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A COMPACTIFICATION OF A MANIFOLD WITH ASYMPTOTICALLY NONNEGATIVE CURVATURE (1)

BY ATSUSHI KASUE

Dedicated to Prof. Shingo Murakami on his 60th birthday

0. Introduction

Cheeger-Gromoll [6] investigated the topological and geometrical properties of a complete manifold M of nonnegative curvature. They proved that such a manifold M has a compact totally convex, totally geodesic submanifold $\mathcal{S}$, which they called a *soul* of M, and moreover M is deffeomorphic to the normal bundle of $\mathcal{S}$ in the tangent bundle of M. The crucial fact for such M is that the Busemann function associated with a ray of M is convex on M. This fact imposes strong restrictions on the topology and geometry of M (*see e.g.*, Cheeger-Ebin [4], Ch. 8, Shiohama [24] for further discussions and the literature). However this class of Riemannian manifolds does not seem to be adequate for the study of the topology and geometry *at infinity* of open manifolds. Actually, if we start with the Riemannian metric g_0 of a complete, noncompact manifold M_0 with nonnegative curvature, and perturbe it slightly or deform the topological structure within a compact subset of M_0, then the resulting Riemannian manifold M′ would keep many geometrical properties of M_0 *at infinity*, but M′ would in general have negative curvature somewhere. From the view point of *geometry at infinity*, it is natural to consider a larger class of (open) Riemannian manifolds. In fact, recently, Abresch [1] has introduced a class of (open) Riemannian manifolds, which are called asymptotically nonnegative curved, and studied the topological structure of such manifolds along the line settled by Gromov [14].

On the other hand, Gromov has defined, in his lectures [3], the Tits' metric on *the points at infinity*, the equivalence classes of rays, of a Hadamard manifold. Moreover,

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he has suggested that there is a counterpart to Tits' metric for nonnegative curvature and proposed several interesting exercises on such manifolds (*cf.* [3], pp. 58-59).

In this paper, motivated by the above works, we shall investigate a class of (open) Riemannian manifolds of asymptotically nonnegative curvature (after Abresch [1])). To be precise, we call a complete, connected, noncompact Riemannian manifold M with base point *o* *asymptotically nonnegative curved*, if there exists a monotone nonincreasing function $k:[0, \infty) \to [0, \infty)$ such that the integral $\int^{\infty} tk(t)\,dt$ is finite and the sectional curvature of M at any point *p* of M is bounded from below by $-k(\mathrm{dis}_M(o,p))$.

Our observation on a manifold M of asymptotically nonnegative curvature begins with the construction of a metric space $M(\infty)$ associated with M. Namely, we call two rays σ and $\gamma\,(\in \mathscr{R}_M)$ of M equivalent if $\mathrm{dis}_M(\sigma(t), \gamma(t))/t$ goes to zero as $t \to +\infty$ and define a distance δ_∞ on the equivalence classes $\mathscr{R}_M/\sim$ by $\delta_\infty([\sigma],[\gamma]) = \lim_{t\to\infty} d_t(\sigma \cap S_t, \gamma \cap S_t)/t$, where S_t denotes the metric sphere of M around a fixed point with radius t and d_t is the inner distance on S_t induced from the distance on M (*see* Section 2 for the details of the results mentioned in this paragraph). It should be pointed out here that M is homeomorphic to the interior of a compact manifold V with boundary ∂V (*cf.* Gromov [14], p. 185, for the statement and [1] for the estimate of the number $\mu(M)$ of the connected components of ∂V in terms of the dimension of M and the lower bound k of the curvature of M). Actually, M is isotropic to the metric balls B_t of large radius t. For the sake of convenience, we call (a neighborhood of) a connected component of ∂V *an end* of M, denoted by $\varepsilon_\alpha(M)$, according to the component ∂V_α of ∂V $(\alpha = 1, \ldots, \mu(M))$. Thus $\delta_\infty([\sigma],[\gamma]) < +\infty$ if and only if σ and γ belong to the same end, namely, $\sigma(t)$ and $\gamma(t)$ go to the same end as $t \to +\infty$. Let us write $M(\infty)$ for the metric space $(\mathscr{R}_M/\sim, \delta_\infty)$ obtained above. Then $M(\infty)$ consists of $\mu(M)$-connected components $M_\alpha(\infty)$ $(\alpha = 1, \ldots, \mu(M))$ and each $M_\alpha(\infty)$ is a compact inner metric space (or length space after Gromov [15]). Moreover it turns out that for large t, there exist Lipschitz maps $\Phi_{t,\infty}: S_t \to M(\infty)$ which enjoy the following properties: $\Phi_{t,\infty}(\sigma(t)) = [\sigma]$ for any ray σ starting at the fixed point (the center of S_t), and $\delta_\infty(\Phi_{t,\infty}(x), \Phi_{t,\infty}(y)) \leqq C(t)\, d_t(x,y)/t$ for any x and $y \in S_t$, where $C(t)$ goes to 1 as $t \to +\infty$. From this observation, it follows that $M(\infty)$ is the Hausdorff limit of a family of the metric spaces $(S_t, d_t/t)$ as $t \to +\infty$.

It would be interesting to see how the geometry of $M(\infty)$ reflects that of M. In Section 4, motivated by Shiohama [23], we shall study Busemann functions on manifolds of nonnegative curvature. The main result of Section 4 is stated as follows:

THEOREM 4.3. — *Let* M *be a complete, noncompact Riemannian manifold of nonnegative sectional curvature. Then for a ray* σ *of* M, *the Busemann function* F_σ *associated with* σ *is an exhaustion function on* M (i.e., *for each* $t \in R$, *the set* $\{x \in M : F_\sigma(x) \leqq t\}$ *is compact*) *if and only if* $\delta_\infty([\sigma],[\gamma]) < \pi/2$ *for any ray* γ *of* M.

In the subsequent papers [20] and [21], we shall continue to discuss some other relationships between the geometry of $M(\infty)$ and that of M.

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1. Preliminaries

In this section, we shall give several preliminary results on comparison theorems and the behavior of geodesics. Throughout this section, M is a connected, complete, noncompact Riemannian manifold of dimension m, ∇ denotes the Levi-Civita connection, and geodesics are assumed to have unit speed, unless otherwise stated.

1.1. We shall begin with some basic notations and definitions. Let us denote by $\mathrm{dis}_M(p, q)$ [resp., $B_t(p)$, $S_p(p)$] the distance (in M) between two points p, q of M (resp., the metric ball around a point p with radius t, the metric sphere around a point p with radius t). A geodesic σ of M defined on $[0, \infty)$ [resp. $(-\infty, \infty)$] is called *a ray* (resp., *a straight line*) if $\mathrm{dis}_M(\sigma(s), \sigma(t)) = |t-s|$ for any s, $t \in [0, \infty)$ [resp. $(-\infty, \infty)$]. We write $\mathscr{R}_M$ (resp., $\mathscr{R}_p$) for all rays of M (resp. all rays of M starting at a point p). The Busemann function associated with a ray $\sigma \in \mathscr{R}_M$ is defined by

$$F_\sigma(x) := \lim_{t \to \infty} t - \mathrm{dis}_M(x, \sigma(t))$$

(*cf. e.g.*, Cheeger-Ebin [4]). After Wu [27], we define a function F_p associated with a family of the metric spheres $\{S_t(p)\}$ around a point p by

$$F_p(x) := \lim_{t \to \infty} t - \mathrm{dis}_M(x, S_t(p)).$$

Then we have the following

Fact 1.1 (*cf.* [4], [24], [27]).

(i) $F_\sigma \leqq F_p \leqq r_p$ on M, and $F_\sigma(\sigma(t)) = F_p(\sigma(t)) = r_p(\sigma(t)) = t$ on $[0, \infty)$, for any $p \in M$ and $\sigma \in \mathscr{R}_p$, where $r_p(x) = \mathrm{dis}_M(p, x)$.

(ii) $F_p(x) = t - \mathrm{dis}_M(x, F_p^{-1}(t))$ for any p, $x \in M$ and $t > 0$ with $F_p(x) < t$.

(iii) A ray $\sigma \in \mathscr{R}_M$ is asymptotic to $\gamma \in \mathscr{R}_M$ if and only if $F_\gamma(\sigma(t)) = t + F_\gamma(\sigma(0))$ for any $t \geqq 0$.

Here a ray σ is called *asymptotic* to a ray γ if there exists a family of distance minimizing geodesics $\{\sigma_n\}_{n=1,2,\ldots}$, each σ_n satisfying $\sigma_n(0) = p_n$ with $\lim_{n \to \infty} p_n = \sigma(0)$ and $\sigma_n(a_n) = \gamma(b_n)$ for some divergent sequence $\{b_n\}$, and they satisfy: $\dot\sigma(0) = \lim_{n \to \infty} \dot\sigma_n(0)$.

Although the above functions r_p, F_σ, F_p are in general only Lipschitz functions on M, it is convenient for us to introduce the following notations:

$$\nabla . r_p(x) := \{ v \in T_x M : |v| = 1, t + r_p(\exp_x - tv) = r_p(x) \, (t \in [0, r_p(x)]) \}$$

$$\nabla . F_p(x) := \{ v \in T_x M : |v| = 1, F_p(\exp_x tv) - t = F_p(x) (t \geqq 0) \}$$
$$\nabla . F_\sigma(x) := \{ v \in T_x M : |v| = 1, F_\sigma(\exp_x tv) - t = F_\sigma(x) (t \geqq 0) \}.$$

1.2. We recall here some definitions and facts used later. *See* [12] and [27] for details.

We begin by the definition of Riemannian convolution smoothing on M. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a nonnegative smooth function that has its support contained in $[-1,1]$, is constant in a neighborhood of 0 and has $\int_{v \in \mathbb{R}^m} \varphi(|v|) = 1$. Given a continuous function $\tau : M \to \mathbb{R}$, define

$$\tau_\varepsilon(p) = \frac{1}{\varepsilon^m} \int_{v \in T_p M} \varphi(|v|/\varepsilon) \tau(\exp_p v) \, d\mu_p(v),$$

where the integration is with respect to the measure μ_p induced on the tangent space $T_p M$ at p by the Riemannian metric of M. For a compact subset A of M, there is a neighborhood of A on which the τ_ε are defined and smooth for all sufficiently small ε.

Let $\tau : M \to \mathbb{R}$ be a continuous function and ξ a constant. The function τ is called ξ-convex at a point p of M if there is a positive constant δ such that the function $q \to \tau - (\xi + \delta) \operatorname{dis}_M(p, q)^2/2$ is convex in a neighborhood of p. If $\eta : M \to \mathbb{R}$ is a continuous function, then τ is called η-convex on M if, for each $p \in M$, τ is $\eta(p)$-convex at p. Moreover τ is a called η-concave on M if $-\tau$ is $(-\eta)$-convex on M. In the similar manner, we can define τ being η-subharmonic or η-superharmonic on M. Let v be a tangent vector at $p \in M$ and $\gamma : (-\varepsilon, \varepsilon) \to M$ a geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Then an extended real number $\nabla^2 . \tau(p : v)$ is defined by

$$\nabla^2 . \tau(p : v) := \liminf_{s \to 0} \frac{1}{s^2} \{ \tau . \gamma(s) + \tau . \gamma(-s) - 2\tau(p) \}.$$

If $\eta : M \to \mathbb{R}$ is a continuous function and if τ is η-convex on M, then $\nabla^2 . \tau(p : v) > \eta(p)$ for any $p \in M$ and every unit vector $v \in T_p M$. Conversely if $\eta_i : M \to \mathbb{R}$ $(i = 1, 2)$ are continuous functions with $\eta_1 > \eta_2$ and if $\nabla^2 . \tau(p : v) \geqq \eta_1(p)$ for any $p \in M$ and every unit vector $v \in T_p M$, then η is η_2-convex on M. Note here that if two continuous functions $\tau_i : M \to \mathbb{R}$ $(i = 1, 2)$ satisfy: $\tau_1 \leqq \tau_2$ and $\tau_1(p) = \tau_2(p)$ at a point p of M, then $\nabla^2 . \tau_1(p : v) \leqq \nabla^2 . \tau_2(p : v)$.

1.3. From now on, we assume that M is a manifold of asymptotically nonnegative curvature. Namely, the sectional curvature K_M of M satisfies:

$$\text{(H.1)} \qquad K_M \geqq -k \circ r_0,$$

where r_0 is the distance function to a fixed point o of M and $k(t)$ is a nonnegative monotone nonincreasing function on $[0, \infty)$ such that the integral $\int^\infty tk(t)\,dt$ is finite. Let us denote by J_k the solution of a classical Jacobi equation:

$$(1.1) \qquad J_k''(t) - k(t) J_k(t) = 0, \quad \text{with} \quad J_k(0) = 0 \quad \text{and} \quad J_k'(0) = 1.$$

Then it is known that

$$(1.2)\qquad \begin{gathered} 1 \leqq J_k'(t) \nearrow J_k'(\infty)(:= \lim_{t\to\infty} J_k'(t)) \leqq \exp \int_0^\infty tk(t)\,dt \\ t \leqq J_k(t) \leqq J_k'(\infty)\,t \end{gathered}$$

(*cf.* Greene-Wu [11: Theorem C]).

