ANNALES SCIENTIFIQUES DE L'É.N.S.

R. MIROLLO
K. VILONEN

Bernstein-Gelfand-Gelfand reciprocity on perverse sheaves

Annales scientifiques de l'É.N.S. 4e série, tome 20, nº 3 (1987), p. 311-323
<http://www.numdam.org/item?id=ASENS_1987_4_20_3_311_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 20, 1987, p. 311 à 324.

BERNSTEIN-GELFAND-GELFAND RECIPROCITY ON PERVERSE SHEAVES (1)

BY R. MIROLLO AND K. VILONEN

0. Introduction

The purpose of this paper is to extend the results of Bernstein-Gelfand-Gelfand on infinite dimensional Lie algebra representations [BGG] to perverse sheaves on a wide class of complex analytic spaces. Our method is to use the inductive construction of perverse sheaves given in [MV1], [MV2]. We give alternative proofs of the theorems in [BGG], and in some sense offer an explanation as to when such results should hold. Our main results are stated in section 1, which follows very closely the introduction to the [BGG] paper.

We thank J. Bernstein and R. MacPherson for bringing these questions to our attention and E. de Shalit for pointing out the reference [Mu] to us.

1. Statement of the main results

Let **k** be a field which will be fixed throughout this paper. Let A be an associative algebra with identity which is finite dimensional as a vector space over **k**. We say that a category is of *Artin type* if it is equivalent to the category of finitely generated A-modules for some A.

Such a category $\mathscr{A}$ has several special properties (*see* e. g. [CR]). It satisfies the Krull-Schmidt and Jordan-Holder theorems. Furthermore, it has a finite number of irreducible objects $L_1, \ldots, L_r$ and each L_i has a unique projective cover P_i. These modules P_i are precisely all the indecomposable projective modules.

Denote by $[M : L_i]$ the number of times the irreducible module L_i occurs in the Jordan-Holder series of M. The matrix $C_{ij}=[P_i : L_j]$ is called the *Cartan matrix* of $\mathscr{A}$. As is pointed out in [BGG] the Cartan matrix turns out to be symmetric in many

(1) Partially supported by N.S.F.

important examples. This is the case for modular representations of finite groups (see [CR]) and modules over the restricted universal enveloping algebra of a semi-simple Lie algebra in characteristic p([H]). A third class of examples is given in [BGG] where the category $\mathcal{O}$ of certain infinite dimensional representations of a complex semi-simple Lie algebra is constructed.

In all these examples a stronger duality principle holds. There is a class of modules $M_1, \ldots, M_l$ such that the modules P_i have a decomposition series with factors isomorphic to the M_j. Let $[P_i : M_j]$ denote the number of times M_j occurs in the decomposition series of P_i. We say that the category $\mathscr{A}$ satisfies *BGG reciprocity* if $[P_i : M_j]=[M_j : L_i]$ for all i and j. In this case $C={}^tDD$, where $D_{ij}=[M_i : L_j]$ is the *decomposition matrix*. In all the above examples we have such a reciprocity. In the case of modular representations $l \neq r$ and the matrix D is not square. In the case of category $\mathcal{O}$ and in our case the matrix D is an upper triangular unimodular square matrix (if we choose a proper ordering for $L_1, \ldots, L_r$). In the category $\mathcal{O}$ the modules M are the Verma modules.

In this paper we want to show that these results are true for the category of perverse sheaves on a wide class of topological spaces. The results in [BBG] can be recovered from ours by applying localization ([BB], [BK]) and the Riemann-Hilbert correspondence ([K], [M]). Here the topological space is the flag manifold of a semi-simple complex group.

Let X be a complex analytic space with a complex analytic Whitney Stratification. Let $\mathscr{S}'$ denote the strata of $\mathscr{S}$ which are not of top dimension. *We make the further assumption that* $\pi_1(S)=0$ *for all* $S \in \mathscr{S}$ *and* $\pi_2(S)=0$ *for all* $S \in \mathscr{S}'$. We will keep this assumption all through this paper. Let P(X) denote the category of perverse sheaves of $\mathbf{k}$-vector spaces) on X which are constructible with respeact to the fixed stratification ([BBD], [MV2]). We recall that P(X) is the subcategory of the bounded derived category of $\mathbf{k}$-sheaves $D^b(X)$ consisting of complexes of $\mathbf{k}$-sheaves $A^\bullet$ on X satisfying:

(0) $H^k(i^* A^\bullet)$ is a local system of finite rank on S

(1) $H^k(i^* A^\bullet)=0$ for $k>-\dim_{\mathbf{C}} S$

(2) $H^k(i^! A^\bullet)=0$ for $k<-\dim_{\mathbf{C}} S$

for all $S \in \mathscr{S}$, where $i : S \to X$ is the inclusion.

It is shown in [BBD] that the category P(X) is an artinian abelian category.

