. We set $l=l_\alpha$, $l'=l_\beta$. When $w=s_\alpha s_\beta$ the term in question, namely, $U(\mathfrak{n}^-)X_{-(\alpha+\beta)}^{l+1}+U(\mathfrak{n}^-)X_{-\beta}$ is already ad X_α stable so there is nothing to prove. When $w=s_\beta s_\alpha$ a similar result applies. Finally when $w=s_\alpha$ (or when $w=s_\beta$) the required result obtains from (**).

Now suppose $\{\alpha, \beta\}$ form a system of type B_2 with β the long root. When $w=s_\alpha$ (or when $w=s_\beta$) the required assertion follows from (**). Except when $w=s_\alpha s_\beta$ for all other values of w the term in question is trivially stable in the appropriate sense and so there is nothing to prove. Finally, suppose $w=s_\alpha s_\beta$. Then we must show that the term in question, namely

$$U(n^-)X_{-\beta}+U(n^-)X^{l+1}_{-(\alpha+\beta)}+U(n^-)X^{l+l'+1}_{-(2\alpha+\beta)}$$

is $\operatorname{ad} X_\alpha$ stable mod $U(n^-)X^{l+2l'+1}_{-\alpha}$. We apply 3.3(i) to the pair $\{\alpha, \beta\}$ with

$$J=U(n^-)X_{-\beta}+U(n^-)X^{l+1}_{-(\alpha+\beta)}+U(n^-)X^{l+2l'+1}_{-\alpha}.$$

Let us show that we can choose $s=l+l'$ in 3.3(i) (and in fact no better). Consider $(\operatorname{ad} X_{-\alpha})^t X^{s+1}_{-\beta}$. We can write this as a sum of terms of the form $X^u_{-(\alpha+\beta)}X^v_{-(2\alpha+\beta)}X^{s-u-v+1}_{-\beta}$. Since $X^{l+1}_{-(\alpha+\beta)}\in J$ we can assume that $u\leqq l$. Now $u+2v=t\leqq l+2l'$ [for the hypothesis of 3.3(i)]. Then $2u+2v\leqq 2l+2l'+u-l\leqq 2l+2l'$ and so $u+v\leqq l+l'$. Thus if $s=l+l'$, the exponent of $X_{-\beta}$ is strictly positive and so the required term lies in J. Then by the conclusion of 3.3(i) we obtain that $(\operatorname{ad} X_\alpha)^t X^{l+l'+1}_{-s_\alpha\beta}\in J$, $\forall t\in\mathbb{N}$, as required.

Now assume $\{\alpha, \beta\}$ form a system of type G_2 with β the long root. The reader may easily check that using (**) only four bad cases remain. These are described in detail below.

1° $w=s_\beta s_\alpha$. We must show that the term

$$U(n^-)X_{-\alpha}+U(n^-)X^{2l+3l'+1}_{-(\alpha+\beta)}+U(n^-)X^{l+3l'+1}_{-(2\alpha+\beta)}$$
$$+U(n^-)X^{l'+1}_{-(3\alpha+\beta)}+U(n^-)X^{l+2l'+1}_{-(3\alpha+2\beta)},$$

is $\operatorname{ad} X_\beta$ stable mod $U(n^-)X^{l+l'+1}_{-\beta}$. Applying 3.3 (ii) to the pair $\{\beta, 3\alpha+\beta\}$ gives the required result.

2° $w=s_\alpha s_\beta$. We must show that the term

$$U(n^-)X_{-\beta}+U(n^-)X^{l+1}_{-(\alpha+\beta)}+U(n^-)X^{2l+3l'+1}_{-(2\alpha+\beta)}$$
$$+U(n^-)X^{l+2l'+1}_{-(3\alpha+\beta)}+U(n^-)X^{l+l'+1}_{-(3\alpha+2\beta)},$$

is $\operatorname{ad} X_\alpha$ stable mod $U(n^-)X^{l+3l'+1}_{-\alpha}$. We apply 3.3(i) to the pair $\{\alpha, \beta\}$ with

$$J=U(n^-)X_{-\beta}+U(n^-)X^{l+1}_{-(\alpha+\beta)}+U(n^-)X^{l+3l'+1}_{-\alpha}.$$

Let us show that we can choose $s=l+2l'$ in 3.3 (i) (and this is the best we can do). Consider $(\operatorname{ad} X_{-\alpha})^t X^{s+1}_{-\beta}$. We can write this as a sum of terms of the form $X^u_{-(\alpha+\beta)}X^v_{-(2\alpha+\beta)}X^w_{-(3\alpha+\beta)}X^{s-u-v-w+1}_{-\beta}$ (arbitrary ordered product). Since $X^{l+1}_{-(\alpha+\beta)}\in J$ we can assume that $u-v\leqq l$. Now $u+2v+3w=t\leqq l+3l'$ [for the hypothesis of 3.3(i)]. Hence $v+w\leqq l'+(1/3)/(l-(u-v))$. Then

$$u+v+2w\leqq l'+\frac{1}{3}(l+2(u-v))+v+w\leqq 2l'+\frac{1}{3}(2l+u-v)\leqq 2l'+l.$$

If $s=2l'+l$ it follows that the exponent of $X_{-\beta}$ exceeds that of $X_{-(3\alpha+\beta)}$ by at least one and so the required term lies in J. We conclude from 3.3(i) that $(\mathrm{ad}\, X_\alpha)^t X^{l+2l'+1}_{-(3\alpha+\beta)} \in J$, $\forall t \in \mathbb{N}$.

To complete the proof for case 2° it now suffices to show that $X^{2l+3l'+1}_{-(2\alpha+\beta)}$ is contained in the left ideal of $U(\mathfrak{m}^-)$ generated by the action of $\mathrm{ad}\, X_\alpha$ on the left ideal $U(\mathfrak{m}^-)X_{-\beta}+U(\mathfrak{m}^-)X^{l+1}_{-(\alpha+\beta)}+U(\mathfrak{m}^-)X^{l+2l'+1}_{-(3\alpha+\beta)}$. Equivalently it is enough to show that

$$X^{2l+3l'+1}_{-(\alpha+\beta)} \in L := U(\mathfrak{n}^-)X_{-\alpha}+U(\mathfrak{n}^-)X_{-(3\alpha+\beta)}+U(\mathfrak{n}^-)X^{l+1}_{-(2\alpha+\beta)}+U(\mathfrak{n}^-)X^{l+2l'+1}_{-\beta}.$$

(This assertion follows by applying the reflection s_α.)

Consider $(\mathrm{ad}\, X_{-\alpha})^{2l+3l'+1} X^{2l+3l'+1}_{-\beta}$. This lies in L and can be expressed as the sum of the required term and terms which lie in $U(\mathfrak{n}^-)X_{-\beta}$. Furthermore, mod L a typical term can be written in the form

$$X^t_{-(2\alpha+\beta)} X^s_{-(\alpha+\beta)} X^r_{-(3\alpha+2\beta)} X^k_{-\beta},$$

where

$$(*) \qquad \begin{aligned} 2l+3l'+1 &= k+2r+s+t, \\ &= 3r+s+2t. \end{aligned}$$

Note that we do not have to include terms involving $X_{-(3\alpha+\beta)}$ since $X_{-(3\alpha+\beta)} \in L$ and cycling such a term to the left brings down out at worst a term in $X_{-(3\alpha+2\beta)}$ and so is just another term of the above form. For a similar reason we can assume that $t \leqq l$. Now consider $(\mathrm{ad}\, X_{-\alpha})^s (\mathrm{ad}\, X_{-(3\alpha+\beta)})^r X^{3l+2l'+1-u}_{-\beta}$. This lies in L as long as $u \leqq l+l'$. We can write it in the form $(\mathrm{ad}\, X_{-\alpha})^s X^r_{-(3\alpha+2\beta)} X^{2l+3l'+1-u-r}_{-\beta}$ and then in the form $X^s_{-(\alpha+\beta)} X^r_{-(3\alpha+2\beta)} X^{2l+3l'+1-u-r-s}_{-\beta}$ modulo terms of higher exponent in $X_{-\beta}$. We require $2l+3l'+1-u-r-s=k$ and so by the first of the equations (*) that $u=r+t$. From the second equation in (*) it is obvious that the worst situation is when t is as large as possible. That is $t=l$ and then $3r+s=3l'+1$. Since r is an integer it follows that $r \leqq l'$ and so $u \leqq l+l'$ as required. By increasing induction on k and r (the latter to take account of the cycling of the terms $X_{-(2\alpha+\beta)}$, $X_{-(3\alpha+\beta)}$ to the left), we conclude that $X^{2l+3l'+1}_{-(\alpha+\beta)} \in L$, as required. (An easier proof of case 2° obtains from 3.11.)

