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HILBERT SCHEME OF SMOOTH SPACE CURVES

LAWRENCE EIN

Denote by $H_{d,g,n}$ the open subscheme of the Hilbert Scheme parametrizing the smooth irreducible curves of degree d and genus g in $\mathbb{P}^n$. The purpose of this paper is to prove that $H_{d,g,3}$ is irreducible when $d \geqq g+3$. We also prove that every irreducible reduced curve in $\mathbb{P}^3$ with $d \geqq P_a+2$ is smoothable in $\mathbb{P}^3$. These results answer two questions proposed by Hartshorne and Hirschowitz ([5], 1.4). I would also like to remark that these results were asserted by Severi with an incomplete proof ([8], p. 370).

Let $\mathscr{C} \to M_{g,m}$ be the universal family of smooth curves over the fine moduli space of genus g curves with level m structure. Suppose $\mathscr{P}ic\,\mathscr{C}$ is the relative Picard scheme. Set $\mathscr{W}_d^r = \{(\mathscr{L}, C) \in \mathscr{P}ic\,\mathscr{C} \mid \mathscr{L}$ is a degree d line bundle on a curve C and $h^0(\mathscr{L}) \geqq r+1\}$. Now suppose that $\mathscr{L}$ is a degree d very ample line bundle with

$$h^0(\mathscr{L}) = r+1 \qquad \text{and} \qquad h^1(\mathscr{L}) = \delta > 0.$$

We show that if Y is an irreducible component of $\mathscr{W}_d^r$ containing the point corresponding to $(\mathscr{L}, C)$, then $\dim Y \leqq 5g-1-4\delta-d$. We also show that the above inequality implies that $H_{d,g,3}$ is irreducible when $d \geqq g+3$. More generally we prove that $H_{d,g,n}$ is irreducible when

$$d > \frac{(2n-3)g+n+3}{n}.$$

I should also point out that Joe Harris has found an example where $H_{d,g,n}$ is reducible when $d \geqq g+n$. Throughout the paper we shall work over the complex numbers.

I would like to thank Mark Green and Rob Lazarsfeld for many helpful discussions.

LEMMA 1. — *Let* E *be a rank m locally free sheaf on a smooth irreducible curve* C. *Let* $X = \mathbb{P}(E)$ *and* $\pi: X \to C$ *be the projection map. We denote by* U *the tautological line bundle of* $\mathbb{P}(E)$. *Suppose* $V \subseteq H^0(U)$ *is a* $r+1$*-dimensional subspace. Then,*

(*a*) *The natural map* $V \otimes \mathcal{O}_X \to U$ *is surjective, if and only if* $V \otimes \mathcal{O}_C \to \pi_* U = E$ *is surjective.*

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(*b*) *Assume that* $|V|$ *gives a birational morphism*

$$f: \quad X \to f(X) = Y \subseteq \mathbb{P}^r. \quad \textit{Set}\, F = \ker(V \otimes \mathcal{O}_C \to E).$$

Then there is an exact sequence,

$$0 \to (\Lambda^m E)^* \otimes \mathcal{O}_C\left(\sum_1^{r-m} p_j\right) \to F \to \sum_1^{r-m} \mathcal{O}_C(-p_j) \to 0$$

where p_j *s' are general points on* C.

Proof. — (*a*) Suppose that $V \otimes \mathcal{O}_X \to U$ is surjective. Let $M = \ker(V \otimes \mathcal{O}_X \to U)$. If $R = \pi^{-1}(x)$ then

$$M|_R \cong \Omega^1_{\mathbb{P}^{m-1}}(1) \oplus (r+1-m)\, \mathcal{O}_{\mathbb{P}^{m-1}}.$$

Hence, $R^1 \pi_* M = 0$. It follows that $V \otimes \mathcal{O}_C \to \pi_* U = E$ is surjective. Conversely, if $V \otimes \mathcal{O}_C \to E$ is surjective, then the composition $V \otimes \mathcal{O}_X \to \pi^* E \to U$ is also surjective.

(*b*) Set $Y = f(X)$. Choose $r-m$ general points $y_1, y_2, \ldots, y_{r-m}$ in Y. We may assume that $\{y_1, y_2, \ldots, y_{r-m}\}$ spans a $(r-m-1)$-plane L in $\mathbb{P}^r$.