LEMMA 1.2. — *Let* M *be as above and* ε *a positive constant. Then the following assertions hold:*

(i) *The distance function* r_0 *to the base point* o *in* (H.1) *is* $\{(1+\varepsilon)(\log J_k)'\}$*-concave on* M *and* $\{(1+\varepsilon)(m-1)(\log J_K)'\}$*-superharmonic on* M.

(ii) $t \to \mathrm{Vol}_m(B_t(0))/J_k(t)^m$ *is monotone nonincreasing.*

(iii) *The function* F_0 *defined in* 1.1 *is* $\left\{-(1+\varepsilon)\int_{F_0}^\infty k(t)\,dt\right\}$*-convex on* $\{p\in M: F_0(p)>0\}$ *if* $k\not\equiv 0$ *near* ∞, *and* $\left\{-\varepsilon-\int_{F_0}^\infty k(t)\,dt\right\}$*-convex there if* $k\equiv 0$ *near* ∞.

Proof. — We shall prove only the last assertion (iii). *See* [17], [18] for the others. In order to prove (iii), we use the method in [27]. Fix a point p of M. For any number t with $F_0(p)<t$, we take a point p_t such that $F_0(p_t)=t$ and $\mathrm{dis}_M(p, p_t)=\mathrm{dis}_M(p, F_0^{-1}(t))$. Set $r_t:=\mathrm{dis}_M(p_t, *)$. Then it follows from Lemma 1 (ii) that $t-r_t\leqq F_0$ near p and $t-r_t(p)=F_0(p)$. This implies that $\nabla^2.(t-r_t)(p)\leqq\nabla^2.F_0(p)$. Hence it is enough to show that if $F_0(p)>0$, then

$$(1.3)\qquad \nabla^2.r_t(p)\leqq\frac{1}{r_t(p)}+\int_{F_0(p)}^\infty k(u)\,du,$$

since the right side of (1.3) goes to $\int_{F_0}^\infty k(u)\,du$ as $t\to\infty$. Let us now prove (1.3). Since $|r_0(p_t)-r_t(q)|\leqq r_0(q)$ for any q of M and $k(t)$ is monotone nonincreasing, the sectional curvature of M at q is bounded from below by $-k(|r_0(p_t)-r_t(q)|)$. It follows that

$$(1.4)\qquad \nabla^2.r_t\leqq(\log J_t)'\circ r_t$$

(*cf.* [17]), where J_t is the solution of an equation:

$$J_t''(u)-k(|r_0(p_t)-u|)\,J_t(u)=0,$$

subject to the initial conditions: $J_t(0)=0$ and $J_t'(0)=1$. On the other hand, we have

$$(1.5)\quad J'_t(s_t)=1+\int_0^{s_t} k(|r_0(p_t)-u|)\,J_t(u)\,du\ (s_t:=r_t(p))$$

$$\leqq 1+J_t(s_t)\int_0^{s_t} k(|r_0(p_t)-u|)\,du$$

$$\leqq 1+J_t(s_t)\int_{F_0(p)}^{\infty} k(u)\,du \qquad \text{if}\quad F_0(p)>0.$$

Thus (1.3) follows from (1.4) and (1.5). This completes the proof of Lemma 1.2.

Remarks. — (i) Let A be a closed subset of M and set $r_A:=\mathrm{dis}_M(A, *)$. Then the same assertion as in Lemma 1.2 (i) holds for r_A (*cf.* [17], [18]).

(ii) Lemma 1.2 (ii) is true under the weaker assumption that the Ricci curvature of M is bounded below by $-(m-1)\,k\circ r_0$ (*cf.* [17], [18]).

(iii) Let $\mathscr{C}=\{A_t\}_{t>0}$ be a divergent family of closed subsets A_t of M. Then a family of Lipschitz functions: $\{\mathrm{dis}_M(A_t, 0)-\mathrm{dis}_M(A_t, *)\}_{t>0}$ is equicontinuous and totally bounded on each compact sets of M. Therefore, we can find a divergent sequence $\{t_n\}$ such that the functions: $\mathrm{dis}_M(A_{t_n}, o)-\mathrm{dis}_M(A_{t_n}, *)$ converge to a Lipschitz function $F_{\mathscr{A}}$ on M, uniformly on compact subsets of M. Then the last assertion of Lemma 1.2 holds for such $F_{\mathscr{A}}$ (*cf.* [27] and the above proof of Lemma 1.2).

1.4. The following version of the Toponogov comparison theorem has been proved by Abresch [1].

Fact 1.3 ([1], Proposition 1). — Let a, $\varepsilon\in(0,1)$ and let $\Delta(p_0, p_1, p_2)$ be a generalized triangle in an asymptotically nonnegative curved manifold M. Suppose moreover that p_2 is the base point o of M in (H.1) and that $\mathrm{dis}_M(p_1, p_2)\leqq(1-\varepsilon)\,\mathrm{dis}_M(p_2, p_0)$. Then the following estimates hold:

(i) $\cos(\sphericalangle \text{ at } o)\geqq\sqrt{1-a^2.\beta^2.\varepsilon^2}\Rightarrow \mathrm{dis}_M(p_0, p_1)\leqq \mathrm{dis}_M(p_2, p_0)-\mathrm{dis}_M(p_1, p_2)\sqrt{1-a^2}$.

(ii) $\cos(\sphericalangle \text{ at } p_1)\geqq-\sqrt{1-a^2}\Rightarrow \mathrm{dis}_M(p_2, p_0)\leqq \mathrm{dis}_M(p_0, p_1)+\mathrm{dis}_M(p_1, p_2)\sqrt{1-a^2.\beta^2.\varepsilon^2}$.

Here the constant β as above should be explained (*cf.* [1]). Let Z_k be the unique solution of an equation: $Z_k''(t)-k(t)Z_k(t)=0$ subject to the conditions: $0<Z_k\leqq 1$ and $Z_k(0)=1$. Then the constant β is by definition the limit of $Z_k(t)$ as $t\to+\infty$ and it is estimated by $\exp-\int_0^{\infty} tk(t)\,dt\leqq\beta\leqq 1$.

1.5. Let us now prove a result which is the starting point of our observation on a manifold M of asymptotically nonnegative curvature.

LEMMA 1.4. — *Let* M *be as above. Then the assertions below hold:* (i) *For any fixed point p of* M, $F_p(x)/r_p(x)$ *converges to* 1 *as x goes to infinity. In particular,* $F_p: M\to R$ *is an exhaustion function on* M, *namely,* $\{x\in M: F_p(x)\leqq t\}$ *is compact for any* $t\in R$.

(ii) *As* $x \in M$ *goes to infinity*,

$$\max(\measuredangle(u, v) : u, v \in \nabla . r_p(x)\} \to 0,$$

$$\max\{\measuredangle(u, v) : u \in \nabla . r_p(x), v \in \nabla . F_p(x)\} \to 0.$$

Proof. — It is enough to show the lemma in case of $p = o$ (the base point in (H. 1)). We first prove the assertion (i). Let $(x_n)_{n=1, 2, \ldots}$ be a sequence of points of M such that $r(x_n)$ $(:= r_0(x_n))$ goes to infinity as $n \to +\infty$. For each n, we have a distance minimizing geodesic $\sigma_n : [0, a_n] \to M$ $(a_n := r(x_n))$ joining o with x_n. Taking a subsequence if necessary, we may assume that σ_n converges to a ray $\sigma_\infty \in \mathscr{R}_o$ starting at o, that is, $\theta_n := \measuredangle(\dot\sigma_n(0), \dot\sigma_\infty(0))$ goes to zero as $n \to +\infty$. Let a, $\varepsilon \in (0,1)$ be chosen arbitrarily. Then for large n, we have

$$\cos\theta_n \geqq \sqrt{1 - a^2 . \beta^2 . \varepsilon^2}.$$

Put $y_n := \sigma_\infty(a_n/(1-\varepsilon))$. Then by Fact 1.3 (i),

$$\text{dis}_M(x_n, y_n) \leqq \frac{a_n}{1-\varepsilon} - a_n\sqrt{1-a^2} \tag{1.6}$$

for large n. It follows from Fact 1.1 (i), (ii) and (1.6) that

$$F_0(x_n) = \frac{a_n}{1-\varepsilon} - \text{dis}_M\left(x_n, F_0^{-1}\left(\frac{a_n}{1-\varepsilon}\right)\right) \geqq \frac{a_n}{1-\varepsilon} - \text{dis}_M(x_n, y_n) \geqq a_n\sqrt{1-a^2}.$$

This implies that

$$\sqrt{1-a^2} \leqq \frac{F_0(x_n)}{r_0(x_n)} (\leqq 1)$$

for large n. Thus we have shown the first assertion, since $\{x_n\}$ and $a \in (0, 1)$ are arbitrary.

Let us prove the second assertion (ii). Put

$$\theta(x) := \max(\measuredangle(u, v) : u, v \in \nabla . r(x)\}.$$

Suppose there exist a constant $c \in (0, 1)$ and a sequence $\{x_n\}$ of points of M such that $a_n = r(x_n)$ goes to infinity as $n \to \infty$ and $\theta(x_n) > 2c > 0$ for any n. Let us take a pair of vectors u_n, v_n of $\nabla . r(x_n)$ such that $\measuredangle(u_n, v_n) > 2c$, and set $\eta_n(t) := \exp_{x_n}(t - a_n) u_n$ and $\xi_n(t) := \exp_{x_n}(t - a_n) v_n$ $(0 \leqq t \leqq a_n)$. We fix an n for a while. Let x be a point of M such that $b = r(x) \geqq a_n/(1-\varepsilon)$ and $\sigma : [0, d] \to M$ $(d := \text{dis}_M(x_n, x))$ a distance minimizing geodesic joining x_n with x. Observe that $\max\{\measuredangle(\dot\sigma(0), u_n), \measuredangle(\dot\sigma(0), v_n)\} > c$. Then for a constant $\varepsilon \in (0, 1)$ and a distance minimizing geodesic $\gamma : [0, b] \to M$ emanating from the base point o such that $\gamma(b)=x$, we see that

$$(1.7)\qquad \begin{aligned} &\measuredangle(\dot\gamma(0), \dot\eta_n(0)) \geqq \delta \quad \text{if} \quad \measuredangle(\dot\sigma(0), u_n) > c, \\ &\measuredangle(\dot\gamma(0), \dot\xi_n(0)) \geqq \delta \quad \text{if} \quad \measuredangle(\dot\sigma(0), v_n) > c, \end{aligned}$$

where δ is a positive constant depending only on c, ε, and the constant β as in Fact 1.3. Actually in the case that $\measuredangle(\dot\sigma(0), u_n) > c$, we apply Fact 1.3 (ii) to the geodesic triangle $\Delta(\delta(b), x_n, 0)$ and obtain

$$(1.8)\qquad b \leqq d + a_n \sqrt{1-a^2 . \varepsilon^4 . \beta^4}.$$

where $a=\sin c$. It turns out from (1.8) and Fact 1.3 (i) that

$$\cos \measuredangle(\dot\gamma(0), \dot\eta_n(0)) \leqq \sqrt{1-a^2 . \varepsilon^4 . \beta^4},$$

and hence

$$\measuredangle(\dot\gamma(0), \dot\eta_n(0)) \geqq \delta := \arccos \sqrt{1-a^2 . \varepsilon^4 . \beta^4}.$$

Similarly it follows that $\measuredangle(\dot\gamma(0), \xi_n(0)) \geqq \delta$ if $\measuredangle(\dot\sigma(0), v_n) > c$. Thus we have shown (1.7). Let us continue the argument to lead a contradiction. Define a set U_n by

$$U_n = \{(u, v) \in T_0 M \times T_0 M : |u| = |v| = 1, \measuredangle(u, \dot\eta_n(0)) < \delta/4, \measuredangle(v, \dot\xi_n(0)) < \delta/4\},$$

where δ is as in (1.7). Then we see from (1.7) that $U_n \cap U_{n'} = \varnothing$ if $n < n'$ and $a_{n'} \geqq a_n/(1-\varepsilon)$. This is a contradiction. Thus it has been proved that $\theta(x)$ goes to zero as $x \in M \to \infty$. Finally we shall show that $\max\{\measuredangle(u, v) : u \in \nabla . r(x), v \in \nabla . F_0\}$ goes to zero as $x \in M$ tends to infinity. This is done by a similar argument to the above one. Suppose there exist a constant $c \in (0, 1)$ and a divergent sequence $\{x_n\}$ of M such that $\measuredangle(u_n, v_n) > c > 0$ for some $u_n \in \nabla . r(x_n)$ and $v_n \in \nabla . F_0(x_n)$. Set $\eta_n(t) := \exp_{x_n} t v_n$ and take a distance minimizing geodesic $\xi_{n,t}$ joining o with $\eta_n(t)$. We consider the case: $r(\eta_n(t)) \geqq r(x_n)/(1-\varepsilon)$ where $\varepsilon \in (0, 1)$. Then applying Fact 1.3 (ii) to the geodesic triangle $\Delta(\eta_n(t), x_n, o)$, we have

$$r(\eta_n(t)) \leqq t + r(x_n) \sqrt{1-a^2 . \varepsilon^2 . \beta^2},$$

where $a=\sin c$. On the other hand, it follows from Fact 1.1 (i), (ii) that

$$r(\eta_n(t)) \geqq F_\sigma(\eta_n(t)) = t + F_0(x_n).$$

The above two inequalities imply that

$$\frac{F_0(x_n)}{r(x_n)} \leqq \sqrt{1-a^2 . \varepsilon^2 . \beta^2} < 1$$

for any n. This contradicts the first assertion of the lemma. Thus we have shown the second assertion. This completes the proof of Lemma 1.4.

