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GENERA OF CURVES
VARYING IN A FAMILY

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Introduction

In this paper, we discuss the possible variation of the geometric genus of a projective algebraic curve varying in a family. The precise statement is in Theorem (1.2), but in informal language this says that if an integral projective curve A of geometric genus g degenerates into B (possibly non-reduced), with associated cycle $\sum_{i=1}^{s} m_i B_i$ [*cf.* (1.1)], where the genus of B_i is g_i, then it holds : $g \geq \sum_{i=1}^{s} \varepsilon_i (m_i g_i - m_i + 1)$, where $\varepsilon_i = 1$ if $g_i > 0$ and zero otherwise. In the case of plane projective curves there is a converse "existence theorem" [*cf.* (2.1)], which may be phrased in geometric language as follows: a plane curve $B = \sum_{i=1}^{s} m_i B_i$ (where $B_1, \ldots, B_s$ are the irreducible components of B and B_i has geometric genus g_i), of degree n, is a specialization of an integral plane curve A, of geometric genus g, having only nodes (ordinary double points) as singularities if and only if the inequality above is satisfied, plus the well-known one $g \leq (1/2)(n-1)(n-2)$. This gives a simple numerical characterization of the boundary points of "Severi's variety" of irreducible plane curves having degree n and a fixed number of nodes [10]. See Theorem (2.3). Here, we work over an algebraically closed field of zero characteristic.

These results (the inequality in the planar case only) appear in a classical paper of G. Albanese [1], unfortunatelly written in an obscure language (*cf.* [11], *p.* 216). Modern papers on the theory are [8] (where the case where B is reduced is discussed) and [9], where a proof of the inequality is presented, and the existence theorem is verified when $g \geq \sum_{i=1}^{r} m_i g_i$ (where $B_1, \ldots, B_r$ are the components of B with $g_i > 0$). However, the proof of the inequality given in [9] [Theorem (1.2) there] contains an error [Lemma (1.5) of [9] is false]. One can give a proof of (1.2) based on the results of [7], a remarkable but technical paper. But in the geometric case of interest to us one can give also a rather short and simple proof based on the semi-stable reduction theorem [as suggested in [9], (1.10)], and this is what is done in paragraph 1 of the present paper.

As pointed out in [9], (2.6), the proof of the Existence Theorem can be reduced to the case $s=1$, $g_1>0$; so this is the crucial case. This is Theorem (2.6) of the present paper, which is proved in paragraph 3. For the sake of completeness, and because we believe that the new presentation is better than that of [9], in paragraph 2 we explain again how the quoted reduction to the case "$s=1$" is accomplished.

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1. The main inequality

(1.1) We work thoughout over a base field K, algebraically closed, of characteristic zero. An algebraic variety will mean a reduced algebraic scheme of finite type over K. A curve will mean a purely one-dimensional algebraic scheme (perhaps non-reduced). If B is a curve, with irreducible components $B_1, \ldots, B_s$, its associated cycle $z(B)$ is defined to be the formal sum $\sum_{i=1}^{n} m_i B_i$, where $m_i = $ length $(\mathcal{O}_{B, z_i})$, where z_i is the generic point of B_i ([6], p. 425). If C is an integral curve (i. e., irreducible and reduced), then its geometric genus $g(C)$ (henceforth referred to as the genus) is the arithmetic genus of the normalization ([6], p. 230).

(1.2) THEOREM. — *Let* $\pi : X \to T$ *be a family of projective curves (i. e., a flat, projective morphism of relative dimension one) such that* T *is integral. Let* A *be the generic geometric fiber,* $t_0 \varepsilon T$ *a closed point,* $B = \pi^{-1}(t_0)$, $z(B) = \sum_{i=1}^{s} m_i B_i$, *where the indices have been chosen so that* $g(B_i) > 0$ *if and only if* $1 \leqslant i \leqslant r$, *for a suitable integer* r, $0 \leqslant r \leqslant s$. *Then, the following inequality holds:*

$$(1.2.1) \qquad\qquad g(A) \geqslant \sum_{i=1}^{r} (m_i g(B_i) - m_i + 1)$$

Clearly, this inequality may be rewritten as:

$$(1.2.2) \qquad\qquad (g(A) - 1) \geqslant \sum_{i=1}^{r} m_i (g(B_i) - 1) + r - 1$$

Proof. — (a) By a suitable base change, we may assume that T is an irreducible smooth curve; in the sequel we shall assume this is the case.

