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SOME EXAMPLES OF HYPERBOLIC EQUATIONS WITHOUT LOCAL SOLVABILITY

BY F. COLOMBINI AND S. SPAGNOLO

1. Introduction

In this paper we illustrate, by means of appropriate examples, various phenomena of lack of local solvability which occur in certain strictly hyperbolic linear equations of second order with non-smooth coefficients, and also in certain weakly hyperbolic equations with smooth coefficients.

As we shall see, the local solvability property may fail for equations of the simple form

$$u_{tt}-(\mathrm{A}(x, t)\,u_x)_x=f(x) \tag{1}$$

with

$$\begin{cases} \mathrm{A}(x, t)\in \mathrm{C}^{0,\alpha}(\mathbb{R}^2) & \text{for all } \alpha<1 \\ \lambda \leqq \mathrm{A}(x, t) \leqq \lambda^{-1} & \text{for some } \lambda>0, \end{cases} \tag{2}$$

and, in some sense, also for equations of the form

$$u_{tt}-a(t)\,u_{xx}=f(x) \qquad (a(t)\geqq \lambda). \tag{3}$$

When $\mathrm{A}(x, t)$ is Lipschitz continuous in t, the local solvability for the equation (1) (near each point of $\mathbb{R}^2$) is a direct consequence of the well-posedness of the corresponding Cauchy problem. The first result of the paper (Theorem 1) is the construction of a function $a(t)\geqq 1/2$, Hölder continuous of any exponent strictly less than one, and two functions of class C^∞ on $\mathbb{R}$, $u_0(x)$ and $u_1(x)$, for which the Cauchy problem

$$\begin{cases} u_{tt}-a(t)\,u_{xx}=0 \\ u(x, 0)=u_0(x), \qquad u_t(x, 0)=u_1(x) \end{cases} \tag{4}$$

has no distribution-solution $u(x, t)$ near any point of the initial line $\{t=0\}$.

([1]) Work partially supported by the G.N.A.F.A. (C.N.R.) and the Ministero P.I.

Secondly, in Theorem 2 we exhibit a 2π-periodic function of class C^∞, $f(x)$, for which the equation (3) [where $a(t)$ is the same as in Theorem 1] has no distribution-solution $u(x, t)$, 2π-periodic in x, on any strip $\{(x, t): |t| < r\}$.

Finally, in Theorem 3 we construct a function $A(x, t)$ satisfying (2), for which the equation (1), with $f(x)=x$, has no C^1 solution near the point (0, 0).

A common starting point for all these examples (as well as for the non-uniqueness examples of [3]) is a result, the Lemma 1 below, concerning ordinary differential equations which ensures the existence of smooth functions $\alpha_\varepsilon(\tau) \geqq 1/2$ such that

$$\alpha_\varepsilon(\tau) \to 1 \qquad \text{for} \quad \varepsilon \to 0$$

and that the initial value problem

$$\begin{cases} w'' + \alpha_\varepsilon(\tau)\, w = 0 \\ w(0)=1, \qquad w'(0)=0 \end{cases}$$

has a solution $w=w_\varepsilon(\tau)$, which decays exponentially for $|\tau| \to \infty$. Indeed, the coefficient $a(t)$ which appears in both Theorems 1 and 2 is constructed by means of the functions $\alpha_\varepsilon(\tau)$; more precisely, it has the form

$$a(t) = \alpha_{\varepsilon_k}(h_k(t-t_k)) \qquad \text{for} \quad t \in I_k,$$

where $\{\varepsilon_k\}$ and $\{h_k\}$ are two suitable sequences of positive numbers, converging respectively to zero and to infinity, while $\{I_k\}$ is a suitable sequence of real intervals with centers at t_k and converging to zero. As regards the coefficient $A(x, t)$ of Theorem 3, it is simply defined as

$$A(x, t) = \frac{a(t)}{a(x)}.$$

We conclude the Introduction with some short comments on Theorems 1, 2 and 3 (see also the Remarks following these Theorems).

Theorem 1 is a refinement of a previous example of [2] (which in turn improves an earlier result of [1]), where a Cauchy problem of type (4) was constructed for which no distribution-solution exists on any strip $\{|t| < \rho\}$. To appreciate better the difference between the result of [2] and Theorem 1 of the present paper, it should be noticed that problem (4), with Hölder continuous coefficient $a(t) \geqq 0$ and C^∞ initial data, has always a unique solution $u(x, t)$ in $C^2(\mathbb{R}_t, (\mathscr{D}^s)'(\mathbb{R}_x))$ where $(\mathscr{D}^s)'$ is some space of Gevrey ultradistributions $(s > 1)$. Now, the result of [2] quoted above consists in an example of a problem of the form (4) whose solution "blows-up as a distribution" at some undetermined point of the initial line $\{t=0\}$, while the solution constructed in Theorem 1 "blows-up as a distribution" everywhere on $\{t=0\}$.

Theorem 2 is again concerned with an equation of the form (3), but now, instead of assigning the initial conditions, we impose to the free term $f(x)$ and to the possible solution $u(x, t)$ a boundary type condition, namely the 2π-periodicity in x. From an abstract point of view, Theorem 2 can be read as follows: there exists a *Hilbert triplet*

(V, H, V′), a family of symmetric and coercive operators $A(t): V \to V'$. Hölder continuous in t, and an element $f_0 \in V'$, in such a way that the "hyperbolic" equation

$$u'' + A(t)\, u = f_0$$

has no local solution at $t=0$. In our example, V is the space of 2π-periodic H^1_{loc} functions on $\mathbb{R}_x$, H is $L^2(0, 2\pi)$ and $A(t) = -a(t)\,\partial_x^2$.

