ANNALES SCIENTIFIQUES DE L'É.N.S.

OLIVIER MATHIEU

Filtrations of *G*-modules

FILTRATIONS OF G-MODULES

By Olivier MATHIEU

Introduction

Let k be an algebraically closed field, let G be a connected semi-simple algebraic group, let B be a Borel subgroup of G, and let $X(B)(X^+(B))$ be the set of characters (respectively dominant characters) of B. For each $\lambda \in X(B)$ let $\mathscr{L}(\lambda)$ be the associated G-equivariant invertible sheaf on G/B. For any $\lambda \in X^+(B)$, set $F(\lambda) = \Gamma(G/B, \mathscr{L}(-\lambda))$. Recall that $F(\lambda)$ can also be defined as the induced module $\mathrm{Ind}_B^G - \lambda$. A filtration of a G-module M is called good if and only if its subquotients are isomorphic to some $F(\lambda)$ (*see* section 1 for forme details, and *see* subsection 1.14 for more references).

When the characteristic of k is zero, every G-module is semi-simple (Weyl's complete reducibility Theorem) and the modules $F(\lambda)$ are exactly the simple modules (Borel-Weil Theorem). Hence M is a direct sum of modules $F(\lambda)$ [with $\lambda \in X^+(B)$]. For a field k of characteristic $p \neq 0$ as we will consider from now on, these results are no longer true. A substitute for the complete reducibility Theorem is the notion of good filtration, as shown by the following theorem:

THEOREM. — (*Donkin*) *Suppose that* G *does not contain any components of type* E_7, E_8 *or that* $p \neq 2$.

1) *For every* λ, $\mu \in X^+(B)$, *the* G*-module* $F(\lambda) \otimes F(\mu)$ *has a good filtration.*

2) *For every semi-simple subgroup* G' *of* G *corresponding to a Dynkin subdiagram, and for every* $\lambda \in X^+(B)$, *the* G'*-module* $F(\lambda)$ *has a good filtration.*

The theorem is due to S. Donkin [D1], but part 1 was previously shown by Wang Jian-Pian for p large [W] (i. e. for $p > C$, where the constant C depends on the Dynkin diagram of G only). It is clear that it suffices to consider the case of a quasi-simple group G. The proof of Donkin's Theorem is based on the classification of quasi-simple algebraic groups and on a case by case analysis. His proof requires long and difficult calculations: e. g. it takes about 45 pages for F_4 only (moreover he used deep results such as Andersen's strong linkage Principle [A3]). S. Donkin also states that the restrictive hypotheses involving E_7 and E_8 are likely to be unnecessary. So it is natural to look for a new approach to the problem, in order to:

a) do without the restrictive hypotheses on E_7, E_8;

b) get a general method and avoid a case-by-case analysis;

c) give general results, not only for $F(\lambda) \otimes F(\mu)$ but also for "natural" subquotients of multiple tensor products.

Such an approach is provided in this paper. We would like to state first our result in its simplest and less technical form:

THEOREM 1. — 1) *For every* $\lambda, \mu \in X^+(B)$, *the* G-*module* $F(\lambda) \otimes F(\mu)$ *has a good filtration*.

2) *For every semi-simple subgroup* G' *corresponding to a Dynkin subdiagram, and for every* $\lambda \in X^+(B)$, *the* G'-*module* $F(\lambda)$ *has a good filtration*.

The result, whose proof does not involve a case-by-case analysis is an answer to points (*a*) and (*b*) of the previous discussion. Moreover Theorem 1 is a particular case of a more general statement (theorem 2 below). Before stating the general result, we need to introduce a few definitions.

Let M be a G-module. A stratification of M is a family $\mathscr{I}$ of G-submodules of M, with $M \in \mathscr{I}$. The elements of $\mathscr{I}$ are called the strata of $\mathscr{I}$. A good filtration of M is called compatible with the stratification $\mathscr{I}$ if its trace over each $N \in \mathscr{I}$ is good.