(b) I claim the inequality (1.2.2) follows if it holds in the case where the general fiber of π is smooth. In fact, assume (1.2.2) to be true for A smooth, and consider any family

$X \to T$ [with T a smooth curve, by (a)]. Then X is an integral surface. Let $X' \overset{\eta}{\to} X$ be the normalization of X and $q = \pi \eta$. One easily checks that q is a flat, projective morphism and that the general fiber of q is smooth (isomorphic to the normalization of the general fiber of π). Let $B' = q^{-1}(t_0)$, and let us write

$$z (B') = \sum_{i=1}^{r} \left(\sum_{j=1}^{h(i)} m_{ij} B_{ij} \right) + \sum r_k B'_k,$$

where $B_{i1}, \ldots, B_{ih(i)}$, $i = 1, \ldots, r$ are all the components of B' such that $\eta (B_{ij}) = B_i$, $i = 1, \ldots, r$ (i. e., $\eta (B'_k) = B_m$, with $m > r$, for all k). Then we have:

$$g - 1 \geqslant \sum_{i=1}^{r} \left(\sum_{j=1}^{h(i)} m_{ij} (g (B_{ij}) - 1) \right) + r - 1$$

by our assumption and the obvious inequality $h(1) + \ldots + h(r) \geqslant r$. By Hurwitz formula, $g (B_{ij}) - 1 \geqslant d_{ij} (g (B_i) - 1)$, where $d_{ij} = \deg (B_{ij} \to B_i)$ (induced by η). Hence we get:

$$(g - 1) \geqslant \sum_{i=1}^{r} \left(\sum_{j=1}^{h(i)} m_{ij} d_{ij} \right) (g (B_i) - 1) + r - 1$$

But $\displaystyle\sum_{j=1}^{h(i)} m_{ij} d_{ij} = m_i$, because $z (B) = \eta_* (z (B'))$ [image as a cycle, cf. [4], Proposition 10.1 or [9], (1.8)].

Hence, we must show (1.2.2) when the general curve A is smooth. First, we discuss:

(c) Case $g (A) \geqslant 2$. In this case, we use the semi-stable reduction theorem $(cf.$ [3]) and some basic facts on surfaces, to obtain a commutative diagram

$$\begin{array}{ccc} Z & \overset{\beta}{\to} & X \\ f \downarrow & & \downarrow \pi \\ U & \overset{p}{\to} & T \end{array}$$

where f is flat, $p^{-1}(t_0) = 0 \varepsilon U$, p is etale off 0, for all $u \varepsilon U$, $u \neq 0$, $f^{-1}(u)$ and $\pi^{-1}(p(u))$ are isomorphic, and $f^{-1}(0) = Z_0$ is a reduced curve, whose only singularities are nodes. Then we have, for $u \varepsilon U$, $u \neq 0$:

$$g (A) = g = g (f^{-1}(u)) = p_a (f^{-1}(u)) = p_a (Z_0),$$

where we use the flatness of f and the smoothness of $f^{-1}(u)$. By a well-known formula, since Z_0 is reduced and connected,

$$p_a (Z_0) \geqslant \sum_{i=1}^{r} \left(\sum_{j=1}^{v(i)} g (E_{ij}) \right) + \sum_k g (E_k),$$

where $E_{i1}, \ldots, E_{i, v(i)}$ are all the components of Z_0 mapping onto B_i, $i = 1, \ldots, r$, the E_k's being the other components. We conclude as in (c):

$$g - 1 \geqslant \sum_{i, j} (g\,(E_{ij}) - 1) + r - 1$$

$$\geqslant \sum_{i=1}^{r} \left(\sum_{j=1}^{v(i)} \rho_{ij}\,(g\,(B_i) - 1) \right) + r - 1$$

