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## SOLUTION OF A COMBINATORIALLY FORMULATED MONODROMY PROBLEM OF EISENBUD AND HARRIS

BY RONALD D. BERCOV AND ROBERT A. PROCTOR <sup>(1)</sup>

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### 1. Introduction

In this paper we will (mostly) solve a finite permutation group problem using combinatorial reasoning and a century old result from the theory of permutation groups. This problem arose when Eisenbud and Harris [1] introduced a combinatorial construction into the study of the following geometric problem: Let  $X$  be a generic compact Riemann surface of genus  $g$ , and consider linear systems of degree  $d$  and dimension  $r$  on  $X$ . If  $g = (r+1)(g-d+r)$ , then only a finite number  $N = N(g, d, r)$  of such linear systems can exist on  $X$ . By varying  $X$  within a suitable Zariski-open subset of the moduli space of compact Riemann surfaces of genus  $g$ , one can obtain monodromy actions on (i. e. permutations of) the set of the  $N$  linear systems on some fixed generic  $X_0$ . It is natural to ask whether the group of all such actions (called the *monodromy group*) is the entire symmetric group  $S_N$ . When combined with the work of Eisenbud and Harris, our main result will imply that these monodromy groups are always at least the alternating groups  $A_N$ .

Eisenbud and Harris explicitly constructed certain monodromy actions and described these actions in combinatorial terms. Let  $m = r+1$  and  $n = g-d+r$ . Then  $m \geq 1$  and  $n \geq 1$ , and  $m$  and  $n$  determine  $g$ ,  $d$ , and  $r$  since we are assuming  $g = (r+1)(g-d+r) = mn$ . Let  $G(m, n)$  denote the monodromy group for the geometric situation indexed by  $g$ ,  $d$ , and  $r$ , and let  $H(m, n)$  denote the subgroup generated by the actions of Eisenbud and Harris. Hence  $H(m, n) \leq G(m, n) \leq S_N$ . We interpret the combinatorial description of [1] in terms of permutations of the set of all  $m \times n$  standard Young tableaux. So then  $N = N(g, d, r) = N(m, n)$  is the number of standard Young tableaux of  $m \times n$  rectangular shape.

When  $m = 2$  Eisenbud and Harris showed that  $H(2, n) = S_N$ . Therefore  $G(2, n) = S_N$ . They then asked whether this approach might work in general: Does  $H(m, n) = S_N$  when

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$m \geq 2$ ? [It is trivial that  $H(m, 1) = S_N$  and  $H(1, n) = S_N$ .] Our results are as follows:

THEOREM 1. — For  $m, n \geq 1$ , the group  $H(m, n)$  is always either the symmetric group  $S_N$  or the alternating group  $A_N$ .

THEOREM 2. — The group  $H(m, 2)$  is the symmetric group  $S_N$  if and only if  $m = 2^i + 2^j - 1$  for some  $i \geq j \geq 0$ .

THEOREM 3. — For  $m, n \geq 3$  and  $mn \leq 108$  the group  $H(m, n)$  is the symmetric group  $S_N$  or the alternating group  $A_N$  according to the Table. (There "S" denotes  $S_N$  and "-" denotes  $A_N$ .)

TABLE

$m \backslash n$	3	4	5	6	7	8	9	0	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	
3	S	S	S	S	S	S	S	S	-	-	-	S	S	S	S	S	S	S	S	-	-	-	-	-	-	-	-	-	-	S	S	S	S	S	S	
4	S	-	S	-	S	-	-	-	-	-	-	-	-	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
5	S	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
6	S	-	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
7	S	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
8	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
9	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
10	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
11	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
12	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
13	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
14	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
15	S	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
16	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
17	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
18	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
19	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
20	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
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22	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
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24	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
25	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
26	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
27	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
28	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
29	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
30	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
31	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
32	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
33	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
34	S	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	
35	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
36	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

Eisenbud and Harris conjectured that the monodromy group  $G(m, n)$  is always the entire symmetric group. The results above imply that  $G(m, n)$  is always at least the alternating group  $A_N$ . Geometers tell us that it is certainly true that  $G(m, n) = G(n, m)$  by use of Serre duality. (A few routine but onerous details need to be checked for this.) We would then have  $G(m, n) = S_N$  whenever  $H(m, n) = S_N$  or  $H(n, m) = S_N$ . Since  $H(2, n) = S_N$  for all  $n$ , but  $H(m, 2) = A_N$  for infinitely many certain  $m$ , we would then have an infinite sequence of examples (in addition to some others provided by the asymmetry of the Table) where the Eisenbud-Harris group  $H(m, n)$  would be known to fall short of the actual monodromy group.