Remark. — Let M be a manifold of asymptotically nonnegative curvature and A a compact subset of M. Let $r_A := \mathrm{dis}_M(A, *)$. Then for each $x \in M$, $\nabla . r_A(x)$ can be defined in the same manner as in 1.1. Moreover the same argument as in the proof of Lemma 1.4 leads us to the following assertion: as $x \in M$ goes to infinity,

$$\max\{\measuredangle(u, v) : u, v \in \nabla . r_A(x)\} \to 0$$

$$\max\{\measuredangle(u, v) : u \in \nabla . r_A(x), v \in \nabla . r_p(x)\} \to 0$$

($p \in M$). Note that if the sectional curvature of M is everywhere nonnegative and A is a soul of M, then

$$\max\{\measuredangle(u, v) : u, v \in \nabla . r_A(x)\} < \pi$$

on $M \setminus A$ (*cf.* Cheeger-Gromoll [6]).

1.6. Here is given a technical but useful fact on smooth approximation of distance functions.

LEMMA 1.5. — *Let* M, o, r_0, F_0, J_k *be as in* 1.1 *and* 1.2. *Then for any large* $t>0$ *and small* $\varepsilon>0$, *there is a constant* $\delta(t, \varepsilon)>0$ *such that the Riemannian mollifier* r_δ *of* $r(:=r_0)$ $(0<\delta\leqq\delta(t, \varepsilon))$ *is well defined on* $B_t(o)$ *and it enjoys the following properties: on* $B_t(o)$,

(i) $|r - r_\delta| \leqq \varepsilon$,

(ii) $1 - \varepsilon - \theta_1(r - \varepsilon) \leqq |\nabla r_\delta| \leqq 1 + \varepsilon$,

(iii) $1 - \varepsilon \leqq |\nabla r_\delta|(x)$ *if* $\mathrm{dis}_M(x, \mathscr{C}_0) \geqq \varepsilon$,

(iv) $\nabla^2 r_\delta \leqq (1 + \varepsilon)(\log J_k)' \circ r_\delta$,

where $\mathscr{C}_o$ *stands for the cut locus of* M *with respect to the base point* o *and* $\theta_1(s) := \max\{\measuredangle(u, v) : u,\ v \in \nabla . r(x),\ r(x) \geqq s\}$. *Moreover the Riemannian mollifier* F_δ *of* $F(:=F_0)$ *is also well defined on* $B_t(o)$ *and satisfies:*

(v) $|F - F_\delta| \leqq \varepsilon$,

(vi) $|\nabla F_\delta - \nabla r_\delta| \leqq \varepsilon + \theta_2(r - \varepsilon)$,

(vii) $\nabla^2 F_\delta \geqq -(1+\varepsilon)\int_F^\infty k(s)\,ds$ *on* $\{x \in B_t(0) : F(x) > 0\}$, *if* $k \not\equiv 0$ *near* $+\infty$;

$\nabla^2 F_\delta \geqq -\varepsilon - \int_F^\infty k(s)\,ds$ *on* $\{x \in B_t(o) : F(x) > 0\}$, *if* $k \equiv 0$ *near* $+\infty$,

where $\theta_2(s) := \max\{\measuredangle(u, v) : u \in \nabla . r(x),\ v \in \nabla . F(x),\ r(x) \geqq s\}$.

Remarks. — (i) Both $\theta_1(s)$ and $\theta_2(s)$ go to zero as $s \to \infty$, because of Lemma 1.4 (ii).

(ii) It turns out from Lemma 1.5(ii) that M is isotopic to the interior of a metric ball of M with large radius (*cf.* [15], p. 185).

(iii) Let M be a manifold of asymptotically nonnegative curvature and fix a point o of M, say the base point in (H.1). Then there are a positive constant t_0 and a nonnegative continuous function $\theta_0(t)$ on $[0, \infty)$ such that $\theta_0(t)$ goes to zero as $t \to +\infty$ and if a geodesic $\sigma:[0, \infty) \to M$ starts at a point $x=\sigma(0)$ with $r_o(x) \geqq t_o$ and $\max\{ \measuredangle(\dot\sigma(0), v): v \in \nabla . r_0(x)\} < \pi - \theta_o(t)$, then σ goes to infinity (actually, $r_o(\sigma(t)) \geqq ct$ for some $c>0$ and any large t). This follows from Lemma 1.5 (vii) (*cf.* 3.6, Step 1).

Proof of Lemma 1.5. – Among the above inequalities, (i) and (v) follow from the definitions of r_δ and F_δ, and furthermore (iv) and (vii) turn out to be true, because of Lemma 1.2 and the results in [10]. We shall now prove the remaining inequalities, refering to [16]. Let c be a positive constant smaller than the injectivity radius of M at any $x \in B_t(o)$. For any pair of points x, y of $B_t(o)$ with $\mathrm{dis}_M(x, y) < c$, we denote by P_{xy} the parallel displacement from x to y along the (unique) minimal geodesic x to y. Then for any $x \in B_t(o)$, we can find a positive number $\delta(x)$ which is smaller than $c/4$ and has the following properties: for any $y \in B_{\delta(x)}$ and $u \in \nabla . r(y)$, there is a vector $v \in \nabla . r(x)$ such that

$$(1.9) \qquad \measuredangle(P_{yx}(u), v) < \frac{\varepsilon}{4}$$

and moreover for any y, $z \in B_{\delta(x)}(x)$ and $u \in T_y M$ with $|u|=1$,

$$(1.10) \qquad \measuredangle(P_{yz}(u), P_{yx} \circ P_{xz}(u)) < \frac{\varepsilon}{4}$$

(*cf.* the proof of Theorem 1.7 in [16]). Set $\delta'(x) = \min\{\varepsilon/2, \delta(x)\}$ and let λ be the Lebesgue number of the covering $\{B_{\delta'(x)}(x)\}$ $(x \in B_t(o))$ of $B_t(o)$. Then for any $x \in B_t(o)$, there is a point x_0 of $B_t(o)$ such that $B_\lambda(x) \subset B_{\delta'(x_0)}(x_o)$, so that for any $y \in B_\lambda(x)$, $u \in \nabla . r(y)$ and $w \in \nabla . r(x)$, we have

$$(1.11) \qquad \measuredangle(w, P_{yx}(u)) < \frac{3}{2}\varepsilon + \Delta(x_o),$$

where $\Delta(x_0) := \max\{ \measuredangle(v, v') : v, v' \in \nabla . r(x_0)\}$. In fact, by (1.9), we see that

$$\measuredangle(w, P_{x_0 x}(v)) = \measuredangle(P_{x x_0}(w), v) < \frac{\varepsilon}{4}$$

for some $v \in \nabla . r(x_0)$ and

$$\measuredangle(P_{x_0 x}(v'), P_{x_0 x} \circ P_{y x_0}(u)) = \measuredangle(v', P_{y x_0}(u)) < \frac{\varepsilon}{4}$$

for some $v' \in \nabla . r(x_0)$. Furthermore by (1.10), we get $\measuredangle(P_{x_0 x} \circ P_{y x_0}(u), P_{xy}(u)) < \varepsilon/4$. Then by the triangle inequality, we obtain (1.11). We observe here that there exists a positive constant $\delta_0 < \lambda$ such that for any $\delta: 0 < \delta < \delta_0$, r_δ and F_δ are well defined and smooth on $B_t(o)$, and they satisfy:

$$(1.12)\qquad \left|\nabla r_\delta(x)-\int_{v\in T_xM}\varphi(|v|/\delta)\,P_{yx}(\nabla r(y))\,d\mu_x(v)\right|<\frac{\varepsilon}{4}$$

$$(1.13)\qquad \left|\nabla F_\delta(x)-\int_{v\in T_xM}\varphi(|v|/\delta)\,P_{yx}(\nabla F(y))\,d\mu_x(v)\right|<\frac{\varepsilon}{4},$$

where $y=\exp_x v$ and φ is as in 1.2 (*cf.* the proof of Theorem 2.2 in [16]). Then it follows from (1.11) and (1.12) that for any $w\in\nabla.r(x)$,

$$(1.14)\qquad |\nabla r_\delta(x)-w|\leqq\left|\nabla r_\delta(x)-\int_{v\in T_xM}\varphi(|x|/\delta)\,P_{yx}(\nabla r(y))\,d\mu_x(v)\right| + \int_{v\in T_xM}\varphi(|v|/\delta)\,|P_{yx}(\nabla r(y))-w|\,d\mu_x(v)<\varepsilon+\Delta(x_0).$$

We note that if $\mathrm{dis}_M(x,\mathscr{C}_0)\geqq\varepsilon$, then x_0 does not belong to $\mathscr{C}_0$, and hence in this case, we have

$$(1.15)\qquad |\nabla r_\delta(x)-w|<\varepsilon.$$

Moreover we have by (1.12), (1.13) and the definition of θ_2,

$$(1.16)\qquad |\nabla F_\delta(x)-\nabla r_\delta(x)|\leqq\left|\nabla F_\delta(x)-\int_{v\in T_xM}\varphi(|v|/\delta)\,P_{yx}(\nabla F(y))\,d\mu_x(v)\right|$$
$$+\int_{v\in T_xM}\varphi(|v|/\delta)\,|P_{yx}(\nabla F(y))-P_{yx}(\nabla r(y))|\,d\mu_x(v)$$
$$+\int_{v\in T_xM}\varphi(|v|/\delta)\,|P_{yx}(\nabla F(y))-P_{yx}(\nabla r(y))|\,d\mu_x(v)$$
$$+\left|\int_{v\in T_xM}\varphi(|v|/\delta)\,P_{yx}(\nabla r(y))\,d\mu_x(v)-\nabla r_\delta(x)\right|<\frac{\varepsilon}{2}+\theta_2(r(x)-\varepsilon).$$

Obviously (1.12), (1.14), (1.15) and (1.16) show the estimates (ii), (iii) and (iv) of Lemma 1.5. This completes the proof of Lemma 1.5.

2. A geometric compactification for a manifold of asymptotically nonnegative curvature and its properties

In this section, based on the observations in Section 1, we shall define a metric space of *points at infinity* of an asymptotically nonnegative curved manifold and state its basic properties. The results of this section will be verified in the next section. Throughout this section, M is a manifold of asymptotically nonnegative curvature.

2.1. The metric sphere $S_t(p)$ around a point p of radius t is not generally a smooth hypersurface of M. However, according to Lemma 1.4 and Lemma 1.5, $\{S_t(p)\}$ (for large t) is a family of Lipschitz hypersurfaces of M which consist of the $\nu(M)$ connected components, where $\nu(M)$ is the number of the ends of M, so that it is possible to introduce the inner distance, denoted by $d_{p,t}$, on $S_t(p)$ induced from the distance $d_M(\ , \ )$ of M restricted to $S_t(p)$. To be precise, we define the length $L(c)$ of a continuous curve $c:[0, a] \to S_t(p)$ by

$$L(c) := \sup_{0=t_0<t_1<\ldots<t_k=a} \sum_{i=0}^{k} \operatorname{dis}_M(c(t_i), c(t_{i+1})) \ (\leqq +\infty),$$

and then, for any pair of points $x, y \in S_t(p)$, the inner distance $d_{p,t}(x, y)$ is defined by

$$d_{p,t}(x, y) := \inf L(c)$$

where c ranges over all continuous paths in $S_t(p)$ joining x to y (*cf.* Gromov [15], Ch. 1). Here $d_{p,t}(x, y)$ is defined to be infinity if x, y do not belong to the same connected component of $S_t(p)$, so that $d_{p,t}(x, y) < +\infty$ if and only if x, y belong to the same connected component of $S_t(p)$ (for large t).

Let us now define an equivalence relation $\sim$ on the set $\mathscr{R}_M$ of all rays of M and a distance δ_∞ on the set of equivalence classes. Two rays σ, $\gamma \in \mathscr{R}_M$ are called *equivalent* and denoted by $\sigma \sim \gamma$ if $\lim_{t\to\infty} \operatorname{dis}_M(\sigma(t), \gamma(t))/t = 0$. We write $[\sigma]$ for the equivalence class of σ. Moreover we introduce a distance δ_∞ on $\mathscr{R}_M/\sim$ by

$$\delta_\infty([\sigma], [\gamma]) := \lim_{t\to\infty} \frac{1}{t} d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p)),$$

where p is any fixed point of M. Then the distance δ_∞ is well defined on $\mathscr{R}_M/\sim$. Actually we have the following

PROPOSITION 2.1. — (i) $\sigma \sim \gamma$ $(\sigma, \gamma \in \mathscr{R}_M)$ *if and only if* $\lim_{t\to\infty} d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p))/t = 0$ *for any fixed point* p *of* M.

