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FRANK MORGAN

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LEAST-VOLUME REPRESENTATIVES OF HOMOLOGY CLASSES IN G (2, $\mathbf{C}^4$)

By FRANK MORGAN

1. Introduction

This paper finds a least-volume representative of every integer homology class in the Grassmannian $G(2, \mathbf{C}^4)$ of complex 2-planes in $\mathbf{C}^4$. Each degree d integral homology group H_d is given in Table 1.0.1 (*cf.* 2.2).

TABLE 1.0.1

The homology groups of $G(2, \mathbf{C}^4)$.

H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}^2$	$\{0\}$	$\mathbb{Z}$	$\{0\}$	$\mathbb{Z}$

The new case, H_4, has rank 2. It is generated by two $\mathbf{C}P^2 s$,

$$\mathbf{C}P^2 = G(2, \mathbf{C}^3) \qquad \text{and} \qquad \mathbf{C}P^{2'} = G(1, \mathbf{C}^3).$$

Both $\mathbf{C}P^2 s$ are volume-minimizing. So is the quaternionic projective space $\mathbf{H}P^1$ of the quaternionic lines in $\mathbf{C}^4 \cong \mathbf{H}^2$. Theorem 2.7 shows that every integral homology class has a least-volume representative that is a linear combination of some two of these three cycles. It is volume-minimizing because it is either complex or quaternionic.

1.1. COMPLEX AND QUATERNIONIC STRUCTURES. — $G(2, \mathbf{C}^4) \cong U_4/U_2 \times U_2$ has an invariant Kahler structure and two invariant quaternionic (or "quaternionic-Kahler") structures, or twice that number if you count their negatives, for a total of six. Equivalently, the tangent space $T_0 G(2, \mathbf{C}^4)$ admits a standard Kahler form ω and two standard quaternionic forms q, q' invariant under $U_2 \times U_2$. This occurs because the holonomy action of $U_2 \times U_2$ on $T_0 G(2, \mathbf{C}^4)$ is contained in certain actions of U_4, $Sp_1 \times Sp_2$ and $Sp_2 \times Sp_1$.

Actually, every $G(2, \mathbf{R}^n)$ has an invariant Kahler form, every $G(2, \mathbf{C}^n)$ has an invariant quaternionic form and every $G(n-2, \mathbf{C}^n)$ has an invariant quaternionic form. $G(2, \mathbf{C}^4)$, which happens to be isometric to $G(2, \mathbf{R}^6)$, falls into all three categories.

By Wirtinger's inequality, $\omega^2/2$ attains its maximum value of 1 on precisely the complex planes. Similarly, q and q' attain their maximum values of 1 on precisely the associated quaternionic lines. It follows from the fundamental theorem of calibrations that complex and quaternionic subvarieties are volume-minimizing.

1.2. THE FUNDAMENTAL THEOREM OF CALIBRATIONS. — *Let* S *be an m-dimensional surface (rectifiable current) in a smoooth, compact, n-dimensional Riemannian manifold. Let* φ *be a closed differential m-form such that for all unit m-planes* ξ,

$$\langle \xi, \varphi(x) \rangle \leqq 1,$$

with equality whenever ξ *is the oriented unit tangent to* S *at* x. *Then* S *is homologically volume-minimizing.*

Remark. — Any closed m-form φ, normalized so that its *comass*

$$\|\varphi\|^* = \sup\{\langle \xi, \varphi(x) \rangle : \xi \text{ is a unit } m\text{-plane}\} \tag{1}$$

equals 1, is called a *calibration.* The planes ξ on which φ attains the value 1, and any surface S with those tangent planes, are said to be *calibrated* by φ. In $G(2, \mathbf{C}^4)$, every 4-dimensional surface S with complex (or quaternionic) tangent planes is calibrated by $\omega^2/2$ (or q or q') and hence volume-minimizing.