3° $w=s_\alpha s_\beta s_\alpha$. We must show that the term

$$U(\mathfrak{n}^-)X_{-\beta}+U(\mathfrak{n}^-)X_{-(\alpha+\beta)}+U(\mathfrak{n}^-)X^{l+3l'+1}_{-(2\alpha+\beta)}+U(\mathfrak{n}^-)X^{l+2l'+1}_{-(3\alpha+\beta)}+U(\mathfrak{n}^-)X^{l'+1}_{-(3\alpha+2\beta)}.$$

is $\mathrm{ad}\, X_\alpha$ stable mod $U(\mathfrak{n}^-)X^{2l+3l'+1}_{-\alpha}$. We apply 3.3(i) to the pair $\{\alpha, \beta\}$ with

$$J=U(\mathfrak{n}^-)X_{-\beta}+U(\mathfrak{n}^-)X_{-(\alpha+\beta)}+U(\mathfrak{n}^-)X^{2l+3l'+1}_{-\alpha}+U(\mathfrak{n}^-)X^{l'+1}_{-(3\alpha+2\beta)}.$$

In the notation and conventions of case 2°, we find that $u+2v+3w=t \leqq 2l+3l'$. Since $X_{-(\alpha+\beta)}$, $X^{l'+1}_{-(3\alpha+2\beta)} \in J$ we can assume $u \leqq l'$. Obviously the worst situation is when u is as large as possible and when w is as small as possible, that is when $u=l'$ and $w=0$. Then $2v \leqq 2l+2l'$ and so $u+v \leqq l+2l'$. This means that we can choose $s=l+2l'$ in 3.3(i) and its conclusion gives the required assertion.

4° $w=s_\alpha s_\beta s_\alpha s_\beta$. We must show that the term

$$U(\mathfrak{n}^-)X_{-\beta}+U(\mathfrak{n}^-)X_{-(\alpha+\beta)}+U(\mathfrak{n}^-)X_{-(2\alpha+\beta)}^{l+1}+U(\mathfrak{n}^-)X_{-(3\alpha+\beta)}^{l+l'+1}+U(\mathfrak{n}^-)X_{-(3\alpha+2\beta)},$$

is ad X_α stable mod $U(\mathfrak{n}^-)X_{-\alpha}^{2l+3l'+1}$. We apply 3.3 (i) to the pair $\{\alpha, \beta\}$ with

$$J=U(\mathfrak{n}^-)X_{-\beta}+U(\mathfrak{n}^-)X_{-(\alpha+\beta)}+U(\mathfrak{n}^-)X_{-(2\alpha+\beta)}^{l+1}+U(\mathfrak{n}^-)X_{-\alpha}^{2l+3l'+1}+U(\mathfrak{n}^-)X_{-(3\alpha+2\beta)}.$$

Inspection shows that we can choose $s=l+l'$ in 3.3 (i) and this gives the required assertion.

This completes the proof of the Theorem. For the purists we remark that it should be possible by further computations of the same type to eliminate the use of (**) and hence derive this result as a consequence.

3.5. It turns out that there is another and perhaps better though less elementary way to view the above computations. Fix $\nu\in P(R)^+$. Recall (2.17) for each $w\in W$ that $\mathscr{D}_w\mathbb{C}_\nu$ is generated over $U(\mathfrak{n})$ by a weight vector $f_{w\nu}$ of weight $w\nu$. Choose $\alpha\in B$ such that $s_\alpha w<w$ and set $n=-(\alpha^\vee, w\nu)$. Then by 2.1,

$$\mathrm{Ann}_{U(\mathfrak{n})}f_{w\nu}=U(\mathfrak{n})s_\alpha(\mathrm{Ann}_{U(\mathfrak{m})}f_{s_\alpha w\nu})+U(\mathfrak{n})X^{n+1}$$

where $\mathfrak{m}$ is the nilradical of $\mathfrak{p}_\alpha$ and $X=X_\alpha$. It follows as in 3.4 (without assuming the Demazure character formula) that

$$(*)\qquad \mathrm{Ann}_{U(\mathfrak{n})}f_{w\nu}=\sum_{\alpha\in R^+}U(\mathfrak{n})X_\alpha^{k_\alpha+1}$$

where

$$k_\alpha:=k_\alpha^\nu(w):=\begin{cases}0: & (\alpha, w\nu)\geqq 0,\\ -(\alpha^\vee, w\nu): & \text{otherwise.}\end{cases}$$

Now the calculations in 3.4 show that there are some redundancies in the right hand side of (*). Since $U(\mathfrak{n})$ is a free $U(\mathfrak{n})$ module and since the weights $w\nu+(k_\alpha+1)\alpha:\alpha\in R^+$ are pairwise distinct these redundancies can be precisely specified in terms of the $\mathfrak{h}$ module structure of $H_1(\mathfrak{n}, \mathscr{D}_w\mathbb{C}_\nu)$. So far we have shown that the weights of this module have multiplicity one and lie in the set

$$\{w\nu+\beta\,|\,(\beta, w\nu)\geqq 0\}\cup\{s_\beta w\nu+\beta\,|\,(\beta, w\nu)\leqq 0\}.$$

3.6. Choose $y, w\in W$. If there exists $\beta\in R^+$ such that $s_\beta w=y$ and $l(y)=l(w)-1$ we write $y\xrightarrow{\beta}w$. We write $y\leqq w$ if there exists a chain $y=w_1\xrightarrow{\beta_1}w_2\to\ldots\to w_n=w$.

Then $\leqq$ is just the Bruhat order on W. Set $S(w)=\{\beta\in R^+\,|\,w\beta\in R^-\}$. It is well known that $s_\beta w<w\Leftrightarrow l(s_\beta w)<l(w)\Leftrightarrow\beta\in S(w^{-1})$. Again choose $w_1, w_2\in W$ and set $T_1=S(w_1)\cap S(w_2^{-1})$, $T_2=S(w_1)\setminus T_1$. One easily checks that

$$(*)\qquad S(w_1 w_2)=(S(w_2)\setminus\{-w_2^{-1}T_1\})\perp\!\!\perp w_2^{-1}T_2.$$

In particular

$$l(w_1 w_2)=\text{card}\, S(w_1 w_2)=\text{card}\, S(w_1)+\text{card}\, S(w_2)-\text{card}\, T_1=l(w_1)+l(w_2)-\text{card}\, T_1.$$

One may view T_1 as a cancellation set; for if $\delta\in T_1$ then $w_1 s_\delta<w_1$, $s_\delta w_2<w_2$ and $w_1 w_2=w_1 s_\delta s_\delta w_2$ is written as a product of shorter elements. Suppose further that $\beta:=-w_1\delta$ is a simple root. Then $S(w_1 s_\delta)=S(w_1)\setminus\{\delta\}$, because $\delta\in S(w_1)$ and $l(w_1 s_\delta)=l(s_\beta w_1)=l(w_1)-1$. Setting $w'_1=w_1 s_\delta$ and $T'_1=S(w'_1)\cap S(w_2^{-1})$ we conclude from $(*)$ that $l(w_1 s_\delta w_2)=l(w_1 w_2)+1$ and so $s_\beta w_1 w_2 \overset{\beta}{\leftarrow} w_1 w_2$. Conversely if $-w_1 T_1\cap B=\varnothing$, then for any $\alpha\in B\cap S(w_1^{-1})$ we have $-w_1^{-1}\alpha\notin T_1$ and so if we write $w'_1=s_\alpha w_1$, $T'_1=S(w'_1)\cap S(w_2^{-1})$, then $l(w'_1)=l(w_1)-1$, yet $T'_1=T_1$; that is the cancellation set for the product $w'_1 w_2$ is the same as that for $w_1 w_2$. This gives the following (perhaps known) Lemma.