$$\geqslant \sum_{i=1}^{r} (g\,(B_i) - 1) \left(\sum_{j=1}^{v(i)} \rho_{ij} \right) + r - 1$$

$$= \sum_{i=1}^{r} m_i\,(g\,(B_i) - 1) + r - 1,$$

where ρ_{ij} is the degree of $E_{ij} \to B_i$ induced by β and we have used Hurwitz formula and the equality of cycles $\beta_* (z\,(Z_0)) = z\,(B)$. This proves the assertion.

(d) We must verify the cases: A is smooth, and its genus is zero or one. The usual semi-stable reduction theorem does not hold in this case, but if $g = g\,(A) = 1$ and there is a section $\sigma : T \to X$, then it holds (because we have a family of abelian varieties of dimension one). Now, we may take a base change $T' \to T$ such that the pulled-back family $X' \to T'$ admits a section σ (e. g., 1 et D be any curve in X such that $\pi\,(D) = T$, T' its normalization; then the composition morphism $T' \to T$ works. We may proceed exactly as in (c) to show (1.2.2) for this family, and this clearly implies the searched inequality for the family $X \to T$ too.

To conclude, we check the case $g = 0$. Let $X \xrightarrow{\pi} T$ be any family whose general fibre A is a smooth rational curve. Clearly in this case (1.2.2) will follow if we verify that $p_a\,(B_i) = 0$ for all i, where $B_1, \ldots, B_s$ are the irreducible components of $B = \pi^{-1}\,(t_0)$. In view of [6] (Proposition 9.8 and Example 2.5.4) and the theory of resolution of singularities, we may assume that X is a smooth, projective surface. Now, for each $i = 1, \ldots, s$ we have a surjection

$$\mathcal{O}_B \to \mathcal{O}_{B_i}$$

which implies, by one-dimensionality, that:

$$H^1\,(B, \mathcal{O}_B) \to H^1\,(B_i, \mathcal{O}_{B_i})$$

is surjective. If we prove that $H^1\,(B, \mathcal{O}_B) = 0$, we are done. But B is a divisor in a regular scheme, hence it is Gorenstein, so its dualizing sheaf ω_B is invertible. By flatness of π, $p_a\,(B) = 0$. Now, $\deg\,(\omega_B) = 2 p_a\,(B) - 2$ (use [6], p. 366, 1.3 and $\omega_B = \mathcal{O}_B \otimes \mathcal{O}_X\,(B) \otimes \omega_X$); hence $\deg\,(\omega_B) = -2$. But then $H^0\,(B, \omega_B) = 0$, hence its dual $H^1\,(B, \mathcal{O}_B)$ is also zero, as needed.

This concludes the proof of Theorem (1.2).

I am indebted to T. Ekedahl of a suggestion to simplify the proof of cases $g = 0, 1$ in above's theorem.

2. An existence theorem

In this section, we discuss a proof of the following result [except for an important auxiliary result, namely Theorem (2.6), which will be shown in paragraph 3].

Here and in the sequel, a nodal curve will mean a reduced curve whose only singularities are nodes, i. e., ordinary double points.

(2.1) THEOREM. — Let $B = \sum_{i=1}^{s} k_i C_i$ be a plane curve of degree n (where $C_1, \ldots, C_s$ are the irreducible components of B), $g_i = g$ (C_i), where $g_i > 0$ if and only if $i \leqslant r \leqslant s$. Let g be an integer such that

$$(2.1.1) \qquad \sum_{i=1}^{r} k_i g_i - k_i + 1) \leqslant g \leqslant \frac{1}{2} (n-1)(n-2)$$

Then, there is a family of plane curves of degree n, $X \to T$ (with T integral) whose general fiber is a nodal irreducible curve of genus g and a special fiber is the curve B.