Theorem 3 is a genuine example of non local solvability (we emphasize that, in order to construct this kind of counter-example, we are forced to go out of the class of equations with coefficients depending only on time, such as (3), *cf.* Remark 1 after Theorem 3). Indeed, Theorem 3 says that for a suitable Hölder continuous and strictly positive function $a(\xi)$, the equation

$$u_{tt} - \left(\frac{a(t)}{a(x)} u_x\right)_x = x \tag{5}$$

has no C^1 solution $u(x, t)$ near the origine (0, 0).

We observe that, by putting

$$u = v_x \qquad \text{or} \qquad u_x = a(x)\, w,$$

we can transform (5) into

$$v_{tt} - \frac{a(t)}{a(x)} v_{xx} = \frac{x^2}{2} \tag{6}$$

or into

$$a(x)\, w_{tt} - a(t)\, w_{xx} = 1. \tag{7}$$

Hence, also the equations (6) and (7) have not the local solvability near the origine.

We also observe that the theory of local solvability of Hörmander, Nirenberg and Trèves (*see* [5], [6], [7]) does not apply to equations (5)-(7), owing to the non regularity of the coefficients of such equations.

Finally, we remark that, after some technical modifications, it is possible to prove a version of Theorems 1 and 2 concerning weakly hyperbolic equations with smooth coefficients; that is to say, there exists an equation like (3), with $a(t)$ of class C^∞ and $\geqq 0$, which presents the same phenomena where the local solvability fails, as illustrated by Theorems 1, 2. It would be interesting to prove an analogous version of Theorem 3, *i. e.* to construct a C^∞ function $A(x, t) \geqq 0$ such that equation (1) is not locally solvable.

NOTATIONS. — Given a real number $s > 1$, we denote by $\mathscr{E}^s(\mathbb{R})$ the space of the *Gevrey functions* of order s and by $\mathscr{D}^s(\mathbb{R})$ the subspace of the Gevrey functions having compact support. The dual space $(\mathscr{D}^s)'(\mathbb{R})$ is the space of the Gevrey *ultradistributions.*

2. Two O.D.E. lemmas

LEMMA 1 (*cf.* also [3]). — *For all* $\varepsilon \in]0, \bar{\varepsilon}[$, *it is possible to find two even real functions,* $\alpha_\varepsilon(\tau)$ *and* $w_\varepsilon(\tau)$, *of class* C^∞ *on* $\mathbb{R}$, *satisfying*

$$\begin{cases} w_\varepsilon'' + \alpha_\varepsilon(\tau)\, w_\varepsilon = 0 \\ w_\varepsilon(0) = 1, \qquad w_\varepsilon'(0) = 0 \end{cases} \tag{8}$$

in such a way that

(9) $\alpha_\varepsilon(\tau)$ *is* 2π-*periodic on* $\{\tau > 0\}$ *and on* $\{\tau < 0\}$

(10) $\alpha_\varepsilon(\tau) \equiv 1$ *in a neighbourhood of* $\tau = 0$

$$|\alpha_\varepsilon(\tau) - 1| \leqq M\varepsilon, \qquad |\alpha_\varepsilon'(\tau)| \leqq M\varepsilon \tag{11}$$

$$\begin{cases} w_\varepsilon(\tau) = p_\varepsilon(\tau)\, e^{-\varepsilon|\tau|} \\ \text{for some } p_\varepsilon(\tau)\ 2\pi\text{-periodic on } \{\tau > 0\} \text{ and on } \{\tau < 0\} \end{cases} \tag{12}$$

$$|w_\varepsilon| + |w_\varepsilon'| + |w_\varepsilon''| \leqq C \tag{13}$$

$$\int_0^{2\pi} w_\varepsilon\, d\tau \geqq \gamma\varepsilon \qquad (\gamma > 0) \tag{14}$$

where M, C *and* γ *are constants independent on* ε.

N.B.: As a consequence of (12) and (8), we have in particular, for all integers $\nu \geqq 0$,

$$w_\varepsilon(\tau) = e^{-\varepsilon|\tau|}, \qquad w_\varepsilon'(\tau) = 0, \qquad w_\varepsilon''(\tau) = -e^{-\varepsilon|\tau|} \quad \text{for } \tau = \pm 2\pi\nu \tag{15}$$

Proof. — Fixed a 2π-periodic function $\rho(\tau) \geqq 0$, of class C^∞, vanishing in a neighbourhood of $\tau = 0$ and satisfying the conditions

$$\int_0^{2\pi} \rho(\tau) \cos^2\tau\, d\tau = \pi \tag{16}$$

$$\int_0^{2\pi} \rho(\tau) \cos^2\tau \sin\tau\, d\tau > 0, \tag{17}$$

we define, for $\tau \geqq 0$

$$\alpha_\varepsilon(\tau) = 1 - 4\varepsilon\rho(\tau)\sin 2\tau + 2\varepsilon\rho'(\tau)\cos^2\tau - 4\varepsilon^2\rho^2(\tau)\cos^4\tau$$

$$w_\varepsilon(\tau) = \cos\tau \,.\, \exp\left[-2\varepsilon \int_0^\tau \rho(s)\cos^2 s\, ds\right]$$

and, for $\tau < 0$,

$$\alpha_\varepsilon(\tau) = \alpha_\varepsilon(-\tau), \qquad w_\varepsilon(\tau) = w_\varepsilon(-\tau).$$