LEMMA. — *Choose $w\in W$, $\gamma\in R^+$ such that $w^{-1}\gamma\in R^+$. If $l(s_\gamma w)<l(w)-1$, then we can find $\beta\in R^+$ such that $(\beta, \gamma)>0$, $\gamma':=s_\beta\gamma\in R^+$ and $w\overset{\beta}{\leftarrow} s_\beta w>s_{\gamma'} s_\beta w\overset{\beta}{\leftarrow} s_\gamma w$.*

Choose a reduced decomposition $w=s_1 s_2 \ldots s_n$: $s_i=s_{\alpha_i}$ of w. Let j be the largest integer >0 such that $s_{j-1} s_{j-2}\ldots s_1\gamma\in R^+$. Then $\gamma=w_1\alpha_j$ and $s_\gamma w=w_1 w_2$ where $w_1=s_1 s_2\ldots s_{j-1}$, $w_2=s_{j+1}\ldots s_n$. Set $T_1=S(w_1)\cap S(w_2^{-1})$ which is non-empty by the hypothesis and $(*)$. Suppose we can find $\delta\in T_1$ such that $\beta:=-w_1\delta$ is a simple root. Then

$$l(s_\beta w)\leqq l(s_\beta w_1)+l(s_j w_2)=l(w_1)-1+l(s_j w_2)=l(w)-1$$

and equality holds by simplicity. Hence $s_\beta w\overset{\beta}{\rightarrow} w$. Set $\gamma'=s_\beta\gamma\in R^+$. Since $s_\gamma w$ and w have different parity $l(s_\gamma w)\leqq l(w)-3$. Consequently,

$$l(s_{\gamma'}\, s_\beta w)=l(s_\beta s_\gamma w)\leqq l(s_\gamma w)+1\leqq l(w)-2\leqq l(s_\beta w)-1$$

and so $s_{\gamma'}\, s_\beta w<s_\beta w$. Again by choice of δ we have $s_\beta w_1 w_2\overset{\beta}{\leftarrow} w_1 w_2$ (as noted above) and so $s_{\gamma'} s_\beta w\overset{\beta}{\leftarrow} s_\gamma w$. Finally $s_\gamma\beta=w_1 s_j w_1^{-1}\beta=-w_1 s_j\delta\in R^-$ for otherwise $l(w_1 s_j s_\delta)<l(w_1 s_\delta)$ and $l(w)=l(w_1 s_j s_\delta s_j w_2)<l(w_1 s_j)+l(w_2)=l(w)$ which is a contradiction. Hence $(\gamma, \beta)>0$.

If $-w_1 T_1\cap B=\varnothing$, choose $\alpha\in S(w_1)$ and set $w'_1=s_\alpha w_1$, $w'=s_\alpha w$, $\gamma'=s_\alpha\gamma$. Then $l(w')=l(w)-1$, $s_{\gamma'}\, w'=s_\alpha w_1 w_2=w'_1 w_2$ and since the cancellation sets coincide (as noted above) we conclude that $l(w)-l(s_\gamma w)=l(w')-l(s_{\gamma'} w')$. The result then obtains by the obvious induction on length.

3.7. Choose $w\in W$, $\nu\in P(R)^+$. By [11], Thm. 2.9, $F(y\nu)$ is a submodule of $F(w\nu)$ if $y\leqq w$ (and ν regular implies only if). When $y\leqq w$ then from the surjection $\mathscr{D}_w\mathbb{C}_\nu\rightarrow F(w\nu)$ we conclude that $y\nu$ is a weight space of $\mathscr{D}_w\mathbb{C}_\nu$. By 2.7 this occurs with multiplicity one.

We use $f_{y\nu}$ to denote a non-zero vector in $(\mathscr{D}_w\mathbb{C}_\nu)_{y\nu}$. If $y=s_\gamma w$, then we can write $f_{y\nu}=X_\gamma^k f_{w\nu}$ where $k=(\gamma^\vee, y\nu)$. This is a slight abuse of notation since we had previously used $f_{y\nu}$ to denote the canonical generator $f'_{y\nu}$ of $\mathscr{D}_y \mathbb{C}_\nu$. From 3.5 (∗) and 2.12 it easily follows that $\mathrm{Ann}_{U(\mathfrak{n})} f'_{y\nu} \subset \mathrm{Ann}_{U(\mathfrak{n})} f_{y\nu}$ and so $f_{y\nu}$ generates an image of $\mathscr{D}_y \mathbb{C}_\nu$ in $\mathscr{D}_w \mathbb{C}_\nu$.

3.8. Fix $\nu \in P(R)^+$, $w \in W$ and define k_α as in 3.5 (∗). Define $S_w \subset R^+$ (or simply, S) such that $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\nu)_{w\nu+(k_\alpha+1)\alpha} \neq 0$ if and only if $\alpha \in S_w$. Then by 3.5 we conclude that

$$\to \bigoplus_{\alpha \in S} U(\mathfrak{n}) X_\alpha^{k_\alpha+1} \otimes \mathbb{C}_{w\nu} \to U(\mathfrak{n}) \otimes \mathbb{C}_{w\nu} \overset{\varphi}{\to} \mathscr{D}_w \mathbb{C}_\nu \to 0$$

extends to a minimal free resolution of $\mathscr{D}_w \mathbb{C}_\nu$.

LEMMA. — *Take $\gamma \in S_w$. If $w^{-1}\gamma \in R^+$ then $k_\gamma = 0$ and $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\nu)_{w\nu+\gamma}$ is spanned by the image of $X_\gamma \otimes f_{w\nu}$. If $w^{-1}\gamma \in R^-$, set $y = s_\gamma w$. Then $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\nu)_{y\nu+\gamma}$ is spanned by the image of $X_\gamma \otimes f_{y\nu}$.*

Both parts are similar. We prove that the second part follows from the first part applied to $\mathscr{D}_{s_\alpha w\nu}$. Let

$$\sum_{\beta \in R^+} X_\beta \otimes m_{\mu-\beta+w\nu} : m_{\mu-\beta+w\nu} \in (\mathscr{D}_w \mathbb{C}_\nu)_{\mu-\beta+w\nu}$$

be a non-zero representative in $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\nu)_{y\nu+\gamma}$. (By the $\mathfrak{h}$ action this is the most general expression we need consider.) We may write $m_{\mu-\beta+w\nu} = a_{\mu-\beta} f_{w\nu}$ where $a_{\mu-\beta} \in U(\mathfrak{n})$ is of weight $\mu-\beta$ and where $\mu-\beta = (k+1)\gamma-\beta : k = k_\gamma$. By the definition of the boundary map

$$d : H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\nu)_{y\nu+\gamma} \to H_0(\mathfrak{n}, \mathrm{Im}(U(\mathfrak{n}) X_\gamma^{k+1} \otimes \mathbb{C}_{w\nu} \to \mathrm{Ker}\, \varphi))$$

we must have

$$(*) \qquad \sum_{\beta \in R^+} X_\beta a_{\mu-\beta} = X_\gamma^{k+1} \bmod \mathfrak{n}(\mathrm{Ann}_{U(\mathfrak{n})} f_{w\nu}).$$

Consider the exact sequence

$$O \to K \to \mathscr{D}_{s_\gamma w\nu} \to \mathscr{D}_{w\nu} \to Q \to 0$$

of $U(\mathfrak{b})$ modules defined in 3.7 and let $\bar{f}_{w\nu}$ denote the image of $f_{w\nu}$ in Q. We have $X_\gamma^k \bar{f}_{w\nu} = \bar{f}_{s_\gamma w\nu} = 0$ and so $X_\gamma^{k+1} \in \mathfrak{n}(\mathrm{Ann}_{U(\mathfrak{n})} \bar{f}_{w\nu})$. It follows from (∗) that the natural map $H_1(\mathfrak{n}, \mathscr{D}_{w\nu})_{w\nu+\mu} \to H_1(\mathfrak{n}, Q)_{w\nu+\mu}$ is zero. Since $K_{s_\gamma w\nu+\gamma} = 0$, it follows that the map $H_1(\mathfrak{n}, \mathscr{D}_{s_\gamma w\nu})_{s_\gamma w\nu+\gamma} \to H_1(\mathfrak{n}, \mathscr{D}_{w\nu})_{s_\gamma w\nu+\gamma}$ is surjective and so the second assertion reduces to the first.