By the uniform position lemma [2], we may assume that

$$L \cap Y = \{y_1, y_2, \ldots, y_{r-m}\}.$$

Furthermore we shall assume that $f^{-1}(y_i) = q_i$ and f is an isomorphism in a neighborhood of q_i. Set

$$Q = \{q_1, q_2, \ldots, q_{r-m}\}.$$

Consider the exact sequence

$$0 \to I_Q \otimes U \to U \to U|_Q \to 0,$$

where I_Q is the ideal sheaf of Q in X. Set $p_i = \pi(q_i)$ and $P = \pi(Q)$. Observe that the restriction map $V \to H^0(U|_Q)$ is surjective.

Let $W = \ker(V \to H^0(U|_Q))$. Observe that the natural map

$$\pi_* U = E \to \pi_*(U|_Q) = \sum_{i=1}^{r-m} \mathcal{O}_{p_i} = \mathcal{O}_P$$

is surjective. Set $E' = \pi_*(I_Q \otimes U)$. Observe that E' is a rank m locally free sheaf and $R^1 \pi_*(I_Q \otimes U) = 0$.

Consider the following diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & & M' & \rightarrow & W \otimes \mathcal{O}_X & \xrightarrow{\alpha} & I_Q \otimes U \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow & & M & \rightarrow & V \otimes \mathcal{O}_X & \rightarrow & U \rightarrow 0 \\ & & \downarrow \beta & & \downarrow & & \downarrow \\ 0 \rightarrow & & \sum_{i=1}^{r-m} I_{q_i} & \rightarrow & H^0(U_{|Q}) \otimes \mathcal{O}_X & \rightarrow & U_{|Q} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

where

$$M = \ker(V \otimes \mathcal{O}_X \rightarrow U) \qquad \text{and} \qquad M' = \ker(W \otimes \mathcal{O}_X \rightarrow I_Q \otimes U).$$

Observe that α is surjective because $f^{-1}(L \cap Y) = Q$.

It follows from the snake lemma β is also surjective.

Let $f_i = \pi^{-1}(p_i) \cong \mathbb{P}^{m-1}$. Consider the exact sequences,

$$0 \rightarrow \mathrm{Tor}_1(I_Q \otimes U, \mathcal{O}_{f_i}) \rightarrow M' \otimes \mathcal{O}_{f_i} \rightarrow W \otimes \mathcal{O}_{f_i} \rightarrow I_Q \otimes U \otimes \mathcal{O}_{f_i} \rightarrow 0,$$

and

$$0 \rightarrow k(q_i) \rightarrow I_Q \otimes U \otimes \mathcal{O}_{f_i} \rightarrow I_{q_i/f_i}(1) \rightarrow 0,$$

where $k(q_i)$ is the residue field of q_i in I_{q_i/f_i} is the ideal sheaf of q_i in f_i. It follows from a local computation that the map

$$H^0(W \otimes \mathcal{O}_{f_i}) \rightarrow H^0(I_Q \otimes U \otimes \mathcal{O}_{f_i})$$

is surjective.

Also observe that $\mathrm{Supp}(\mathrm{Tor}_1(I_Q \otimes U, \mathcal{O}_{f_i})) \subset q_i$.

Hence $H^1(M' \otimes \mathcal{O}_{f_i}) = 0$. M' is torsion free and it is flat over C. It follows from the theorem of base changes that $R^1 \pi_* M' = 0$.

There is the following diagram:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \pi_* M' & \rightarrow & W \otimes \mathcal{O}_C & \rightarrow & E' & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & F & \rightarrow & V \otimes \mathcal{O}_C & \rightarrow & E & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & \sum_{i=1}^{r-m} \mathcal{O}_C(-p_i) & \rightarrow & H^0(U|_Q) \otimes \mathcal{O}_C & \rightarrow & \sum_{i=1}^{r-m} \mathcal{O}_{p_i} & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

This showed that $F \to \sum_{i=1}^{r-m} \mathcal{O}(-p_i)$ is surjective.