Most or all algebraic geometers would expect that the monodromy group  $G(m, n)$  is always the entire symmetric group. In Section 6 we show that the strong tendency for  $H(m, n)$  to be the alternating group can be viewed as an expected byproduct of the nature of the actions defined by Eisenbud and Harris. Theorem 1 reduces the original problem to showing that there is always at least one odd permutation in the monodromy group. It is probably quite difficult to determine whether a geometrically constructed permutation is odd or even. The most obvious odd permutations are the transpositions. But these are probably the hardest to construct, since one must pointwise fix  $N - 2$  objects. Determining which  $H(m, n)$  for  $m, n \geq 3$  are equal to  $S_N$  seems irrelevant to solving the original monodromy problem in general. Hence we believe that this paper brings things to a high state of completion until some completely new ideas are introduced.

We now describe the combinatorial context of the problem. Let

$$L(m, n) = \{ (a_1, a_2, \dots, a_m) : n \geq a_1 \geq a_2 \geq \dots \geq a_m \geq 0, a_i \in \mathbf{Z} \},$$

and consider the set of all paths of  $mn$  steps from  $(0, 0, \dots, 0)$  to  $(n, n, \dots, n)$  which stay within  $L(m, n)$ . Let  $N = N(m, n)$  be the cardinality of this set. The monodromy actions of Eisenbud and Harris permute the  $N$  paths. We describe these actions in terms of standard Young tableaux in Section 2. In the special case  $n = 2$ , after interchanging the roles of  $m$  and  $n$  above, the paths at hand are just the "Catalan paths" in the plane. Here

$$N = C_m = \frac{1}{m+1} \binom{2m}{m}$$

is the  $m$ -th Catalan number. The Catalan paths which have a corner at a point lying on the line  $x + y = 2k$  can be grouped into pairs in an obvious way. The following result, which is closely related to Theorem 2, is a combinatorial consequence of our algebraic methods.

**THEOREM 4.** — *There are an even number of pairs of Catalan paths from  $(0, 0)$  to  $(m, m)$  with corners on the line  $x + y = 2k$ ,  $0 < k < m$ , unless  $m = 2^i$ ; then there are an odd number of such pairs.*

The result from the theory of permutation groups which we use is Theorem 15.1 of [4].

PROPOSITION 1 (Bochert, 1889). — *Let  $H$  be a 2-transitive permutation group acting on  $N$  objects. If there is some non-identity element of  $H$  which moves fewer than  $(1/3)N - (2/3)\sqrt{N}$  objects, then  $H$  is either the alternating group  $A_N$  or the symmetric group  $S_N$ .*

This proposition is used to obtain Theorem 1: We show that  $H(m, n)$  is 2-transitive in Section 3 and then exhibit elements satisfying the degree bound in Section 4. In Sections 5 and 6 the computations which determine the parity of the Eisenbud-Harris generators are described, thereby proving Theorem 2, 3, and 4. In Section 6 we also explain the rarity of symmetric groups in the Table. This numerical phenomenon is closely related to the numerical fact that representations of the symmetric group are rarely odd dimensional.