Proof. — Let T be homologous to S. Then

$$\operatorname{vol} S = \int_S \varphi = \int_T \varphi \leqq \operatorname{vol} T.$$

Therefore S is homologically volume-minimizing.

1.3. MASS AND COMASS. — The comass norm 1.2 (1) on m-covectors and the dual mass norm on m-vectors play important roles in the theory of calibrations. This paper includes a complete description of these norms on $H^4 G(2, \mathbf{C}^4)$ and $H_4 G(2, \mathbf{C}^4)$ as Corollaries 2.9 and 2.8.

1.4. REFERENCES. — Surveys with historical remarks appear in [Mo1] and [Mo2]. For basic definitions and results, also see [Mo3] and [HL].

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2. Least volume representatives in $G(2, \mathbf{C}^4)$

2.1. DEFINITIONS. — Let $\mathbf{C}$ and $\mathbf{H} = \mathbf{C} + j\mathbf{C}$ denote the complex and quaternionic fields. $\mathbf{C}^4 = \mathbf{H}_1 \oplus \mathbf{H}_2$, $\mathbf{H}_1 = \mathbf{C}_1 \oplus j\mathbf{C}_1$, $\mathbf{H}_2 = \mathbf{C}_2 \oplus j\mathbf{C}_2$. The quaternionic-linear isometry of $\mathbf{H}_1 \oplus \mathbf{H}_2$ which switches $\mathbf{H}_1$ and $\mathbf{H}_2$ induces an isometry σ_0 of $G(2, \mathbf{C}^4)$. The Hodge $*$ operator gives another isometry of $G(2, \mathbf{C}^4)$. Let $\sigma = \sigma_0 \circ *$.

$G(2, \mathbf{C}^4) \cong U_4/U_2 \times U_2$. At the complex 2-plane ξ_0 in $\mathbf{H}_1$, the tangent space $T_0 G(2, \mathbf{C}^4) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$. A homomorphism $A \in \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$ is the tangent vector to the curve in $G(2, \mathbf{C}^4)$ given by graph (tA). Since σ leaves $\mathbf{H}_1$ invariant, it induces an isometry of $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$, namely, $A \to A^T$.

We will need to consider a number of cycles in $G(2, \mathbf{C}^4)$: the subgrassmannians

$$\begin{aligned} \mathbf{C}P^1 &= \{\xi : j\mathbf{C}_1 \subset \xi \subset \mathbf{H}_1 \oplus \mathbf{C}_2\} \cong G(1, \mathbf{C}^2), \\ \mathbf{C}P^2 &= G(2, \mathbf{C}^3) \quad \text{where} \quad \mathbf{C}^3 = \mathbf{H}_1 \oplus \mathbf{C}_2 \\ &= \{\xi : \xi \subset \mathbf{H}_1 \oplus \mathbf{C}_2\}, \\ \mathbf{C}P^{2'} &= \sigma(\mathbf{C}P^2) = \{\xi : j\mathbf{C}_1 \subset \xi\} \cong G(1, \mathbf{C}^3), \end{aligned}$$

quaternionic projective space

$$\mathbf{H}P^1 = G(1, \mathbf{H}_1 \oplus \mathbf{H}_2),$$

a product

$$\mathbf{C}P^1 \times \mathbf{C}P^1 = G(1, \mathbf{C}_1 \oplus \mathbf{C}_2) \times G(1, j\mathbf{C}_1 \oplus j\mathbf{C}_2);$$

and the Schubert cycle

$$C^6 = (\xi : \dim \xi \cap \mathbf{C}_1 \oplus \mathbf{C}_2 \geqq 1\}.$$

TABLE 2.1.1

Cycles in $G(2, \mathbf{C}^4)$ and their tangent spaces at ξ_0.