Clearly $\alpha_\varepsilon(\tau) \equiv 1$ and $w_\varepsilon(\tau) \equiv \cos\tau$ near the origine, hence α_ε and w_ε are of class C^∞ on $\mathbb{R}$. Moreover, it is easily to check that (8), (11) and (13) are fulfilled, while the periodicity properties (9) and (12) are an easy consequence of (16). As to the lower estimate (14), this follows from (17); indeed

$$\left[\frac{d}{d\varepsilon}\int_0^{2\pi} w_\varepsilon \, d\tau\right]_{\varepsilon=0} = \int_0^{2\pi} \rho(\tau)\cos^2\tau\sin\tau\, d\tau > 0. \quad \square$$

LEMMA 2. — *Let $\varphi(t)$ be a solution of the equation*

$$\varphi'' + h^2 a(t)\varphi = 0 \qquad (t \in \mathbb{R})$$

where $h \in \mathbb{Z}$ and $a(t)$ is a strictly positive function of class C^1, and let us consider the "energy functions"

$$E_\varphi(t) = h^2 |\varphi(t)|^2 + |\varphi'(t)|^2$$
$$\tilde{E}_\varphi(t) = h^2 a(t) |\varphi(t)|^2 + |\varphi'(t)|^2.$$

Then, for all t_1 and t_2, the following estimates hold

(18) $$E_\varphi(t_2) \leqq E_\varphi(t_1) . \exp\left| h \int_{t_1}^{t_2} |1 - a(t)|\, dt \right|$$

(19) $$\tilde{E}_\varphi(t_2) \leqq \tilde{E}_\varphi(t_1) . \exp\left| \int_{t_1}^{t_2} \frac{|a'(t)|}{a(t)}\, dt \right|.$$

Proof. — It is sufficient to differentiate the energy functions and then apply the Gronwall's lemma. $\square$

3. The main results

THEOREM 1. — *There exists a function $a(t)$ such that*

(20) $$\frac{1}{2} \leqq a(t) \leqq \frac{3}{2}$$
$$a(t) \in C^{0,\alpha}(\mathbb{R}) \quad \textit{for all } \alpha < 1$$

and two C^∞ functions $u_0(x)$, $u_1(x)$ for which the Cauchy problem

(21) $$\begin{cases} u_{tt} - a(t) u_{xx} = 0 \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \end{cases}$$

has no solution $u(x, t)$ in $C^2([-r, r], \mathscr{D}'(]\bar{x}-r, \bar{x}+r[))$, for any $\bar{x} \in \mathbb{R}$ and $r > 0$.

Proof. — We firstly define, by the aid of Lemma 1, the function $a(t)$. To this purpose, let us consider the sequences

(22) $$\rho_k = 4\pi . 2^{-k}, \; h_k = 2^{2^{Nk}}, \qquad \varepsilon_k = h_k^{-1} . (\log h_k)^3$$

where N is a fixed integer > 1 so large [with respect to the constant M of (11)] that the following inequalities hold for every integer $k \geqq 1$:

$$\varepsilon_k \leqq \frac{1}{2M} \tag{23}$$

$$4M.\sum_{j=1}^{k-1} \varepsilon_j h_j \rho_j \leqq \varepsilon_k h_k \rho_k \tag{24}$$

$$2M.\sum_{j=k+1}^{\infty} \varepsilon_j \rho_j \leqq \varepsilon_k \rho_k \tag{25}$$

(in order to make clear that such inequalities are true for large N, we observe that $\{\varepsilon_k\}$ is a decreasing sequence for $N \geqq 2$ and ε_1 tends to zero for $N \to \infty$, moreover $\varepsilon_j h_j \rho_j (\varepsilon_k h_k \rho_k)^{-1} = (A)_N^{j-k}$ with A_N converging to ∞ for $N \to \infty$; finally we have $\varepsilon_j \rho_j (\varepsilon_k \rho_k)^{-1} \leqq \delta_N^{j-k}$ for some δ_N converging to zero).

Then, we put

$$t_k = \frac{\rho_k}{2} + \sum_{j=k+1}^{\infty} \rho_j \tag{26}$$

and

$$I_k = \left[t_k - \frac{\rho_k}{2},\, t_k + \frac{\rho_k}{2} \right], \tag{27}$$

so that the intervals I_k and I_{k+1} are contiguous and $\{I_k\} \to \{0\}$ for $k \to \infty$, and we define

$$a(t) = \begin{cases} \alpha_{\varepsilon_k}(h_k(t-t_k)) & \text{for } t \in I_k \\ 1 & \text{for } t \in \mathbb{R} \setminus \left(\bigcup_{j=1}^{\infty} I_j \right) \end{cases}$$

where $\alpha_\varepsilon(\tau)$ are the functions introduced in Lemma 1.

Let us notice that $h_k \varepsilon_k (4\pi)^{-1}$ is an integer, so that, by (9)-(10), we have

$$a(t) \equiv 1 \text{ in a neighbourhood of } t_k \pm \frac{\rho_k}{2}. \tag{28}$$

On the other hand, (11) implies that

$$|1 - a(t)| \leqq M \varepsilon_k \quad \text{in } I_k \tag{29}$$

thus, by (23), we see that $1/2 \leqq a(t) \leqq 3/2$.