## 2. Definitions and actions

The set  $L(m, n)$  of  $n$ -tuples can be made into a partially ordered set by componentwise comparison. Then the paths described in Section 1 become maximal chains. This poset has been studied by Richard Stanley and others for purely combinatorial reasons. Here it occurs as the poset of Schubert subvarieties of the Grassmannian  $G_{m-1, m+n-1}$  of  $\mathbf{P}^r$ 's in  $\mathbf{P}^{g-d+2r}$  [or  $m$ -dimensional subspaces of an  $(m+n)$ -dimensional vector space]. The maximal chains of Schubert cycles occurring in [1] can be indexed by standard Young tableaux of  $m \times n$  rectangular shape as follows. In the notation of [1] (which is inessential here), if we are given a chain

$$\sigma_{0,0,\dots,0} \supset \dots \supset \sigma_{\alpha^{(c-1)}} \supset \sigma_{\alpha^{(c)}} \supset \dots \supset \sigma_{n,n,\dots,n}$$

let  $T_{ij} = c$  if  $\alpha_k^{(c)} = \alpha_k^{(c-1)}$  for all  $k$  such that  $0 \leq k \leq m-1 = r$  except for  $k = i-1$ , where  $j = \alpha_k^{(c)} = \alpha_k^{(c-1)} + 1$ . Doing this for the  $mn$  values of  $c$  from 1 to  $mn$  yields a standard Young tableaux  $T$ , viz.

$$\{T_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\} = \{1, 2, 3, \dots, mn\},$$

with  $T_{i,j} < T_{i+1,j}$  and  $T_{i,j} < T_{i,j+1}$ . Let  $X(m, n)$  be the set of all such rectangular tableaux and let  $N = N(m, n) = |X(m, n)|$ .

Here we replace the notation  $x_{c,a}$  of [1] with  $\pi_{b,c}$ , where  $2 \leq b \leq mn-1$  and  $1 \leq c \leq n-1$ . Then in the language of tableaux the monodromy actions of Eisenbud and Harris act from the right on the set  $X(m, n)$  as follows: If  $T \pi_{b,c} = U$ , then  $U$  is obtained from  $T$  by interchanging the entries  $b$  and  $b+1$  if and only if they were in different rows and columns in  $T$  and they were exactly  $c$  columns apart. In symbols: If  $T_{ij} = b$ ,  $T_{kl} = b+1$ ,  $i \neq k$ , and  $|l-j| = c$ , then  $U_{ij} = b+1$ ,  $U_{kl} = b$ , and  $U_{pq} = T_{pq}$  elsewhere. If  $i = k$  or  $|l-j| \neq c$  then  $U = T$ .

Let  $S_N$  be the symmetric group on  $X(m, n)$ , and let  $H(m, n)$  be the subgroup of  $S_N$  generated by the  $\pi_{b,c}$ . Then the problem posed by Eisenbud and Harris is:

*Problem.* — Does  $H(m, n) = S_N$ ?

### 3. Two-transitivity

Order the elements of  $X(m, n)$  by reading the entries of a tableau down the columns from left to right, and then ordering the resulting  $mn$ -tuples lexicographically. Then  $A$  is the minimum element of  $X(m, n)$ , where  $A_{ij} = (j-1)m + i$ ; and  $Z$  is the maximum element, where  $Z_{ij} = (i-1)n + j$ . Suppose  $T_{ij} = b$  and  $T_{kl} = b+1$  with  $j-l = c > 0$  and  $i \neq k$ . Then  $U = T\pi_{b,c}$  precedes  $T$  in the order, viz.  $U < T$ . The tableau  $A$  is the only tableau for which no  $b$  lies to the right of  $b+1$ ; all other tableaux can be moved toward  $A$  with some  $\pi_{b,c}$ . So  $H(m, n)$  acts transitively on  $X(m, n)$ . (This was known to Eisenbud and Harris.)

LEMMA 1. — *The action of  $H(m, n)$  on  $X(m, n)$  is 2-transitive.*

*Proof.* — Given any  $T$  such that  $A < T < Z$  we will move the ordered pair  $(T, Z)$  to  $(T', Z)$  with  $T' < T$ . This implies 2-transitivity since the action is transitive. Note that the only  $\pi_{b,c}$  which move  $Z$  are those with  $c = n-1$  and  $b \equiv 0 \pmod n$ .

Let  $k$  be the largest  $i$  for which  $T_{i,1} = i$ . Consider two cases.

(1)  $k = m$ . The first column of  $T$  is minimal. The first entry of  $T$  which differs from that for  $A$  will be an entry  $b+1$  for which  $b$  occurs  $c$  columns to the right in  $T$  where  $c < n-1$ . Then  $T\pi_{b,c} = T' < T$  and  $Z\pi_{b,c} = Z$ .