THEOREM 1.1. — *The category* P(X) *is of Artin type and its Cartan matrix is symmetric.*

We will prove this theorem in Section 2. We remark here that it suffices to prove that P(X) has enough projectives to conclude that it is of Artin type. This follows from the following well-known

LEMMA 1.2. — *Let* $\mathscr{A}$ *be an artinian, abelian category with enough projectives, finitely many irreducibles and* Hom(A, A′) *having a structure of a finite dimensional* $\mathbf{k}$*-vector space for all* A, $A' \in \mathscr{A}$. *Then* $\mathscr{A}$ *is of Artin type over* $\mathbf{k}$.

Proof. – Let $L_1, \ldots, L_m$ be the irreducible objects and choose projectives $P_i \twoheadrightarrow L_i$. Then $P = \oplus P_i$ is a projective generator, and $\mathscr{A}$ is equivalent to the category of (right) A-modules where $A = \mathrm{Hom}(P, P)$ (*see* [B]).

Next we impose a further condition on the space X. We assume that $\bar{S} - S$ is a Cartier divisor in $\mathscr{S}$ (or empty) for all $S \in \mathscr{S}$. For all $S_i \in \mathscr{S}$ we define perverse sheaves M_i as follows:

$$M_i = j_! \mathbf{k}_{S_i} [\dim_{\mathbf{C}} S_i]$$

where $j : S_i \to X$ is the inclusion and $\mathbf{k}_{S_i}$ *is the constant sheaf on* S_i. This makes sense by Lemma 3.1. Let $l(X) = \max\{\dim_{\mathbf{C}} X - \dim_{\mathbf{C}} S \mid S \in \mathscr{S}\}$.

We say that an object $M \in P(X)$ has a *p-filtration* if it has a filtration whose quotients are M_i's.

THEOREM 1.3. — *In the category* P(X) *every projective object has a p-filtration, the* BGG *reciprocity is satisfied and* P(X) *has projective dimension* $\leqq 2\, l(X)$.

We will prove this theorem in Section 3.

To recover the results of [BGG] from ours we remark that given a complex semi-simple Lie Algebra g the category $\mathcal{O}_0$ is equivalent to P(X), where $X = G/B$ is the flag manifold with stratification by the Schubert cells.

Remark 1.4. – The condition $\pi_1(S) = 0$ for $S \in \mathscr{S}'$ and $\pi_2(S) = 0$ for all $S \in \mathscr{S}'$ can be replaced by the following weaker condition. Let X be a complex manifold with Whitney stratification $\mathscr{S}$. We call the stratification $\mathscr{S}$ a *good Whitney stratification* if all the projections $\pi_s : \tilde{\Lambda} \to S$ are fibre bundles, where $\tilde{\Lambda}_s = T_s^* X - \bigcup_{\substack{s' \neq s \\ s' \in \mathscr{S}}} \overline{T_{s'}^* X}$. For a good Whitney stratification $\mathscr{S}$ the condition we need becomes that the map $\alpha : \pi_1(\pi_s^{-1}(x)) \to \pi_1(\tilde{\Lambda}_s)$ is an isomorphism. The crucial point here is that this condition implies that $\pi_1(S) = 0$. By different arguments one can show that for the results of this paper to remain true it suffices to assume that α is injective with Coker $\alpha = \pi_1(S)$ finite and char $(\mathbf{k}) = 0$.

2. Construction of projectives and the symmetry of the Cartan matrix

In this section we will prove Theorem 1.1. We first apply the results of [MV1] and [MV2] to reduce it to an algebraic problem. We start by recalling the main construction of these papers.

Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories, $F : \mathscr{A} \to \mathscr{B}$ a right exact functor, $G : \mathscr{A} \to \mathscr{B}$ a left exact functor and $T : F \to G$ a natural transformation. Then we define a category $\mathscr{C}(F, G; T)$ as follows. Its objects are pairs $(A, B) \in \mathrm{Ob}\,\mathscr{A} \times \mathrm{Ob}\,\mathscr{B}$ together with a commutative diagram

$$\begin{array}{ccc} FA & \xrightarrow{TA} & GA \\ & {\scriptstyle m}\searrow \quad \nearrow{\scriptstyle n} & \\ & B & \end{array}$$

The morphisms are pairs $(f, g) \in \mathrm{Mor}\,\mathscr{A} \times \mathrm{Mor}\,\mathscr{B}$ such that the appropriate prism commutes. The category $\mathscr{C}(F, G; T)$ has a natural abelian category structure [MV2].

Recall that we have a complex analytic space X with a fixed stratification $\mathscr{S}$ satisfying $\pi_1(S)=0$ for all $S \in \mathscr{S}$ and $\pi_2(S)=0$ for all $S \in \mathscr{S}'$. We denote by $\mathscr{V}$ the category of finite dimensional **k**-vector spaces. Under these hypotheses we have

THEOREM 2.1 ([MV1], [MV2]). — *The category* P(X) *can be constructed by iterating the* $\mathscr{C}(F, G; T)$ *construction starting with* $\mathscr{A}=\mathscr{V}$ *and always using* $\mathscr{B}=\mathscr{V}$.