(ii) *For any pair of rays* σ, γ *and a fixed point* p *of* M, *there exists the limit:* $\lim_{t\to\infty} d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p))/t$, *which is independent of the reference point* p.

(iii) *The inclusion* $\iota: \mathscr{R}_p/\sim \to \mathscr{R}_M/\sim$ *is bijective for any* $p \in M$.

We write $M(\infty)$ for the metric space $(\mathscr{R}_M/\sim, \delta_\infty) = (\mathscr{R}_p/\sim, \delta_\infty)$. Note *here that* $\delta_\infty([\sigma], [\gamma]) < +\infty$ *for* σ, $\gamma \in \mathscr{R}_M$ *if and only if* σ, γ belong to the same end of M, namely, there is a large number t_0 such that if $t \geqq t_0$, $\sigma(t)$ and $\gamma(t)$ belong to the same connected component of $M \setminus B_{t_0}(p)$, which is homeomorphic to $S_{t_0}(p) \times [t_0, \infty)$. Here we write $M_\alpha(\infty)$ $(\alpha = 1, \ldots, \nu(M))$ for the component of $M(\infty)$ corresponding to the end $\mathscr{E}_\alpha(M)$ of M.

2.2. It is possible to introduce the metric space $M(\infty) = M_1(\infty) \cup \ldots \cup M_{\nu(M)}$ in a different fashion with the aid of the following

PROPOSITION 2.2. — *Take the base point o in* (H.1) *and fix a large number* t_0. *Then for any pair of numbers s, t with* $t_0 \leqq s \leqq t$, *there exists a map* $\Phi_{s,t} : S_s(o) \to S_t(o)$ *such that*

(i) $$\frac{d_t(\Phi_{s,t}(x), \Phi_{s,t}(y))}{J_k(t)} \leqq \frac{d_s(x, y)}{J_k(s)}$$

where $d_t = d_{o,t}$ *and* $J_k(t)$ *is as in* 1.2.

(ii) $$\Phi_{t,u} \circ \Phi_{s,t} = \Phi_{s,u} \qquad (t_o \leqq s \leqq t \leqq u),$$

(iii) $$\Phi_{s,t}(\sigma(s)) = \sigma(t) \qquad \textit{for any} \quad \sigma \in \mathcal{R}_0.$$

Let $\bigcup_{t \geqq t_0} S_t(o)$ denote the disjoint union of $\{S_t(o)\}_{t \geqq t_0}$ and call two elements $x_s \in S_s(o)$ and $x_t \in S_t(o)$ *equivalent if* $\lim_{u \to \infty} d_u(\Phi_{s,u}(x_s), \Phi_{s,u}(x_t))/u = 0$. Then we can define a distance δ_∞ on the set of equivalence classes $[x_t]$ $(x_t \in S_t(o))$ by

$$\delta_\infty([x_s], [x_t]) := \lim_{u \to \infty} \frac{1}{u} d_u(\Phi_{s,u}(x_s), \Phi_{t,u}(x_t)).$$

Then the metric space $M(\infty) = (\mathcal{R}_M/\sim, \delta_\infty) = (\mathcal{R}_o/\sim, \delta_\infty)$ is identified with the metric space $(\bigcup_{t \geqq t_0} S_t(o)/\sim, \delta_\infty)$ through the natural correspondence:

$$[\sigma] \in \mathcal{R}_o/\sim \;\to [\sigma(t)] \in \bigcup_{t \geqq t_0} S_t(o)/\sim.$$

Define a map $\Phi_{t,\infty} : S_t(o) \to M(\infty)$ $(t \geqq t_0)$ by $\Phi_{t,\infty}(x) := [x]$. Then we have the following

PROPOSITION 2.3:

(i) $$\delta_\infty(\Phi_{t,\infty}(x), \Phi_{t,\infty}(y)) \leqq \frac{J'_k(\infty)\, t}{J_k(t)} \frac{d_t(x, y)}{t},$$

where $J'_k(\infty)$ *and* J_k *are as in* 1.2.

(ii) *For any* $x_\infty \in M(\infty)$, *the diameter of* $\mathcal{F}_t(x_\infty) := \Phi_{t,\infty}^{-1}(x_\infty)$ *in* $S_t(o)$ *with respect to the distance* $(1/t)\, d_t$ *goes to zero as* $t \to +\infty$.

(iii) *For any pair of points* x_∞, y_∞ *of* $M(\infty)$,

$$\delta_\infty(x_\infty, y_\infty) = \lim_{t \to \infty} \frac{1}{t} d_t(\mathcal{F}_t(x_\infty), \mathcal{F}_t(y_\infty)).$$

In particular, for each component $M_\alpha(\infty)$ *of* $M(\infty)$,

$$\operatorname{diam}(M_\alpha(\infty)) = \lim_{t \to \infty} \operatorname{diam}\left(S_{\alpha,t}(o), \frac{1}{t} d_t\right) < +\infty$$

$(\alpha = 1, \ldots, \nu(M))$, *where* $S_{\alpha,t}(o)$ *stands for the component of* $S_t(o)$ *corresponding to* $M_\alpha(\infty)$.

(iv) *The Hausdorff distance between* $(S_{\alpha,t}(o), (1/t)\, d_t)$ *and* $M_\alpha(\infty)$ $(\alpha = 1, \ldots, \nu(M))$ *goes to zero as* $t \to +\infty$. *Especially* $(M_\alpha(\infty))$ *are compact inner metric spaces* (*or length spaces in* [5]).

Here we recall the definition of Hausdorff distance on metric spaces (*cf.* [15], Ch. 3). Given a metric space Z and subsets A, B of Z, the Hausdorff distance in Z between A and B is defined by $d_H^Z(A, B) := \inf\{\varepsilon > 0 : \mathrm{dis}_Z(a, B) < \varepsilon$ for $a \in A$, $\mathrm{dis}_Z(A, b) < \varepsilon$ for $b \in B\}$. Given two metric spaces X, Y, the Housdorff distance between them is defined by $d_H(X, Y) := \inf d_H^Z(f(X)), g(Y))$, where Z, $f: X \to Z$, and $g: Y \to Z$, respectively, range over all metric spaces, distance preserving maps from X to Z, and distance preserving maps from Y to Z. Note that if $d_H(X, Y) = 0$ for compact metric spaces X and Y, then X is isometric to Y (*cf.* [15], Ch. 3, Proposition 3.6).

Making use of the above family of maps $\Phi_{t,\infty} : S_t(o) \to M(\infty)$, we can give a compactification $\bar{M}$ of M. More precisely, as a set, $\bar{M}$ is the disjoint union of M and $M(\infty)$, and the topology is generated by the following collection of subsets U: U is an open subset of M or $U = \bigcup_{s \geq t} \Phi_{s,\infty}^{-1}(V) \cup V$, where V is an open set of $M(\infty)$ and t is so large that the maps $\Phi_{s,\infty}$ $(s \geqq t)$ are defined. Remark that $\bar{M}$ satisfies "*Ball Convergence Criterion*" in Donnely-Li [8], namely, if $x_n \in M$ is a sequence with $x_n \to \bar{x} \in M(\infty)$ then for all $t > 0$, $B_t(x_n) \to \bar{x}$.

2.3. There is another way to define a distance $\measuredangle_\infty$ on $\mathscr{R}_M/\sim$ (or $\mathscr{R}_p/\sim$) which coincides with the distance stated in [3] when M has nonnegative curvature everywhere (*i.e.*, $k \equiv 0$). Let us here define it and state its properties. The distance $\measuredangle_\infty$ on $\mathscr{R}_M/\sim$ is defined as follows:

$$\measuredangle_\infty([\sigma], [\gamma]) := \lim_{t \to \infty} 2 \arcsin \frac{1}{2t} \mathrm{dis}_M(\sigma(t), \gamma(t))$$

Then, we have the following

PROPOSITION 2.4. — The above distance $\measuredangle_\infty$ on $\mathscr{R}_M/\sim$ is well defined and $\measuredangle_\infty = \min\{\pi, \delta_\infty\}$.

Before concluding this section, we shall mention a result on smooth approximation of the metric spheres of M with bounded curvature, under certain additional conditions to Hypothesis (H.1).

Let M be as before and suppose, in addition to (H.1), that M satisfies

$$\text{(H.2)} \qquad \kappa_M := \limsup_{t \to \infty} t^2 K(t) < +\infty,$$

where $K(t) := \sup$ (the sectional curvature of M at points x with $\mathrm{dis}_M(o, x) \geqq t\}$ and o is a fixed point of M, say the base point in (H.1). Obviously κ_M is independent of the choice of the reference point o. The following theorem will be proved in [20].

THEOREM 2.5. — Under the conditions (H.1) and (H.2), for large t, there exists a smooth hypersurface S'_t of M which has the following properties:

(i) $(1/t)\max\{\max_{x\in S_t(0)} \mathrm{dis}_M(x, S'_t), \max_{y\in S'_t} \mathrm{dis}_M(S_t(o), y)\}\to 0$ as t goes to infinity.

(ii) There is a Lipschitz homeomorphism $\varphi_t: S'_t\to S_t(o)$ satisfying

$$e^{-\varepsilon(t)}\leqq \frac{d_t(\varphi_t(x), \varphi_t(y))}{d'_t(x, y)}\leqq e^{+\varepsilon(t)},$$

where $\varepsilon(t)$ tends to zero as $t\to\infty$ and d_t (resp. d'_t) denotes the inner distance on $S_t(o)$ (resp. S'_t).

(iii) The second fundamental form α_t of S'_t is estimated by

$$\{-a\sqrt{\kappa_M}\tan a\sqrt{\kappa_M}-\varepsilon(t)\}g_M\leqq t\alpha_t\leqq\left\{1+\frac{1}{a}+\varepsilon(t)\right\}g_M,$$

where a is a constant such that $0<a<\pi/2\sqrt{\kappa_M}$. Moreover if

$$\lim_{t\to\infty} \mathrm{Vol}_m(B_t(o)\cap \mathscr{E}_\alpha(M))/t^m>0$$

for some end $\mathscr{E}_\alpha(M)$ of M, then one has a smooth approximation $\hat{S}_t$ with (i) and (ii) as above, whose second fundamental form $\hat{\alpha}_t$ enjoys the following estimates:

$$(1-\varepsilon(t))g_M\leqq t\hat{\alpha}_t\leqq(1+\varepsilon(t))g_M,$$

on $\hat{S}_t\cap\mathscr{E}_\alpha(M)$.

Theorem 2.5 says in particular that under the conditions (H.1) and (H.2), $M(\infty)$ is the limit (with respect to the Hausdorff distance) of a family of compact $(m-1)$-dimensional Riemannian manifolds $\{M_t\}$ which are bounded in diameter and in curvature, and moreover when $\lim_{t\to\infty}\mathrm{Vol}_m(B_t(o)\cap\mathscr{E}_\alpha(M))/t^m>0$ for some end $\mathscr{E}_\alpha(M)$ of M, the volume of the connected component of M_t converging to $M_\alpha(\infty)$ may be assumed to have a positive lower bound uniformly in t. These facts would clarify much more the geometry of M at infinity. We refer the reader to, *e.g.*, [15], Ch. 8, [9], [13], etc.

3. Proofs of the results in Section 2

The purpose of this section is to verify the results stated in Section 2. Throughout this section, M is an m-dimensional Riemannian manifold of asymptotically nonnegative curvature. We use the same notations as in the previous sections.

3.1. Let us begin by constructing the maps $\Phi_{s,t}: S_s(o)\to S_t(o)$ in Proposition 2.2, where o is the base point in (H.1).