(2)  $k < m$ . Let  $T_{k+1,1} = r$ . Define  $p$  and  $q$  by  $T_{p,q} = r-1$ . In each case below we define  $\sigma \in H(m, n)$  and  $T' = T\sigma$  such that  $Z\sigma = Z$ ,  $T'_{k+1,1} < T_{k+1,1}$ , and  $T'_{i,1} = T_{i,1} = i$  for  $1 \leq i \leq k$ .

If  $r-1 \not\equiv 0 \pmod n$  or if  $q \neq n$ , then let  $\sigma = \pi_{r-1, q-1}$ . So  $T'_{k+1,1} = r-1$ . Otherwise  $r-1 \equiv 0 \pmod n$  and  $q = n$ , i. e.  $T_{p,n} = r-1$ . Note that  $p$  must be  $\leq k$ . Consider three cases.

(a)  $T_{p, n+1-i} = r-i$  for  $n \geq i \geq 1$ .

Note that  $p \leq k$  implies  $T_{p,1} = p$  which implies that  $T_{p,n} = p+n-1$  which is impossible unless  $p=1$  since  $T$  is a standard Young tableau. Then  $T_{1,2} = 2$  implies  $T_{2,1} \neq 2$  which implies  $k=1$  and  $T_{2,1} = n+1$ . If  $m=2$ , then  $T=Z$  contrary to assumption. So assume  $m \geq 3$ . Now  $T$  and  $Z$  have identical first rows, so we can use induction on  $m$  to assume  $T_{3,1} = n+2$ . Now set  $\sigma = \pi_{n, n-1} \pi_{n+1, n-1} \pi_{n, n-1}$ . Then  $T'_{2,1} = n < T_{2,1} = n+1$ .

(b)  $T_{p, n+1-i} = r-i$  for  $n-1 \geq i \geq 1$  and  $T_{p,1} \neq r-n$ .

If  $T_{p-1, n} = r-n$  then proceed as in (c) below. Otherwise  $T_{u,1} = r-n$  with  $u > p$ . But then  $p=1$  as in (a) above. Since all of the numbers from 1 to  $r$  occur in the first column or first row and  $T_{1,n} = r-1$ , there are two possible locations for  $r+1$  in  $T$ . If  $T_{k+2,1} = r+1$ , then set  $\sigma = \pi_{r-1, n-1} \pi_{r, n-1} \pi_{r-1, n-1}$ . If  $T_{2,2} = r+1$ , set  $\sigma = \pi_{r,1} \pi_{r-1,1} \pi_{r,1} \pi_{r-1,1}$  if  $n=2$ , otherwise for  $n \geq 3$  set  $\sigma = \pi_{r,1} \pi_{r-1, n-2} \pi_{r, n-1} \pi_{r-1,1}$ . In all cases  $T'_{k+1,1} = r-1$ .

(c)  $T_{p, n+1-i} = r-i$  for  $s \geq i \geq 1$  where  $n-2 \geq s \geq 1$  and  $T_{p, n+1-s-1} \neq r-s-1$ . If  $s \geq 2$  then either  $T_{p-1, n} = r-s-1$  or  $T_{u,v} = r-s-1$  with  $u > p$  and  $1 < v < n+1-s$ . (If  $v=1$  then  $r-s-1 < r$  implies  $u \leq k$  implying that  $1, 2, \dots, r-s-1$  are all used in the first column, leaving no numbers for  $T_{p,2}, \dots, TT_{p, n-s}$ .) Act with either  $\pi_{r-s-1, s-1}$  or  $\pi_{r-s-1, n+1-s-v}$  to replace  $T$  with a new tableau  $T$  such that  $T_{p, n+1-s+1} = r-s+1$  but

$T_{p, n+1-s} \neq r-s$ . Repeat this until  $s$  has been decreased to 1, i. e. until  $T_{p, n-1} \neq r-2$ . Consider two cases.