Finally, from the estimate

$$|a_{\varepsilon_k}|_{C^{0,\alpha}(\mathbb{R})} \leqq M \varepsilon_k (2\pi)^{1-\alpha} \qquad (0 < \alpha \leqq 1),$$

which follows from (9) and the second of (11), we derive that

$$|a|_{C^{0,\alpha}(I_k)} \leqq M\,\varepsilon_k (2\pi)^{1-\alpha} h_k^\alpha \qquad (0 < a \leqq 1)$$

(where $|\,.\,|$ denotes here the Hölder semi-norm). Now, by (22), we have

$$\sup_k \varepsilon_k h_k^\alpha < \infty \qquad \text{for all} \quad \alpha < 1,$$

hence, taking (28) into account, we conclude that $a(t) \in C^{0,\alpha}(\mathbb{R})$ for all $\alpha < 1$.

Let us now define the initial data $u_i(x)$ of problem (21), by setting

$$u_0(x) = \bar{u}(x, 0), \qquad u_1(x) = \bar{u}_t(x, 0), \tag{30}$$

where

$$\bar{u}(x, t) = \sum_{k=1}^{\infty} \varphi_k(t)\, e^{i h_k x}, \tag{31}$$

$\varphi_k(t)$ being the solution of

$$\begin{cases} \varphi_k'' + h_k^2\, a(t)\, \varphi_k = 0 \\ \varphi_k(t_k) = \eta_k, \qquad \varphi_k'(t_k) = 0 \end{cases} \tag{32}$$

with

$$\eta_k = e^{(\log h_k)^2}. \tag{33}$$

Thus, $\bar{u}(x, t)$ is a (formal) solution of the equation (21), *i. e.*

$$\bar{u}_{tt} - a(t)\, \bar{u}_{xx} = 0. \tag{34}$$

The rest of the proof will consist in showing that $\bar{u}$ satisfies, in addition to (34), the following properties:

(35) $\bar{u} \in C^2(\mathbb{R}_t, (\mathscr{D}^s)'(\mathbb{R}_x)), \qquad \forall s > 1$

(36) $\bar{u}(.\,, 0)$ and $\bar{u}_t(.\,, 0)$ are C^∞ functions on $\mathbb{R}_x$

(37) $\bar{u}(x, t)$ is a C^∞ function on $\mathbb{R}^2 \setminus \{t=0\}$

(38) $\bar{u} \notin \mathscr{D}'(Q(\bar{x}, r)), \qquad \forall \bar{x} \in \mathbb{R}, \quad \forall r > 0$

where we put

$$Q(\bar{x}, r) =]\bar{x}-r, \bar{x}+r[\times]-r, r[.$$

The conclusion of Theorem 1 is a direct consequence of the previous properties, more exactly of (35), (36) and (38) ((37) will only be used to prove the Remark subsequent this Theorem). Indeed, if problem (21) had a solution u which belongs to $C^2([-r, r], \mathscr{D}'(]\bar{x}-r, \bar{x}+r[))$ for some $\bar{x}$ and $r > 0$, then by an uniqueness result of [1]

(Theorem 6) such a solution should coincide with $\bar{u}$ on a neighbourhood of $(\bar{x}, 0)$, in contradiction with (38).

Proof of (35). — In view of the Paley-Wiener's theorem for the Fourier series, we'll estimate the growth of $\varphi_k(t)$ [*see* (32)] for $k \to \infty$, by the aid of Lemma 2.

Firstly, we estimate, for $t \leqq t_k$, the energy function

$$E_{\varphi_k}(t) = h_k^2 \,|\varphi_k(t)|^2 + |\varphi'_k(t)|^2$$

and we get, by (18), (29), (25), (33) and (22),

$$\begin{aligned} E_{\varphi_k}(t) &\leqq h_k^2 \,.\, \eta_k^2 \,.\, \exp\left[M\, h_k\left(\varepsilon_k \frac{\rho_k}{2} + \sum_{j=k+1}^{\infty} \varepsilon_j \rho_j\right)\right] \\ &\leqq h_k^2 \,.\, \eta_k^2 \,.\, \exp(C\, h_k\, \varepsilon_k\, \rho_k) \\ &\leqq h_k^2 \,.\, \exp[2(\log h_k)^2 + \tilde{C}(\log h_k)^3] \\ &\leqq \exp(C_s\, h_k^{1/s}) \end{aligned}$$

for all $s \geqq 1$ and $t \leqq t_k$.

Then, we estimate the second energy function

$$\tilde{E}_{\varphi_k}(t) = h_k^2\, a(t)\, |\varphi_k(t)|^2 + |\varphi'_k(t)|^2,$$

using (19), (28), (24) and the inequalities $|a'(t)| \leqq M\varepsilon_k$ on I_k, $a(t) \geqq 1/2$, and we obtain

$$\begin{aligned} \tilde{E}_{\varphi_k}(t) &\leqq h_k^2 \,.\, \eta_k^2 \,.\, \exp\left[2M\left(\sum_{j=1}^{k-1} h_j \varepsilon_j \rho_j + h_k \varepsilon_k \frac{\rho_k}{2}\right)\right] \\ &\leqq \exp(C_s\, h_k^{1/s}) \end{aligned}$$

for all $s \geqq 1$ and $t \geqq t_k$.

In conclusion we have, for all $t \in \mathbb{R}$ and all $s \geqq 1$,

$$|\varphi_k(t)| + |\varphi'_k(t)| \leqq \exp(C_s\, h_k^{1/s})$$

thus, taking (32) into account, we find (35).