By a similar reduction the first assertion leads to (∗) with $k=0$. For each $\beta \in R^+$, let (β, ρ) denote its order. Suppose $a_{\gamma-\beta} \notin \mathrm{Ann}_{U(\mathfrak{n})} f_{w\nu}$ with β of minimal order. Then $|\gamma-\beta+w\nu| \leqq |\nu|$ and so $(\beta, w\nu) \geqq (\gamma, w\nu)$ with equality only if $\beta = \gamma$. We show that $a_{\gamma-\beta}$ can be written as a sum of monomials with factors X_δ where $\delta = \gamma$ or

$(\delta, w\nu) > (\gamma, w\nu)$. Otherwise shifting X_δ to the left (thereby gaining only monomials of lower degree) we can write the monomial as $X_\delta b_{\mu-\beta-\delta}$. Our assumption on δ implies that $|\beta+\gamma-\beta-\delta+w\nu| > |\nu|$ and so $X_\beta b_{\gamma-\beta-\delta} f_{w\nu}=0$. Then $d(X_\beta \wedge X_\delta \otimes b_{\gamma-\beta-\delta} f_{w\nu}) = [X_\beta, X_\delta] \otimes b_{\gamma-\beta-\delta} f_{w\nu} - X_\beta \otimes X_\delta b_{\gamma-\beta-\delta} f_{w\nu}$, so the unwanted term can be eliminated. Induction on order shows that the assertion holds for all $\beta \in R^+$. We conclude that each $X_\beta a_{\gamma-\beta}$ can be written as a sum of monomials $X_{\beta_1} X_{\beta_2} \ldots X_{\beta_{n+1}}$ with $\sum \beta_i = \gamma$ and $(\beta_i, w\nu) > (\gamma, w\nu)$ *or* $\beta_i = \gamma$. Thus either $\beta=\gamma$ (which is the required solution) or $n<0$ which is impossible. This proves the Lemma.

3.9. Fix $\nu \in P(R)^+$, $w \in W$. Define S as in 3.8. Call $\mathfrak{g}$ simply-laced if all roots have the same length, e. g. in type A_n.

PROPOSITION. — Fix $w \in W$, $\gamma \in R^+$.

(i) *If* $w^{-1}\gamma \in R^-$, *then* $s_\gamma w\nu + \gamma$ *is a weight of* $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\lambda)$ *only if* $l(s_\gamma w) = l(w)-1$.

(ii) *Assume* $\mathfrak{g}$ *simply-laced. If* $w^{-1}\gamma \in R^+$, *Then* $s_\gamma w\nu+\gamma$ *is a weight of* $H_1(\mathfrak{n}, \mathscr{D}_w \mathbb{C}_\lambda)$ *only if* $l(s_\gamma w) = l(w)+1$.

Both parts are similar (surprisingly!). Consider the first. Assume $l(s_\gamma w) < l(w)-1$ and let β, γ' be defined by the conclusion of 3.6. We have $(\beta, \gamma') < 0$ and so the pair β, γ' generate a root system of rank 2. We have $\gamma = \gamma' + n\beta : n \in \mathbb{N}^+$ and we set $\delta = \gamma' + (n-1)\beta$, $y = s_\gamma w$. Then $s_\beta y > y$ and so

$$(y\nu+\beta, y\nu+\beta) = (\nu, \nu) + (\beta, \beta) + 2(y^{-1}\beta, \nu) > (\nu, \nu).$$

Hence $X_\beta f_{y\nu}=0$ by 2.12. Similarly if $s_\delta y > y$, then $X_\delta f_{y\nu}=0$. Let d be the boundary operator. Then $d(X_\beta \wedge X_\delta \otimes f_{y\nu}) = X_\gamma \otimes f_{y\nu}$ in this case and we conclude that $\gamma \notin S$. This part of the argument applies in a very similar manner to the second part.

If $(\gamma', \gamma') \geqq (\beta, \beta)$, then we always have $s_\gamma \delta = -\beta$. Then

$$y^{-1}\delta = w^{-1} s_\gamma \delta = -w^{-1}\beta \in R^+$$

and so we have $s_\delta y > y$ as required. (For example in G_2, we have $n=3$, so $\delta = \gamma' + 2\beta$ and $s_{\gamma'+3\beta}(\gamma'+2\beta) = -\beta$.) It remains to consider the case $(\gamma', \gamma') < (\beta, \beta)$. Set $\gamma' = \alpha_1$, $\beta = \alpha_2$. Then $\gamma = s_{\alpha_2}\alpha_1 = \alpha_1+\alpha_2$, so $n=1$ and $\delta = \alpha_1$. We can suppose that $s_\delta y < y$ (this can happen). Summarizing

$$(*) \qquad (\alpha_1+\alpha_2, y\nu) \geqq 0, \qquad (\alpha_1, y\nu) \leqq 0, \qquad (\alpha_2, y\nu) \geqq 0.$$

If $(\alpha_2, y\nu)=0$, then $(\alpha_1, y\nu)=0$ and in this case $d(X_{\alpha_1} \wedge X_{\alpha_2} \otimes f_{y\nu}) = X_\gamma \otimes f_{y\nu}$ as required. Otherwise $k := (\alpha_2^\vee, y\nu) > 0$ and recalling that $s_\beta s_\gamma w < w$ we have $f := f_{s_{\alpha_2} y\nu} \in \mathscr{D}_w \mathbb{C}_\lambda$. Since $(\alpha_2, y\nu) \geqq 0$ we have $X_{\alpha_2} f_{y\nu} = 0$ and so

$$(**) \qquad d(X_{\alpha_1} \wedge X_{\alpha_2} \otimes f_{y\nu}) = X_{\alpha_1+\alpha_2} \otimes f_{y\nu} + X_{\alpha_2} \otimes X_{\alpha_1} f_{y\nu}.$$

Again, up to a non-zero scalar, $X_{\alpha_2}^k f = f_{y\nu}$ and $X_{\alpha_1} f = 0$. Setting $g = X_{\alpha_2}^{k-1} f$ we obtain $X_{\alpha_1} f_{y\nu} = k X_{\alpha_1+\alpha_2} g$. Thus

$$(***) \qquad d(X_{\alpha_1+\alpha_2} \wedge X_{\alpha_2} \otimes g) = -X_{\alpha_1+\alpha_2} \otimes f_{y\nu} + \frac{1}{k} X_{\alpha_2} \otimes X_{\alpha_1} f_{y\nu}.$$

Since $k>0$ we obtain from (**) and (***) that $X_\gamma \otimes f_{y\nu} \in \operatorname{Im} d$ as required. This part of the argument fails for the second part which indeed can fail (in say type G_2).

3.10. To see the importance of 3.9 suppose we have shown that $\mathscr{D}_w \mathbb{C}_\nu \tilde{\to} F(w\nu)$. Choose $\beta \in R^+$ such that $s_\beta w \overset{\beta}{\to} w$, set $y=s_\beta w$ and consider the exact sequence

$$(*) \qquad 0 \to F(y\nu) \to F(w\nu) \to Q \to 0$$

and let $\bar{e}_{w\nu}$ denote the image of $e_{w\nu}$ in Q. Define S as in 3.8 and set $k=k_\beta$ in (*) of 3.8.

LEMMA :

$$\operatorname{Ann}_{U(\mathfrak{n})} \bar{e}_{w\nu} = \operatorname{Ann}_{U(\mathfrak{n})} e_{w\nu} + U(\mathfrak{n}) X_\beta^k.$$

From (*) we have an exact sequence

$$(**) \qquad \to H_1(\mathfrak{n}, F(y\nu)) \to H_1(\mathfrak{n}, F(w\nu)) \overset{\psi}{\to} H_1(\mathfrak{n}, Q) \to H_0(\mathfrak{n}, F(y\nu)) \to 0.$$

Take $\gamma \in S \cap S(w^{-1})$. By 3.10 we have $l(s_\gamma w) = l(w)-1 = l(y)$. If $s_\gamma w \neq y$ then $e_{s_\gamma w\nu}$ is not a weight vector in $F(y\nu)$ and we conclude from 3.9 that the restriction of ψ to the $s_\gamma w\nu+\gamma$ weight space is injective. Conversely if $s_\gamma w = y$ then $\gamma=\beta$ and $X_\beta^k e_{w\nu} = e_{y\nu}$, so $X_\beta^k \bar{e}_{w\nu}=0$. In this case the restriction of ψ is the zero map.

Take $\gamma \in S \cap (R^+ \setminus S(w^{-1}))$. Since $e_{w\nu}$ is not a weight vector in $F(y\nu)$ we again conclude from 3.9 that the restriction of ψ to the $w\nu+\gamma$ weight space is injective. This shows that $H_1(\mathfrak{n}, Q)$ may be computed from $H_1(\mathfrak{n}, F(w\nu))$ and gives in particular the assertion of the Lemma.