Now

$$\operatorname{rank}(\pi_* M') = 1 \qquad \text{and} \qquad \pi_* M' \cong (\wedge^m E')^* = (\wedge^m E)^* \otimes \mathcal{O}_C(P).$$

Remark. — The above construction is inspired by the techniques of Gruson and *et. al.* [6].

The fine moduli space of smooth irreducible genus g curves with level m structure is denoted by $M_{g,m}$. Suppose that $\mathscr{C} \to M_{g,m}$ is the universal family of curves. Let $\mathscr{P}ic\,\mathscr{C}$ be the relative Picard scheme. Set,

$$\mathscr{W}^r_d = \{(\mathscr{L}, C) \,\varepsilon\, \mathscr{P}ic\,\mathscr{C} \mid \deg \mathscr{L} = d \qquad \text{and} \qquad h^0(\mathscr{L}) \geqq r+1\}.$$

For the rest of the paper we shall use the following notations. We shall denote by C, a smooth irreducible genus g curve. $\mathscr{L}$ is a degree d line bundle on C. We shall assume $h^0(\mathscr{L}) = r+1$, $h^1(\mathscr{L}) = \delta > 0$, and $|\mathscr{L}|$ has no base points. We denote by f the natural map:

$$f\colon\ C \to f(C) \underset{\text{def}}{=} C' \subseteq \mathbb{P}(H^0(\mathscr{L})) = \mathbb{P}^r.$$

Suppose that $\mathcal{O}(1)$ is the tautological line bundle of $\mathbb{P}(H^0(\mathscr{L}))$. $P^1(\mathcal{O}(1))$, the first principal part of $\mathcal{O}(1)$, is isomorphic to $H^0(\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}^r}$. Set $M = f^*(\Omega^1_{\mathbb{P}^r}(1))$ and $P^1(\mathscr{L})$ = first principal part of $\mathscr{L}$. There is the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & M & \to & H^0(\mathscr{L}) \otimes \mathcal{O}_C & \to & \mathscr{L} & \to & 0 \\ & & \downarrow{\scriptstyle df} & & \downarrow & & \| & & \\ 0 & \to & K \otimes \mathscr{L} & \to & P^1(\mathscr{L}) & \to & \mathscr{L} & \to & 0, \end{array}$$

where K is the canonical sheaf of C. Observe that $P^1(\mathscr{L}) \otimes K \otimes \mathscr{L}^{-1} \cong P^1(K)$. Hence there is the following diagram:

$$(1.\text{A}) \qquad \begin{array}{ccccccccc} 0 & \to & M \otimes K \otimes \mathscr{L}^{-1} & \to & H^0(\mathscr{L}) \otimes K \otimes \mathscr{L}^{-1} & \to & K & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & K^2 & \to & P^1(\mathscr{L}) \otimes K \otimes \mathscr{L}^{-1} & \to & K & \to & 0. \end{array}$$

Consider the map:

$$(1.\text{B}) \qquad \mu\colon\ H^0(\mathscr{L}) \otimes H^0(K \otimes \mathscr{L}^{-1}) \to H^0(P^1(\mathscr{L}) \otimes K \otimes \mathscr{L}^{-1}).$$

$H^0(P^1(\mathscr{L}) \otimes K \otimes \mathscr{L}^{-1})$ is naturally isomorphic to the cotangent space of $\mathscr{P}ic\,\mathscr{C}$ at the point $(\mathscr{L}, C)$. The image of μ is the annihilator of the Zariski tangent space of $\mathscr{W}^r_d$ at the point $(\mathscr{L}, C)$. See [1] for more details.

THEOREM 2. — *Suppose that $\mathscr{L}$ is a very ample degree d line bundle on a smooth irreducible curve* C *such that $h^0(\mathscr{L}) = r+1$ and $h^1(\mathscr{L}) = \delta > 0$, where $r \geqq 3$. Then,*

(*a*) *rank* $(\mu) \geqq 3\delta - 2 + r = 4\delta + d - g - 2$. (1.B).

(*b*) *If* Y *is an irreducible component of* $\mathscr{W}^r_d$ *containing the point* $(\mathscr{L}, C)$, *then* $\dim Y \leqq 5g - 4\delta - d - 1$.

(*c*) *Let* N *be the normal sheaf of* C *in* $\mathbb{P}(H^0(\mathscr{L}))$. *Then* $h^1(N) \leqq (r-2)(\delta-1)$.