Proof of (36) *and* (37). — By (8) and the definition of $a(t)$, we can explicitely write the Fourier coefficients $\varphi_k(t)$ of $\bar{u}(x, t)$ [*see* (31) and (32)] as

$$\varphi_k(t) = w_{\varepsilon_k}(h_k(t - t_k)) \qquad (t \in I_k)$$

where $w_\varepsilon(\tau)$ are the functions defined in Lemma 1. Therefore, noticing that $h_k \rho_k (4\pi)^{-1}$ are integers, we get by (15) and (28) the equalities

$$E_{\varphi_k}\left(t_k \pm \frac{\rho_k}{2}\right) = \tilde{E}_{\varphi_k}\left(t_k \pm \frac{\rho_k}{2}\right) = h_k^2\, \eta_k^2 \exp(-h_k \varepsilon_k \rho_k).$$

Proceeding as above, we then find, for $t \leqq t_k - \rho_k/2$,

$$E_{\varphi_k}(t) \leqq h_k^2 . \eta_k^2 . \exp(-h_k \varepsilon_k \rho_k) . \exp\left(M h_k \sum_{j=k+1}^{\infty} \varepsilon_j \rho_j\right)$$

while for $t \geqq t_k + (\rho_k/2)$,

$$\tilde{E}_{\varphi_k}(t) \leqq h_k^2 . \eta_k^2 . \exp(-h_k \varepsilon_k \rho_k) . \exp\left(2 M \sum_{j=1}^{k-1} h_j \varepsilon_j \rho_j\right),$$

and hence, by (24), (25), (33) and (22),

$$|\varphi_k(t)| + |\varphi_k'(t)| \leqq h_k^2 . \eta_k^2 . \exp\left(-\frac{1}{2} h_k \varepsilon_k \rho_k\right)$$
$$\leqq h_k^2 . \exp[2(\log(h_k)^2 - 2\pi 2^{-k}(\log h_k)^3]$$

i. e.

(39) $$|\varphi_k(t)| + |\varphi_k'(t)| \leqq C_p h_k^{-p}, \qquad \forall p > 0, \quad \forall t \in \mathbb{R} \setminus I_k.$$

The last estimate ensure that

$$\bar{u} \in C^2(]0, +\infty[; C^\infty(\mathbb{R}_x)) \cap C^2(]-\infty, 0[; C^\infty(\mathbb{R}_x))$$

and hence (36). To obtain (37), we must only observe, in addition, that $\bar{u}_{tt} = a(t)\bar{u}_{xx}$ with $a(t)$ of class C^∞ outside $\{t=0\}$.

Proof of (38). — It is immediate at this point to conclude that

$$\bar{u} \notin C([0, r], \mathscr{D}'(\mathbb{R}_x)), \qquad \forall r > 0;$$

indeed, for $k \to \infty$, we have [by (31), (32) and (33)]

$$\int_0^{2\pi} \bar{u}(x, t_k) . \frac{e^{-ih_k x}}{\sqrt{\eta_k}} dx = \sqrt{\eta_k} \to \infty$$

whereas

$$\frac{e^{-ih_k x}}{\sqrt{\eta_k}} \to 0 \quad \text{in } C^\infty([0, 2\pi]).$$

It would also be easy to show that

(40) $$\bar{u} \notin \mathscr{D}'(I(x, \pi) \times I(0, \rho)), \qquad \forall x, \forall \rho \ (^2).$$

([2]) Here and in the following, we shall briefly denote by $I(\xi, \rho)$ the open real interval with center ξ and radius ρ.

Actually (38) is stronger than (40), and we shall prove it by a *duality argument*.

Let us fix $\bar{x} \in \mathbb{R}_x$, $r > 0$ and a 2π-periodic Gevrey function $\chi(x)$ such that

$$\int_0^{2\pi} \chi(x)\, dx = 1 \tag{41}$$

$$\operatorname{supp}(\chi) \cap \mathrm{I}(\bar{x}, \pi) \subseteqq \mathrm{I}(\bar{x}, r/4), \tag{42}$$

and let us consider the "dual problem"

$$\begin{cases} (v_k)_{tt} - a(t)(v_k)_{xx} = 0 \\ v_k(x, t_k) = 0, \qquad (v_k)_t(x, t_k) = \chi_k(x) \end{cases} \tag{43}$$

where $k = 1, 2, 3, \ldots$ and

$$\chi_k(x) = \chi(x)\, e^{-ih_k x}. \tag{44}$$

We observe that the function χ_k belongs to $\mathscr{E}^{\bar{s}}(\mathbb{R}_x)$ for some $\bar{s}$, while $a(t)$ belongs to $C^{0,\alpha}$ for all $\alpha < 1$; thus, by an existence theorem of [1] (Theorem 4,ii), problem (43) is uniquely solvable in $C^2(\mathbb{R}_t, \mathscr{E}^{\bar{s}}(\mathbb{R}_x))$. Moreover, by the finite speed of propagation property, we have, for $k \geqq \bar{k}(r)$,

$$\operatorname{supp}(v_k(t, .)) \cap \mathrm{I}(\bar{x}, \pi) \subseteqq \mathrm{I}\left(\bar{x}, \frac{r}{2}\right) \qquad \text{for} \quad t \in \mathrm{I}_k. \tag{45}$$

Let us now take, for each k, a C^∞ function $\theta_k(t)$ such that

$$\theta_k(t) = \begin{cases} 0 & \text{for} \quad t \leqq t_k - \dfrac{\rho_k}{2} \\ 1 & \text{for} \quad t \geqq t_k \end{cases} \tag{46}$$

$$|\theta_k^{(p)}(t)| \leqq C_p \rho_k^{-p} \quad \text{for all } p \in \mathbb{N}, \tag{47}$$

and let us multiply each term of the equation (34) by $\theta_k(t) v_k(x, t)$ and each term of the equation (43) by $\theta_k(t) \bar{u}(x, t)$. By integrating on the square $Q(\bar{x}, r) = \mathrm{I}(\bar{x}, r) \times \mathrm{I}(0, r)$ and taking into account the initial conditions in (43), we then find (after some computation), for $k \geqq \bar{k}(r)$,

$$\iint_{Q(\bar{x}, r)} \bar{u} \,.\, (2(v_k)_t \theta_k' + v_k \theta_k'')\, dx\, dt = \int_{\mathrm{I}(\bar{x}, \pi)} \bar{u}(x, t_k)\, \chi_k(x)\, dx. \tag{48}$$