Let *n* be an integer, let $\lambda = (\lambda_1, \ldots, \lambda_n) \in X(B)^n$ and let $\mathscr{L} = \mathscr{L}(\lambda_1, \ldots, \lambda_n)$ be the associated invertible sheaf on $(G/B)^n$ associated with λ. Let H be a Cartan subgroup of B and let W be the Weyl group $N_G(H)/H$. For every $w = (w_1, \ldots, w_n) \in W^n$, set $S_w = \overline{B w_1 B} \times^B \ldots \times^B \overline{B w_n B}/B$. The varieties S_w, called generalized Schubert varieties, are subvarieties of $(G/B)^n$. We call generalized Schubert scheme any union of generalized Schubert varieties. We will still denote by $\mathscr{L}$ be the restriction of $\mathscr{L}$ to any generalized Schubert scheme.

Let G' be a semi-simple subgroup corresponding to a Dynkin subdiagram, let $Q = G' . B$ be the corresponding parabolic subgroup, and let S be a Q-invariant generalized Schubert scheme. Note that $\mathscr{L}$ is a Q-equivariant invertible line bundle on S. Set $M = \Gamma(S, \mathscr{L})$. The G'-module M has two natural stratifications.

1) *The Geometrical Stratification:* Let S', S'' be two Q-invariant generalized Schubert scheme (we assume that S'' can also be the empty set). Suppose that $S' \supseteq S''$ and $S' \supseteq S$. We have the restriction morphisms $a : \Gamma(S', \mathscr{L}) \to \Gamma(S'', \mathscr{L})$ and $b : \Gamma(S', \mathscr{L}) \to \Gamma(S, \mathscr{L})$. The geometrical stratification of M is the set of submodules $b(\operatorname{Ker} a)$.

2) *The Arithmetical Stratification:* The scheme S and the invertible sheaf $\mathscr{L}$ are actually defined over $\mathbb{Z}$. Hence there exists a stratification, called the arithmetical stratification $0 \subseteq M(0) \subseteq M$, where M(0) is the group of sections which come from $\mathbb{Z}$ (some refinements are also possible).

In this paper, we will be interested only in the geometrical stratification: but it could also be interesting to work with the arithmetical stratification in order to extend the results over $\mathbb{Z}$. Also, it is interesting to work with parabolic analogs of generalized Schubert schemes and related schemes: in this paper we will state our results for a larger class of schemes, called the class of $\mathscr{S}$-schemes. In the introduction, we will restrict ourself to generalized Schubert schemes. Our main result is:

THEOREM 2. — *The* G'-*module* $\Gamma(S, \mathscr{L})$ *has a good filtration compatible with the geometrical stratification*.

We would like to illustrate Theorem 2 with two examples:

Example 1. — Set $S = (G/B)^2$ and let λ, μ be two dominant characters of B. Set $M = F(\lambda) \otimes F(\mu)$ and set $\mathscr{L} = \mathscr{L}(-\lambda, -\mu)$. By theorem 2 the G-module M has a good filtration (hence Theorem 2 generalizes Theorem 1). For every G-invariant generalized Schubert scheme Σ of S, let $K(\Sigma)$ be the kernel of $\Gamma(S, \mathscr{L}) \to \Gamma(\Sigma, \mathscr{L})$. Hence by Theorem 2, for $\Sigma \subseteq \Sigma'$ (Σ' being the empty set or a generalized Schubert scheme), the quotients $K(\Sigma)/K(\Sigma')$ have a good filtration (note that for p large, the result was already proved by P. Polo [P1]). But the length of a good filtration of $K(\Sigma)/K(\Sigma')$ can be arbitrary large (when λ, μ are large), so the strata $K(\Sigma)$ give very little information about the good filtrations of M. Fortunately it is possible to get more geometrical strata because it is possible to embed S in $(G/B)^{n+2}$ in many ways: indeed for each $w \in W^{n+2}$ in which 1 occurs n times and in which the maximal element ω of W occurs 2 times, the generalized Schubert variety S_w is canonically isomorphic to S. Then, for $G = SL(m)$, it is easy to prove that some towers of the geometrical stratification are good filtrations.