(2.2) This can be rephrased in terms of "Severi's varieties", which will be heavily used in the proof. Namely, let $V_{n,\delta}$ be the closure in $\mathbb{P}^N$, $N = (1/2) n(n+3)$, of the set of points corresponding to plane curves of degree n having δ nodes and no more singularities. In general we shall use the same letter to indicate a curve and the corresponding point of $\mathbb{P}^N$. The basic facts on Severi varieties that we need are summarized in [8] (3.1), see also [10] for the details (however, here we shall use a slightly different notation).

It is known that $V_{n,\delta}$ has a unique irreducible component whose general point corresponds to an irreducible nodal curve [5]. This will be denoted by $S_{n,\delta}$.

If C is a (possibly reducible) nodal curve having δ nodes, we define $g(C) : p_a(C) - \delta$. If C has s irreducible components then it holds: $g(C) = \sum_{i=1}^{s} g(C_i) - s + 1$.

We shall use sometimes a "dual" notation:

$$V_{n,\delta} := V_n^{(g)}, \qquad S_{n,\delta} := S_n^{(g)}, \quad \text{if } g = (1/2)(n-1)(n-2) - \delta.$$

Clearly, the conclusion of Theorem (2.1) is equivalent to the assertion: $B \in S_n^{(g)}$; and Theorems (1.2) and (2.1) imply:

(2.3) THEOREM. — A plane curve $B = \sum_{i=1}^{s} k_i C_i$ [as in (2.1)] is in $S_n^{(g)}$ if and only if g satisfies the inequalities (2.1.1).

We present next some auxiliary results, needed in the proof of Theorem (2.1).

(2.4) LEMMA. — Assume $k_i C_i$ is a plane curve of degree n_i, with C_i irreducible and $k_i \geqslant 1$, such that $k_i C_i \in S_{n_i}^{(h_i)}$ (for a suitable h_i), $i = 1, \ldots, s$. Let h be an integer satisfying:

$$\sum_{i=1}^{s} h_i \leqslant h \leqslant \frac{1}{2} (n-1)(n-2); \qquad n = \sum_{i=1}^{s} n_i.$$

Then

$$\sum_{i=1}^{s} k_i C_i \in S_n^{(h)}.$$

Proof. — One considers the morphism:

$$S_{n_1}^{(h_1)} \times \ldots \times S_{n_s}^{(h_s)} \xrightarrow{\varphi} \mathbb{P}^N, \qquad N = \frac{1}{2} n \, (n+3)$$

given by $\varphi \, (D_1, \ldots, D_s) = D_1 + \ldots + D_s$, $D_i \, \varepsilon \, S_{n_i}^{(h_i)}$ for all i. It is known (e. g., *cf.* [12]) that the general point of the image corresponds to a nodal curve of genus $e = (\sum_i h_i) - s + 1$, i. e., Im $(\varphi) \subset V_n^{(e)}$. If D is a general curve of an irreducible component V_1 of $V_n^{(e)}$ containing Im (φ), we may choose a set of nodes S of D, of cardinality h-e, such that if $S' = \{P/P \text{ is a node of D and } P \notin S\}$, then D-S' is connected. Then [*cf.* [8], (3.1)], the "assignment" of the points of S' determines an irreducible component V_2 of $V_{n, \delta}$ ($\delta =$ card (S')) with irreducible general member E (i. e., this component is $S_{n, \delta}$), such that $V_1 \subset V_2$. An easy calculation shows: $g \, (E) = h$, i. e., $\sum_i k_i C_i \, \varepsilon \, \text{Im} \, (\varphi) \subset S_{n, \delta} = S_n^{(h)}$. This proves the Lemma.

(2.5) LEMMA. — *Let* C *be any integral curve of degree m in* $\mathbb{P}^2$, *with* $g \, (C) = 0$. *Then, for any integer* $k > 0$, $k \, C \, \varepsilon \, S_{km}^{(0)}$.