We shall see that, if $\bar{u}(x, t)$ is a distribution on $Q(\bar{x}, r)$, (48) is impossible for k large. To this end, we estimate the right hand term of (48), using (31), (32), (44) and

(39) (for $p=1$):

$$\int_{I(\bar{x},\pi)} \bar{u}(x, t_k)\chi_k(x)\,dx = \sum_{j \geqq 1} \varphi_j(t_k) \int_{I(\bar{x},\pi)} \chi(x) e^{i(h_j-h_k)x}\,dx$$

$$= \eta_k + \sum_{\substack{j \geqq 1 \\ j \neq k}} \varphi_j(t_k) \int_{I(\bar{x},\pi)} \chi(x) e^{i(h_j-h_k)x}\,dx$$

$$\geqq \eta_k - \sum_{j \geqq 1} C_1 h_j^{-1} \int_{I(\bar{x},\pi)} |\chi(x)|\,dx$$

$$\geqq \eta_k - \tilde{C}_1.$$

Hence, going back to (48), we conclude that

$$\frac{1}{\sqrt{\eta_k}} \iint_{Q(\bar{x},r)} \bar{u}.\, w_k\,dx\,dt \to +\infty \qquad (\text{as } k \to \infty) \tag{49}$$

where we have put, for brevity,

$$w_k = 2(v_k)_t\,\theta'_k + v_k\,\theta''_k. \tag{50}$$

Now, we estimate the growth of w_k as $k \to \infty$ by using the inequalities

$$|\partial_x^q \partial_t^p v_k(x, t)| \leqq C_{p,q} h_k^{p+q-1} \qquad (p, q \in \mathbb{N}) \tag{51}$$

which follow from the fact that v_k is the solution of a problem like (43), in which

$$|a^{(p)}(t)| \leqq C_p h_k^p \qquad \text{for} \quad t \in I_k$$
$$|\chi_k^{(q)}(x)| \leqq C_q h_k^q.$$

Introducing (51) and (47) [we remark that $\rho_k^{-1} \leqq h_k$ by (22)] in (50), noticing that the functions $w_k(x, t)$ have equi-compact supports in $Q(\bar{x}, r)$ [by (45) and (46)] and remembering that $\eta_k = \exp[(\log h_k)^2]$, we then find

$$\left\{\frac{1}{\sqrt{\eta_k}} w_k\right\} \to 0 \quad \text{in } \mathcal{D}(Q(\bar{x}, r)) \qquad (\text{as } k \to \infty). \tag{52}$$

Clearly, (49) and (52) exclude that $\bar{u}(x, t)$ can be a distribution on the square $Q(\bar{x}, r)$, so that, by the arbitrariness of $\bar{x}$ and r, (38) is proved.

This completes the proof of Theorem 1. □

Remark. — In point of fact, we have proved a stronger result than the one claimed in Theorem 1, namely that there exists a function $a(t)$ satisfying (20) such that there is a solution $\bar{u}(x, t)$ of the equation (21), belonging to $C^2(\mathbb{R}_t, (\mathcal{D}^s)'(\mathbb{R}_x))$ for all $s > 1$, which is a C^∞ function outside the line $\{t=0\}$ but not a distribution near any point of this line.

Thus, in particular, we have

$$\mathcal{D}'\text{-sing supp}\,(\bar{u})=\{t=0\}.$$

THEOREM 2. — *There exists a function* $a(t)$ *satisfying* (20) *and a* 2π*-periodic function* $f(x)$ *of class* C^∞, *for which the equation*

$$u_{tt}-a(t)\,u_{xx}=f(x) \tag{53}$$

has no solution $u(x, t)$ *in* $C^2([-r, r], \mathcal{D}'_{2\pi}(\mathbb{R}_x))$, *for any* $r>0$ (*where* $\mathcal{D}'_{2\pi}$ *is the space of* 2π*-periodic distributions*).

Proof. – Let us take the parameters ρ_k, h_k, ε_k and the function $a(t)$ just as in the beginning of the proof of Theorem 1 [*see* (22)], hence in particular

$$a(t)=\alpha_{\varepsilon_k}(h_k(t-t_k)) \qquad \text{for} \quad t\in I_k$$

where $\alpha_\varepsilon(\tau)$ are the functions of Lemma 1 and $I_k=[t_k-\rho_k/2,\ t_k+\rho_k/2]$.

Then, let us define the function $f(x)$ as

$$f(x)=\sum_{h=-\infty}^{+\infty} c_h\,.\,e^{ihx}$$

with

$$c_h=e^{-(\log h)^2} \tag{54}$$

(we observe that $f(x)$ is a C^∞, but not a Gevrey function).