- (i)  $T_{p-1, n} = r-2$ . Set  $\sigma = \pi_{r-1, n-1} \pi_{r-2, n-1} \pi_{r-1, n-1}$ . Then  $T'_{k+1, 1} = r-2$ .
- (ii)  $T_{u, v} = r-2$  with  $u > p$  and  $v < n$ . If  $v=1$  then  $T_{k, 1} = r-2 = k$  and therefore we must really be in case (b) above with  $n=2$ . If  $v > 1$  then set  $\sigma = \pi_{r-2, n-v} \pi_{r-1, v-1}$ . Then  $T'_{k+1, 1} = r-1$ .

This concludes case (2) and the proof of Lemma 1. ■

#### 4. Degree bound

Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_p)$  be a partition of  $t$ ; i. e.  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \geq 0$  and  $\lambda_1 + \lambda_2 + \dots + \lambda_p = t$ . Consider the (Ferrers or Young) diagram for  $\lambda$ , which has  $\lambda_i$  boxes in the  $i$ -th row. The conjugate  $\lambda'$  of  $\lambda$  is the partition of  $t$  obtained by reading off the column lengths  $\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_q > 0$ , where  $q = \lambda_1$ . Let  $f_\lambda$  be the number of standard Young tableaux on the diagram for  $\lambda$ . Let  $h_{i, j}$  be the "hook length" at the  $(i, j)$  square in the diagram, viz.  $h_{i, j} = \lambda_i + \lambda'_j - i - j + 1$ . Then (Ex. I. 5. 2 of [2])

$$f_\lambda = \frac{t!}{\prod_{i, j} h_{i, j}}$$

In particular, if  $\lambda$  is a perfect  $m \times n$  rectangle then  $N(m, n) = f_\lambda$ ; explicitly

$$N(m, n) = \frac{(mn)!}{\prod_{1 \leq i \leq m} (n+i-1)_n}$$

where  $(a)_b = a(a-1)(a-2) \dots (a-b+1)$ .

If  $m=1$  or  $n=1$  then  $N=1$  and  $H(m, n)$  is trivially  $S_1$ . If  $m=2$  or  $m=3$  and  $n=2$ , then  $\pi_{n, n-1}$  or  $\pi_{3, 1}$  respectively are transpositions and we need not appeal to Proposition 1: Lemma 1 can be used to create any transposition. (This is how Eisenbud and Harris handled the  $m=2$  case.) If  $m=4$  and  $n=2$ , then  $N=14$  and  $(1/3)N - (2/3)\sqrt{N}$  is approximately 2.17, forcing us to find a transposition. Random computer search yielded the transposition 3434323434342 (where 3 denotes  $\pi_{3, 1}$ , etc.) of length 13 in the Eisenbud-Harris generators. We believe this is the shortest length for a transposition in  $H(4, 2)$ .

If  $N \geq 36$  then  $(2/9)N$  may be used as the bound in Proposition 1. Clearly  $N(m, n) \geq 42 > 36$  when  $m \geq 5$  and  $n=2$  or when  $m \geq 3$  and  $n \geq 3$  because  $N(5, 2) = N(3, 3) = 42$ .

LEMMA 2a. — *If  $m \geq 3$  and  $n \geq 3$ , then the generator  $\pi_{4, 2}$  moves fewer than  $(2/9)N$  tableaux.*

*Proof.* — This generator affects only those tableaux  $T$  where the entries 4 and 5 are two columns apart. This happens exactly when  $T_{1, 3} = 4$  and  $T_{3, 1} = 5$  or vice versa. Hence

$q=4 f_\lambda$  paths are moved by  $\pi_{4,2}$  where  $\lambda$  is the  $m$ -tuple  $(n, n, \dots, n, n-1, n-1, n-3)$ . The hook lengths  $3, 4, 5, 6, \dots, (n+2)$  and  $3, 4, 6, 7, \dots, (m+2)$  (5 should not be included twice) occur as hooks in the expression for  $N$  but not in  $f_\lambda$ , and conversely for the hooks  $1, 2, \dots, (n-3)$  and  $1, 2, \dots, (m-3)$ . So then

$$\frac{q}{N} = \frac{1}{5} \frac{(n+2)_5 (m+2)_5}{(mn)_5}.$$

Then showing

$$(mn)_5 \geq (n+2)_5 (m+2)_5$$

will imply  $q/N \leq 1/5 < 2/9$ . But the inequality for  $(mn)_5$  is equivalent to

$$5m^2n^2(m-n)^2 + 20(m-n)^2 + 2m^2n^2 + 8 \geq 4(m^2-n^2)^2 + 10mn,$$

which is easily verified for  $m, n \geq 3$ . ■

LEMMA 2*b*. — If  $m=5$  and  $n=2$ , then  $[\pi_{5,1}\pi_{6,1}]^3$  moves fewer than  $(2/9)N$  tableaux. If  $m \geq 6$  and  $n \geq 2$ , then  $[\pi_{2,1}\pi_{3,1}]^3$  moves fewer than  $(2/9)N$  tableaux.