3.11. In certain favourable circumstances 3.10 can be used to show that the quotient $F(w\nu)/F(y\nu)$ takes the form $F(w\nu') \otimes \mathbb{C}_{w(\nu-\nu')}$ for some $\nu' \in P(R)^+$. For example take $w=w_0$, $y=w_0 s_\alpha : \alpha \in B$. Then $\mathscr{D}_{w_0} \mathbb{C}_\nu \tilde{\to} F(w_0\nu)$ by 2.13 and then by 3.9 (or Kostant's Theorem) $S \subset B$ and in fact equality holds. Assume $(\nu, \alpha)>0$ to avoid trivialities. Now choose $\nu' \in P(R)^+$ such that for all $\beta \in B$

$$(*) \qquad -(\nu', w_0\beta^\vee) = \begin{cases} -(\nu, w_0\beta^\vee) : & w_0\beta \neq \alpha, \\ -(\nu, w_0\beta^\vee) - 1 : & w_0\beta = \alpha. \end{cases}$$

(i. e. $\nu-\nu'$ is the fundamental weight associated to α).

Then from 3.10 one easily checks that one has an exact sequence

$$0 \to F(w_0 s_\alpha \nu) \to F(w_0\nu) \to F(w_0\nu') \otimes \mathbb{C}_{w_0(\nu-\nu')} \to 0$$

of $U(\mathfrak{b})$ modules. Now the $\mathfrak{n}$ homology group of the second two terms are completely known (by Kostant's Theorem) and so $H_i(\mathfrak{n}, F(w_0 s_\alpha \nu))$ can be explicitly computed. In particular $H_1(\mathfrak{n}, F(w_0 s_\alpha \nu))$ can be determined and from this we can show (as in 2.21

using 2.18) that $\mathscr{D}_{w_0 s_\alpha}\mathbb{C}_\nu \xrightarrow{\sim} F(w_0 s_\alpha \nu)$. Finally, we remark that resulting determination of Ann $e_{w_0 s_\alpha \nu}$ can be used to simplify the calculations in 3.4. In particular the worst case in G_2 (namely case 2°) and the only non-trivial case in B_2 can be so resolved.

The above analysis fails as an inductive method for computing the $H_*(\mathfrak{n}, F(w\nu))$ because the quotient Q usually fails to take a nice form. When rank $\mathfrak{g}>2$ this is because one can have card $(S \cap S(w^{-1}))>$ rank $\mathfrak{g}$, yet it can even fail in B_2 and G_2 because the equations analogous to $(*)$ do not yield a solution in $P(R)$.

3.12. For each $w \in W$, set

$$S_-(w)=\{\gamma \in R^+ \mid s_\gamma w \xrightarrow{\gamma} w\}, \qquad S_+(w)=\{\gamma \in R^+ \mid w \xrightarrow{\gamma} s_\gamma w\}.$$

Take $\nu \in P(R)^+$ and define S_w as in 3.8 (with respect to $\mathscr{D}_w\mathbb{C}_\nu$). For $\mathfrak{g}$ simply-laced, then by 3.10 $S_w \subset S_-(w) \cup S_+(w)$ and it is natural to conjecture that equality holds. Here some difficulties occur if ν is not regular for then the definition of S_w is ambiguous and one can have $S_-(w) \neq S_-(w')$ even though $w\nu = w'\nu$ (e.g. take $w=s_\alpha$: α simple, $w'=1$ in type A_2). To avoid this difficulty we assume $\nu \in P(R)^+$ regular from now on.

LEMMA. — *Assume that* $\mathscr{D}_w\mathbb{C}_\nu \xrightarrow{\sim} F(w\nu)$, $\forall w \in W$ *(for example if* ν *is sufficiently large). If* $S_w \supset S_-(w)$, $\forall w \in W$, *then* $S_y \supset S_+(y)$, $\forall y \in W$.

Choose $\beta \in S_+(y)$, set $w=s_\beta y$ and consider the exact sequence $(**)$ of 3.10. We showed that the restriction of ψ to the $s_\beta w\nu+\beta$ weight space is the zero map. By the hypothesis $\beta \in S_w$, and so $\beta \in S_y$ as required.

3.13. Fix $\nu \in P(R)^+$, $w \in W$ and define k_β: $\beta \in R^+$ as in 3.5. To show that $S_w \supset S_-(w)$ we must show that for each $\gamma \in S_-(w)$ that

$$(*) \qquad X_\gamma^{k_\gamma+1} \notin \sum_{\beta \in S_+} U(\mathfrak{n})X_\beta + \sum_{\beta \in S_- \setminus \{\gamma\}} U(\mathfrak{n})X_\beta^{k_\beta+1}.$$

We can almost prove this. In fact we have the:

LEMMA. — *Assume* ν *regular. Then for all* $\gamma \in S_-$

$$(**) \qquad X_\gamma^{k_\gamma} \notin \sum_{\beta \in S_+} U(\mathfrak{n})X_\beta + \sum_{\beta \in S_- \setminus \{\gamma\}} U(\mathfrak{n})X_\beta^{k_\beta}.$$

Set $y=s_\gamma w$. By 2.12 and the regularity of ν, we conclude that $y\nu$ is not a weight subspace of any $\mathscr{D}_z\mathbb{C}_\nu : z=s_\beta w : \beta \in S_-(w)$. Recalling 3.7 it follows that $f_{y\nu}$ has a non-zero image in

$$\mathscr{D}_w\mathbb{C}_\nu / \sum_{\beta \in S_- \setminus \{\gamma\}} \operatorname{Im}(\mathscr{D}_{s_\beta w}\mathbb{C}_\nu \to \mathscr{D}_w\mathbb{C}_\nu).$$

Now for each $z \in W$ let $\bar{f}_{z\nu}$ denote the image of $f_{z\nu}$ in the above quotient. We have $X_\beta \bar{f}_{w\nu}=0$ if $\beta \in S_+$ and by the remarks in 3.7 we have

$$X_\beta^{k_\beta}\bar{f}_{w\nu}=\bar{f}_{s_\beta w\nu}=0, \qquad \forall \beta \in S_- \setminus \{\gamma\}.$$

Yet $X_{\gamma}^{k_{\gamma}} \bar{f}_{y\nu} \neq 0$ as noted above. This proves the Lemma.

3.14. In certain favourable circumstances (perhaps always for ν regular) we can choose $\nu' \in P(R)^{+}$ such that $k_{\gamma}^{\nu'} = k_{\gamma}^{\nu} + 1$ and $k_{\beta}^{\nu'} \leqq k_{\beta}^{\nu} + 1$, $\forall \beta \in S_{-} \setminus \{\gamma\}$. If so then (**) (for ν′) implies (*) (for ν). For example we have the:

COROLLARY. — *Assume* ν *regular and* $\mathfrak{g}$ *simple of type* A_n *(Cartan notation). Then* $S_w \supset S_{-}(w)$, *for all* $w \in W$. *In particular* $S_w = S_{+}(w) \perp\!\!\perp S_{-}(w)$, *for* ν *sufficiently large.*

Choose $\gamma \in S_{-}(w)$ and let ω be a fundamental weight such that $(\omega, w^{-1}\gamma^{\vee}) = -1$. Then $0 \geqq (\omega, w^{-1}\beta^{\vee}) \geqq -1$, $\forall \beta \in S_{-}(w)$ and it suffices to take $\nu' = \omega + \nu$.

Remark. — We have now obtained more precise results which will be described elsewhere.

4. A counterexample of Kac

4.1. As pointed out by Kac one does not have to go very far to find a counterexample to [1], Prop. 11. Indeed let $\mathfrak{g}$ be of type $\mathfrak{sl}(3)$ with α_1, α_2 a choice of simple roots. Take the six-dimensional representation with highest weight $2\omega_1$ [where ω_1 is the fundamental weight satisfying $(\omega_1, \alpha_1) = 1$, $(\omega_1, \alpha_2) = 0$]. The weights in this module all have multiplicity one. Let e be a vector of weight $2\omega_1 - \alpha_1$. This is not an extreme weight vector. Set $Y_i = X_{-\alpha_i}$, $Z = X_{-\alpha_1-\alpha_2}$. Let $\mathfrak{m}$ be the subalgebra of $\mathfrak{n}^{-}$ with basis Y_1, Z. One checks that $X_{\alpha_2} e = 0$ and $Y_2 e \neq 0$, $Y_2^2 e = 0$. Yet

$$\operatorname{Ann}_{U(\mathfrak{n}^-)} e \supsetneq U(\mathfrak{n}^-) \operatorname{Ann}_{U(\mathfrak{m}^-)} e + U(\mathfrak{n}^-) Y_2^2 =: J,$$

in contradiction to the assertion of [1], Prop. 11. Indeed one notes that an appropriate linear combination of $Y_1 Y_2$ and $Y_2 Y_1$ must lie in $\operatorname{Ann}_{U(\mathfrak{n}^-)} e$ [because the $2\omega_1 - 2\alpha_1 - \alpha_2$ weight subspace of $E(2\omega_1)$ has multiplicity 1]; yet such a linear combination cannot lie in J (by weight space decomposition and the fact that $Ze \neq 0$). One may also remark that $U(\mathfrak{n}^-)e/U(\mathfrak{n}^-)Y_2 e$ is isomorphic to $\mathbb{C}e \oplus \mathbb{C}Y_1 e$ and the latter does not admit an appropriate $\mathfrak{sl}(2)$ module structure as would implied by the analysis following [1], Prop. 11.