Assume, by contradiction, that equation (53) has a solution $u(x, t)$ belonging to $C^2([-r, r], \mathcal{D}'_{2\pi}(\mathbb{R}_x))$ for some $r>0$. Therefore, we can write

$$u(x, t)=\sum_{h=-\infty}^{+\infty} \varphi_h(t)\,e^{ihx}$$

for some φ_h satisfying

$$\varphi''_h+a(t)\,h^2\,\varphi_h=c_h \tag{55}$$

and

$$|\varphi_h(t)|+|\varphi'_h(t)|\leqq C_0\,h^m, \quad \text{for some } C_0 \text{ and } m. \tag{56}$$

Let us now introduce the functions

$$\psi_k(t)=w_{\varepsilon_k}(h_k(t-t_k)), \tag{57}$$

where the $w_\varepsilon(\tau)$'s are the functions appearing in Lemma 1, and let us observe that [by (8)]

$$\psi''_k+a(t)\,h_k^2\,\psi_k=0 \quad \text{in } I_k. \tag{58}$$

Noticing that

$$h_k \frac{\rho_k}{2} = 2\pi\nu_k, \quad \text{for some } \nu_k \in \mathbb{N},$$

we see that

$$\psi_k\left(t_k \pm \frac{\rho_k}{2}\right) = e^{-\varepsilon_k h_k \rho_k/2}, \qquad \psi_k'\left(t_k \pm \frac{\rho_k}{2}\right) = 0. \tag{59}$$

Moreover, by (14), we have the estimate

$$\int_{I_k} \psi_k(t)\,dt \geqq 2\gamma\varepsilon_k h_k^{-1} \qquad (\gamma > 0). \tag{60}$$

Such an estimate can be derived as follows:

$$\begin{aligned} \int_{I_k} \psi_k(t)\,dt &= h_k^{-1} \int_{-2\pi\nu_k}^{2\pi\nu_k} w_{\varepsilon_k}(\tau)\,d\tau \\ &= 2h_k^{-1} \int_0^{2\pi\nu_k} w_{\varepsilon_k}\,d\tau \\ &= 2h_k^{-1} \sum_{j=0}^{\nu_k-1} \int_{2\pi j}^{2\pi(j+1)} w_{\varepsilon_k}\,d\tau \\ &= 2h_k^{-1} . \int_0^{2\pi} w_{\varepsilon_k}\,d\tau . \sum_{j=0}^{\nu_k-1} e^{-\varepsilon_k 2\pi j} \\ &\geqq 2h_k^{-1}\gamma\varepsilon_k, \end{aligned}$$

where we have used the eveness of $w_{\varepsilon_k}(\tau)$ and the equality

$$\int_{2\pi j}^{2\pi(j+1)} w_{\varepsilon_k}(\tau)\,d\tau = e^{-\varepsilon_k 2\pi j} \int_0^{2\pi} w_{\varepsilon_k}(\tau)\,d\tau,$$

which is a consequence of (12).

Putting in duality equation (55) with (58), we then find the equality

$$[\psi_k \varphi_{h_k}' - \psi_k' \varphi_{h_k}]_{t=t_k-\rho_k/2}^{t=t_k+\rho_k/2} = c_{h_k} \int_{I_k} \psi_k\,dt,$$

and hence, by (56), (59), (54) and (60), the inequality

$$C_0 h_k^m e^{-\varepsilon_k h_k \rho_k/2} \geqq e^{-(\log h_k)^2}\, 2\gamma\varepsilon_k h_k^{-1},$$

which becomes false for $k \to \infty$, by our definition of ε_k, h_k and ρ_k [*see* (22)].

This completes the proof of Theorem 2. □

THEOREM 3. — *There exists a function* $A(x, t)$ *such that*

$$A(x, t) \in C^{0,\alpha}(\mathbb{R}_x \times \mathbb{R}_t), \quad \textit{for all } \alpha < 1, \qquad (61)$$
$$\frac{1}{3} \leqq A(x, t) \leqq 3$$

for which the equation

$$u_{tt} - (A(x, t) u_x)_x = x \qquad (62)$$

has no C^1*-solution* $u(x, t)$ *on any neighbourhood of* $(0, 0)$.

Proof. — Fixed ρ_k, h_k, ε_k, I_k and $a(t)$ as in the proof of Theorem 1, let us define

$$A(x, t) = \frac{a(t)}{a(x)}. \qquad (63)$$

Then, let us introduce the functions

$$v_k(x, t) = \psi_k'(x) \psi_k(t) \qquad (64)$$

where the ψ_k's are defined as in the proof of Theorem 2 [*see* (57)]. As a direct consequence of (58), we see that $v_k(x, t)$ solves the equation

$$(v_k)_{tt} - (A(x, t)(v_k)_x)_x = 0 \quad \text{in } Q_k \qquad (65)$$

where

$$Q_k = I_k \times I_k \qquad (66)$$

(*i. e.* Q_k is the square of $\mathbb{R}_x \times \mathbb{R}_t$ with center at the point $(x, t) = (t_k, t_k)$ and side equal to ρ_k).

Let us now assume, by contradiction, that there exists a C^1-solution $u(x, t)$ of the equation (62) on some neighbourhood W of (0, 0); by pairing equation (62) with (65), we then find that, for k large with respect to W,

$$\int_{\partial Q_k} [(u_t v_k - u(v_k)_t) \nu_t - A(x, t)(u_x v_k - u(v_k)_x) \nu_x] \, d\sigma = \iint_{Q_k} x v_k \, dx \, dt \qquad (67)$$

where (ν_t, ν_x) is the exterior normal to ∂Q_k.