Proof. — This follows from Theorem 3.3 and section 7 of [MV2] using the hypothesis that $\pi_1(S)=0$ for all $S \in \mathscr{S}$, and $\pi_2(S)=0$ for all $S \in \mathscr{S}'$.

So in order to prove Theorem 1.1, it suffices to prove theorems about the categories $\mathscr{C}(F, G; T)$ which arise from perverse sheaves. *For simplicity, we assume from now on that all abelian categories* $\mathscr{A}$ *or* $\mathscr{C}(F, G; T)$ *under discussion come from iterating the* $\mathscr{C}(F, G; T)$ *construction beginning with* $\mathscr{V}$ *and always using* $\mathscr{B}=\mathscr{V}$. *Such categories have a natural* **k**-*vector space* structure on their Hom sets.

GENERAL FACTS ABOUT $\mathscr{C}(F, G; T)$'s. — There are several interesting functors relating $\mathscr{A}$, $\mathscr{V}$ and the $\mathscr{C}(F, G; T)$ built from $F \xrightarrow{T} G : \mathscr{A} \to \mathscr{V}$. First, given an object

$$N=(A, B, m, n) \in \mathscr{C}(F, G; T))$$

we have restrictions of N to $\mathscr{A}$ and $\mathscr{B}$:

$$N|_{\mathscr{A}}=A, \qquad N|_{\mathscr{B}}=B.$$

The restriction functors are exact. In fact, a complex $N^\bullet \in \mathscr{C}(F, G; T)$ is exact if and only if $N^\bullet|_{\mathscr{A}}$ and $N^\bullet|_{\mathscr{B}}$ are exact. This follows immediately from the description of kernels and cokernels in $\mathscr{C}(F, G; T)$ in [MV2].

There are three functors $\hat{F}$, $\hat{T}$ and $\hat{G}$ from $\mathscr{A}$ to $\mathscr{C}(F, G; T)$. If $A \in \mathscr{A}$ we set

$$\hat{F}A = \begin{array}{ccc} FA & \xrightarrow{TA} & GA, \\ & {\scriptstyle Id}\searrow \quad \nearrow{\scriptstyle TA} & \\ & FA & \end{array} \quad \hat{T}A = \begin{array}{ccc} FA & \xrightarrow{TA} & GA, \\ & {\scriptstyle TA}\searrow \quad \nearrow{\scriptstyle incl.} & \\ & \mathrm{Im}\,TA & \end{array} \quad \hat{G}A = \begin{array}{ccc} FA & \xrightarrow{TA} & GA. \\ & {\scriptstyle TA}\searrow \quad \nearrow{\scriptstyle Id} & \\ & GA & \end{array}$$

There are obvious maps $\hat{F} \to \hat{T} \to \hat{G}$. Note that these functors $\hat{F}$, $\hat{T}$ and $\hat{G}$ correspond to the functors ${}^p j_!$, ${}^p j_{!*}$ and ${}^p j_*$ in P(X) (*see* example 4.6 in [MV2]).

The functor $\hat{F}$ is right exact and the functor $\hat{G}$ is left exact. We also have

LEMMA 2.2. — *For* $N \in C(F, G; T)$ *and* $A \in \mathscr{A}$ *we have*

$$\operatorname{Hom}(\hat{F}A, N) = \operatorname{Hom}(A, N|\mathscr{A})$$

and

$$\operatorname{Hom}(N, \hat{G}A) = \operatorname{Hom}(N|\mathscr{A}, A).$$

This lemma implies that $\hat{F}$ preserves projectives and $\hat{G}$ preserves injectives.

IRREDUCIBLE OBJECTS. — We wish to describe all the irreducible objets in any $\mathscr{C}(F, G; T)$ built on $\mathscr{A}, \mathscr{V}$.

PROPOSITION 2.3. — *The category* $\mathscr{C}(F, G; T)$ *has the following irreducible objects:*

1.

$$\begin{array}{ccc} FO & \longrightarrow & GO \\ & \searrow \quad \nearrow & \\ & k & \end{array}$$

2. $\hat{T}L$, *where* $L \in \mathscr{A}$ *is irreducible.*

Hence any $\mathscr{C}(F, G; T)$ has finitely many nonisomorphic irreducible objects.

Proof. — Suppose

$$\begin{array}{ccc} FA & \longrightarrow & GA \\ & {}_m\searrow \quad \nearrow_n & \\ & B & \end{array}$$

is irreducible, $A \neq 0$. Then m is surjective and n is injective, because otherwise there would be nontrivial maps of N to or from the irreducible

$$\begin{array}{ccc} FO & \longrightarrow & GO \\ & \searrow \quad \nearrow & \\ & \mathbf{k} & \end{array}$$

Hence $N = \hat{T}(N|_{\mathscr{A}}) = \hat{T}A$. We need to show that $A \in \mathscr{A}$ is irreducible. Let $A' \to A$ be any nonzero map. Then $\hat{T}A' \to \hat{T}A$ is a nonzero map, and so must be surjective. Hence $A' \to A$ is surjective, so A is irreducible.