4.2. We can also describe the difficulty in the proof of [1], Prop. 11, from the above example. Let $\mathfrak{s}$ denote the $\mathfrak{sl}(2)$ subalgebra generated by X_{α_2}, $X_{-\alpha_2}$. Since $U(\mathfrak{n})e$ is an $\mathfrak{s}$ module, so is $U(\mathfrak{n}^-)/\operatorname{Ann}_{U(\mathfrak{n}^-)} e$. By [1], Lemma 6, $U(\mathfrak{n}^-)/J$ admits an $\mathfrak{s}$ module structure so that the canonical projection $\varphi : U(\mathfrak{n}^-)/J \to U(\mathfrak{n}^-)/\operatorname{Ann}_{U(\mathfrak{n}^-)} e$ is a map of $\mathfrak{s}$ modules. Restricted to the generating subspace $V := U(\mathfrak{m}^-)/\operatorname{Ann}_{U(\mathfrak{m}^-)} e$ this map is in injective by construction. Yet one cannot conclude that φ itself is injective even if one further remarks that V is $\mathfrak{b}$ stable. In fact $U(\mathfrak{s})V$ is the direct sum of a one-dimensional module E_0, a two dimensional module E_1, a three dimensional module E_2 and the map φ has kernel E_0. Yet V is formed from the highest weight spaces of E_1, E_2 and a linear combination of the zero weight subspaces of E_0, E_2, so $\varphi|_V$ is injective.

4.3. We may also use this example to illustrate the discussion in 2.17. Let F denote the $U(\mathfrak{n})$ submodule of $E:=E(2\omega_1)$ generated by $f=Y_2e$ and set $\alpha=\alpha_2$. From the embedding $F \subsetneq E$ we obtain by functoriality a map $\mathscr{D}_\alpha F \to \mathscr{D}_\alpha E \xrightarrow{\sim} E$. One checks, however, that this map is not an embedding. On the other hand, one does have an embedding of $\mathscr{D}_\alpha F$ in the largest finite dimensional $U(\mathfrak{g})$ quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\alpha)} \mathscr{D}_\alpha F$ which is isomorphic to $E(2\omega_1) \oplus E(\omega_2)$.

5. Derived functors and Bott's theorem

5.1. Fix $\alpha \in B$. Because K does not have enough projectives it is quite tricky to define the derived functor of $\mathscr{D}_\alpha$. Instead of taking projective resolutions we shall use the resolutions used in 2.15. Via our main theorem this will also allow us to define the derived functors of the $\mathscr{D}_w$. We then prove a version of Bott's Theorem for $\mathscr{D}_{w_0}$ and a strengthening of a version of Demazure's vanishing Theorem ([1], 5.5, Cor. 2) for $\mathscr{D}_w$. The latter applies to studying tensor products $E \otimes F(w\lambda)$, where E is a finite dimensional $U(\mathfrak{g})$ module. Throughout this section E_i, λ_i denote respectively (a family) of finite dimensional $U(\mathfrak{g})$ modules, respectively of elements of $P(R)^+$.

5.2. Fix $F \in \operatorname{Ob} K$. By 2.14 we have an exact sequence (possibly infinite)

$$(*) \qquad \ldots \to E_i \otimes \mathbb{C}_{\lambda_i} \to \ldots \to E_1 \otimes \mathbb{C}_{\lambda_1} \to F \to 0$$

of $U(\mathfrak{b})$ modules. We shall call $(*)$ a *standard resolution*. We remark that one may always assume the λ_i to be sufficiently large. The analysis of 2.14 also shows that for any short exact sequence

$$0 \to F' \to F' \to F'' \to 0$$

we obtain an exact commuting diagram of the form

$$(D) \qquad \begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & E_i' \otimes \mathbb{C}_{\lambda_i} & \to \ldots \to & E_1' \otimes \mathbb{C}_{\lambda_1} & \to & F' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & E_i \otimes \mathbb{C}_{\lambda_i} & \to \ldots \to & E_1 \otimes \mathbb{C}_{\lambda_1} & \to & F & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & E_i'' \otimes \mathbb{C}_{\lambda_i} & \to \ldots \to & E_1'' \otimes \mathbb{C}_{\lambda_1} & \to & F'' & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

Moreover, say for λ_1 sufficiently large, we can choose say E_1 to be the finite dimensional $U(\mathfrak{g})$ module with highest weights $-w_0\lambda_1+\Omega(F)$ and then $E_i=E_i' \oplus E_i''$. Recall that $\mathscr{D}_\alpha$ is a right exact functor which commutes with tensoring by finite dimensional $U(\mathfrak{g})$ modules (2.5). Thus from $(*)$ we obtain a complex

$$(**) \qquad \ldots \to E_i \otimes \mathscr{D}_\alpha \mathbb{C}_{\lambda_i} \to \ldots \to E_1 \otimes \mathscr{D}_\alpha \mathbb{C}_{\lambda_1} \to 0.$$

LEMMA. — *The cohomology* $H^*(E_i \otimes \mathscr{D}_\alpha \mathbb{C}_{\lambda_i})$ *of* (**) *is independent of the standard resolution chosen.*

It is enough to show that given short exact sequences

$$0 \to M \to E \otimes \mathbb{C}_\lambda \to F \to 0,$$
$$0 \to M' \to E' \otimes \mathbb{C}_{\lambda'} \to F \to 0,$$

one has $\operatorname{Ker}(\mathscr{D}_\alpha M \to E \otimes \mathscr{D}_\alpha \mathbb{C}_\lambda) \xrightarrow{\sim} \operatorname{Ker}(\mathscr{D}_\alpha \mathbb{C}_\lambda)$, where the map is a composition of connecting homomorphisms.

Taking $\mu \in P(R)^+$ sufficiently large we obtain from (D) an exact commuting diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M_1 & \to & E_1 \otimes \mathbb{C}_\mu & \to & F & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M_2 & \to & E_2 \otimes \mathbb{C}_\mu & \to & E \otimes \mathbb{C}_\lambda & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & M_3 & \to & E_3 \otimes \mathbb{C}_\mu & \to & M & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Furthermore, by the remark following (D) we may replace the right hand column by the primed quantities without changing the first row. Now apply $\mathscr{D}_\alpha$ to the above diagram taking account of 2.5 and 2.8 (iv). This gives the exact commuting diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & L_1 & \to & \mathscr{D}_\alpha M_1 & \to & E_1 \otimes \mathscr{D}_\alpha \mathbb{C}_\mu & \to & \mathscr{D}_\alpha F & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & L_2 & \to & \mathscr{D}_\alpha M_2 & \to & E_2 \otimes \mathscr{D}_\alpha \mathbb{C}_\mu & \to & E \otimes \mathscr{D}_\alpha \mathbb{C}_\lambda & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & L_3 & \to & \mathscr{D}_\alpha M_3 & \to & E_3 \otimes \mathscr{D}_\alpha \mathbb{C}_\mu & \to & \mathscr{D}_\alpha M & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & & L & & \\
 & & & & & & & & \uparrow & & \\
 & & & & & & & & 0 & &
\end{array}$$

[Here we used that M_1, $E_1 \otimes \mathbb{C}_\mu$ can be taken to be F_3 in 2.8 (iv).] By a standard result (*cf* [14], Prop. 2.10) we obtain an exact sequence

$$0 \to L_3 \to L_2 \to L_1 \xrightarrow{d} \mathscr{D}_\alpha M \to E \otimes \mathscr{D}_\alpha \mathbb{C} \to \mathscr{D}_\alpha F \to 0$$

where d is the boundary (or connecting) homomorphism. If we further observe that $E \otimes \mathbb{C}_\lambda$, M can be taken to be F_3 in 2.8 (iv) we obtain $L_2 = L_3 = 0$ and so we deduce an isomorphism $L_1 \xrightarrow{\sim} L$ as required.