Proof. — Consider the natural embedding,

Let N^* be the conormal sheaf of C in $\mathbb{P}^r$. There is the following exact sequence:

$$0 \to N^* \otimes \mathscr{L} \to H^0(\mathscr{L}) \otimes \mathscr{O}_C \to P^1(\mathscr{L}) \to 0.$$

Consider the natural map

$$F: \quad \mathbb{P}(P^1(\mathscr{L})) \to T \subseteq \mathbb{P}^r.$$

T is the tangent surface of C, and F is a birational morphism. By Lemma 1,

$$h^1(N) = h^0(N^* \otimes K) \leqq \sum_{i=1}^{r-2} h^0(K \otimes \mathscr{L}^{-1}(-p_i)) + h^0\left(\mathscr{L}^{-3}\left(\sum_{i=1}^{r-2} p_i\right)\right) = (r-2)(\delta-1).$$

But $H^0 N^* \otimes K) = \ker \mu$. Thus

$$\operatorname{rank}(\mu) \geqq (r+1)\delta - (r-2)(\delta-1) = 3\delta - 2 + r = 4\delta + d - g - 2.$$

Since the image of μ is the annihilator of the Zariski tangent space of $\mathscr{W}^r_d$ at $(\mathscr{L}, C)$, it follows that,

$$\dim Y \leqq (4g-3) - (4\delta + d - g - 2) = 5g - 4\delta - d - 1.$$

COROLLARY 3. — *Assume that* $r \geqq 3$ *and*

$$f: \quad C \to f(C) = C' \subseteq \mathbb{P}(H^0(\mathscr{L}))$$

is a birational map. Furthermore assume either f *is unramified or* $P_a(C') < g + 3d - (r-2)$. *Then,*

(*a*) rank $(\mu) \geqq 4\delta + d - g - 2$.

(*b*) *If* Y *is an irreducible component of* $\mathscr{W}^r_d$ *containing the point* $(\mathscr{L}, C)$, *then* $\dim Y \leqq 5g - 1 - 4\delta - d$.

Proof. — Consider the natural map

$$\varphi: \quad H^0(\mathscr{L}) \otimes \mathscr{O}_C \to P^1(\mathscr{L}).$$

Set

$$E = \operatorname{Im}(\varphi), \qquad N^* \otimes \mathscr{L} = \ker(\varphi) \qquad \text{and} \qquad D = \operatorname{cok}(\varphi).$$

Observe that cok φ is equal to $\operatorname{cok}(df: f^* \Omega_{\mathbb{P}^r} \otimes \mathscr{L} \to \Omega^1_C \otimes \mathscr{L})$.

It follows that cok φ is isomorphic to $\Omega^1_C \otimes \mathscr{L} \otimes \mathscr{O}_R$, where R is the ramification divisor.

Let $X = \mathbb{P}(E)$. Consider the natural map

$$F: \quad X \to F(X) = T \subseteq \mathbb{P}(H^0(\mathscr{L})).$$

T is the closure of the tangent surface of the smooth part of C′. $F: X \to T$ is birational. Now

$$P_a(C') - g = \sum_{p \in C} \text{length}\,(\mathcal{O}_{P, C/\mathcal{O}_{f(p), C'}}).$$

Observe that

$$\deg R = \sum_{p \in C} \text{length}\,(I_{P, C/\mathcal{O}_{P, C} \cdot I_{f(P), C'}}).$$

It follows that $\deg R \leqq P_a(C') - g$. By Lemma 1, we can

LEMMA 1. — *We can construct the following exact sequence:*

$$0 \to \mathcal{L}^{-3} \otimes \mathcal{O}_C(R) \otimes \mathcal{O}_C\left(\sum_{i=1}^{r-2} p_i\right) \to N^* \otimes K \to \sum_{i=1}^{r-2} K \otimes \mathcal{L}^{-1}(-p_i) \to 0.$$

Since $\deg(R) \leqq P_a - g$, it follows from our assumption

$$h^0\left(\mathcal{L}^{-3}\left(R + \sum_{i=1}^{r-2} p_i\right)\right) = 0.$$

Thus $\dim \ker \mu = h^0(N^* \otimes K) \leqq (r-2)(\delta - 1)$. As in Theorem 2, we conclude that $\text{rank}\, \mu \geqq 4\delta + d - g - 2$ and $\dim Y \leqq 5g - 1 - 4\delta - d$.