Step 1. — Let us take a small number $\varepsilon>0$ and large numbers t_0, t_1 such that $t_0<t_1$ and $|\nabla r_\delta|>1/2$ on $A(t_1, t_0)(:=\overline{B_{t_1}(o)}\setminus B_{t_0}(o))$, where r_δ is the Riemannian mollifier of the distance r to o and $0<\delta\leqq\delta(t,\varepsilon)$ (*cf.* Lemma 1.5). For a point $x\in A(t_1,t_0)$, we denote by $\lambda_\delta(x;\tau)$ $(\tau\in[a,b]; a<0<b)$ the maximal integral curve of the vector field $\nabla r_\delta/|\nabla r_\delta|^2$ on $A(t_1,t_0)$ such that $\lambda_\delta(x;0)=x$. Let $\eta(s)$ be a smooth regular curve in M defined on an interval I such that $0\in I$ and $r_\delta(\eta(s))\equiv r_\delta(\eta(0))$. Set $X(s,\tau):=(\partial/\partial s)\lambda_\delta(\eta(s);\tau)$ and $Y(s,\tau):=(\partial/\partial\tau)\lambda_\delta(\eta(s);\tau)$. Then we have

$$(3.1)\quad \frac{\partial}{\partial\tau}\log|X|=\frac{\langle\nabla_Y X, Y\rangle}{|X|^2}=\frac{\langle\nabla_X Y, X\rangle}{|X|^2}$$
$$=\frac{\langle\nabla_X\nabla r_\delta, X\rangle}{|\nabla r_\delta|^2|X|^2}\leqq\frac{(1+\varepsilon)}{|\nabla r_\delta|^2}\frac{\partial}{\partial\tau}\log J_k(\tau+r_\delta(\eta(0)))$$

by Lemma 1.5 (iv). We set here $I_\varepsilon:=\{\tau\in[0,l]: \mathrm{dis}_M(\lambda_\delta(\eta(0);\tau),\mathscr{C}_0)\leqq\varepsilon\}$ $(l\leqq b)$, where $\mathscr{C}_0$ is the cut locus of M with respect to o. Then, since $|\nabla r_\delta|(\lambda_\delta(\eta(0);\tau))\geqq 1-\varepsilon$ for $\tau\in I_\varepsilon$ by Lemma 1.5 (iii), we get by (3.1)

$$(3.2)\quad \frac{|X(0,\tau)|}{|X(0,0)|}\leqq\exp\left(2(\varepsilon+1)\int_{I_\varepsilon}\frac{\partial}{\partial\rho}\log J_k(\rho+r_\delta(\eta(0)))\,d\rho\right)$$
$$\times\left[\frac{J_k(r_\delta(\lambda_\delta(\eta(0);\tau)))}{J_k(r_\delta(\lambda_\delta(\eta(0);0)))}\right]^{2\varepsilon/(1-\varepsilon)}\frac{J_k(r_\delta(\lambda_\delta(\eta(0);\tau)))}{J_k(r_\delta(\lambda_\delta(\eta(0);0)))}$$

$(\tau\in[0,l])$. Note that

$$(3.3)\qquad \lim_{|I_\varepsilon|\to 0}\exp\left(2(\varepsilon+1)\int_{I_\varepsilon}\frac{\partial}{\partial\rho}\log J_k(\rho+r_\delta(\eta(0)))\,d\rho\right)=1,$$

$$(3.4)\qquad \lim_{\varepsilon\to 0}\left[\frac{J_k(r_\delta(\lambda_\delta(\eta(0);\tau))}{J_k(r_\delta(\lambda_\delta(\eta(0);0))}\right]^{2\varepsilon/(1-\varepsilon)}=1.$$

Step 2. — For any pair of numbers s, t with $t_0\leqq s\leqq t\leqq t_1$, we can define a map $\Phi_{\delta;s,t}$: $S_s(o)\to S_t(o)$ as follows: for a point x of $S_s(o)$, $\Phi_{\delta;s,t}(x)$ is the point of $S_t(o)$ where the integral curve $\lambda_\delta(x;*)$ intersects $S_t(o)$. Then it follows from (3.2) and Lemma 1.5 that $\{\Phi_{\delta;s,t}\}$ $(0<\delta\leqq\delta(t_1,\varepsilon))$ is a totally bounded, equicontinuous family of maps from $S_s(o)$ onto $S_t(o)$. Hence we have a sequence $\{\delta_n\}$ with $\delta_n\searrow 0$ as $n\to\infty$ and a Lipschitz map $\Phi_{s,t}: S_s(o)\to S_t(o)$ such that $\{\Phi_{\delta_n;s,t}\}$ converges to $\Phi_{s,t}$ as $n\to\infty$. Observe that for any distance minimizing geodesic $\sigma: [0,t]\to M$ joining o to a point of $S_t(o)$,

$$\Phi_{s,t}(\sigma(s))=\sigma(t).$$

Moreover, taking a subsequence of $\{\delta_n\}$ if necessary, we see that the choice of $\{\delta_n\}$ is independent of s, t and t_1, namely, for any pair of numbers s, t with $t_0\leqq s\leqq t$, $\{\Phi_{\delta_n;s,t}\}$ converges to a Lipschitz map $\Phi_{s,t}: S_s(o)\to S_s(o)$ as $\delta_n\to 0$. Clearly $\{\Phi_{s,t}\}$ has the following property:

$$\Phi_{t,u}\circ\Phi_{s,t}=\Phi_{s,u}$$

$(t_0\leqq s\leqq t\leqq u)$.

Step 3. — We are now in a position to show

$$\frac{d_t(\Phi_{s,t}(x),\Phi_{s,t}(y))}{J_k(t)}\leqq\frac{d_s(x,y)}{J_k(s)} \tag{3.5}$$

for any pair of s, t with $t_0\leqq s\leqq t$ and every pair of points x, y of $S_s(o)$, where d_t denotes the inner distance on $S_t(o)$ induced from the distance of M. For any pair of points x, y of a connected component of $S_s(o)$, we can take a Lipschitz curve $\eta:[0,l]\to S_s(o)$ such that $\eta(0)=x$, $\eta(l)=y$ and $d_s(\eta(u),\eta(v))=|u-v|$. Since $S_s(o)$ is not necessary smooth, we approximate it by a smooth hypersurface $S'_{\delta'}:=\{z\in M: r_{\delta'}(z)=s\}$, where δ' is sufficiently small. Let us denote by $\varphi_{\delta'}$ the projection from $S_s(o)$ onto $S'_{\delta'}$ along the integral curves of $\nabla r_{\delta'}/|\nabla r_{\delta'}|^2$, and set $x'=\varphi_{\delta'}(x)$ and $y'=\varphi_{\delta'}(y)$. Let $\eta':[0,l']\to S'_{\delta'}$ be a minimal geodesic in $S'_{\delta'}$ joining x' to y'. Observe that the length l' of η' (=the distance in $S'_{\delta'}$ between x' and y') goes to l as $\delta'\to 0$. We take here a (small piece of) smooth hypersurface U in $S'_{\delta'}$ such that η' intersects orthogonally U at $x'=\eta'(0)$, and we write V for the domain of a smooth family of geodesics $\eta'_p:[0,l']\to S'_{\delta'}$ which start at p of U and point to the same direction as η'. Let us here choose a small positive constant ε and a sequence $\{\varepsilon_n\}$ of positive numbers with $\varepsilon_n\to 0$ as $n\to\infty$. We may assume that $0<\delta_n\leqq\delta(t_1,\varepsilon_n)$, where $\{\delta_n\}$ is as in Step 2 and $\delta(t_1,\varepsilon_n)$ is as in Lemma 1.5. For a point z' of $S'_{\delta'}$, we set

$$I_n(z'):=\{\tau\in[0,\tau_n(z')]:\lambda_{\delta_n}(z';\tau)\in\mathscr{C}_{0,\varepsilon_n}\},$$

where $\tau_n(z')$ is defined by $\lambda_{\delta_n}(z';\tau_n(z'))\in S_t(o)$ and $\mathscr{C}_{0,\varepsilon_n}:=\{x\in M:\mathrm{dis}_M(x,\mathscr{C}_0)\leqq\varepsilon_n\}$. Observe that the m-dimensional Hausdorff measure $\mathscr{H}^m(\mathscr{C}_{0,\varepsilon_n}\cap A(t,s))$ goes to zero as $\varepsilon_n\to 0$. We define subsets $K_{\varepsilon,n}$ of $S'_{\delta'}$ by $K_{\varepsilon,n}:=\{z'\in S'_{\delta'}:|I_n(z')|\geqq\varepsilon\}$. Then by the Chebyshev's inequality, we have

$$\mathscr{H}^{m-1}(K_{\varepsilon,n})\leqq\frac{1}{\varepsilon}\int_{S'_{\delta'}}|I_n(z')|\leqq\frac{c_1}{\varepsilon}\mathscr{H}^m(\mathscr{C}_{0,\varepsilon_n}\cap A(t,s))$$

where c_1 is a positive constant independent of n. This implies that

$$\mathscr{H}^{m-1}(K_{\varepsilon,n})\to 0\quad\text{as } n\to\infty. \tag{3.6}$$

For a point p of U, set $\tilde{I}_{\varepsilon,n}(p):=\{u\in[0,l']:\eta'_p(u)\in K_{\varepsilon,n}\}$. Then by the Chebyshev's inequality again, we obtain

$$\mathscr{H}^{m-2}(\{p\in U:|\tilde{I}_{\varepsilon,n}(p)|\geqq\varepsilon\})\leqq\frac{1}{\varepsilon}\int_U|\tilde{I}_{\varepsilon,n}(p)|\leqq\frac{c_2}{\varepsilon}\mathscr{H}^{m-1}(K_{\varepsilon,n}\cap V), \tag{3.7}$$

where c_2 is a positive constant independent of n. It is not hard to see from (3.6) and (3.7) that

$$\liminf_{n \to \infty} |\tilde{I}_{\varepsilon, n}(p)| \leqq \varepsilon \tag{3.8}$$

for almost all p of U. Let us take a point p of U with (3.8) and a subsequence n' such that

$$\lim_{n' \to \infty} |\tilde{I}_{\varepsilon, n'}(p)| \leqq \varepsilon.$$

We can define a family of curves $\xi_{p, n'}(u)$ $(u \in [0, l'])$ in $S_t(o)$ by $\xi_{p, n'}(u) := \lambda_{\delta_{n'}}(\eta'_p(u); \tau_{n'}(\eta'_p(u)))$. Set $l_{p, n'} :=$ the length of $\xi_{p, n'}$. Then it follows from (3.2), (3.3), (3.4) and (3.8) that

$$l_{p, n'} \leqq l' \left[(1 + O(n'))(1 + O(\varepsilon)) \frac{J_k(t)}{J_k(s)} + O(\varepsilon) \right], \tag{3.9}$$

where $O(n')$ [resp., $O(\varepsilon)$] stands for a constant which goes to zero as $n' \to \infty$ (resp., $\varepsilon \to 0$). Since $\xi_{p, n'}(0)$ (resp., $\xi_{p, n'}(l')$) converges to $\Phi_{s, t}(\varphi_{\delta'}^{-1}(p))$ (resp., $\Phi_{s, t}(\varphi_{\delta'}^{-1}(\eta'_p(l')))$ as $n' \to \infty$, and further $d_s(x, \varphi_{\delta'}^{-1}(p)) < \varepsilon$ and $d_s(y, \varphi_{\delta'}^{-1}(\eta'_p(l'))) < \varepsilon$ if p is sufficiently close to x', we have by (3.2) and (3.9)

$$d_t(\Phi_{s, t}(x), \Phi_{s, t}(y)) \leqq \frac{J_k(t)}{J_k(s)} (1 + O(\varepsilon_{n'}))(1 + O(\varepsilon)) l' + O(\varepsilon) + O(n').$$

Thus, letting n' go to infinity and ε and δ' go to zero, we have shown the required inequality (3.5). This completes the proof of Proposition 2.2.

3.2. We shall now show the following

LEMMA 3.1. — *Let* M *be as above and* σ, γ *two rays of* M.
(i) *If* σ *is asymptotic to* γ, *then,*

$$\frac{1}{t} \operatorname{dis}_M(\sigma(t), \gamma(t)) \to 0,$$

$$\frac{1}{t} d_t(\sigma \cap S_t(p), \gamma \cap S_t(p)) \to 0,$$

as t *goes to infinity, where* p *is a fixed point of* M.
(ii) *If* σ *and* γ *are equivalent, i. e.,* $\lim_{t \to \infty} \operatorname{dis}_M(\sigma(t), \gamma(t))/t = 0$, *then,*

$$\frac{1}{t} d_t(\sigma \cap S_t(p), \gamma \cap S_t(p)) \to 0,$$

as t *goes to infinity.*

Proposition 2.1 is an immediate consequence of Proposition 2.2 (which has been proved in 3.1) and Lemma 3.1.

Proof of Lemma 3.1. — Suppose first that σ is asymptotic to γ. We take two positive constants ε, $\delta \in (0, 1)$ and a divergent sequence $\{t_n\}_{n=1,2,\ldots}$. Let $\sigma_{\varepsilon, n}: [0, l_{\varepsilon, n}] \to M$ be distance minimizing geodesics such that $\sigma_{\varepsilon, n}(0) = \sigma(\varepsilon)$, $\sigma_{\varepsilon, n}(l_{\varepsilon, n}) = \gamma(t_n)$. Then Fact 1.1 (iii) implies that $\{\sigma_{\varepsilon, n}(t)\}$ converges to $\sigma(t+\varepsilon)$ as $n \to \infty$. Hence we have

$$\theta_n := \measuredangle_{\sigma(\varepsilon)}(\dot{\sigma}_{\varepsilon, n}(0), \dot{\sigma}(\varepsilon)) \to 0, \tag{3.10}$$

as n goes to infinity. Let us apply Fact 1.3 (i) to the geoedesic triangle $\Delta_n := (\sigma(\varepsilon), \gamma(t_n), \sigma((l_{\varepsilon, n}+\varepsilon)/(1-\delta)))$. Then we get

$$\mathrm{dis}_M(\gamma(t_n), \sigma((l_{\varepsilon, n}+\varepsilon)/(1-\delta))) \leqq \frac{l_{\varepsilon, n}+\varepsilon}{1-\delta} - l_{\varepsilon, n}\sqrt{1-a_n^2} \tag{3.11}$$

where $a_n := \beta^{-1}\delta^{-1}\sin\theta_n$ and β is a constant depending only on M and $\sigma(\varepsilon)$ (*cf.* 1.3). It follows from (3.10), (3.11) and $\lim_{n \to \infty} l_{\varepsilon, n}/t_n = 1$ that

$$\limsup_{n \to \infty} \frac{\mathrm{dis}_M(\gamma(t_n), \sigma(t_n))}{t_n} \leqq \frac{2\delta}{1-\delta}.$$

Since δ and $\{t_n\}$ are taken arbitrarily, we see that σ and γ are equivalent. It is easy to see from the argument in 3.1 that if γ and σ are equivalent, then $\lim_{t \to \infty} d_t(\gamma \cap S_t(p), \sigma \cap S_t(p))/t = 0$. This completes the proof of Lemma 3.1.