Proof. — If $k = 1$, this is classical (one considers the m-tuple embedding of $\mathbb{P}^1$ in $\mathbb{P}^m$, via $(x_0 : x_1) \to (\ldots : x_0^{m-i} x_1^i : \ldots)$, $\Gamma \subset \mathbb{P}^m$. Then any integral plane curve of genus zero is a projection of Γ, and a general projection will be nodal). If $k > 1$, one uses the result just gotten and Lemma (2.4), with $C_i = C$, $i = 1, \ldots, k = s$, $h_i = 0$ (all i) and $h = \sum_i h_i$.

(2.6) THEOREM. — *Let* C *be an integral curve of degree m and genus* $g > 0$, 1 *an integer,* $h = kg - k + 1$. *Then* $k \, C \, \varepsilon \, S_{km}^{(h)}$.

The proof of (2.6) will be presented in paragraph 3.

(2.7) *Proof of Theorem* (2.1). — Given $B = \sum_{i=1}^{s} k_i C_i$ as in (2.1), by (2.5) and (2.6) we obtain $k_i C_i \, \varepsilon \, S_{k_n n_i}$ if $i > r$ (resp. $\varepsilon \, S_{k_i n_i}^{(h_i)}$, $h_i = k_i g_i - k_i + 1$, if $i \leq r$). Then, Lemma (2.4), with $h = g$, $h_i = 0$ if $i > r$, $h_i = k_i g_i - k_i + 1$ if $i \leq r$, implies the result.

3. Proof of Theorem (2.6)

In this proof we need several known facts, which we recall next.

(3.1) Given a family $p X \to S$ of smooth curves, admitting a section, and a positive integer d, there is a morphism $\pi : \mathcal{V}_d^2 \, (p) \to S$, where each point $\alpha \, \varepsilon \, \mathcal{V}_d^2 \, (p)$ naturally corresponds to a triple (L, Λ, σ), where L is a line bundle over $C = p^{-1} \, (\pi \, (\alpha))$, Λ is a three-dimensional subspace of $H^0 \, (C, L)$ [i. e., (L, Λ) is a "g_d^2"] and σ is a basis of Λ, up

to homothety. Let $\tilde{\mathscr{V}}_d^2$ be the open subset of $\mathscr{V}_d^2$ of points corresponding to g_d^2's without base points. Then, a point of $\tilde{\mathscr{V}}_d^2$ can be identified to a morphism $\varphi : C \to \mathbb{P}^2$ whose image (as a cycle) $\varphi_* (C)$ has degree d (cf. [2], §2).

(3.2) Fix $g > 1$. Let C be a curve of genus g, p_0 its corresponding point in the moduli variety $\mathscr{M}_g$. Then, there is a neighborhood U of p_0 in $\mathscr{M}_g$, a finite ramified cover $\eta : S \to U$, with S smooth, a family $p : Z \to S$ of smooth projective curves of genus g, such that $\eta (s)$ is the moduli point of $p^{-1} (s)$ for each $s \varepsilon S$, also p admits a section ([2], §4).

(3.3) It is known that each irreducible component of $\mathscr{V}_d^2 (p)$ [with p as in (3.2)] has dimension at least $3d + g - 1$ (here, $g > 1$). We briefly recall the proof, essentially given in [2]. Let $\mathscr{G}_d^2 (p)$ be the space parametrizing triples (s, L, Λ), with $s \varepsilon S$, L a line bundle on $p^{-1} (s)$ and Λ a 3-dimensional subspace of $H^0 (p^{-1} (s), L)$. Then, there is a natural fibration $\mathscr{V}_d^2 \to \mathscr{G}_d^2$ (we omit "p"), whose fibers have dimension 8. So, it suffices to see that each component of $\mathscr{V}_d^2$ has dimension $\geqslant 3d + g - 9$. But $\mathscr{G}_d^2$ is covered by opens as follow. If $\mathrm{Pic}^d (p) \to S$ parametrizes pairs (s, L), $s \varepsilon S$, L a line bundle of degree d on $p^{-1} (s)$ then $\mathrm{Pic}^d (p)$ can be covered by affines U such that on each one there is a