*Proof.* — The tableaux (or paths) in the case  $n=2$  are treated in detail in the next section. In the language we will use there, by Lemma 3 we will see that  $[\pi_5\pi_6]^3$  and  $[\pi_2\pi_3]^3$  consist of  $C_2C_2$  and  $C_1C_{m-2}$  transpositions respectively. Hence  $2C_2C_2=8$  and  $2C_1C_{m-2}=2C_{m-2}$  paths are moved. But  $N=42$  and  $N=C_m$  in the two cases. But  $8 < (2/9)N$ , and it is also easy to see that  $2C_{m-2} < (2/9)C_m$  when  $m \geq 6$ . ■

Once the above proof has been understood by reading Section 5, then Proposition 1, Lemmas 1, 2*a*, 2*b*, and the remarks preceding Lemma 2*a* can be combined to provide a proof of Theorem 1.

### 5. Parity of generators when $n=2$

When  $n=2$  it is more convenient to use  $L(2, m)$  to represent the paths in  $X(m, 2)$  than  $L(m, 2)$ : Given an element  $(2, 2, \dots, 2, 1, 1, \dots, 1, 0, 0, \dots, 0)$  of  $L(m, 2)$ , define a corresponding element  $(x, y)$ ,  $m \geq x \geq y \geq 0$ , of  $L(2, m)$  by letting  $x$  be the number of 2's and 1's and  $y$  be the number of 2's. Then  $X(m, 2)$  is represented by the set of "Catalan paths" (never rising above the diagonal  $x=y$ ) in the plane from  $(0, 0)$  to  $(m, m)$ . Since  $n=2$ , we have  $\pi_{b,c}$  only for  $c=1$ . We will denote  $\pi_{b,1}$  by  $\pi_b$ . If a path has an East-then-North corner at a point lying on the line  $x+y=b$  and this corner does not have coordinates  $(k+1, k)$ , then it is interchanged by  $\pi_b$  with its sister path which has a North-then-East corner  $\sqrt{2}$  to the northwest. Any path not having such a corner corresponds to a tableau with  $b$  and  $b+1$  in the same row [when the corner is at  $(k, k+1)$ ], or in the same column [when the path passes straight through a point  $(x, b-x)$ ]. Therefore  $\pi_b$  is a product of  $r(m, b)$  transpositions, where  $r(m, b)$  is the number of Catalan paths with corners on the line  $x+y=b$  not of the form  $(k+1, k)$  when  $b=2k+1$ .



At this point it is essential for the reader to draw a picture. It is easy to see that  $\pi_2$  fixes all paths except those passing through  $(2, 1)$ . Also note that  $\pi_b \pi_{b+1}$  cycles the three paths contained in any  $2 \times 1$  rectangle lying between  $x + y = b - 1$  and  $x + y = b + 2$ . Furthermore  $\pi_b \pi_{b+1}$  fixes any paths which are straight through these levels. Thus  $[\pi_b \pi_{b+1}]^3$  affects only paths near the  $x = y$  border. This element interchanges paths passing through  $(b/2, (b/2) - 1)$  and  $((b/2) + 1, (b/2) + 1)$  when  $b$  is even and  $((b - 1)/2, (b - 1)/2)$  and  $((b + 1)/2 + 1, (b - 1)/2 + 1)$  when  $b$  is odd. Let

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

the  $n$ -th Catalan number, which is the number of Catalan paths from  $(0, 0)$  to  $(n, n)$ . We have just proved.