5.3. We let $\mathscr{D}_\alpha^i F$ denote the ith cohomology of (**). Although $E \otimes \mathbb{C}_\lambda : \lambda \in P(R)^+$ is not projective in K, we still obtain taking λ sufficiently large and using the comment following (D) an analogue of the "nine Lemma". For example we presented the argument of 5.2 so that it is clear that from any short exact sequence

$$0 \to F' \to F \to F'' \to 0$$

in K we obtain a long exact sequence

$$0 \to \mathscr{D}_\alpha^1 F' \to \mathscr{D}_\alpha^1 F \to \mathscr{D}_\alpha^1 F'' \to \mathscr{D}_\alpha^0 F' \to \mathscr{D}_\alpha^0 F \to \mathscr{D}_\alpha^0 F'' \to 0$$

in K_α. (In particular $\mathscr{D}_\alpha^i = 0 : i \geqq 2$.) Again we deduce from 2.5 that $\mathscr{D}_\alpha^i$ commutes with tensoring by finite dimensional $U(\mathfrak{p}_\alpha)$ modules.

5.4. For each $\mu \in P(R)$ we let $E_\alpha(\mu)$ denote the unique simple finite dimensional quotient of the Verma module $M_\alpha(\mu) := U(\mathfrak{p}_\alpha) \otimes_{U(\mathfrak{b})} \mathbb{C}_\mu$. [In particular $E_\alpha(\mu) = 0$ if $(\mu, \alpha) < 0$.]

LEMMA. — *Take* $\lambda \in P(R)$ *with* $(\lambda, \alpha) \geqq 0$. *Then*:

(i) $\mathscr{D}_\alpha^i \mathbb{C}_{-\lambda} = 0$, $\forall i \in \mathbb{N}$ *if* $(\lambda, \alpha^\vee) = 1$.

(ii) $\mathscr{D}_\alpha^0 \mathbb{C}_\lambda \tilde{\to} \mathscr{D}_\alpha^1 \mathbb{C}_{s_\alpha . \lambda} \tilde{\to} E_\alpha(\lambda) \tilde{\to} \mathscr{D}_\alpha^0 E_\alpha(\lambda)$.

(iii) $\mathscr{D}_\alpha^i F = 0$, $\forall i \geqq 2$, $\forall F \in \operatorname{Ob} K$.

(iv) $\operatorname{ch} \mathscr{D}_\alpha^0 F - \operatorname{ch} \mathscr{D}_\alpha^1 F = \Delta_\alpha \operatorname{ch} F$.

(v) $\mathscr{D}_\alpha^1 F = 0$ *if* F *is a submodule of* $E_\alpha(\mu) \otimes \mathbb{C}_\lambda$.

(i) Since $\mathbb{C}_\mu : (\mu, \alpha) = 0$ admits the structure of a one-dimensional $U(\mathfrak{p}_\alpha)$ module, we reduce to the case when $-s_\alpha \lambda \in P(R)^+$. Then the assertion follows from the exact sequence

$$0 \to \mathbb{C}_{-s_\alpha \lambda} \to E_\alpha(-s_\alpha \lambda) \to \mathbb{C}_{-\lambda} \to 0$$

and successive applications of 2.8 (ii), (iv).

(ii) We use Demazure's trick [8] for proving Bott's theorem. We have exact sequences

$$(*) \qquad 0 \to \mathbb{C}_\lambda \to E_\alpha(\lambda) \to E_\alpha(\lambda - \alpha + \rho) \otimes \mathbb{C}_{-\rho} \to 0,$$

$$(**) \qquad 0 \to E_\alpha(\lambda) \to E_\alpha(\lambda + \alpha - \rho) \otimes \mathbb{C}_{\rho - \alpha} \to \mathbb{C}_{s_\alpha . \lambda} \to 0,$$

of $U(\mathfrak{b})$ modules. From (*), (i) and 2.5 we obtain $\mathscr{D}_\alpha^i \mathbb{C}_\lambda \tilde{\to} \mathscr{D}_\alpha^i E_\alpha(\lambda)$. Similarly from (**) we obtain $\mathscr{D}_\alpha^{i+1} \mathbb{C}_{s_\alpha . \lambda} \tilde{\to} \mathscr{D}_\alpha^i E_\alpha(\lambda)$. Recalling that $E_\alpha(\lambda) \tilde{\to} \mathscr{D}_\alpha^0 E_\alpha(\lambda)$, this gives (ii).

(iii) This has already been observed in 5.2 but we may check its consistency as follows. We have seen that $\mathscr{D}_\alpha^{i+1} \mathbb{C}_{s_\alpha . \lambda} \tilde{\to} \mathbb{C}_\lambda$. Now $\mathscr{D}_\alpha^i \mathbb{C}_\lambda = 0$, $\forall i > 0$ and so $\mathscr{D}_\alpha^i \mathbb{C}_\mu = 0$, $\forall i > 1$, $\forall \mu \in P(R)$. Since any $F \in \operatorname{Ob} K$ admits a finite filtration with quotients isomorphic to the $\mathbb{C}_\mu : \mu \in P(R)$ this proves (iii).

(iv) follows from (ii), (iii) and the definition of Δ_α.

(v) follows from 2.8 (iv).

5.5. We shall now define the derived functors $\mathscr{D}^i_w: w \in W$ by induction on $l(w)$ by successive applications of [13], Prop. 3.1, and our main Theorem (2.21). Suppose that $\mathscr{D}^i_w$ is defined for $l(w)<l$, and we have shown that $\mathscr{D}^i_w$ is a covariant functor which commutes with tensoring by finite dimensional $U(\mathfrak{g})$ modules and satisfies $\mathscr{D}^i_w \mathbb{C}_\lambda=0$, $\forall i>0$, $\forall \lambda \in P(R)^+$ sufficiently large. Choose $\alpha \in B$ such that $l(s_\alpha w)>l(w)$, set $y=s_\alpha w$ and consider the functor $\mathscr{D}_\alpha \mathscr{D}_w$ which we can identify with $\mathscr{D}_y$ (recall 2.15). Fix $F \in \operatorname{Ob} K$. By our hypothesis the objects $E_i \otimes \mathbb{C}_{\lambda_i}$ in a standard resolution of F can be assumed to be $\mathscr{D}_w$ acyclic (take the λ_i sufficiently large). Furthermore, they are transformed by $\mathscr{D}_w$ to objects namely $E_i \otimes \mathscr{D}_w \mathbb{C}_{\lambda_i}$ which are $\mathscr{D}_\alpha$ acyclic (for λ_i sufficiently large). Indeed $\mathscr{D}_w \mathbb{C}_{\lambda_i} \xrightarrow{\sim} F(w\lambda_i)$ by 2.21 and so

$$\mathscr{D}^j_\alpha(E_i \otimes \mathscr{D}_w \mathbb{C}_{\lambda_i}) \cong E_i \otimes \mathscr{D}^j_\alpha F(w\lambda_i)=0, \qquad \forall j>0$$

by 5.4 (iii), (iv). Thus [13], Prop. 3.1, applies and we conclude that $\mathscr{D}^i_y$ exists and satisfies the required properties. More precisely let $D(K)$ denote the derived category of K defined as complexes of objects in K identified when they have the same cohomology. Then the left derived functor $L(\mathscr{D}_w)$ of $\mathscr{D}_w$ is defined as an exact functor on $D(K)$ and for any $F \in \operatorname{Ob} K$ viewed as the complex $0 \to 0 \to \ldots \to F \to 0$ we obtain $\mathscr{D}^i_w(F)=H^i(L(\mathscr{D}_w)F)$. Furthermore ([13], Prop. 3.1) one has $L(\mathscr{D}_\alpha)L(\mathscr{D}_w)=L(\mathscr{D}_\alpha \mathscr{D}_w)$ and so $L(\mathscr{D}_y)$ and hence $\mathscr{D}^i_y$ is independent of the reduced decomposition chosen for y. Again we easily deduce that $\mathscr{D}^i_y$ commutes with tensoring by finite dimensional $U(\mathfrak{g})$ modules and that $\mathscr{D}^i_y \mathbb{C}_\lambda=0$, $\forall i>0$, $\forall \lambda \in P(R)^+$ sufficiently large. Again if $y=w_1 w_2$ where $w_1, w_2 \in W$ satisfy $l(w_1)+l(w_2)=l(y)$ then $\mathscr{D}^i_y F=0$ if $\mathscr{D}^i_{w_2} F=0$, $\forall i \geqq 0$ because $H^i(L(\mathscr{D}_{w_2})F)=0$ so $L(\mathscr{D}_{w_2})F$ and hence $L(\mathscr{D}_y)F=L(\mathscr{D}_{w_1})L(\mathscr{D}_{w_2})F$ are exact. Again $(\mathscr{D}^i_y F)=\mathscr{D}^i_{w_1}(\mathscr{D}^0_{w_2} F)$ if $\mathscr{D}^i_{w_2} F=0$, $\forall i>0$. From 5.4 (iv) one obtains

$$\sum_{i=0}^{l(w)} (-1)^i \operatorname{ch} \mathscr{D}^i_w F=\Delta_w \operatorname{ch} F.$$