3.3. *We shall now prove Proposition* 2.3. — The first three assertions are direct consequences of the previous results. It remains to show that the Hausdorff distance $d_H(M_\alpha(\infty), M_{\alpha, t})$ goes to zero as $t \to \infty$, where $M_{\alpha, t} := (S_t(o) \cap \mathscr{E}_\alpha(M), (1/t)\, d_t)$. Let ε be any small positive number and $\Lambda = \{p_1, \ldots, p_\mu\}$ a 2ε-lattice of $M_\alpha(\infty)$ with gap ε, namely, $\delta_\infty(p_i, p_j) \geqq \varepsilon\, (i \neq j)$ and $\delta_\infty(x, \Lambda) \leqq 2\varepsilon$ for any $x \in M_\alpha(\infty)$. We assume that p_i $(i = 1, \ldots, \mu)$ are represented by rays σ_i starting at o. Set $\Lambda(t) := \{\sigma_1(t), \ldots, \sigma_\mu(t)\}$. Then $\Lambda(t)$ defines a $(2\varepsilon + \varepsilon(t)\}$-lattice in $M_{\alpha, t}$ with gap $\varepsilon - \varepsilon(t)$, where $\varepsilon(t)$ goes to zero as $t \to \infty$. Moreover we have

$$\frac{J_k(t)}{J'_k(\infty)\, t}\, \delta_\infty(p_i, p_j) \leqq \frac{1}{t}\, d_t(\sigma_i(t), \sigma_j(t)) \leqq \left\{ 1 + \frac{\varepsilon(t)}{\varepsilon} \right\} \delta_\infty(p_i, p_j).$$

This shows that $d_H(M_\alpha(\infty), M_{\alpha, t})$ goes to zero as $t \to \infty$ (*cf.* [15], Ch. 3, Proposition 3.5).

3.4. We shall here give the proof of Proposition 2.4 which is devided into 4 steps.

Step 1. — Let σ and γ be two rays of M which belong to the same end of M. For simplicity, we assume that they start at the same point, say the base point o in (H.1). We fix constants a, b with $0 < a < b$. For any large number t, we take a sufficiently small positive constant δ_t which goes to zero as $t \to \infty$. Then $S'_{\delta_t} := \{x \in M : r_{\delta_t}(x) = at\}$

approximates $S_{at}(o)$. Set $p_t := \sigma \cap S'_{\delta_t}$ and $q_t := \gamma \cap S'_{\delta_t}$, and let $\eta_t : [0, l_t] \to S'_{\delta_t}$ be a distance minimizing geodesic in S'_{δ_t} joining p_t to q_t $\{l_t :=$ the length of $\eta_t)$. Observe that

$$\frac{l_t}{at} \to \delta_\infty([\sigma], [\gamma]) \tag{3.12}$$

as $t \to \infty$, because of Proposition 2.3 (iii). We have now a (piece of) smooth surface $\Sigma_t : (u, s) \to M$ $(a \leqq u \leqq b;\ 0 \leqq s \leqq l_t/t)$ defined by $\Sigma_t(u, s) := \lambda_{\delta_t}(\eta_t(st); (u-a)t)$. Set $g_t := t^{-2} \Sigma_t^* g_M$, where g_M stands for the Riemannian metric of M. Then it follows from the definition of Σ_t and Lemma 1.5 that

$$g_t\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial s}\right) \equiv 0, \tag{3.13}$$

$$g_t\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) \to 1 \text{ uniformly as } t \to \infty. \tag{3.14}$$

Moreover we observe that

$$g_t\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(u, s) \leqq c_t \frac{u^2}{a^2} \tag{3.15}$$

for any (u, s), and

$$\int_0^{l_t/t} \sqrt{g_t\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(u, s)}\, ds \to u\delta_\infty([\sigma], [\gamma]) \tag{3.16}$$

as t goes to infinity, where c_t is independent of (u, s) and it converges to 1 as $t \to \infty$. In fact, by Lemma 1.5, we have

$$\frac{\partial}{\partial u} \log g_t\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) \leqq 2\left[1 + \frac{2\varepsilon_t + \theta_1(ut - \varepsilon_t)}{1 - \varepsilon_t - \theta_1(ut - \varepsilon)}\right] \frac{\partial}{\partial u} \log J_k(ut),$$

where θ_1 and J_k are as in Lemma 1.5 and ε_t goes to zero as $t \to \infty$. This implies (3.15). On the other hand, we have

$$\liminf_{t \to \infty} \int_0^{l_t/t} \sqrt{g_t\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(u, s)}\, ds \geqq \liminf_{t \to \infty} \frac{d_{ut}(\sigma(ut), \gamma(ut))}{t} = u\,\delta_\infty([\sigma], [\gamma]). \tag{3.17}$$

Hence (3.16) follows from (3.15) and (3.17). Consequently, we can assert that for a fixed u,

$$g_t\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right)(u, s) \to \frac{u^2}{a^2} \qquad \text{(almost all } s)$$

as t goes to infinity, if $\delta_\infty([\sigma], [\gamma]) > 0$.

Step 2. — Let us now define the metric space associated with $M(\infty)$. For any pair of points (t, p), (t', p') of $[0, \infty) \times M(\infty)$, we set

$$\Delta_\infty((t, p), (t', p')) := \sqrt{t^2 + t'^2 - 2tt' \cos \hat{\delta}_\infty(p, p')},$$

where $\hat{\delta}_\infty(p, p') := \min\{\pi, \delta_\infty(p, p')\}$. Then we get a metric space $([0, \infty) \times M(\infty), \Delta_\infty)$ and write $\mathscr{C}(M(\infty))$ for it. Making use of Propositions 2.2 and 2.3, we have a map $\Phi: M \setminus B_{t_0}(o) \to \mathscr{C}(M(\infty))$ defined by $\Phi(x) := (r(x), \Phi_{r(x), \infty}(x))$ which satisfies: $\Phi^{-1}(\{t\} \times M(\infty)) = S_t(o)$ $(t \geqq t_0)$, and for any Lipschitz curve $\eta: [0, l] \to M \setminus B_{t_0}(o)$,

$$\text{the length of } \Phi \circ \eta \leqq \mu(\mathrm{dis}_M(o, \eta([0, l]))) \times \text{ the length of } \eta. \tag{3.18}$$

Here $\mu(t)$ satisfies:

$$\mu(t) \geqq 1 \quad \text{and} \quad \lim_{t \to \infty} \mu(t) = 1. \tag{3.19}$$

Step 3. — In this step, we shall show that for two rays σ, γ of M,

$$\liminf_{t \to \infty} \measuredangle(\sigma(t), \gamma(t)) \geqq \delta_\infty([\sigma], [\gamma]), \tag{3.20}$$

where $\measuredangle(\sigma(t), \gamma(t)) := 2 \arcsin(\mathrm{dis}_M(\sigma(t), \gamma(t))/2t)$. Obviously it is enough to show (3.20) in case that σ and γ start at the same point, say the base point o in (H.1). Let $\eta_t: [0, 1] \to M$ be a geodesic joining $\eta_t(0) = \sigma(t)$ to $\eta_t(1) = \gamma(t)$ with $|\dot{\eta}_t| \equiv \mathrm{dis}_M(\sigma(t), \gamma(t))$. We first consider the case that $\mathrm{dis}_M(o, \eta_t([0, 1]))$ goes to infinity as $t \to \infty$. Set $\hat{\eta}_t := (1/t)\Phi \circ \eta_t: [0, 1] \to \mathscr{C}(M(\infty))$ (for large t), namely, $\hat{\eta}_t(s) := ((1/t) r(\eta_t(s)), \Phi_{r(\eta_t(s)), \infty}(\eta_t(s)))$. Then, $\{\hat{\eta}_t\}$ defines a family of Lipschitz curves in $\mathscr{C}(M(\infty))$ such that $\hat{\eta}_t(0) = (1, [\sigma])$, $\hat{\eta}_t(1) = (1, [\gamma])$ and

$$\text{the length of } \quad \hat{\eta}_t \leqq \mu(\mathrm{dis}_M(o, \eta_t([0, 1]))) \cdot \frac{\mathrm{dis}_M(\sigma(t), \gamma(t))}{t}.$$

Thus $\{\hat{\eta}_t\}$ are equicontinuous and totally bounded. This implies that for any divergent sequence $\{t_n\}$, there exists a subsequence $\{t_{n'}\}$ of $\{t_n\}$ such that $\{\hat{\eta}_{t_{n'}}\}$ converges to a Lipschitz curve $\hat{\eta}_\infty: [0, 1] \to \mathscr{C}(M(\infty))$ joining $(1, [\sigma])$ to $(1, [\gamma])$ with

$$\text{the length of } \quad \hat{\eta}_\infty \leqq \liminf_{t_{n'} \to \infty} \frac{\mathrm{dis}_M(\sigma(t_{n'}), \gamma(t_{n'}))}{t_{n'}}.$$

Hence we have

$$\begin{aligned}\liminf_{t \to \infty} \measuredangle(\sigma(t), \gamma(t)) &= 2 \arcsin\left[\liminf_{t \to \infty} \frac{\mathrm{dis}_M(\sigma(t), \gamma(t))}{2t}\right] \\ &\geqq 2 \arcsin(\text{the length of } \hat{\eta}_\infty) \\ &\geqq 2 \arcsin \Delta_\infty((1, [\sigma]), (1, [\gamma])) = \hat{\delta}_\infty([\sigma], [\gamma]).\end{aligned}$$

It remains to prove (3.20) in the case that $\sup \mathrm{dis}_M(o, \eta_{t_n}([0,1]))$ is finite for some divergent sequence $\{t_n\}$. In this case, we have a straight line $\eta: (-\infty, +\infty) \to M$ such that $\eta^+: [0, \infty) \to M$ $(\eta^+(t) := \eta(t))$ is asymptotic to γ and $\eta^-: [0, \infty) \to M$ $(\eta^-(t) := \eta(-t))$ is asymptotic to σ. This implies that $\mathrm{dis}_M(\sigma(t), \gamma(t))/2t$ goes to 1 as $t \to \infty$, because of Lemma 3.1 (i) and $\mathrm{dis}_M(\eta^+(t), \eta^-(t))/2t \equiv 1$. Hence we have

$$\lim_{t \to \infty} \measuredangle(\sigma(t), \gamma(t)) = \pi \geqq \hat{\delta}_\infty([\sigma], [\gamma]).$$

Thus we have shown (3.20).

Step 4. — In this step, making use of the observation in Step 1, we shall show that, given two rays σ, γ of M,

$$\limsup_{t \to \infty} \measuredangle(\sigma(t), \gamma(t)) \leqq \hat{\delta}_\infty([\sigma], [\gamma]). \tag{3.21}$$

Obviously it is enough to prove (3.21) in the case that σ, γ belong to the same end and start at the same point, say the base point o in (H.1). Moreover we may assume that $\delta_\infty([\sigma], [\gamma]) < \pi$. In what follows, we use the same notation as in Step 1 (the constant a, b there are assumed to satisfy: $0 < a < \cos \delta_\infty([\sigma], [\gamma])/2 < 1 < b$). For sufficiently large t, we consider smooth curves $\xi_t: [0, l_t/t] \to M$ defined by

$$\xi_t(s) := \Sigma_t((\cos l_t/2t)/(\cos(s/a - l_t/2t)), s).$$

Then it is clear from the definition of Σ_t that $\lim_{t \to \infty} \mathrm{dis}_M(\xi_t(0), \sigma(t))/t = 0$ and $\lim_{t \to \infty} \mathrm{dis}_M(\xi_t(l_t/t), \gamma(t))/t = 0$. Moreover it turns out from (3.13), (3.14) and (3.15) that

$$\limsup_{t \to \infty} \frac{1}{t} \text{. the length of } \xi_t \leqq 2 \sin \delta_\infty([\sigma], [\gamma])/2.$$

Thus we have

$$\limsup_{t \to \infty} \frac{\mathrm{dis}_M(\sigma(t), \gamma(t))}{t} \leqq \limsup_{t \to \infty} \frac{1}{t} \text{. the length of } \xi_t \leqq 2 \sin \delta_\infty([\sigma], [\gamma])/2.$$

This proves (3.21). The first assertion of Proposition 2.4 follows from (3.20) and (3.21).

4. Busemann functions on a manifold of nonnegative curvature

In this section, we shall study a complete, connected, noncompact Riemannian manifold M of nonnegative sectional curvature and the behavior of Busemann functions on M, motivated by Shiohama [23].