Q.E.D.

REPRESENTABILITY OF FUNCTORS

PROPOSITION 2.4. — *Assume* $\mathscr{A}$ *has enough injectives. Then any left exact functor* $G : \mathscr{A} \to \mathscr{V}$ *is representable by an object* $R \in \mathscr{A}$.

Proof. — This follows from Grothendieck's pro-representability theorem [Mu]. However we give a simple direct proof. G is exact when restricted to injective objects of $\mathscr{A}$. It suffices to show that there exists $R \in \mathscr{A}$ and a natural isomorphism $\mathrm{Hom}(R, I) = GI$ for I injective in $\mathscr{A}$. If $I \in \mathscr{A}$, let $h_I = \mathrm{Hom}(I, .)$. Given $v \in GI$ there exists a natural map

$$h_v : \quad h_I \to G$$

defined by $(h_v N)(f) = (G f) v$, $f \in \mathrm{Hom}(I, N)$. If $N = I$ then $(h_v I)(\mathrm{Id}_I) = v$. Choose a spanning set $v_1, \ldots, v_r$ for GI. Then we get a natural map

$$\varphi_I = \bigoplus_i h_{v_i} : h_{I^r} \to G.$$

φ_I has the property that $\varphi_I(I) : h_{I^r}(I) \to GI$ is surjective. Let $I_1, \ldots, I_m$ be the indecomposable injectives, corresponding to the irreducibles objects of $\mathscr{A}$. Consider the sum $I = \bigoplus_k I_k^{r_k}$ and

$$\varphi = \oplus \varphi_{I_k} : h_I = \bigoplus_k h_{I_k^{r_k}} \to G.$$

Then $\varphi : h_I \to G$ has the property that $\varphi(J) : h_I(J) \to GJ$ is surjective for *any* injective $J \in \mathscr{A}$. Let $G' = \ker(h_I \xrightarrow{\varphi} G)$. Then G' is also exact, so there exists $\varphi' : h_{I'} \twoheadrightarrow G'$. Hence we have produced a 2-step resolution of G:

$$h_{I'} \to h_I \to G \to 0.$$

By Yoneda's lemma, we get a map $I \xrightarrow{\alpha} I'$. Let $R = \ker \alpha$. Then R represents G because for any injective J,

$$\mathrm{Hom}(R, J) \cong \mathrm{Hom}(I, J)/\mathrm{Im}\,\mathrm{Hom}(I', I) \cong h_I(J)/\mathrm{Im}\ h_{I'}, (J) \cong GJ.$$

Q.E.D.

Remark. — A similar statement holds for right exact functors $F : \mathscr{A} \to \mathscr{V}$: If $\mathscr{A}$ has enough projectives, there exists an object $S \in \mathscr{A}$ st

$$FN \cong \mathrm{Hom}(N, S)^*$$

where * is the dual in the sense of **k**-vector spaces.

PROJECTIVES AND INJECTIVES IN $\mathscr{C}(F, G; T)$. — The following proposition together with Lemma 1.2 establishes the first part of Theorem 1.1.

PROPOSITION 2.5. — *Suppose $\mathscr{A}$ has enough projectives (injectives). Then any $\mathscr{C}(F, G; T)$ built on $\mathscr{A}$, $\mathscr{V}$ has enough projectives (injectives).*

Proof. — We must show that any object $N \in \mathscr{C}(F, G; T)$ is covered by a projective. We can assume N is irreducible. If $N = \hat{T}A$, $A \in \mathscr{A}$ irreducible, $A' \to A$ a projective covering of A, then $\hat{F}A' \to \hat{T}A$ is a projective cover of N.

So it suffices to cover the new irreducible

$$\begin{array}{ccc} FO & \longrightarrow & GO \\ & \searrow \quad \nearrow & \\ & \mathbf{k} & \end{array}$$

By Proposition 2.4 G is representable. Suppose $G \cong \mathrm{Hom}(R, .)$. Form the object P:

$$\begin{array}{ccc} FR & \longrightarrow & GR = \mathrm{Hom}(R, R) \\ & {}^{m}\searrow \quad \nearrow_{n} & \\ & FR \oplus \mathbf{k} & \end{array}$$

where $m = (\mathrm{Id}, 0)$, $n|_{FR} = TR$, $n(0,1) = \mathrm{Id}_R$. For any $N \in \mathscr{C}(F, G; T)$

$$\mathrm{Hom}(P, N) \cong N|_{\mathscr{V}};$$

i. e., a map $P \to N$ is uniquely determined by the image of the element (0,1) in $N|_{\mathscr{V}}$. Since $|_{\mathscr{V}}$ is exact, P is projective. Clearly, P covers the new irreducible. Hence $\mathscr{C}(F, G; T)$ has enough projectives. A similar proof works for injectives.

Q.E.D.