LEMMA 3. — *The generators  $\pi_2$  and  $\pi_{2m-2}$  are products of  $C_{m-1}$  transpositions. The element  $[\pi_b \pi_{b+1}]^3$  is a product of  $C_k C_{m-k-1}$  transpositions when  $b = 2k$  or  $b = 2k + 1$ .*

When  $n!$  is expressed as a product of powers of primes, the exponent of 2 is  $n - \text{bin}(n)$ , where  $\text{bin}(n)$  is the number of 1's occurring in the binary expansion of  $n$ . Hence  $C_m$  is odd exactly when  $m = 2^i - 1$  for some  $i \geq 0$ .

THEOREM 2'. — *The generators  $\pi_b$  for  $H(m, 2)$  are always even unless  $m = 2^i + 2^j - 1$  for some  $i \geq j \geq 0$ . If  $m = 2^i$  (i. e.  $j = 0$ ), then all  $\pi_b$  are odd. If  $m = 2^i + 2^j - 1$  with  $j > 0$ , then only  $\pi_b$  with  $b = 2^{j+1} - 1$  and  $b = 2^{i+1} - 1$  are odd.*

*Proof.* — The quantity  $C_k C_{m-k-1}$  is odd only when  $k = 2^i - 1$  and  $m - k - 1 = 2^j - 1$  for some  $i, j \geq 0$ . Then  $m = 2^i + 2^j - 1$ . If  $m \neq 2^i + 2^j - 1$ , then  $m \neq 2^i$ , so  $C_{m-1}$  is even and  $\pi_2$  is even. Then  $C_k C_{m-k-1}$  being even implies that  $[\pi_b \pi_{b+1}]^3$  is even for  $b \geq 2$ . Therefore all  $\pi_b$  are even. Suppose  $m = 2^i$ . Then  $\pi_2$  is odd. Since  $C_k C_{m-k-1}$  is odd only for  $k = 0$  or  $m - 1$ , this means that  $[\pi_b \pi_{b+1}]^3$  is even for  $2 \leq b \leq 2m - 2$ , and so all  $\pi_b$  are odd. If  $m = 2^i + 2^j - 1$  with  $j \geq 1$ , then  $\pi_2$  and  $\pi_{2m-2}$  are even. And  $[\pi_b \pi_{b+1}]^3$  is odd exactly when  $k = 2^i - 1$  or  $2^j - 1$ , i. e. when  $b = 2^{i+1} - 2, 2^{i+1} - 1, 2^{j+1} - 2, 2^{j+1} - 1$ . This forces  $\pi_b$  to be odd when  $b = 2^{i+1} - 1$  or  $2^{j+1} - 1$  and even otherwise. ■

The combinatorial version of Theorem 2' is:

THEOREM 4' *The number of pairs of Catalan paths from  $(0, 0)$  to  $(m, m)$  with corners on the line  $x + y = b$  not of the form  $(k + 1, k)$  is even unless  $m = 2^i + 2^j - 1$  for some  $i \geq j \geq 0$ . If  $m = 2^i$  (i. e.  $j = 0$ ), then there are always an odd number of pairs. If  $m = 2^i + 2^j - 1$  with  $j > 0$ , then there are an odd number of pairs exactly when  $b = 2^{i+1} - 1$  or  $b = 2^{j+1} - 1$ .*

### 6. Parity of generators when $n \geq 3$

The generator  $\pi_{b,c}$  interchanges the entries  $b$  and  $b + 1$  in all tableaux wherein these entries are in different rows and  $c \geq 1$  columns apart. Given such locations for  $b$  and  $b + 1$ , let  $\lambda$  and  $\mu$  be the regions occupied by the entries  $1, 2, 3, \dots, b - 1$  and  $b + 2, b + 3, \dots, mn$  respectively. Here  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$  and  $\mu = (\mu_1, \dots, \mu_m)$  are partitions of  $b - 1$  and  $mn - b - 1$  respectively, where  $\mu_i$  is the number of entries in the  $(m - i + 1)$ th

row which are larger than  $b + 1$ . Let  $b$  and  $b + 1$  be in the  $i$ -th and  $j$ -th rows, and let  $f_\lambda$  and  $f_\mu$  be the numbers of standard Young tableaux on the diagrams  $\lambda$  and  $\mu$ . Then  $f_\lambda f_\mu$  is the number of pairs of paths passing through the points  $\lambda$  and  $\lambda' = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j + 1, \dots, \lambda_m)$ . Thus  $\pi_{b,c}$  is the product of

$$\sum_{\lambda, i, j} f_\lambda f_\mu$$

disjoint transpositions, where the sum runs over all partitions  $\lambda$  of  $b$  and choices of two squares just outside  $\lambda$  which lie in different rows and  $c$  columns apart. Computing this quantity (mod 2) for all  $\pi_{b,c}$  for each  $H(m, n)$  yielded the Table.