Cycle S	Real dimension	$T_0 S$
$\mathbf{C}P^1$	2	$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}_1, \mathbf{C}_2) = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}\right\}$
$\mathbf{C}P^2$	4	$\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{C}_2) = \left\{\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}\right\}$
$\mathbf{C}P^{2'}$	4	$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}_1, \mathbf{H}_2) = \left\{\begin{bmatrix} \alpha & 0 \\ \beta & 0 \end{bmatrix}\right\}$
$\mathbf{H}P^1$	4	$\mathrm{Hom}_{\mathbf{H}}(\mathbf{H}_1, \mathbf{H}_2) = \left\{\begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}\right\}$
$\mathbf{C}P^1 \times \mathbf{C}P^1$	4	$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}_1, \mathbf{C}_2) \oplus \mathrm{Hom}_{\mathbf{C}}(j\mathbf{C}_1, j\mathbf{C}_2) = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}\right\}$
C^6	6	$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}_1, \mathbf{C}_2) \oplus \mathrm{Hom}_{\mathbf{C}}(j\mathbf{C}_1, \mathbf{H}_2) = \left\{\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}\right\}$

Table 2.1.1 lists these cycles S and their unit tangents $T_0 S$ at ξ_0 in $T_0 G(2, \mathbf{C}^4) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$.

TABLE 2.2.1

The integral homology of $G(2, \mathbf{C}^4)$.

$H\,G(2, \mathbf{C}^4)$

Degree	Homology	Generators
0.	$\mathbf{Z}$	$\{\xi_0\}$
1.	$\{0\}$	
2.	$\mathbf{Z}$	$\mathbf{C}P^1$
3.	$\{0\}$	
4.	$\mathbf{Z}^2$	$\mathbf{C}P^2$, $\mathbf{C}P^{2'}$
5.	$\{0\}$	
6.	$\mathbf{Z}$	C^6
7.	$\{0\}$	
8.	$\mathbf{Z}$	$G(2, \mathbf{C}^4)$

2.2. PROPOSITION. — *The integral homology of* $G(2, \mathbf{C}^4)$ *is given by the table* 2.2.1.

Proof. — See Chern [C], § 8, p. 74, which deduces the homology of $G(2, \mathbf{C}^4)$ [which he calls Gr(3,1)] from a Schubert cell decomposition. The generators listed in our table are precisely his Schubert cycles (00), (01), (11), (02), (12), (22), respectively.

2.3. KAHLER AND QUATERNIONIC STRUCTURES. — As a complex vectorspace, $T_0 G(2, \mathbf{C}^4) = \mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$ has a Kahler form ω. Since ω is invariant under a $U_4 \supset U_2 \times U_2$, it extends to an invariant Kahler form on $G(2, \mathbf{C}^4)$ that exhibits the Kahler structure of $G(2, \mathbf{C}^4)$. $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$ is a quaternionic vectorspace in two different ways, by pre- or post-right-multiplication by quaternions α, *i.e.*, $(\alpha A)(x)$ is either $A(x\alpha)$ or $(A(x))\alpha$. (Left multiplication would not yield a complex-linear map). These multiplications yield two commuting actions of the imaginary unit quaternions $\cong SU_2$ on $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$.

Pre-right-multiplication by an imaginary unit quaternion u defines an orthogonal complex structure J_u on $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$, with associated Kahler form ω_u. By Wirtinger's inequality, for any real 4-plane ξ in $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$, $\langle \xi, \omega_u^2/2 \rangle \leqq 1$, with equality if and only if ξ is a complex 2-plane for the complex structure J_u. Define a quaternionic calibration

$$q = \operatorname*{ave}_u \omega_u^2/2,$$

which incidentally equals $(1/3)(\omega_i^2/2 + \omega_j^2/2 + \omega_k^2/2)$. Then $\langle \xi, q \rangle \leqq 1$, with equality if and only if ξ is a complex 2-plane for every complex structure J_u, *i.e.*, if and only if ξ is a quaternionic line under pre-right-multiplication by quaternions.

Similarly, for post-right multiplication by quaternions there is a second quaternionic calibration q'. Incidentally, $q' = \sigma^{\#} q$.