The open set of the Hilbert scheme corresponding to smooth irreducible degree d genus g curves in $\mathbb{P}^3$ is denoted by $H_{d, g, 3}$. If $X \in H_{d, g, 3}$, then $\chi(N_{X/\mathbb{P}^3}) = h^0(N_{X/\mathbb{P}^3}) - h^1(N_{X/\mathbb{P}^3}) = 4d$.

As in [7], one can show that each irreducible component of $H_{d, g, 3}$ has dimension greater or equal to $4d$.

THEOREM 4. — *If* $d \geqq g + 3$, *then* $H_{d, g, 3}$ *is irreducible.*

Proof. — There is an irreducible open set of $H_{d, g, 3}$ corresponding to nonspecial curves $(h^1(\mathcal{O}_C(1)) = 0)$ ([5], 6.2). Suppose for contradiction that $H_{d, g, 3}$ is reducible. Then there is an irreducible component W of $H_{d, g, 3}$ such that the general curve C in the family W satisfies

$$h^0(\mathcal{O}_C(1)) = r + 1 \qquad \text{and} \qquad h^1(\mathcal{O}_C(1)) = \delta > 0.$$

We denote by $H^m_{d, g, 3}$ the Hilbert scheme of degree d genus g smooth irreducible curves in $\mathbb{P}^3$ with level m structure. Let W_m be an irreducible component of $H^m_{d, g, 3}$ which maps onto W. Then $\dim W = \dim W_m$.

There is a natural map from

$$h: \quad W_m \to \mathcal{W}^r_d \subseteq \mathcal{P}ic\,\mathcal{C}.$$

Let Y be an irreducible component of $\mathscr{W}_d^r$ containing $h(W_m)$. Let x be a general point of W_m, then

$$\dim h^{-1} h(x) \leqq \dim G(4, d+1+\delta-g) + \dim \operatorname{Aut} \mathbb{P}^3$$

where $G(4, d+1+\delta-g)$ is the Grassman variety of 4 dimensional subspaces in a $d+1+\delta-g$-dimensional vector space. Then

$$\dim W = \dim W_m \leqq \dim h^{-1} h(x) + \dim Y \leqq 4d - 1$$

by Theorem 2. This is a contradiction. Hence, $H_{d, g, 3}$ is irreducible.

Remark. — In [4], Harris has proved that $H_{d, g, 3}$ is irreducible while $d > 5/4 g + 1$.

Suppose that C′ is an irreducible reduced degree d curve in $\mathbb{P}^3$. Let

$$N_{C'/\mathbb{P}^3} = \mathscr{H}om\, \mathcal{O}_{\mathbb{P}^3}(I_{C'}, \mathcal{O}_{C'}) = \mathscr{H}om\, \mathcal{O}_{C'}(I_{C'}/I_{C'}^2, \mathcal{O}_{C'})$$

be the normal sheaf of C′.

LEMMA 5. — $\chi(N_{C'/\mathbb{P}^3}) = h^0(N_{C'/\mathbb{P}^3}) - h^1(N_{C'/\mathbb{P}^3}) = 4d$. *Hence every irreducible component of the Hilbert scheme containing C′ has dimension greater or equal to* $4d$.

Proof. — C′ is locally Cohen Macaulay. We can construct an exact sequence:

$$0 \to E_2 \to E_1 \to I_{C'} \to 0$$

where E_1 and E_2 are locally free sheaves on $\mathbb{P}^3$.