The prevalence of the alternating group in the Table is not at all surprising when one considers the question of when the number of transpositions in  $\pi_{b,c}$  is odd. In order for this to happen, there must be an odd number of terms in the above sum where  $f_\lambda$  and  $f_\mu$  are simultaneously odd. We will see that it is rare for just one  $f_\lambda$  to be odd. Furthermore, one of  $\lambda$  or  $\mu$  must have  $t$  squares, where  $t \geq (mn - 2)/2$ .

Let  $\lambda$  be a partition of  $t$  with  $p$  parts, viz.  $t = \lambda_1 + \lambda_2 + \dots + \lambda_p$ . According to Example I. 1. 1 of [2],

$$f_\lambda = \frac{t! \prod_{i < j} (\theta_i - \theta_j)}{p \prod_{i=1}^p (\theta_i)!},$$

where  $\theta_i = \lambda_i + p - i$ . Then  $f_\lambda$  is odd only when

$$\binom{p}{2} + \text{bin}(t) = \sum_{i < j} \text{two}(\theta_i - \theta_j) + \sum_{i=1}^p \text{bin}(\theta_i),$$

where  $\text{bin}(t)$  is as in Section 5, and  $\text{two}(k)$  is the exponent of 2 in the prime factorization of  $k$ . This condition seems to be especially hard to satisfy when the number of parts  $p$  is not small, say  $p \geq 5$ , and when the partition is "fat". By this we mean  $t \geq pq/2 - 1$ , where  $q = \lambda_1$ . Of the 891,042 partitions  $\lambda$  with  $p, q \geq 5$  and  $11 \leq t \leq 63$  and  $t \geq pq/2 - 1$ , only 108 have odd  $f_\lambda$ . (And all of these had  $t \leq pq/2 + 3$ , supporting the belief that a fat partition is very unlikely to have odd  $f_\lambda$ .) With only 2 exceptions which can be treated by hand, every pair of partitions  $\lambda, \mu$  which arises in the transposition count calculation for  $m, n \geq 8$  has at least  $\lambda$  or  $\mu$  satisfying  $p, q \geq 5$  and  $t \geq (pq - 1)/2$ . Therefore only on the basis of the preliminary data above concerning the parity of  $f_\lambda$  for  $\lambda$  meeting such conditions, we would expect to find few odd Eisenbud-Harris generators after the full  $\sum f_\lambda f_\mu$  computations are made for cases with  $m, n \geq 8$ . This is because any  $\lambda$  with  $f_\lambda$  odd must be paired with a  $\mu$  with  $f_\mu$  odd in order to have any effect, and such a pairing is very unlikely. These considerations lead us to believe that the rarity of  $S_N$  in the middle region of the Table is a byproduct of the particular construction of Eisenbud and Harris, and that this rarity does not reflect any geometric phenomenon.

We note in passing that  $f_\lambda$  is the dimension of the  $\lambda$ -th irreducible representation of the symmetric group  $S_p$ , where  $|\lambda| = t$ . John McKay and Ian Macdonald [3] have shown

that the number of partitions  $\lambda$  of  $t$  for which  $f_\lambda$  is odd is  $2^{k_1+k_2+\dots}$  if  $t=2^{k_1}+2^{k_2}+\dots$  with  $k_1 < k_2 < \dots$ . This can be compared with the Hardy-Ramanujan estimate  $(4\sqrt{3}t)^{-1} e^{\pi\sqrt{273}\sqrt{t}}$  for the number of all partitions of  $t$  to see how rare odd  $f_\lambda$  are for a given value of  $t$ .

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