morphism of free sheaves $\mathcal{O}_U^m \overset{\alpha}{\to} \mathcal{O}_U^n$, where $m - n = d - g + 1$, with the property that for each $u = (s, L) \varepsilon U$, the induced map $\alpha (u) : K^m \to K^n$ (K is the base field) has cokernel isomorphic to $H^{-1} (p^{-1} (s), L)$. Thus, α induces a morphism $\beta : U \to M = \{n \times m$ matrices over K$\}$, and we have $\beta \times \mathrm{id} : U \times G \to M \times G$, G being the grassmanian of 3-planes in K^m. Let

$$V = \{(A, \Lambda) \varepsilon M \times G / \Lambda \subset \mathrm{Ker} (A)\}.$$

V is defined by $3n$ equations in $M \times G$, and $(\beta \times \mathrm{id})^{-1} (V)$ (defined by $3n$ equations in $U \times G$) is isomorphic to an open $\mathscr{U}$ of $\mathscr{G}_d^2$; such opens cover $\mathscr{G}_d^2$. Thus, each component of $\mathscr{G}_d^2$ has dimension at least $\dim (U \times G) - 3n = (3g - 3 + g) + 3 (m - 3) - 3n = 3d + g - 9$, as claimed.

(3.4) In case $g = 1$, there is a similar construction. In this case we take $S = \mathbb{A}^1$ and $X \to S$ to be the classical family with (affine) equation:

$$y^2 = x (x - 1) (x - \lambda), \qquad (cf. [6]).$$

We may find a lower bound for the dimensions of the irreducible components of $\mathscr{V}_d^2 (p)$ as in (3.3), the only difference is that now $\dim U = \dim S + g = 1 + 1 = 2$; hence the dimension of each component of $\mathscr{G}_d^2 (p)$ is no less than

$$2 + 3 (m - 3) - 3n = 2 + 3 (m - n) - 9 = 3d - 7;$$

hence each component of $\mathscr{V}_d^2 (p)$ has dimension $\geqslant 3d - 7 + 8 = 3d + 1$.

(3.5) Recall that given a smooth curve C, a point $Q \varepsilon C$ and an integer $q > 0$, then there is a bijection between isomorphism classes of finite etale covers $f : E \to C$ of degree q and conjugation classes of subgroups N of $\pi_1 (C; Q)$ of index q. From this one easily concludes that the collection of such isomorphism classes of covers is finite, and that given such a C, there is always an etale cover of it of degree q (if $g (C) > 0$).

(3.6) Recall the following theorem of Zariski (*cf.* [12]). Consider in $\mathbb{P}^N$, $N = (1/2) d (d+3)$ (parametrizing plane curves of degree d) an irreducible subvariety V, whose points generically correspond to reduced curves of geometric genus g. Then, dim $V \leqslant 3 d + g - 1$, and if equality holds then the general member corresponds to a nodal curve.

(3.7) *Proof of Theorem* (2.6). — Consider our integral curve C of degree n and genus g. Let us consider an etale cover $\bar{\varphi}_0 : E_0 \to \bar{C}$ (where $\bar{C}$ is the normalization of C) of degree k. By Hurwitz' Theorem, the genus of E_0 is $g' = kg - k + 1$, and the linear system of linear sections of C pulls back to a g_d^2 on E_0, $d = kn$, without base points. In other words, using the construction of (3.2) or (3.4) (with $U \subset \mathcal{M}_g$ centered at $p_0 = $ moduli point of E_0, in case $g > 1$), we have $(E_0, \varphi_0) \in \tilde{\mathcal{V}}_d^2$ ($d = kn$, $\varphi_0 : E_0 \to \mathbb{P}^2$ induced by $\bar{\varphi}_0$).

We have a naturally defined morphism $\psi : \tilde{\mathcal{V}}_d^2 \to \mathbb{P}^N$ ($N = (1/2) d (d+3)$), where $\psi (E, \varphi)$ is the point corresponding to the divisor $\varphi_* (E) \subset \mathbb{P}^2$. Let X be an irreducible component of $\tilde{\mathcal{V}}_d^2$ containing (E_0, φ_0), $X' = \psi (X) \subset \mathbb{P}^N$. Clearly, $\psi (E_0, \varphi_0) = k C$; if we prove that the general member of X' corresponds to a nodal curve of genus g', then we are done. Let $\psi_0 = \psi_{|X}$. We consider two cases separately.