We shall prove that (67) becomes false for $k \to \infty$. In fact, from the definition of ψ_k [*see* (57)] and the properties (13) and (15) of w_ε, we derive that

$$\begin{cases} |\psi_k^{(j)}| \leqq C h_k^j & \text{in } I_k \\ |\psi_k^{(j)}| \leqq h_k^j e^{-\varepsilon_k h_k \rho_k/2} & \text{on } \partial I_k \end{cases} \qquad (j = 0, 1, 2) \qquad (68)$$

and hence, by (64),

$$|v_k| + |(v_k)_t| + |(v_k)_x| \leqq C_1 . h_k^2 e^{-\varepsilon_k h_k \rho_k/2} \quad \text{on } \partial Q_k.$$

If we introduce the last estimate in the left hand term of (67) and we take into account that $u \in C^1(W)$ and $W \supset Q_k$ (for k large), we get

$$\left| \iint_{Q_k} x v_k \, dx \, dt \right| \leqq C_2 h_k^2 e^{-\varepsilon_k h_k \rho_k/2}. \tag{69}$$

On the other hand, by (64) we have

$$\left| \iint_{Q_k} x v_k \, dx \, dt \right| = - \iint_{Q_k} \psi_k(x) \psi_k(t) \, dx \, dt + \int_{\partial Q_k} x \psi_k(x) \psi_k(t) \, d\sigma,$$

and, by (60) and (68) respectively,

$$\iint_{Q_k} \psi_k(x) \psi_k(t) \, dx \, dt = \left(\int_{I_k} \psi_k(\xi) \, d\xi \right)^2 \geqq 4 \gamma^2 \varepsilon_k^2 h_k^{-2}$$

$$\int_{\partial Q_k} x \psi_k(x) \psi_k(t) \, d\sigma \leqq C_3 e^{-\varepsilon_k h_k \rho_k/2}.$$

Thus, going back to (69), we find the inequality

$$\gamma^2 \varepsilon_k^2 h_k^{-2} \leqq C_4 h_k^2 e^{-\varepsilon_k h_k \rho_k/2}$$

which is false for k large, after our choice of the parameters [*see* (22), where $N > 1$].

This concludes the proof of Theorem 3. □

Remark 1. — The coefficient $A(x, t)$ in equation (62) is a Hölder continuous function of any exponent $\alpha < 1$; this is, in some sense, the best regularity allowed in order to have an example of non-local solvability for an equation such as

$$u_{tt} - (A(x, t) u_x)_x = f(x, t), \tag{70}$$

under the hyperbolicity condition $A(x, t) \geqq \nu > 0$.

In fact, if $A(x, t)$ is Lipschitz continuous near the origine with respect to one of its variables, then (70) is locally solvable at (0, 0). This is obvious if A is Lipschitz continuous in t, since in such a case we can solve the Cauchy problem for (70) assigning the initial data at $t=0$. On the other side, if A is Lipschitz continuous in x, we can solve the Cauchy problem

$$\left\{ \begin{array}{l} v_{xx} - \left(\dfrac{v_t}{A} \right)_t = - \left(\dfrac{g}{A} \right)_t \\ v = v_0(t), \qquad v_x = v_1(t) \quad \text{at } \{x=0\} \end{array} \right. \tag{71}$$

where $g(x, t)$ is any function such that $g_x = f$ and $v_0(t)$, $v_1(t)$ are arbitrarily taken in $C^\infty(\mathbb{R}_t)$ (we notice that (71) is uniquely solvable, since $1/A$ is Lipschitz continuous in x while g/A and $(g/A)_x$ are bounded near the origine). Now, a simple computation shows that the function

$$u(x, t) = \int_0^t v_x(x, \xi)\, d\xi + \psi(x)$$

is a solution of (70), provided that

$$\psi'(x) = \left. \frac{v_t + g}{A} \right|_{t=0}.$$

Remark 2. — By modifying the coefficient $A(x, t)$ of equation (62) outside of the set $\bigcup_{k=1}^{\infty} Q_k$, where Q_k are the squares introduced in the proof of Theorem 3 [*see* (66)], we can construct another equation like (62), say

$$u_{tt} - (\tilde{A}(x, t)\, u_x)_x = x, \tag{73}$$

where $\tilde{A}(x, t)$ is a function of class C^∞ on $\mathbb{R}^2 \setminus \{(0, 0)\}$ which satisfies again (61), in such a way that, given an arbitrary neighbourhood W of the origine, there is no distribution $u(x, t)$ in W which solves (73) in $W \setminus \{(0, 0)\}$ ([3]).

Clearly, such a result improves (slightly) Theorem 3.

Erratum. — In some preliminary announcements of the present paper (*Pitman Research Notes in Math.*, 158, 1987, pp. 202-219, and *Proceedings of the "Saint Jean de Monts journées"*, 1987, n° VIII), we stated Theorem 3 in a more general form, by claiming the existence of a positive function $A(x, t)$ of the form (63) for which the equation (70) is not locally solvable near the origin, not only for $f = x$ (as proved in theorem 3) but also for every f for which $f_x(0, 0) \neq 0$. Unfortunately, this result is false in this generality.

REFERENCES

[1] F. COLOMBINI, E. DE GIORGI and S. SPAGNOLO, *Sur les équations hyperboliques avec des coefficients qui ne dépendent que du temps* (*Ann. Sc. Norm. Sup. Pisa*, Vol. 6, 1979, pp. 511-559).

[2] F. COLOMBINI, E. JANNELLI and S. SPAGNOLO, *Well-Posedness in the Gevrey Classes of the Cauchy Problem for a Non-strictly Hyperbolic Equation with Coefficients Depending on Time* (*Ann. Sc. Norm. Sup. Pisa*, Vol. 10, 1983, pp. 291-312).

([3]) We observe that the coefficient $\tilde{A}(x, t)$ of equation (73), as well as the coefficient $A(x, t)$ of (62), cannot be a smooth function near the origine (*see* Remark 1); thus one cannot speak of *distribution-solution* on W.