Example 2. — Set $S = (G/B)^3$ and let λ, μ, ν be three dominant characters of B. Set $M = F(\lambda) \otimes F(\mu) \otimes F(\nu)$ and set $\mathscr{L} = \mathscr{L}(-\lambda, -\mu, -\nu)$. Let $\Sigma', \Sigma'' \subseteq G/B$ be ordinary Schubert varieties. Set: $\Sigma = G \times^B (\Sigma' \times \Sigma'')$. The variety Σ, which is called double Schubert variety, is naturally embedded in S. A double Schubert scheme is an union of double Schubert varieties (we define similarly triple Schubert schemes [M2]). For every double Schubert scheme Σ in S, let $K(\Sigma)$ be the kernel of $\Gamma(S, \mathscr{L}) \to \Gamma(\Sigma, \mathscr{L})$. Hence by theorem 2, the modules $K(\Sigma)$ have good filtrations (for p large and for a special double Schubert scheme Σ (*see* 1.8 for the definition), the result was already proved by a general method: see [M3], theorem 1). There was another approach to theorem 1, based on conjectures involving B-modules (*see* A. Joseph [J], [M2, 3], P. Polo [P1, 2, 3], W. van der Kallen [vdK1]). For example the following conjecture would imply (by Kempf's theorem [K]) point 1 of theorem 1:

Conjecture (*Polo's Conjecture* (C2) [P1]): *Let λ be an antidominant character of B, and let M be a strong B-module (following the terminology of [M2]). Then the B-module $M \otimes \lambda$ is strong.*

The triple tensor products considered in the example are of special interest because we proved that the existence of good filtrations on $K(\Sigma)$ for special double Schubert schemes Σ implies the Joseph, Polo and van der Kallen conjectures on filtrations of B-modules (*see* section 5 and [M2]). Note that we used implicitly that double Schubert schemes are generalized Schubert schems in S (lemma 5.2). Triple Schubert schemes are no longer generalized Schubert schemes: this can explain why tensor products of strong modules are not necessarly strong, as pointed out by W. van der Kallen [vdK2].

Organization of the Proof. — The proof of Theorem 2 requires the notion of Frobenius splittings (an notion due to V.B. Mehta, S. Ramanan and A. Ramanathan: *see* [MR1], [RR], [R1, 2]). By the Mehta, Ramanan and Ramanathan theory, the varieties $(G/B)^n$ have many Frobenius splittings. However, we select a special one which we call the canonical Frobenius splitting of $(G/B)^n$. The new and main result here is the compatibility of the canonical Frobenius splitting and the good filtrations: the result is indeed

surprising because the canonical Frobenius splitting is not G-invariant but it is only H-invariant.

We now briefly describe the organization of the paper. Section 1 is devoted to the usual background (the Subsection 1.14 is a short survey on good filtrations). The two main notions are the canonical Frobenius splitting (Section 2) and the canonical filtrations (Section 3). The proof of the main result (Theorem 3 in Section 4) is based on three key lemmas. The first one (Lemma 2.4) states commutation relations satisfied by the canonical Frobenius splitting. Lemma 3.3 gives a criterion for the existence of good filtrations for commutative reduced algebras: it uses a simple result on finite homeomorphisms. The commutation relations imply that the canonical Frobenius splitting is compatible with some canonical filtrations (it is the third key Lemma 4.3). As proved previously in [M2], the main result implies some statements for filtrations of B-modules. These consequences are stated in Section 5.

The results of the paper were announced in [M4]. We would like to thank S. Donkin, M. Duflo, J. Humphreys and D. N. Verma for their encouragements and some helpful discussions.

Remark added at revision time. — Let h be the highest coroot. Say that a dominant weight μ satisfies (*) if and only if μ is a sum of fundamental weight ω such that $\langle \omega | h \rangle \leq 3$. Note that for a group G of classical type or of type E_6, G_2 every dominant weight satisfies (*).

1) In a recent talk Peter Littelmann indicated that he had proved that the modules $K(\Sigma)$ of our example 1 have a good filtration whenever λ satisfies (*).

2) In a recent preprint (with the same title as [P3]) P. Polo shows that for any strong module M and for any dominant weight λ satisfying (*) the module $M \otimes -\lambda$ is strong: the result is a little bit better than it was announced in [P3], and contains many cases of corollary 1 (moreover P. Polo proved similar statement for classes $\mathscr{W}$, $\mathscr{H}$ and $\mathscr{K}$).

3) In the same preprint, P. Polo announces a different proof of our theorem 1 (point 1) for E_7 over a field characteristic 2.

4) In a recent preprint (Good bases for G-modules) we use our corollary 1 in order to prove a statement conjectured by I. Gelfand, A. Zelevinsky and K. Baclawski.