To complete the proof of Theorem 1.1 we have to prove the symmetry of the Cartan matrix. However this symmetry is not true for $\mathscr{C}(F, G; T)$'s in general. The category of perverse sheaves P(X) has an involution $A \to A^*$ given by Verdier duality which satisfies

(*a*) $\mathrm{Hom}(A_1, A_2^*) \cong \mathrm{Hom}(A_2, A_1^*)$,

(*b*) $L^* \cong L$ for L irreducible.

Condition (*b*) holds because $\pi_1(S) = 0$ for all strata S. If $\mathscr{L}$ is a complex link of S at a point ([MV2], [GM]) then $F(A) = H^{-d-1}(\mathscr{L}, A)$ and $G(A) = H_c^{-d-1}(\mathscr{L}, A)$. By Verdier duality we then have

$$F(A) = G(A^*)^* \qquad \text{and} \qquad T(A^*) = T(A)^*$$

which means that the representing objects S and R for F and G satisfy $S = R^*$.

Motivated by these considerations we develop a notion of duality for $\mathscr{C}(F, G; T)$'s.

Duality. — Let $\mathscr{A}$ be an abelian category. A *duality* on $\mathscr{A}$ is by definition a contravariant functor $A \to A^*$ st.

(*a*) If $A, B \in \mathscr{A}$, $\mathrm{Hom}(A, B^*) \cong \mathrm{Hom}(B, A^*)$ naturally;

(*b*) * is fully faithful.

Condition (*b*) is equivalent to

(*b′*) The natural map $A \to A^{**}$ is an equivalence of categories.

Suppose $\mathscr{A}$ has a duality *. We wish to extend * to $\mathscr{C}(F, G; T)$. However, we need some conditions on the representing objects for F and G.

We assume that $S = R^*$ and that the diagram

$$\begin{array}{ccc} \mathrm{Hom}(R, A)^* & \xrightarrow{(TA)^*} & \mathrm{Hom}(A, R^*) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A^*, R^*)^* & \xrightarrow[T(A^*)]{} & \mathrm{Hom}(R, A^*) \end{array}$$

commutes: i. e., $T(A^*) = (TA)^*$ under the above identifications.

If

$$N = \begin{array}{ccc} FA & \xrightarrow{TA} & GA \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

we can let N^* be the object

$$\begin{array}{ccc} (GA)^* & \xrightarrow{(TA)^*} & (FA)^* \\ & \searrow \quad \nearrow & \\ & B^* & \end{array} = \begin{array}{ccc} \mathrm{Hom}(R, A)^* & \xrightarrow{(TA)^*} & \mathrm{Hom}(A, R^*). \\ & \searrow \quad \nearrow & \\ & B^* & \end{array}$$

Then $*$ is a duality on $\mathscr{C}(F, G; T)$ extending the duality on $\mathscr{A}$. Note that $^*\hat{F} = \hat{G}^*$, $^*\hat{T} = \hat{T}^*$ and $*$ fixes the new irreducible object. Hence if $*$ on $\mathscr{A}$ fixes irreducible objects in $\mathscr{A}$, $*$ on $\mathscr{C}(F, G; T)$ will also fix irreducible objects in $\mathscr{C}(F, G; T)$.

Symmetry of the Cartan Matrix. — Let $L_1, \ldots, L_r$ be the distinct irreducibles in $\mathscr{A}$, and $P_i \twoheadrightarrow L_i$ *the projective covers of* L_i. Note that the L_i's have the property that $\dim_k \mathrm{Hom}(L_i, L_j) = \delta_{ij}$.

Consider the Grothendieck group $K(\mathscr{A})$. This is a free abelian group with basis $[L_1]$, $\ldots$, $[L_r]$. We have by definition

$$[N] = \sum_{j=1}^{r} [N : L_j][L_j], \qquad N \in \mathscr{A}.$$

Note that $[N : L_j] = \dim_k \mathrm{Hom}(P_j, N)$, because both sides are additive functions on $K(\mathscr{A})$ which agree when $N = L_i$.

Recall that the Cartan matrix of $\mathscr{A}$ is $C_{ij} = [P_i : L_j]$. We are interested in the *symmetry* of C_{ij}.

The category of perverse sheaves $P(X)$ is constructed by iterating the $\mathscr{C}(F, G; T)$ construction where $F = \mathrm{Hom}(., S)^*$ and $G = \mathrm{Hom}(R, .)$ are represented by dual objects $S = R^*$. Since $*$ fixes irreducibles, $[R^*] = [R]$ in the Grothendieck group. Therefore the following proposition shows that the Cartan matrix for $P(X)$ is symmetric, and completes the proof of Theorem 1.1.

PROPOSITION 2.6. — *The Cartan matrix of* $\mathscr{C}(F, G; T)$ *is symmetric precisely when the Cartan matrix of* $\mathscr{A}$ *is symmetric and* $[R]=[S]$ *in* $K(\mathscr{A})$.

Proof. — In $\mathscr{C}(F, G; T)$ let $\hat{L}_i = \hat{T}L_i$, $\hat{P}_i = \hat{F}P_i$, $1 \leqq i \leqq r$,

$$\begin{array}{ccccc} \hat{L}_{r+1} = FO & \longrightarrow & GO, & \hat{P}_{r+1} \longrightarrow \hat{L}_{r+1} \\ & \searrow \; \mathbf{k} \; \nearrow & & \end{array}$$

its projective cover.