Both q and q' are invariant under both pre- and post- right-multiplication by quaternions, which give the standard representation of $SU_2 \times SU_2$. Of course they are also invariant under the standard action of U_1 on $\mathrm{Hom}_{\mathbf{C}}(\mathbf{H}_1, \mathbf{H}_2)$ as a complex vectorspace. Hence q and q' are invariant under $U_1(SU_2 \bullet SU_2) = U_2 \bullet U_2$. Therefore q and q' extend to U_4-invariant forms on G (2, $\mathbf{C}^4$) that exhibit two quaternionic (or "quaternionic-Kahler") structures on $G(2, \mathbf{C}^4)$.

2.4. Proposition. — *$\mathbf{C}\,P^1$, $\mathbf{C}\,P^2$, $\mathbf{C}\,P^{2'}$, $\mathbf{C}\,P^1 \times \mathbf{C}\,P^1$ and C^6 are all complex analytic. $\mathbf{H}\,P^1$ and $-\mathbf{C}\,P^2$ are quaternionic for the first quaternionic structure. $-\mathbf{H}\,P^1$ and $-\mathbf{C}\,P^{2'}$ are quaternionic for the second quaternionic structure. Therefore all are homologically volume-minimizing.*

Proof. — Since $\mathbf{C}\,P^1$, $\mathbf{C}\,P^2$, $\mathbf{C}\,P^{2'}$, and $\mathbf{C}\,P^1 \times \mathbf{C}\,P^1$ are orbits of subgroups of U_4, it suffices to check that the tangent space at ξ_0 is complex, which is apparent from Table 2.1.1.

Let ξ be a generic point in C^6, so that ξ meets $\mathbf{C}_1 \oplus \mathbf{C}_2$ in a complex line ξ_1 and $*\xi$ meets $j\mathbf{C}_1 \oplus j\mathbf{C}_2$ in a complex line ξ_2. Let $g \in U_2(\mathbf{C}_1 \oplus \mathbf{C}_2) \times U_2(j\mathbf{C}_1 \oplus j\mathbf{C}_2)$ such that $g\xi_1 = \mathbf{C}_1$, $g\xi_2 = j\mathbf{C}_1$, and hence $g\xi = \xi_0$. One checks that $T_0 g C^6 = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}_1, \mathbf{C}_2) \oplus \mathrm{Hom}_{\mathbf{C}}(j\mathbf{C}_1, \mathbf{H}_2)$, which is complex. Therefore C^6 is complex analytic.

The statements about quaternionic structure follow immediately from Table 2.1.1, at least up to orientations. One checks that the quaternionic structures on $\mathbf{C}\,P^2$ and $\mathbf{C}\,P^{2'}$ induce orientations opposite to the canonical ones induced by the complex structure. Finally, one checks that the two quaternionic structures induce opposite orientations of $\mathbf{H}\,P^1$; we choose the first.

An explanation of the relation to volume minimization appears in the introduction 1.1, 1.2.

2.5. Corollary. — *$\mathbf{C}\,P^1$ and $\mathbf{C}^6$ and their multiples give least-volume representatives of all the integral homology classes in $H_2\,G(2, \mathbf{C}^4)$ and $H_6\,G(2, \mathbf{C}^4)$.*

Proof. — They give the homology by 2.2 and minimize volume by 2.4.

2.6. Proposition. — *The intersection numbers of various cycles in $H_4\,G(2, \mathbf{C}^4)$ are given by Table 2.6.1.*

Table 2.6.1

Intersection numbers in H_4 G (2, $\mathbf{C}^4$).