Consider the following exact sequences:

$$0 \to \mathscr{H}om(I_{C'}, \mathcal{O}_{\mathbb{P}^3}) \to E_1^* \xrightarrow{\varphi_1} E_2^* \to \omega_{C'}(4) \to 0,$$

$$0 \to \mathscr{H}om(I_{C'}, \mathcal{O}_{C'}) \to E_1^*|_{C'} \xrightarrow{\varphi_2} E_2^*|_{C'} \to \mathscr{E}xt^1(I_{C'}, \mathcal{O}_{C'}).$$

Observe that $\varphi_2 = \varphi_1 \otimes \mathcal{O}_{C'}$. Thus

$$\operatorname{Cok} \varphi_2 = \operatorname{Cok} \varphi_1 \otimes \mathcal{O}_{C'} = \omega_{C'}(4).$$

Observe that

$$c_1(E_1^*) = c_1(E_2^*) \qquad \text{and rank} \quad E_1^* = 1 + \operatorname{rank} E_2^*.$$

It follows from the Rieman-Roch theorem,

$$\chi(N_{C'/\mathbb{P}^3}) = \chi(E_1^*|_{C'}) + \chi(\omega_{C'}(4)) - \chi(E_2^*|_{C'}) = 1 - P_a + \chi(\omega_{C'}) + 4d = 4d.$$

C′ is codimension two Cohen-Macaulay. It follows that there is no local obstructions to the deformations of C′ ([3], 5.1). Hence the obstructions to the deformations of C′ in $\mathbb{P}^3$ is given by $H^1(N_{C'/\mathbb{P}^3})$. As in [7], one can show that this implies the inequality of dimension as claimed.

THEOREM 6. — *Suppose that* X *is an irreducible reduced degree* d *curve in* p^3. *If* $d \geqq P_a(X) + 2$, *then* X *is smoothable.*

Proof. — Let W be an irreducible component of the Hilbert scheme containing the point corresponding to X. If the general member of W is smooth, then X is smoothable. Assume for contradiction that a general curve C′ in W is singular. Let $S \to W$ be the universal family of curves. Let $p: \tilde{S} \to S \to W$ be the normalization of S. Let $U \subseteq W$ be the open set where p is smooth. Suppose the normalization of C′ is a smooth curve of genus g. We can construct a variety U_m étale over U such that there is a map $h: U_m \to \mathscr{P}ic\, d\mathscr{C}$. We shall divide the proof into five cases. Consider the normalization map $\pi\ C \to C'$. Set $\pi^* \mathcal{O}_{C'}(1) = \mathcal{O}_C(1)$.

Since $g < P_a(C')$, $\deg \mathcal{O}_C(1) \geqq g+3$.

Case 1. — Assume that $g=0$.

Then $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(d)$. C′ is obtained by projecting the d-uple embedding of $\mathbb{P}^1$. The generic projection gives a smooth curve. Thus,

$$\dim W < \dim G(4,\ d+1) + \dim \mathrm{Aut}\, \mathbb{P}^3 - \dim \mathrm{Aut}\, \mathbb{P}^1 = 4d.$$

Case 2. — Assume that $g=1$.

As in Case 1, we can prove that

$$\dim W < \dim G(4,\ d) + \dim \mathrm{Aut}\, \mathbb{P}^3 - \dim \mathrm{Aut}\, C + \dim \mathscr{P}ic\, \mathscr{C} = 4d.$$

Case 3. — Assume that $g \geqq 2$, $\dim h(U_m) = \dim \mathscr{P}ic\, \mathscr{C} = 4g-3$, and $h^1(\mathcal{O}_C(1)) = 0$.

The generic line bundle of degree $d \geqq g+3$ is very ample. Let x be a general point of U_m. Then $\dim h^{-1} h(x) < \dim G(4,\ d+1-g) + \dim \mathrm{Aut}\, \mathbb{P}^3$.

Hence, $\dim W = 4g+3+\dim h^{-1} h(x) < 4d$.

Case 4. — Assume that $h^1(\mathcal{O}_C(1)) = 0$, $g \geqq 2$, and $\dim h(U_m) < 4g-3$, in this case

$$\dim W = \dim U_m = \dim h^{-1} h(x) + \dim h(U_m) < \dim G(4,\ d+1-g) + \dim \mathrm{Aut}\, \mathbb{P}^3 + (4g-3) \leqq 4d.$$

Case 5. — Assume that $g \geqq 2$, and $h^1(\mathcal{O}_C(1)) = \delta > 0$.

Using Corollary 3, we can show that

$$\dim W = \dim U_m \leqq 4d-1,$$

as in Theorem 2.

In each of the five cases, we show that $\dim W < 4d$.

This is impossible. Thus a general curve in W is smooth.