Let $\hat{C}_{ij}$ be the Cartan matrix of $\mathscr{C}(F, G; T)$. If $i, j \leqq r$ then by adjunction $\hat{C}_{ij} = \dim \operatorname{Hom}(\hat{F}P_j, \hat{F}P_i) = \dim \operatorname{Hom}(P_j, P_i) = C_{ij}$. So we need only check that that $\hat{C}_{i,\, r+1} = \hat{C}_{r+1,i}$, $1 \leqq i \leqq r$. Write the new projective

$$\begin{array}{ccc} \hat{P} = FR & \longrightarrow & GR \\ & \searrow \quad \nearrow & \\ & FR \oplus \mathbf{k} & \end{array}$$

in terms of $\hat{P}_i$'s:

$$\hat{P} = \hat{P}_{r+1} \oplus \bigoplus_{i=1}^{r} \hat{P}_i^{\alpha_i}$$

Then $\dim \operatorname{Hom}(\hat{P}_{r+1}, \hat{P}_i) = \dim \operatorname{Hom}(\hat{P}, \hat{P}_i) - \sum_{j=1}^{r} \alpha_j C_{ij}$, and $\dim \operatorname{Hom}(\hat{P}_i, \hat{P}_{r+1}) = \dim \operatorname{Hom}(\hat{P}_i, \hat{P}) - \sum_{j=1}^{r} \alpha_j C_{ji}$. So we need to compare $\operatorname{Hom}(\hat{P}_i, \hat{P})$ and $\operatorname{Hom}(\hat{P}, \hat{P}_i)$. By adjunction

$$\dim \operatorname{Hom}(\hat{P}_i, \hat{P}) = \dim \operatorname{Hom}(P_i, R) = [R : L_i]$$

and

$$\dim \operatorname{Hom}(\hat{P}, \hat{P}_i) = \dim FP_i = \dim \operatorname{Hom}(P_i, S)^* = [S : L_i].$$

So $\hat{C}_{ij}$ is symmetric $\Leftrightarrow [R]=[S]$ in $K(\mathscr{A})$.

Q.E.D.

3. The BGG reciprocity

In this section we will give a proof of theorem 1.3. We start with some topological considerations. As in the previous section, after this the rest of the proof is purely algebraic.

We recall that we have a complex analytic space X with an analytic stratification $\mathscr{S}$ satisfying the Whitney conditions. As before we assume that $\pi_1(S)=0$ for all $S \in \mathscr{S}$ and

$\pi_2(S)=0$ for all $S\in\mathscr{S}'$. From now on we assume furthermore that $\bar{S}-S$ is a Cartier divisor in $\bar{S}$ (or empty) for for all $S\in\mathscr{S}$. (If X is algebraic this means that $S\to X$ is affine.)

LEMMA 3.1. — *Let* $S\in\mathscr{S}$ *and* $j: S\to X$ *be the inclusion. Then* $j_!\,\mathbf{k}_S[\dim_{\mathbf{C}} S]$ *is perverse.* (*See* [BBD] 4.1.3 for the algebraic case).

Proof. — Clearly $j_!\,\mathbf{k}_S[\dim_{\mathbf{C}} S]$ satisfies the first perversity condition. It remains to check the second condition or equivalently the first perversity condition for the dual $Rj_*\,\mathbf{k}_S[\dim_{\mathbf{C}} S]$. Let $d=\dim_{\mathbf{C}} S$.

Cutting by a normal slice reduces the problem to the case of a point stratum $S'=\{x\}\subset\bar{S}$. Let $i:\{x\}\to X$. Then if B is a small neighborhood of x in X,

$$H^k(i^*Rj_*\,\mathbf{k}_S[d])=H^{k+d}(Rj_*\,\mathbf{k}_S)_x\simeq H^{k+d}(B,\,Rj_*\,\mathbf{k}_S)\cong H^{k+d}(B\cap S,\,\mathbf{k})\cong 0 \qquad \text{for } k>0$$

because $B\cap S$ is a Stein manifold of dimension d.

Q.E.D.

For any stratum $S_k\in\mathscr{S}$ we define the object M_k by $M_k=j_!\,\mathbf{k}_{S_k}[\dim_{\mathbf{C}} S_k]$, where $j: S_k\to X$ is the inclusion.

The construction of the objects M_k can be done inductively as follows. Let $X\subset\hat{X}$ such that X is stratified by $S_1,\ldots,S_{r-1}$ and let $\hat{X}-X=S_r$. We assume that $\dim S_k\geqq\dim S_h$ if $k\leqq h$. Let $\hat{j}: X\to\hat{X}$, $j_k: S_k\to X$ and $\hat{j}_k: S_k\to\hat{X}$ be the inclusions. Then if we denote $\hat{M}_k=\hat{j}_!\,\mathbf{k}_{S_k}[\dim_{\mathbf{C}} S_k]$ we have $\hat{M}_k=\hat{j}_!\,M_k$ for $k\leqq r-1$. Or because $\hat{M}_k$ is perverse we can phrase this as $\hat{M}_k={}^p\hat{j}_!\,M_k$ for $k\leqq r-1$.