Cycles	$\mathbf{C}\,P^2$	$\mathbf{C}\,P^{2'}$	$\mathbf{H}\,P^1$	$\mathbf{C}\,P^1 \times \mathbf{C}\,P^1$
$\mathbf{C}\,P^2$	1	0	-1	1
$\mathbf{C}\,P^{2'}$	0	1	$+1$	1
$\mathbf{H}\,P^1$	-1	$+1$	2	0
$\mathbf{C}\,P^1 \times \mathbf{C}\,P^1$	1	1	0	2

Proof. — First consider $\mathbf{C}\,P^2 = G(2, \mathbf{C}^3)$, where $\mathbf{C}^3 = \mathbf{H}_1 \oplus \mathbf{C}_2$. By perturbing $\mathbf{C}_2$ to a nearby complex line in $\mathbf{H}_2$, one sees that the self-intersection number is ± 1. Since

$\mathbf{C}P^2$ and its perturbation are both complex, it must be $+1$. Similarly the self-intersection number of $\mathbf{C}P^{2'}$ is $+1$.

$\mathbf{C}P^{2'}$ is isotopic to $\{\xi : j\mathbf{C}_2 \subset \xi\}$, which has no intersection with $\mathbf{C}P^2$.

$\mathbf{H}P^1$ meets $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ in the single point ξ_0. Since $\mathbf{H}P^1$ shares the first quaternionic structure with $-\mathbf{C}P^2$, and $-\mathbf{H}P^1$ shares the second quaternionic structure with $-\mathbf{C}P^{2'}$, the intersection numbers are -1 and $+1$.

$\mathbf{C}P^1 \times \mathbf{C}P^1$ meets $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ in the single point ξ_0. Since all three are complex, the intersection numbers are $+1$.

Since $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ form a basis for the homology, the other intersection numbers follow as consequences.

The following theorem is the main result of this paper.

2.7. THEOREM. — *Least volume representatives of every homology class in* $H_4G(2, \mathbf{C}^4)$. *In Figure* 2.7.1 *each of the six vertices of the hexagon is homologically*

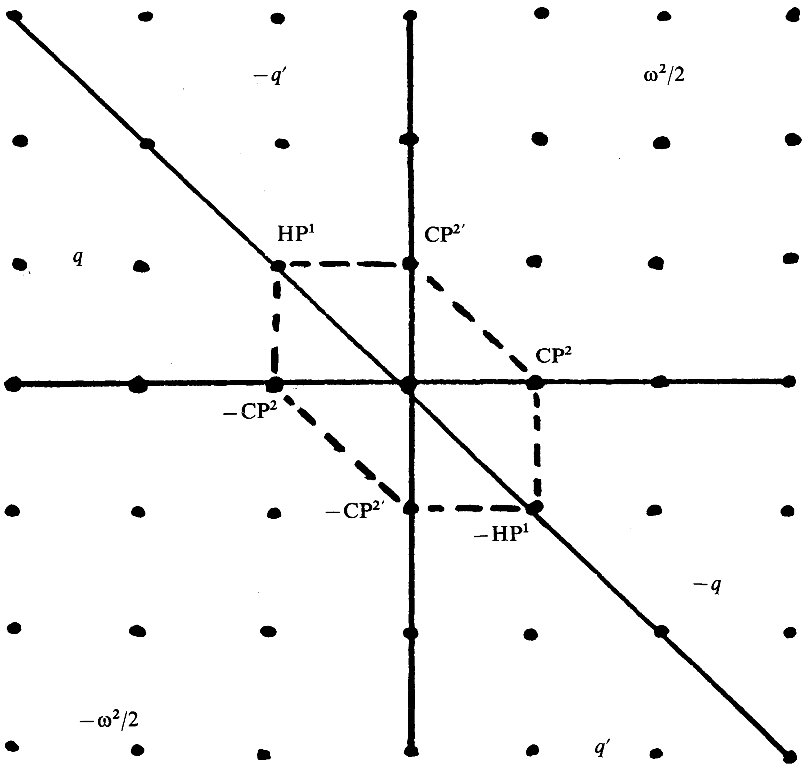

Fig. 2.7.1. — Least-volume representatives of every homology class in $H_4\,G\,(2, \mathbf{C}^4)$.

volume-minimizing. Indeed, each nonnegative integral linear combination of two adjacent vertices is homologically volume-minimizing.