LEMMA 7. — *Assume* $f: C \to C' \subseteq \mathbb{P}(H^0(\mathscr{L})) = \mathbb{P}^r$ *is a birational map. Also assume that* $d \geqq g$.

(*a*) *Consider the multiplication map:*

$$\mu_0:\quad H^0(\mathscr{L}) \otimes H^0(K \otimes \mathscr{L}^{-1}) \to H^0(K).$$

Then rank $(\mu_0) \geqq 2\delta + r - 1 = 3\delta + d - g - 1$.

(*b*) $\delta \leqq \frac{2g+1-d}{3}$.

Proof. — Consider the exact sequence:

$$0 \to \mathrm{M} \to \mathrm{H}^0(\mathscr{L}) \otimes \mathscr{O}_{\mathrm{C}} \to \mathscr{L} \to 0 \qquad \text{when} \quad \mathrm{M} = f^* \Omega^1_{\mathbb{P}^r}(1).$$

By Lemma 1, we can construct an exact sequence:

$$0 \to \mathscr{L}^{-1} \otimes \mathscr{O}\left(\sum_{i=1}^{r-1} p_i\right) \to \mathrm{M} \to \sum_{i=1}^{r-1} \mathscr{O}(-p_i) \to 0.$$

Observe that,

$$h^1\left(\mathrm{K} \otimes \mathscr{L}^{-2} \otimes \mathscr{O}\left(\sum_{i=1}^{r-1} p_i\right)\right) = h^0\left(\mathscr{L}^2 \otimes \mathscr{O}\left(-\sum_{i=1}^{r-1} p_i\right)\right)$$

$$= 2d+1-g-(r-1) = -\chi\left(\mathrm{K} \otimes \mathscr{L}^{-2} \otimes \mathscr{O}\left(-\sum_{i=1}^{r-1} p_i\right)\right).$$

Thus

$$h^0\left(\mathrm{K} \otimes \mathscr{L}^{-2} \otimes \mathscr{O}\left(\sum_{i=1}^{r-1} p_i\right)\right) = 0.$$

Hence,

$$h^0(\mathrm{M} \otimes \mathrm{K} \otimes \mathscr{L}^{-1}) = \dim \ker \mu_0 \leqq (r-1)(\delta - 1).$$

Thus rank $\mu_0 \geqq 3\delta + d - g - 1$. Since $g \geqq \operatorname{rank}(\mu_0)$, it follows that $\delta \leqq (2g+1-d)/3$.

THEOREM 8. — *Let* $\mathrm{H}_{d,g,n}$ *be the open set of the Hilbert scheme of smooth irreducible degree d genus g curves in* $\mathbb{P}^n$ $(n \geqq 3)$. *If* $d > ((2n-3)g+n+3)/n$, *then* $\mathrm{H}_{d,g,n}$ *is irreducible.*

Proof. — Let C be a smooth irreducible degree d genus curve in $\mathbb{P}^n$. Then $\chi(\mathrm{N}_{\mathrm{C}/\mathbb{P}^n}) = (n+1)d + (n-3)(1-g)$.

It follows that the dimension of each irreducible component of $\mathrm{H}_{d,g,n}$ is at least $(n+1)d+(n-3)(1-g)$. Assume that $\mathrm{H}_{d,g,n}$ has an irreducible component W such that the general curve in the family satisfies the property $h^0(\mathscr{L}) = r+1$ and $h^1(\mathscr{L}) = \delta > 0$.

Then,

$$\dim \mathrm{W} \leqq 5g - 1 - 4\delta - d + \dim \mathrm{G}(n+1, r+1)$$

$$+ \dim \operatorname{Aut} \mathbb{P}^n = 5g - 2 - 4\delta - d + (n+1)(\delta + d - g + 1),$$

Since

$$\delta \leqq \frac{2g+1-d}{3} \qquad \text{and} \qquad d > \frac{(2n-3)g+n+3}{n},$$

it follows that dim $W < (n+1)d + (n-3)(1-g)$ which is a contradiction.

Remark. — The above result is an improvement of a theorem of Joe Harris. In ([4], p. 72), Harris proved that $H_{d,\,g,\,n}$ is irreducible while $d > \frac{(2n-1)g}{n+1} + 1$.

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