5) We would like to thank W. van der Kallen for some nice comments of the preprint, and P. Littelmann and A. Zelevinsky for recent discussions.

1. Generalities, notations, conventions

This section is devoted to the main definitions and notations. For the general background on algebraic groups used in the paper, we would like to refer especially to [J2], [H2].

1.1. Let k be an algebraically closed field of caracteristic $p \neq 0$.

1.2. We will assume that all the schemes are separated, reduced and of finite type over k. We will use the term *variety* only for those which are irreducible. For a

smooth variety X, we will denote by ω_X the sheaf of top differential forms on X. For a codimension 1 subscheme Y, we will denote by $\omega_X(Y)$ the sheaf of top differential forms with at most simple poles along Y.

1.3. Let K be an arbitrary algebraic group. We will denote by X(K) its character group. Let $\mathcal{U}(K)$ be the ring of left invariant differential operators on K and let $\mathcal{C}(K)$ be the category of algebraic K-modules. A K-scheme (respectively a K-algebra) is a scheme X (respectively an algebra R) endowed with an action of the algebraic group K.

1.4. Let G be a connected simply connected semisimple algebraic group, let B be a Borel subgroup of G, let U be the unipotent radical of B, let H be a Cartan subgroup of B, let W be the Weyl group: $N_G(H)/H$, and let I be the Dynkin diagram of G. Note that we have an isomorphism $X(H) \simeq X(B)$. We will call *weights* the elements of X(H). Let $X^+(H)$ be the set of dominant weights. For any H-module M and any $\lambda \in X(H)$, let M_λ be the set of vectors in M of weight λ. With each vertex $i \in I$, one associates a simple root $\alpha_i \in X(H)$. For any $s \in \mathbb{N}$, $i \in I$, the vector space $\mathcal{U}(U)_{s\alpha_i}$ is one dimensional. Let $e_i^{(s)}$ be a nonzero vector of $\mathcal{U}(U)_{s\alpha_i}$. It is possible to choose the elements $e_i^{(s)}$ such that $e_i^{(0)} = 1$, and $\delta e_i^{(s)} = \sum\limits_{s'+s''s} e_i^{(s')} . e_i^{(s'')}$ [where δ is the comultiplication of $\mathcal{U}(U)$].

LEMMA 1.1. — *The elements $e_i^{(s)}$ ($i \in I$, $s \in \mathbb{N}$) generate the algebra $\mathcal{U}(U)$. Hence a vector subspace N of a U-module M is a U-submodule if an only if N is stable by the elements $e_i^{(s)}$.*

Proof. — The algebra $\mathcal{U}(U)$ is the reduction modulo p of the Kostant-Chevalley form (*see* [B] [Ko]). The lemma follows [B] (Lemma 5, ch. VIII, § 12).

1.5. Let $n \in \mathbb{N}$ and let $u = (u_1, \ldots, u_n)$ be an element of W^n. The variety $S_u = \overline{B u_1 B} \times^B \ldots \times^B \overline{B u_n B}/B$ is called a generalized Schubert variety. When $u_1 = \ldots = u_n = \omega$ (where ω is the maximal element of W), the variety S_u is naturally isomorphic to $(G/B)^n$. Indeed we will systematically identify $(G/B)^n$ with the generalized Schubert variety $G \times^B \ldots \times^B G/B$. Hence all the generalized Schubert varieties $S_u (u \in W^n)$ are subvarieties of $(G/B)^n$ (*see* [M1]). By definition, a generalized Schubert scheme is a union of Schubert varieties.

1.6. Let $\overline{W}$ be the free monoid generated by formal symbols s_i, $i \in I$. When the n-tuple u of W^n is a product of simple reflexions, it can be identified with an element of $\overline{W}$ (still denoted by u). Then the associated Schubert variety is denoted by $D(u)$, and it is a Demazure variety [D].

1.7. Let $u = (u_1, \ldots, u_n)$ be an element of W^n and let $v_1, \ldots, v_n$ be a reduced decomposition of $u_1, \ldots, u_n$. Set $v = v_1 \ldots v_n$ (the product is calculated in $\overline{W}$). There exists a natural morphism $\pi : D(v) \to S_u$, which is proper and birational [M1]. Since the Demazure variety is smooth, the morphism π is called the Demazure desingularization (although π is not a strict desingularization). For ordinary Schubert varieties, the morphism π was introduced by M. Demazure in [D].