5.6. Let $w_0 \in W$ denote the unique element of greatest length. We have the following analogue of Bott's Theorem.

THEOREM. — *Fix $\mu \in P(R)$. If $\mu+\rho$ is not regular, then $\mathscr{D}^i_{w_0} \mathbb{C}_\mu=0$, $\forall i$. Otherwise there exists a unique $w \in W$ such that $w(\mu+\rho)-\rho$ is dominant and we have*

$$\mathscr{D}^i_{w_0} \mathbb{C}_\mu=\begin{cases} 0: & i \neq l(w), \\ E(w.\mu): & i=l(w). \end{cases}$$

We can follow Demazure's proof of Bott's Theorem [8]. In particular, recall the notation and the exact sequences (*), (**) of 5.4. By 5.4 (i) and 5.5 we have

$$\mathscr{D}^i_{w_0}(E_\alpha(\nu) \otimes \mathbb{C}_{-\rho})=0, \qquad \forall \nu \in P(R).$$

Take $\lambda \in P(R)$ with $(\lambda, \alpha) \geqq 0$. Then by (*) and the above $\mathscr{D}^i_{w_0} \mathbb{C}_\lambda \xrightarrow{\sim} \mathscr{D}^i_{w_0}(E_\alpha(\lambda))$ whereas by $(**)$ we have $\mathscr{D}^{i+1}_{w_0}\mathbb{C}_{s_\alpha . \lambda} \tilde{\rightarrow} \mathscr{D}^{i}_{w_0}(E_\alpha(\lambda))$. Then $\mathscr{D}^{i+1}_0 \mathbb{C}_{s_\alpha . \lambda} \tilde{\rightarrow} \mathscr{D}^{i}_{w_0}\mathbb{C}_\lambda$, $\forall i \in \mathbb{Z}$ (taking $\mathscr{D}^{i}_{w_0}=0$, $i<0$ by convention) and $\forall \alpha \in B$ satisfying $(\alpha^\vee, \lambda) \geqq -1$. Yet by 5.4 (iii) and 5.5 we obtain $\mathscr{D}^{l}_{w_0}=0$, $\forall l > l(w_0)$ and then the assertion of the Theorem results by a standard argument.

5.7. From 5.6 we obtain that $\mathscr{D}^{i}_{w_0}\mathbb{C}_\lambda=0$, $\forall i>0$, $\forall \lambda \in P(R)^+$. One may ask if it is possible to deduce that $\mathscr{D}^{i}_{w}\mathbb{C}_\lambda=0$, $\forall i>0$, $\forall \lambda \in P(R)^+$ by backward induction on $l(w)$ and hence deduce Demazure's character formula from 5.5. This leads to the same combinatorial difficulty encountered in the proof of 2.21. Recall the definition of ρ_w (2.21). One has.

THEOREM. — *Assume $\mu \in P(R)$ sufficiently large. Then for all $w \in W$ one has $\mathscr{D}^{i}_{w}\mathbb{C}_\mu=0$ if $i>l(y)$ where $y\mu \in P(R)^+$.*

Take $\alpha \in B$. If $\alpha \in S(w)$ (so then $\mathscr{D}_w=\mathscr{D}_{ws_\alpha}\mathscr{D}_\alpha$ with $l(ws_\alpha)<l(w)$) then exactly as in 5.6 we obtain that $\mathscr{D}^{i}_{w}\mathbb{C}_\mu \tilde{\rightarrow} \mathscr{D}^{i+1}_{w}\mathbb{C}_{s_\alpha . \mu'}$, whenever $(\alpha, \mu) \geqq 0$ and both terms are zero when $(\alpha^\vee, \mu)=-1$. If $\alpha \notin S(w)$, set $z=ws_\alpha$. We have $l(z)=l(w)+1$.

Assume $(\alpha, \mu) \geqq 0$. Then $\mathscr{D}^{i}_{z}\mathbb{C}_\mu \cong \mathscr{D}^{i}_{w}(\mathscr{D}_\alpha \mathbb{C}_\mu) \cong \mathscr{D}^{i}_{w}(E_\alpha(\mu))$. [Note $E_\alpha(\mu)=\mathbb{C}_\mu$ if $(\alpha, \mu)=0$.] Now take μ sufficiently large and assume $y\mu \in P(R)^+$. Then $\mathscr{D}^{i}_{z}\mathbb{C}_\mu=0$: $i>l(y)$ by the induction hypothesis. Substitution in 5.4 $(*)$ gives $\mathscr{D}^{i+1}_{w}(E_\alpha(\mu-\alpha+\rho)\otimes \mathbb{C}_{-\rho}) \tilde{\rightarrow} \mathscr{D}^{i}_{w}\mathbb{C}_\mu$, $\forall i>l(y)$. On the other hand from 5.4 $(**)$ [with μ replaced by $\mu'=\mu-\alpha$ where we note that $s_\alpha . \mu'=s_\alpha \mu$ and since $E_\alpha(\mu-\alpha+\rho)\otimes \mathbb{C}_{-\rho} \cong E_\alpha(\mu-\rho)\otimes \mathbb{C}_{\rho-\alpha}$] we obtain

$$\mathscr{D}^{i+1}_{w}(E_\alpha(\mu-\alpha+\rho)\otimes \mathbb{C}_{-\rho}) \tilde{\rightarrow} \mathscr{D}^{i+1}_{w}\mathbb{C}_{s_\alpha \mu}, \quad \forall i>l(y).$$

We conclude that $\mathscr{D}^{i+1}_{w}\mathbb{C}_{s_\alpha \mu} \tilde{\rightarrow} \mathscr{D}^{i}_{w}\mathbb{C}_\mu$, $\forall i>l(y)$. Now set $y'=w_0 y$. One has $l(y')=l(w_0)-l(y)$. From our two isomorphisms we conclude that

$$\mathscr{D}^{i+l(y')}_{w}\mathbb{C}_{y'(\mu+\rho_{w^{-1}})-\rho_{w^{-1}}} \tilde{\rightarrow} \mathscr{D}^{i}_{w}\mathbb{C}_\mu,$$

$\forall i>l(y)$. Yet $i+l(y')>l(w_0) \geqq l(w)$ and so by 5.4 (iii) the left hand side is zero. This proves the Theorem.

5.8 The result obtained is analogous but more precise than Demazure's vanishing Theorem ([1], 5.5), which corresponds to the case $\mu \in P(R)^+$. We give a nice application of the latter which indicates the existence of a rich theory analogous to the translation theory in the $\mathcal{O}$ category (*see* [3], 7.6.14, for example).

LEMMA. — *Take $\lambda \in P(R)^+$ and E a finite dimensional $U(\mathfrak{g})$ module. Then for all $w \in W$ and all λ sufficiently large $E \otimes F(w\lambda)$ admits a filtration with quotients being the $F(w(\lambda+\mu)): \mu \in \Omega(E)$ ordered as in* [3], 7.6.14.

Under the hypothesis $F(w\lambda) \cong \mathscr{D}_w \mathbb{C}_\lambda$. So

$$E \otimes F(w\lambda) \cong E \otimes \mathscr{D}_w \mathbb{C}_\lambda \cong \mathscr{D}_w(E \otimes \mathbb{C}_\lambda).$$

Now $E \otimes \mathbb{C}_\lambda$ admits a filtration with quotients isomorphic to the $\mathbb{C}_{\lambda+\mu}: \mu \in \Omega(E)$ ordered as in [3], 7.6.14. For λ sufficiently large $\lambda+\mu \in P(R)^+$ is also sufficiently large. Then the result obtains by the vanishing of the $\mathscr{D}^i_w \mathbb{C}_{\lambda+\mu}: i>0$, and the isomorphism $\mathscr{D}_w \mathbb{C}_{\lambda+\mu} \tilde{\to} F(w(\lambda+\mu))$.

Remarks. — Presumably we should only need that $\lambda+\mu \in P(R)^+$, $\forall \mu \in \Omega(E)$ for the result. When $\lambda+\mu \notin P(R)^+$ one expects the quotients to involve the other $F(y\nu): y \in W$, $\nu \in P(R)^+$.

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