Remarks. — The proof shows that each of the six regions of Figure 2.7.1 is either complex analytic or quaternionic for the indicated Kahler or quaternionic forms.

Least volume representatives are not generally unique, even up to congruence. For example, $\mathbf{C}P^1 \times \mathbf{C}P^1 \sim \mathbf{C}P^2 + \mathbf{C}P^{2'}$, and both are complex analytic.

The regularity results of F. J. Almgren [A] guarantee that a 4-dimensional homologically volume-minimizing surface is a smooth submanifold except for a singular set of dimension at most 2. Note that $\mathbf{C}P^2 + \mathbf{C}P^{2'}$ has the 2-dimensional singular set $\mathbf{C}P^1$, where $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ intersect.

Proof. – Note that since $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ generate the integral homology (Proposition 2.2) and $\mathbf{H}P^1 \sim -\mathbf{C}P^2 + \mathbf{C}P^{2'}$ by Proposition 2.6, Figure 2.7.1 gives an accurate picture of $H_4\, G(2, \mathbf{C}^4)$.

By Proposition 2.4, each adjacent pair is simultaneously complex or quaternionic, with the calibrations indicated in Figure 2.7.1. Therefore all nonnegative linear combinations are homologically volume-minimizing (*cf.* 1.1, 1.2).

2.8. Corollary. — *The hexagon of Figure* 2.7.1 *gives the unit mass ball in* $H_4\, G(2, \mathbf{C}^4) \cong \{U_2 \times U_2\text{-invariant 4-vectors in } T_0\, G(2, \mathbf{C}^4)\}$. *Each cycle* S *stands for the* $U_2 \times U_2$ *average of its unit tangent* 4*-plane at* ξ_0.

Proof. – Let S_1, S_2 be cycles at adjacent vertices, let ζ_1, ζ_2 be their unit tangent 4-planes at ξ_0, let $\overline{\zeta_1}$, $\overline{\zeta_2}$ be their $U_2 \times U_2$ averages, and let φ be the common calibration of S_1 and S_2. Of course $\overline{\zeta_1}$, $\overline{\zeta_2}$, and any convex linear combination $\lambda_1 \overline{\zeta_1} + \lambda_2 \overline{\zeta_2}$ have mass at most 1. But since φ is $U_2 \times U_2$ invariant.

$$\varphi(\lambda_1 \overline{\zeta_1} + \lambda_2 \overline{\zeta_2}) = \lambda_1 + \lambda_2 = 1.$$

Therefore convex linear combinations of adjacent vertices have mass 1, and the unit mass ball is the pictured hexagon.

2.9. Corollary. — *Figure* 2.9.1 *gives the unit comass ball in* $H^4\, G(2, \mathbf{C}^4) \cong U_4$*-invariant differential* 4*-forms on* $G(2, \mathbf{C}^4)$.

Proof. – The unit comass ball is just the polar or dual of the unit mass ball. Its vertices calibrate the corresponding sides of the unit mass ball.

Remarks. – c_2, c_2' denote the second Chern calibrations, defined as dual to $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$ ([C], § 8, *cf.* [MS], Problem 14–D, p. 171). c_2 calibrates $\mathbf{C}P^2$ (because both $\omega^2/2$ and $-q$ do). Similarly, c_2' calibrates $\mathbf{C}P^{2'}$. The first Chern calibration is just the Kahler form ω.

p denotes the first Pontryagin calibration, characterized as a positive multiple of $-c_2 + c_2'$ (*cf.* [MS], Cor. 15.5, p. 177). It calibrates $\mathbf{H}P^1$. Note that our calibrations have been normalized to have unit comass.

As self-dual 4-forms in $\mathbf{R}^8$, $\omega^2/2$, q, c_2, and p appear as types (2,0), (3,0), (3,2), and (3,3) of [DHM], Chapter 3.