Let us begin with the following

Fact 4.1 (Toponogov [26], Lemma 19). Let M be as above and $\sigma_i : [0, l_i] \to M$ ($i=1,2$) two distance minimizing geodesics starting at the same point. For each (t_1, t_2) with $0 < t_i \leqq l_i$ ($i=1,2$), let $\Delta(t_1, t_2)$ be the triangle sketched on $\mathbb{R}^2$ whose edge lengths are t_1, t_2 and $\mathrm{dis}_M(\sigma_1(t_1), \sigma_2(t_2))$, and denote by $\theta(t_1, t_2)$ the angle of $\Delta(t_1, t_2)$ opposite to the edge of length $\mathrm{dis}_M(\sigma_1(t_1), \sigma_2(t_2))$. Then $(t_1, t_2) \to \theta(t_1, t_2)$ is monotone nonincreasing in the following sense: $\theta(t_1, t_2) \leqq \theta(s_1, s_2)$ if $s_1 \leqq t_1$ and $s_2 \leqq t_2$.

Before showing the first result of this section, we note that M has at most two ends and further if M has two ends, then M splits isometrically into $N \times \mathbb{R}$, where N is compact (*cf.* [5], [25]).

PROPOSITION 4.2. — *Let* M *be as above and suppose* M *has one end. Then the two distances* $\sphericalangle_\infty$ *and* δ_∞ *on* $M(\infty)$ *defined in Section* 2 *coincide. Moreover the following conditions are mutually equivalent:*

(i) *The diameter of* $M(\infty)$ *is equal to* π.

(ii) M *contains a straight line.*

(iii) M *splits isometrically into* $M' \times \mathbb{R}$.

(iv) *The isometry group of* M *is non compact.*

Proof. — Suppose $\delta_\infty([\sigma], [\gamma]) \geqq \pi$ for some σ and $\gamma \in \mathscr{R}_M$. Then by Proposition 2.4, $\sphericalangle_\infty([\sigma], [\gamma]) = \pi$. We claim that $\delta_\infty([\sigma], [\gamma]) = \pi$ and M splits isometrically into $\mathbb{R} \times N$. In fact, we may assume that $\sigma(0) = \gamma(0)$. Then by Fact 4.1, $2 \arcsin(\mathrm{dis}_M(\sigma(t), \gamma(t))/2t)$ is a monotone nonincreasing function in t and converges to $\pi = \sphericalangle_\infty([\sigma], [\gamma])$. This implies that $\mathrm{dis}_M(\sigma(t), \gamma(t)) = 2t$, namely the geodesic $\xi : \mathbb{R} \to M$ defined by $\xi(t) = \sigma(t)$ for $t \geqq 0$ and $\xi(t) = \gamma(-t)$ for $t \leqq 0$ gives a line on M. Therefore it turns out from the Toponogov splitting theorem that M splits isometrically into $\mathbb{R} \times N$ along ξ and $\delta_\infty([\sigma], [\gamma]) = \pi$. Now the proposition follows from the above observation and Corollary 6.2 in [6].

Let us now prove the following

THEOREM 4.3. — *Let* M *be a complete, noncompact Riemannian manifold of nonnegative sectional curvature. Then for a ray* σ *of* M, *the Busemann function* F_σ *associated with* σ *is exhaustion function on* M (i.e., *for each* $t \in \mathbb{R}$, *the set* $\{x \in M : F_\sigma(x) \leqq t\}$ *is compact) if and only if* $\delta_\infty([\sigma], [\gamma]) < \pi/2$ *for any ray* γ *of* M.

Proof. — Take two points $[\sigma]$, $[\gamma]$ of $M(\infty)$. We may assume that $\sigma(0) = \gamma(0)$. For any u, $s \geqq 0$, we define $\theta(u, s)$ by $\mathrm{dis}_M(\sigma(s), \gamma(u))^2 = u^2 + s^2 - 2\, us \cos\theta(u, s)$. Suppose that $u \leqq s$. Then by Fact 4.1, $\theta(u, u) \geqq \theta(u, s) \geqq \theta(s, s)$ and $\lim_{u \to \infty} \theta(u, u) = \lim_{s \to \infty} \theta(s, s) = \sphericalangle_\infty([\sigma], [\gamma])$. Therefore we have

$$F_\sigma(\gamma(u)) = \lim_{s \to \infty} s - \mathrm{dis}_M(\sigma(s), \gamma(u))$$

$$= \lim_{s \to \infty} s\left(1 - \sqrt{1 + u^2 s^{-2} - 2us^{-1}\theta(u, s)}\right) = u \cos\theta(u, \infty).$$

Obviously this shows the theorem, since $\sphericalangle_\infty([\sigma], [\gamma]) = \lim_{u \to \infty} \theta(u, \infty)$.

Remark. — The above proof of Theorem 4.3 says that

$$\measuredangle_{\infty}([\sigma],[\gamma])=\lim_{u\to\infty}F_{\sigma}(\gamma(u))/u.$$

Moreover we can give another description of the distance $\measuredangle_{\infty}$ on $M(\infty)$ as follows. Let σ and γ be two rays of M. For each $t>0$, we can take a ray σ_t which emanates from $\gamma(t)$ and which is asymptotic to σ. We claim now that

$$\measuredangle_{\infty}([\sigma],[\gamma])=\lim_{t\to\infty}\measuredangle_{\gamma(t)}(\dot\gamma(t),\dot\sigma_t(0)) \tag{4.2}$$

In fact, it is obvious from Lemma 3.1 and Fact 4.1 that

$$\measuredangle_{\infty}([\sigma],[\gamma])\leqq\measuredangle_{\gamma(t)}(\dot\gamma(t),\dot\sigma_t(0))$$

Hence it is enough to show that

$$\measuredangle_{\gamma(t)}(\dot\gamma(t),\dot\sigma_t(0))\leqq\measuredangle(\sigma(t),\gamma(t))+\delta_t, \tag{4.3}$$

where δ_t goes to zero as $t\to\infty$ and $\measuredangle(\sigma(t),\gamma(t))=2\arcsin\{\mathrm{dis}_M(\sigma(t),\gamma(t))/2t\}$ (*cf.* Proposition 2.4). (4.3) is verified by refering to the argument of Shiohama [23], p. 287. For simplicity, we assume that $\sigma(0)=\gamma(0)$. Then by the definition of σ_t being asymptotic to σ (t is fixed), there exists a family of minimizing geodesics $\{\sigma_{t,n}\}_{n=1,2,\ldots}$ such that the starting points $q_{t,n}=\sigma_{t,n}(0)$ converge to $\gamma(t)$, as $n\to\infty$, the initial vectors $\dot\sigma_{t,n}(0)$ approach $\dot\sigma_t(0)$ as $n\to\infty$ and $\sigma_{t,n}(a_{t,n})=\sigma(d_{t,n})$ with $\lim_{n\to\infty}d_{t,n}=\infty$. Let $\gamma_{t,n}:[0,b_{t,n}]\to M$ be the unique minimizing geodesics joining $\sigma(0)$ to $q_{t,n}$ (which are assumed to be sufficiently close to $\gamma(0)$). We take the triangle $\Delta_{t,n}$ sketched on $\mathbb{R}^2$ whose edge lengths are $d_{t,n}$, $a_{t,n}$, $b_{t,n}$. Let us denote by $\delta_{t,n}$, $\alpha_{t,n}$, and $\beta_{t,n}$, respectively, the edge angle of $\Delta_{t,n}$ opposite to the edge of lengths $d_{t,n}$, $a_{t,n}$ and $b_{t,n}$. Then it turns out from Fact 4.1 that for large n,

$$\measuredangle_{q_{t,n}}(\dot\sigma_{t,n}(0),\dot\gamma_{t,n}(b_{t,n}))\leqq\pi-\delta_{t,n}=\alpha_{t,n}+\beta_{t,n}\leqq\theta_{t,n}+\beta_{t,n},$$

where

$$\theta_{t,n}:=2\arcsin\{\mathrm{dis}_M(q_{t,n},\sigma(b_{t,n}))/2b_{t,n}\}.$$

Since $\lim_{n\to\infty}\beta_{t,n}=0$ and $\lim_{n\to\infty}\theta_{t,n}=\measuredangle(\sigma(t),\gamma(t))$, we have (4.3). This completes the proof of (4.2).

As direct consequences of Theorem 4.3, we have the two results below.

COROLLARY 4.4. — *Let* M *be as in Theorem* 4.3. *Then every Busemann function is an exhaustion function on* M, *if the diameter of* $M(\infty)$ *is less than* $\pi/2$.

COROLLARY 4.5. — *Let* M *be as above. Suppose that* $M(\infty)$ *is a circle. Then:*

(i) *The diameter of* $M(\infty)$ *is less than* $\pi/2$ *if and only if every Busemann function is an exhaustion function on* M.

(ii) *The diameter of* $M(\infty)$ *is greater than or equal to* $\pi/2$ *if and only if every Busemann function is a nonexhaustion function on* M.

In particular, these two statements hold for the case: $\dim M = 2$.

Remarks. — (i) Let M be as in Theorem 4.3. Then it was conjectured by Shiohama [23], p. 282, that M could not admit both exhaustion and nonexhaustion Busemann functions simultanuously. Actually he proved it in the case: $m = \dim M = 2$. However it is not true in general for the case: $m \geqq 4$. For example, let M_i $(i = 1, 2)$ be complete, noncompact manifolds of nonnegative curvature such that $\operatorname{diam}(M_i(\infty))$ $(i = 1, 2)$ are sufficiently small. Then the product manifold $M = M_1 \times M_2$ admits both exhaustion and nonexhaustion Busemann functions simultanuously (*cf.* Section 5).

(ii) Let M be a manifold of asymptotically nonnegative curvature. Suppose that M has one end and the diameter of $M(\infty)$ is less than π. Then it follows from Proposition 2.4 that M admits no straight lines. Moreover we see that the isometry group I(M) of M is compact. Actually, if I(M) is not compact, then $I(M) \cdot p$ is unbounded for any $p \in M$, and hence the sectional curvature of M must be nonnegative everywhere. Thus by Proposition 4.2, we see that the diameter of $M(\infty)$ is equal to π.

5. Examples

We consider first Riemannian products of manifolds with nonnegative curvature. Let $M_i (i = 1, 2)$ be complete, noncompact Riemannian manifolds of nonnegative curvature and M the Riemannian product of M_1 and M_2. Then we have the natural inclusions $M_i(\infty) \subset M(\infty)$ $(i = 1, 2)$. It is easy to see that if $p_i \in M_i(\infty)$ $(i = 1, 2)$, then $\delta_\infty(p_1, p_2) = \pi/2$, and if $p \in M(\infty)$, then there are $p_i \in M_i(\infty)$ $(i = 1, 2)$ such that p lies on the distance minimizing curve in $M(\infty)$ joining p_1 to p_2.

Example 5.1. — Let $M_i (i = 1, \ldots, k)$ be complete, noncompact Riemannian manifolds of nonnegative curvature such that for each i, $M_i(\infty)$ consists of a single point. Then $(M_1 \times \ldots \times M_k)(\infty)$ is isometric to the part of the unit sphere: $\{(x_1, \ldots, x_k) \in S^{k-1}(1) : x_i \geqq 0 \ (i = 1, \ldots, k)\}$.

We shall here give the following

PROPOSITION 5.2. — *Let* M *be a complete, noncompact Riemannian manifold and suppose the sectional curvature of* M *is bounded from below by* $c/r^2 \log r$ *outside a compact set, where* c *is a positive constant and* r *denotes the distance to a fixed point of* M. *Then* $\dim M(\infty) = 0$, i. e., $M(\infty)$ *consists of a finite number of points.*

Proof. — Let us take a continuous function $\hat{k}$ on $[0, \infty)$ such that the sectional curvature of M is bounded from below by $\hat{k} \circ r$ and $\hat{k}(t) = c/t^2 \log t$ for large t. Let $J_{\hat{k}}$ be the solution of an equation: $J_{\hat{k}}'' + \hat{k} J_{\hat{k}} = 0$, with $J_{\hat{k}}(0) = 0$ and $J_{\hat{k}}'(0) = 1$. Then by the lemma below, we see that $J_{\hat{k}}(t)/t$ goes to zero as $t \to \infty$. This implies that given two rays σ, γ of M starting at the same point o of M and belong to the same end, we have

$$\lim_{t\to\infty}\frac{1}{t}d_t(\sigma(t),\gamma(t))=\lim_{t\to\infty}\frac{J_{\bar{k}}(t)}{t}\cdot\frac{d_t(\sigma(t),\gamma(t))}{J_{\bar{k}}(t))}=0,$$

since $t\to d_t(\sigma(t),\gamma(t))/J_{\bar{k}}(t)$ is monotone nonincreasing for large t (*cf.* the proof of Proposition 2.2). This completes the proof of Proposition 5.2.