1.8. Let $u = (u_1, \ldots, u_n)$ be an element of W^n. Set $\Sigma = S_{u_1} \times \ldots \times S_{u_n}$. The variety $\Sigma' = G \times^B \Sigma$ is called a multiple Schubert variety. The variety Σ can be identified with

the subvariety $B/B \times \Sigma$ in $(G/B)^{n+1}$, and Σ' can be identified with the G-orbit of Σ in $(G/B)^{n+1}$. By definition, a multiple Schubert scheme is an union of multiple Schubert varieties in $(G/B)^{n+1}$. For $n = 2, 3$, multiple Schubert schemes are called double, triple Schubert schemes. The schemes $G \times^B (S \times S')$ are called special double Schubert schemes whenever S is a Schubert variety and S' is a Schubert scheme.

1.9. Let X be a B-scheme, let $\mathcal{M}$ be a coherent B-equivariant sheaf over X and let $\lambda \in X(B)$. We denote by $\mathcal{M}[\lambda]$ the sheaf $\mathcal{M}$ where the B-action is shifted by λ. Accordingly when M is a B-module, we set $M[\lambda] = M \otimes \lambda$.

1.10. Let $S \supseteq S'$ be generalized Schubert schemes, let X be a B-scheme and let $\pi \colon S \to X$ be a morphism of B-schemes. Suppose that: $\pi_* \mathcal{O}_S = \mathcal{O}_X$. Then the scheme $\pi(S')$ is called a $\mathscr{S}$-scheme. Let $\Sigma \supseteq \Sigma'$ be two other generalized Schubert schems, let $\tau \colon \Sigma \to Y$ be a morphism of B-schemes with $\pi_* \mathcal{O}_\Sigma = \mathcal{O}_Y$. Suppose $S \subseteq \Sigma$, $S' \subseteq \Sigma'$. Let $f \colon X \to Y$ be a morphism of B-schemes which is compatible with the inclusion $S \subseteq \Sigma$. Then the induced morphism $g \colon \pi(S') \to \tau(\Sigma')$ is called a morphism of $\mathscr{S}$-schemes. Let Q be a standard parabolic subgroup of G and let G' be its Levi component. When S, S' and X are actually Q-schemes and when the morphism π is a morphism of Q-schemes, we say that $\pi(S)$ is a G'-$\mathscr{S}$-scheme (or we say that it is a Q-$\mathscr{S}$-scheme).

1.11. Let V be a vector space. A stratification of V is a family $\mathscr{I}$ on subspaces of V, such that $V \in \mathscr{I}$. A filtration of V is a family $\mathscr{F}$ of totally ordered (for the inclusion relation) subspaces of V. In particular a tower (i.e. a maximal totaly ordered family) of a stratification $\mathscr{I}$ is a filtration. Let K be an algebraic group. Usually we will consider V being a K-module: in that case a stratification or a filtration will contain K submodules only. For technical reasons, we will consider only filtrations $(\mathscr{F}_x V)$ indexed by real numbers $x \in \mathbb{R}$ (i.e. we have: $\mathscr{F}_x V \subseteq \mathscr{F}_y V$ for $x \leq y$). In that case, we set $\mathscr{F}_x^- V = \bigcup_{y < x} \mathscr{F}_y$ and $\mathscr{H}_x V = \mathscr{F}_x V / \mathscr{F}_x^- V$. In this paper, we will consider only finite filtrations. Hence the non-zero vector spaces among the family vector spaces $\mathscr{H}_x V (x \in \mathbb{R})$ are the ordinary subquotients of the filtration $\mathscr{F}$.

1.12. For every $\lambda \in X(B)$, we will denote by $\mathscr{L}(\lambda)$ the associated invertible sheaf on G/B. We recall that $\Gamma(G/B, \mathscr{L}(\lambda))$ is non zero if and only if λ is antidominant (Borel-Weil Theorem). For a dominant λ, set $F(\lambda) = \Gamma(G/B, \mathscr{L}(-\lambda))$.