(i) $g' > 1$. Then we claim that ψ_0 is generically finite-to-one. In fact, if $(E', \varphi') \in \tilde{\mathcal{V}}_d^2$ is such that $\psi (E', \varphi') = \psi (E_0, \varphi_0)$, then $\varphi'_* (E') = k C$, i.e., $\varphi' : E' \to C'$ is generically k-to-one. Since also $g (E') = g'$, by applying Hurwitz' formula to the maps $\bar{\varphi}_0 : E_0 \to \bar{C}$ and $\bar{\varphi} : E' \to \bar{C}$ induced by φ_0 and φ' respectively, we see that $\bar{\varphi}'$ is also etale. Hence, Remark (3.5) (on etale covers), the finiteness of the group of automorphisms of C [note that necessarily $g (C) > 1$] and the finiteness of the morphism $S \to U \subset \mathcal{M}_g$ (3.2) imply that there are finitely many such pairs (E', φ'), i.e., that $\psi^{-1} (\psi (E_0, \varphi_0))$ is finite.

From this, we'll see next that the general point of $\psi (X) = X'$ corresponds to a reduced curve. Were it not the case, i.e., were it of the form $q D$, D reduced and $q > 1$, then the correspondence $q D \to D$ sets up a birrational equivalence between X' and an irreducible subvariety V of $\mathbb{P}^M$ [where $M = (1/2) (d/q) (d/q + 3)$], the space parametrizing plane curves of degree d/q. Since $q D = \beta_* (E_1)$, for some $(E_1, \beta) \varepsilon X$, by Hurwitz' formula the genus g_1 of D satisfies $g_1 \leqslant g'$. But then, according to a Theorem of Zariski [*cf.* (3.6)], dim $X' = $ dim $V \leqslant 3 (d/q) + g_1 - 1 < 3 d + g' - 1$. Hence, since generically the fibers of ψ_0 are finite, dim $X < 3 d + g' - 1$, a contradiction [*cf.* (3.3)]. Thus, for (E, φ) generic in X, $\varphi : E \to \mathbb{P}^2$ is birrational onto its image A. That A must be nodal is checked with a dimension count similar to the previous one: were A not nodal, then again by (3.6), dim $X = $ dim $X' < 3 d + g' - 1$, contradiction. This concludes case (i).

(ii) $g' = 1$. Here, $g = 1$ and $3 d + g' - 1 = 3 d$. In this case, we have: dim $\psi^{-1} (\psi (E_0, \varphi_0)) = 1$. This is obtained by the same argument as in Case (i), the only difference now is that Aut $(\bar{C})$ is one-dimensional. As before, were the general member of X' a point corresponding to a multiple curve $q D$, D of degree d/q, we'd get: dim $X' = $ dim V, where $V \subset \mathbb{P}^M$ parametrizes curves of degree d/q, generically of genus $\leqslant 1$. By Zariski's Theorem, dim $\leqslant 3 (d/q) < 3 d$, if $q > 1$. So, the dimension of X will be $< 3 d + 1$, a contradiction [*cf.* (3.4)]. Hence, the general member of X' corresponds to an integral curve, of genus 1. As before, it must be nodal, otherwise Zariski's inequality would imply: dim $X < 3 d + 1$, a contradiction.

(3.8) In case $k = 1$, Theorem (2.6) gives another proof of the theorem that says that any integral plane curve of degree d is a specialization of a nodal plane curve of the same degree and genus (cf. [8], §4).

(3.9) In paragraphs 2 and 3, we haven't made essential use of the "irreducibility theorem" (i.e., the existence of a unique component of $V_{n, \delta}$ whose general point corresponds to an irreducible curve). Minor modifications of the given arguments (which makes them somewhat more complicated) allow us to bypass that theorem, if we prefer.

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