If we interpret this in terms of the $\mathscr{C}(F, G; T)$ via theorem 2.1 we get that $\hat{M}_k=\hat{F}(M_k)$ for $k\leqq r-1$ and $\hat{M}_r=\hat{L}_r$.

LEMMA 3.2. — *We have* $L^1\hat{F}(M_k)=0$.

Proof. — It suffices to show that given any exact sequence $0\to N'\to N\to M_k\to 0$ the sequence $0\to\hat{F}N'\to\hat{F}N\to\hat{F}M_k\to 0$ is exact. Because ${}^pj_!\,M_k=j_!\,M_k$ we have an exact sequence $0\to{}^pj_!\,N'\to{}^pj_!\,N\to{}^pj_!\,M_k\to 0$ in P(X), but this is just the exact sequence $0\to\hat{F}N'\to\hat{F}N\to\hat{F}M_k\to 0$.

Remark. — Because we are using a *fixed* stratification in our definition of P(X) it is *not* true that $\mathrm{Ext}^k(A, B)$ is the same in P(X) and in $D^b(X)$. It is however, clearly true for $k=0,1$.

We now turn to algebra. We make the additional hypothesis that $L^1\hat{F}(M_k)=0$ at every stage of the construction of our $\mathscr{C}(F, G; T)$. We also assume duality at every stage.

Recall [BBG] that we say that N has a *p-filtration* if there is a filtration $N_1\subset N_2\subset\ldots$ such that $N_k/N_{k+1}\cong M_i$ for some i. We will start by proving a lemma about the existence of p-filtrations which in particular shows that every projective object has a p-filtration.

LEMMA 3.3. — *Let* $\mathscr{C}$ *be a category which is constructed by iteration with the above hypotheses. Then* N *has a p-filtration if and only if* $\mathrm{Ext}^1(M_i, N^*)=0$ *for all* i.

Proof. — We proceed by induction. Assume that it is true for $\mathscr{A}$ and construct a $\mathscr{C}(F, G; T)$ from $\mathscr{A}$. Suppose $\mathrm{Ext}^1(\hat{M}_i, N^*)=0$ for $i=1, \ldots, r+1$.

To calculate $\mathrm{Ext}^1(\hat{M}_{r+1}, \hat{N}^*)$ we use the resolution

$$0 \to \hat{F}R \to \hat{P} \to \hat{M}_{r+1} \to 0.$$

This gives

$$\mathrm{Hom}(\hat{P}, \hat{N}) \to \mathrm{Hom}(\hat{F}R, \hat{N}^*) \to \mathrm{Ext}^1(\hat{M}_{r+1}, \hat{N}^*) \to 0.$$

Let $\hat{N}=FA \overset{m}{\to} B \overset{n}{\to} GA$.

Then $\mathrm{Hom}(\hat{P}, \hat{N}^*)=B^*$, and

$$\mathrm{Hom}(\hat{F}R, \hat{N}^*)=\mathrm{Hom}(R, \hat{N}^*|_{\mathscr{A}})=G(A^*)=(FA)^*.$$

Therefore $\mathrm{Ext}^1(\hat{M}_{r+1}, \hat{N}^*) \cong \mathrm{Coker}(m^*)=\mathrm{Ker}(m)^*$.

Hence m in an injection. It follows from this that we have a short exact sequence

$$0 \to \hat{F}(\hat{N}\,|\,\mathscr{A}) \to \hat{N} \to \hat{M}_{r+1}^{\oplus q} \to 0, \qquad q \geqq 0.$$

So it is enough to show that $F(\hat{N}\,|\,\mathscr{A})$ has a p-filtration or since $L^1\hat{F}(M_i)=0$ that $\hat{N}\,|\,\mathscr{A}$ has a p-filtration. But $L^1\hat{F}M_i=0$ means we have

$$\mathrm{Ext}^1(M_i, \hat{N}^*\,|\,\mathscr{A})=\mathrm{Ext}^1(\hat{M}_i, \hat{N}^*)=0.$$

and therefore $\hat{N}^*\,|\,\mathscr{A}$ has a p-filtration.

For the converse it suffices to check that $\mathrm{Ext}^1(\hat{M}_i, \hat{M}_j^*)=0$. By duality $\mathrm{Ext}^1(\hat{M}_i, \hat{M}_j^*)=\mathrm{Ext}^1(\hat{M}_j, \hat{M}_i^*)$. If either i or j is $\leqq r$, then $\mathrm{Ext}^1(\hat{M}_i, \hat{M}_j^*)=0$ by adjunction and the vanishing of $L^1\hat{F}R$. And $\mathrm{Ext}^1(\hat{M}_{r+1}, \hat{M}_{r+1}^*)=0$ as before.