LEMMA. — *Let $\hat{k}$ be a continuous function on $[0,\infty)$ such that $\hat{k}(t)\geqq 0$ for large t and $\int^{\infty} t\hat{k}(t)\,dt=\infty$. Let J be the solution of an equation: $J''+\hat{k}J=0$, with $J(0)=0$ and $J'(0)=1$. Suppose that J is positive on $(0,\infty)$. Then $J(t)/t$ tends to zero as $t\to\infty$.*

Proof. — We assume that $\hat{k}\geqq 0$ on $[a,\infty)$ for some a. Then for any $t>a$, we have

$$J'(t)(t-a)-J(t)=\int_a^t\{J'(s)(s-a)-J(s)\}\,ds-J(a)=\int_a^t-\hat{k}(s)J(s)(s-a)\,ds-J(a)<0.$$

This shows that $J(t)/(t-a)$ is monotone nonincreasing on $[a,\infty)$. Suppose that $b:=\lim_{t\to\infty}J(t)/t>0$. Then $J(t)\geqq b(t-a)$ on $[a,\infty)$, and hence we have

$$-\int_a^t\hat{k}(s)J(s)\,ds\leqq-\int_a^t b(s-a)\hat{k}(s)\,ds.$$

The right side of the above inequality goes to $-\infty$ as $t\to\infty$, so that $J'(t)\left(=J'(a)-\int_a^t\hat{k}(s)J(s)\,ds\right)$ goes to $-\infty$ as $t\to\infty$. This contradicts the assumption that $J(t)>0$ on $(0,\infty)$. Thus we have shown that $J(t)/t$ tends to zero as $t\to\infty$. This completes the proof of Lemma.

Let us next consider a Riemannian submersion $\pi:\hat{M}\to M$, where $\hat{M}$ is a manifold of asymptotically nonnegative curvature (and hence so is M, since π is curvature nondecreasing (*cf.* O'Neill [22])). Let us denote by $M^{\pi}(\infty)$ the set of equivalence classes containing the *horizontal* rays of $\hat{M}$. Then the projection can be naturally extended to a map $\bar{\pi}:\hat{M}\cup M^{\pi}(\infty)\to M\cup M(\infty)$. We write π_∞ for the restriction of $\bar{\pi}$ to $M^{\pi}(\infty)$. Then it turns out that π_∞ is a distance nonincreasing map from $M^{\pi}(\infty)$ onto $M(\infty)$. In what follows, we assume that π *has compact fibres.* In this case, it is not hard to see that $M^{\pi}(\infty)$ coincides with $\hat{M}(\infty)$ and moreover that for each pair of points p, q of $M(\infty)$, the distance between them in $M(\infty)$ is equal to the distance between the two fibres $\pi_\infty^{-1}(p)$ and $\pi_\infty^{-1}(q)$ in $\hat{M}(\infty)$. In particular, π_∞ gives rise to an isometry between $\hat{M}(\infty)$ and $M(\infty)$, if $\operatorname{diam}(\pi^{-1}(x))/\operatorname{dis}_M(x,o)$ goes to zero as $x\in M$ tends to infinity, where o is a fixed point of M. We remark that for a slight perturbation of the Riemannian submersion $\pi:\hat{M}\to M$, we have the same conclsuions as above. Actually it is natural to consider "an asymptotically Riemannian submersion" in an appropriate sense. However we do not go into details here.

Let us now consider a group H of isometries of a manifold $\hat{M}$ with asymptotically nonnegative curvature and suppose that H acts freely on M so that the orbit space $M=\hat{M}/H$ is a manifold and basis of a principal fibration $H \to \hat{M} \to M$ with natural projection π. Since H acts by isometries, the metric of $\hat{M}$ projects down to a complete metric for M with respect to which π becomes a Riemannian submersion. Since H also acts isometrically on $\hat{M}(\infty)$, we see that $M(\infty)=\hat{M}(\infty)/H$, if H is compact.

Example 5.2. – Let $G=SO(m+1)$ with bi-invariant metric and $H=SO(m)$ acting on Euclidean space $\mathbb{R}^m$ by rotations. Then $M=G\times\mathbb{R}^m/H$ is the tangent bundle of the sphere S^m, where H acts diagonally on $G\times\mathbb{R}^m$, and $M(\infty)$ consists of only one point.

Example 5.3. – Consider the unit sphere $S^3(1)$ of dimension 3 in $\mathbb{H}$ (Quaternion field) as a Lie group with multiplication in $\mathbb{H}$. Let $\{Z_1, Z_2, Z_3\}$ be a left invariant, orthonormal frame field on $S^3(1)$ such that $[Z_1, Z_2]=2Z_3$, $[Z_2, Z_3]=2Z_1$, $[Z_3, Z_1]=2Z_2$ (*cf.* e.g., [4], 3.35). We denote by $\theta_i\,(i=1,2,3)$ the dual forms of Z_i and consider a Riemannian metric G on $\mathbb{R}^4$ of the form:

$$G=dr^2+f^2(r)\,\theta_1^2+f^2(r)\,\theta_2^2+g^2(r)\,\theta_3^2,$$

where $f(r)$, $g(r)$ are smooth functions on $[0,\infty)$ which are chosen later. Let π be a tangent 2-plane spanned by unit vectors X, Y which are orthogonal. Without loss of generality, we may assume that $G(Y, \partial/\partial r)=0$. Then the sectional curvature $K(\pi)$ for the plane π is given by

$$\begin{aligned}K(\pi)=-\frac{f''}{f}x_0^2y_1^2-\frac{f''}{f}x_0^2y_2^2-\frac{g''}{g}x_0^2y_3^2\\
+f^{-2}(4-3g^2f^{-2}-f'^2)(x_1^2y_2^2+x_2^2y_1^2)\\
+f^{-1}g^{-1}(g^3f^{-3}-g'f')(x_1^2y_3^2+x_2^2y_3^2+x_3^2y_1^2+x_3^2y_2^2)\\
+6f^{-3}(g'f-gf')(x_0x_1y_2y_3-x_0x_2y_1y_3),\end{aligned}$$

where $x_0=G(X, \partial/\partial r)$, $x_i=G(X, Z_i)/f\ (i=1,2)$, $x_3=G(X, Z_3)/g$, $y_i=G(Y, Z_i)/f\ (i=1,2)$ and $y_3=G(Y, Z_3)/g$. We set here $f(r)=\lambda r\,(0<\lambda<2)$ and $g(r)=r^2/(1+r^2)$ for large r. Then $M=(\mathbb{R}^4, G)$ is a manifold of asymptotically nonnegative curvature such that $M(\infty)$ is isometric to the 2-sphere of constant curvature λ^{-2}.

The following example shows that certain minimal submanifolds in $\mathbb{R}^n$ belong to a class of manifolds with asymptotically nonnegative curvature.

Example 5.4 (Anderson [2]). – Let M be a complete minimal submanifold of Euclidean space such that the total scalar curvature: $\int_M |\alpha_M|^m$ is finite, where $m=\dim M$ and α_M denotes the second fundamental form of M. Then if $m\geqq 3$, $|\alpha_M|$ is bounded from above by c/r^m for some constant c, where r is the distance function to a fixed point of M. In this case, $M(\infty)$ consists of a finite number of the $(m-1)$-spheres of constant curvature 1.

Before concluding this section, we shall mention a result on the volume growth of metric balls of a manifold M with asymptotically nonnegative curvature.

PROPOSITION 5.5. — *Let* M *be as above. Suppose that the sectional curvature is bounded from above by a positive constant. Then for each end* $\mathscr{E}_\alpha$ *of* M ($\alpha=1, \ldots, \mu(\mathrm{M})$), *one has*

$$\liminf_{t \to \infty} \frac{\log \mathrm{Vol}_m(\mathrm{B}_t(p) \cap \mathscr{E}_\alpha(\mathrm{M}))}{\log t} \geqq 1+\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty),$$

where p *is a fixed point of* M *and* $\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty)$ *stands for the Hausdorff dimension of the connected component* $\mathrm{M}_\alpha(\infty)$ *of* $\mathrm{M}(\infty)$ *corresponding to* $\mathscr{E}_\alpha(\mathrm{M})$.

Proof. — Let us first observe that for any ray σ starting at the base point o in (H.1),

$$(5.1) \quad \text{the injectivity radius of M at } \sigma(t) \geqq \frac{a}{[\log(2+t)]^b},$$

where a, b are positive constants independent of σ. (5.1) can be verified by applying the argument of Cheng-Li-Yau [7], Theorem 1. Since Proposition 5.5 is obvious when $\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty)=0$, we assume that $\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty)$ is positive. Let us take a positive constant μ with $\mu<\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty)$. Given a positive constant ε, let $\{x_1, \ldots, x_n\}$ be finite points of $\mathrm{M}_\alpha(\infty)$ such that $\hat{\mathrm{B}}_\varepsilon(x_i) \cap \hat{\mathrm{B}}_\varepsilon(x_j)=\varnothing$ $(i \neq j)$, where $\hat{\mathrm{B}}_\varepsilon(x)$ denotes the metric ball of $\mathrm{M}_\alpha(\infty)$ centered at x with radius ε and further $\{x_1, \ldots, x_n\}$ is maximal among the finite points with the above property. Then $\mathrm{M}_\alpha(\infty)$ is covered by $\{\mathrm{B}_{2\varepsilon}(x_i)\}_{i=1, \ldots, n}$, and hence we have

$$n(4\varepsilon)^\mu \geqq \Sigma_{i=1}^\mu [\mathrm{diam}(\mathrm{B}_{2\varepsilon}(x_i))]^\mu \geqq \mathrm{C}_\mu \mathscr{H}^\mu_{4\varepsilon}(\mathrm{M}_\alpha(\infty)),$$

where C_μ is a constant depending only on μ. Since $\mu<\dim_{\mathrm{H}} \mathrm{M}_\alpha(\infty)$, $\lim_{\varepsilon \to 0} \mathscr{H}^\mu_{4\varepsilon}(\mathrm{M}_\alpha(\infty))=\mathscr{H}^\mu(\mathrm{M}_\alpha(\infty))=\infty$. This implies that $n \geqq \varepsilon^{-\mu}$ if $\varepsilon \leqq \varepsilon(\mu, \mathrm{M}_\alpha(\infty))$, where $\varepsilon(\mu, \mathrm{M}_\alpha(\infty))$ is a positive constant depending only on μ and $\mathrm{M}_\alpha(\infty)$. Now let us take $\varepsilon=1/k$ $(k=1, 2, \ldots)$ and let $\{x_{k,1}, \ldots, x_{k,n(k)}\}$ be finite points of $\mathrm{M}_\alpha(\infty)$ chosen as above. Set $p_{k,i}=\sigma_{k,i}(k)$, where $\{\sigma_{k,i}\}$ are rays emanating from 0 such that $[\sigma_{k,i}]=x_{k,i}$. Then we have a constant $c<1$ which satisfies

$$\frac{1}{k} \mathrm{dis}_{\mathrm{M}}(p_{k,i}, p_{k,j}) \geqq c\, \delta_\infty(x_{k,i}, x_{k,j})$$

for large k and any $i, j: 1 \leqq i, j \leqq n(k)$ [*cf.* 3.1 step 1; Proposition 2.3(i)]. This shows that $\mathrm{dis}_{\mathrm{M}}(p_{k,i}, p_{k,j}) \geqq 2c$ $(i \neq j)$ and hence we have

$$(5.2) \qquad \mathrm{B}_{c/2}(p_{k,i}) \cap \mathrm{B}_{c/2}(p_{k,j})=\varnothing \ (i \neq j).$$

Set $A_{\alpha,k}:=(B_{k+1/2}(o)\setminus B_{k-1/2}(o))\cap \mathscr{E}_\alpha(M)$. Then by (5.1), (5.2) and the assumption that the sectional curvature of M is bounded from above by a constant, say Λ^2, we have

$$\text{(5.3)}\quad \mathrm{Vol}_m(A_{\alpha,k})\geqq \sum_{i=1}^{n(k)} \mathrm{Vol}_m(B_{c/2}(p_{k,i}))$$

$$\geqq \omega_{m-1}\, n(k)\int_0^{c_k}\left[\frac{\sin \Lambda u}{\Lambda}\right]^{m-1} du\left(c_k:=\frac{a}{[\log(2+k)]^2}\right)$$

$$\geqq \omega_{m-1}\, k^{\mu}\int_0^{c_k}\left[\frac{\sin \Lambda u}{\Lambda}\right]^{m-1} du.$$

Then it turns out from (5.3) that

$$\liminf_{t\to\infty}\frac{\log \mathrm{Vol}_m(B_t(p)\cap \mathscr{E}_\alpha(M))}{\log t}$$

$$=\liminf_{t\to\infty}\frac{\log \mathrm{Vol}_m(B_t(o)\cap \mathscr{E}_\alpha(M))}{\log t}$$

$$\geqq \liminf_{t\to\infty}\frac{\log \displaystyle\int^{[t]} u^{\mu}[\log(2+u)]^{-bm}\,du}{\log t}\geqq 1+\mu.$$

Since μ is any constant less than $\dim_H M_\alpha(\infty)$, we get the required inequality. This completes the proof of Proposition 5.5.

Remarks. — (i) It is clear from Proposition 2.2 that $\dim_H M(\infty)\,(:=\max \dim_H M_\alpha(\infty))$ is less than or equal to $m-1$, without the additional condition that the curvature of M is bounded from above.

(ii) In Proposition 5.5, the equality does not hold in general (*cf.* Proposition 5.2).

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