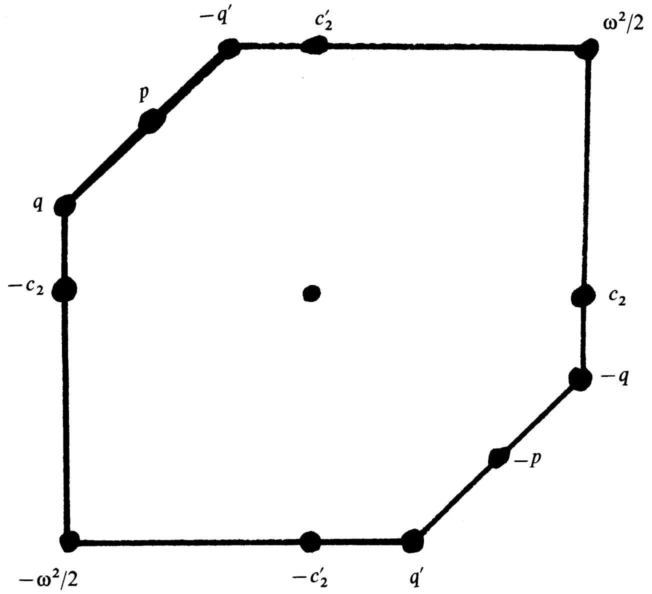

Fig. 2.9.1. — The unit comass ball in H^4 G (2, $\mathbf{C}^4$).

It happens that $G(2, \mathbf{C}^4)$ is isometric to $G(2, \mathbf{R}^6)$. The universal Euler calibrations in $H^2 G(2, \mathbf{R}^6)$ and $H^4 G(2, \mathbf{R}^6)$ correspond to $\pm\omega$ and p. The first universal Pontryagin calibration in $H^4 G(2, \mathbf{R}^6)$ corresponds to $\omega^2/2$.

2.10. *Remark.* — $H_4 G(m, \mathbf{C}^n)$ for $m \geqq 2$, $n \geqq m+2$ continues to be generated by $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$. The various calibrations in $H^4 G(2, \mathbf{C}^4)$, such as $\omega^2/2$, q, and q', extend by averaging to calibrations $\tilde{\omega}^2/2$, $\tilde{q}$, $\tilde{q}'$ in $H^4 G(2, \mathbf{C}^4)$. $\tilde{\omega}^2/2$ always calibrates $\mathbf{C}P^2$ and $\mathbf{C}P^{2'}$, so that their nonnegative [nonpositive] linear combinations still give least-volume representatives of the homology classes in the first [third] quadrant of Figure 2.7.1. For $m=2$, $\tilde{q}$ still calibrates $-\mathbf{C}P^2$, $\mathbf{H}P^1$, and the associated subquadrants of Figure 2.7.1; for $m>2$, the question is open. Similarly, for $n=m+2$, $\tilde{q}'$ still calibrates $-\mathbf{C}P^{2'}$, $-\mathbf{H}P^1$, and the associated subquadrants of Figure 2.7.1; for $n>m+2$, the question is open.

2.11. $G(2, \mathbf{C}^5)$. — The next case, $G(2, \mathbf{C}^5)$ has nontrivial homology in even degrees:

	H_2	H_4	H_6	H_8	H_{10}
	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$
Generators.	$\mathbf{C}P^1$	$\mathbf{C}P^2$, $\mathbf{C}P^{2'}$	$\mathbf{C}P^3$, C^6	$G(2, \mathbf{C}^4)$, C^8	C^{10}

The generators, including certain Schubert cycles C^6, C^8, C^{10}, are all complex analytic varieties. Therefore these generators and nonnegative (or nonpositive) integral linear combinations of them are volume-minimizing. Some of the other classes have known quaternionic (and hence volume-minimizing) representatives (*cf.* Remark 2.10). However, the whole story is far from known.

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F. MORGAN,
Department of Mathematics,
Williams College,
Williamstown, MA 01267.