Q.E.D.

Next we give a proof of the BGG reciprocity. Assume that we have constructed a category $\mathscr{C}$ by iteration, where $F(A)=G(A^*)^*$, $(TA)^*=TA^*$, and $L^1\hat{F}(M_k)=0$ for all k at each stage of the iteration.

Note that the decomposition matrix $D=[M_i : L_j]$ is unipotent upper triangular and therefore the M_i form a basis for $K(\mathscr{C})$. Let $E=[P_i : M_j]$, where $[P_i]=\sum [P_i : M_j][M_j]$ *in* $K(\mathscr{C})$. Since the P_i have a p-filtration the matrix E has positive entries.

THEOREM 3.4 (BGG Reciprocity). — *We have* $E={}^tD$ *and therefore* $C={}^tDD$.

Proof. — We proceed by induction. Let E and D be the decomposition matrices of $\mathscr{A}$ and $\hat{E}$ and $\hat{D}$ the corresponding matrices in $\mathscr{C}(F, G; T)$. Because the P_i have p-filtrations and $L^1\hat{F}(M_j)=0$ we have

$$\hat{E}_{ij}=E_{ij} \qquad \textit{if} \quad 1 \leqq i, j \leqq r$$

and

$$E_{i,\, r+1}=0 \qquad \text{if} \quad 1 \leqq i \leqq r.$$

So to prove the proposition we must only check that

$$\hat{D}_{r+1,\,i}=\hat{E}_{i,\,r+1} \qquad \text{if} \quad 1 \leqq i \leqq r+1,$$

i. e.

$$[\hat{P}_{r+1} : \hat{M}_i]=[\hat{M}_i : \hat{L}_{r+1}].$$

We have the short exact sequence $0 \to \hat{F}R \to \hat{P} \to \hat{L}_{r+1} \to 0$ where R represents G and $\hat{P}$ is the new projective constructed in paragraph 2. Write

$$\hat{P}=\hat{P}_{r+1} \oplus \bigoplus_{i=1}^{r} \hat{P}_i^{\alpha_i}$$

as before. Then if $1 \leqq i \leqq r$

$$[\hat{P}_{r+1} : \hat{M}_i]=[\hat{P} : \hat{M}_i]-\sum_{j=1}^{r} \alpha_j [\hat{P}_j : \hat{M}_i]=[R : M_i]-\sum_{j=1}^{r} \alpha_j E_{ji};$$

$$\begin{aligned}[\hat{M}_i : \hat{L}_{r+1}] &= \dim \operatorname{Hom}(\hat{P}_{r+1}, \hat{M}_i) \\ &= \dim \operatorname{Hom}(\hat{P}, \hat{M}_i)-\sum_{j=1}^{r} \alpha_j \dim \operatorname{Hom}(\hat{P}_j, \hat{M}_i) \\ &= \dim \hat{F}M_i-\sum_{j=1}^{r} \alpha_j D_{ij} \\ &= \dim \operatorname{Hom}(M_i, R^*)-\sum_{j=1}^{r} \alpha_j D_{ij}.\end{aligned}$$

So we must show that $[R : M_i]=\dim \operatorname{Hom}(M_i, R^*)$. We have $\operatorname{Ext}^1(M_i, M_j^*)=0$ and $\dim \operatorname{Hom}(M_i, M_j^*)=\delta_{ij}$ (this can easily be established by induction). Using this and the fact that R has a p-filtration we see that $[R : M_i]=\dim \operatorname{Hom}(M_i, R^*)$.

Q.E.D.

We will conclude by proving that the projective dimension of $P(X) \leqq 2\,l(X)$, where

$$l(X)=\dim_{\mathbf{C}} X-\min\{\dim_{\mathbf{C}} S \mid S \in \mathscr{S}\}.$$

We define another length function $l(k)$ by induction as follows. $l(1)=0$. Suppose $\mathscr{A}$ has objects $M_1, \ldots, M_r$ and $\mathscr{C}=\mathscr{C}(F, G; T)$ is constructed with representing object R. Let $l(r+1)=l(r)$ if R has a decomposition series with M_k such that $l(k)<l(r)$, $l(r+1)=l(r)+1$ otherwise. Note that if X has strata $S_1, S_2, \ldots$ then $l(k) \leqq \operatorname{codim}_{\mathbf{C}} S_k$. Let $l(\mathscr{C})=\max_{k \geqq 1} l(k)$. Then $l(\mathscr{C}) \leqq l(X)$.

LEMMA 3.5. — *We have p. d.* $M_i \leqq l(i)$.

Proof. — We proceed by induction. Construct $\mathscr{C}(F, G, T)$ from $\mathscr{A}$. Since $\mathscr{A}$ has finite projective dimension by induction, $L^q \hat{F}(M_i)=0$ for all $q>0$, i. e. the modules M_i are $\hat{F}$-acylic. Hence p. d. $\hat{F}M_i=$ p. d. M_i. Therefore it is enough to prove the result for