1.13. Let V be a finite dimensional G-module and let $\mathscr{F}_x V$ be a filtration of V. The filtration is called a $\mathscr{G}$-filtration (respectively: good filtration) if its subquotients are direct sum of modules $F(\lambda)$ [where $\lambda \in X^+(H)$] [respectively: are isomorphic to some $F(\lambda)$, for various $\lambda \in X^+(H)$]. Hence every $\mathscr{G}$-filtration has a good refinement. However it will be more convenient to work with $\mathscr{G}$-filtrations, because $\mathscr{G}$-filtrations have functorial properties (see [D2], [F]).

Let Z be a G-scheme, let $\mathcal{M}$ be a G-equivariant coherent sheaf on Z and let (Z_α) be a family of G-invariant subschemes of Z. Set $M = \Gamma(Z, \mathcal{M})$ and for every index α let $K(\alpha)$ be the kernel of the restriction map $\Gamma(Z, \mathcal{M}) \to \Gamma(Z_\alpha, \mathcal{M})$. A good filtration $\mathscr{F}$ of the G-module M is called compatible with the family (Z_α) if the trace of $\mathscr{F}$ on each submodule $K(\alpha)$ is a good filtration.

1.14. Let $\lambda \in X(H)$ be a dominant weight. The dual $L(\lambda)$ of $F(\lambda)$ is called a Weyl module, because it satisfies Weyl character formula. Let $V(\lambda) = \mathscr{U}(G) \otimes_{\mathscr{U}(B)} \lambda$ be the Verma module of highest weight λ. The module $L(\lambda)$ can be described as the maximal quotient of $V(\lambda)$ which carries a structure of G-module. A filtration of a G-module M whose subquotients are Weyl modules is called a Weyl filtration (so Weyl filtrations are duals of good filtrations). Some of the following remarks are borrowed from the survey paper [D3]. For the best of our knowledge, Point 1 of Theorem 1 was first conjectured in an unpublished paper of J. Humphreys [H1]. The paper of Upadhyaya [U] was the first published paper devoted to Weyl filtrations. The basic properties of Weyl filtrations appear in Jantzen's paper [J1]. As we saw in introduction, the works of Wang Jian-Pian [W] and S. Donkin [D1] were a motivation for further studies of good filtrations: actually S. Donkin strongly conjectured that his theorem should be true without the restrictive hypotheses (see remarks after Theorem 3 in [D3]). Thus E. Friedlander [F] and S. Donkin [D2] pointed out that good filtrations can be realized in a functorial way. Actually, the filtrations defined in section 3 differ a little bit (note also Donkin's papers [D4], [D5] where some G-algebras are studied). D. N. Verma pointed out that the module $L(\lambda) \otimes V(\mu)$ has a filtration whose subquotients are Verma modules, and he suggested that the filtration should induce a Weyl filtration on $L(\lambda) \otimes L(\mu)$. The idea of using the algebra A in section 4 is adapted from Verma's suggestion.

In [M1], we proved a conjecture on the representation of compact groups (the conjecture, generally attributed to Parthasaraty, Ranga-Rao and Varadarajan was independently proved by S. Kumar [Ku]; for the group $SU(n)$, the result is due to P. Polo [P1]; later some special cases were reproved by M. McGovern [MG] and P. Littelmann). As by-product of our proof, we showed that some modules $F(\xi)$ appear as natural subquotients in $F(\lambda) \otimes F(\mu)$. These subquotients are called PVR components. The result supported the idea that Donkin theorem should be true for E_7, E_8 as well. Let λ, $\mu \in X^+(H)$, let $\mathscr{L} = \mathscr{L}(-\lambda, -\mu)$ be the corresponding sheaf over $(G/B)^2$ and let Σ be the closure of a G-orbit on $(G/B)^2$. Set $M(\Sigma) = \Gamma(\Sigma, \mathscr{L})$ and let $K(\Sigma)$ be the kernel of the restriction map $F(\lambda) \otimes F(\mu) \to M(\Sigma)$. Using some vanishing results of [M1], [MR2] and a case-by-case analysis, P. Polo showed that for large p the modules $K(\Sigma)$ and $M(\Sigma)$ have good filtrations [P1]. Later P. Littelmann pointed out that, for a classical group G, Polo's result can be deduced from his work [L] using the monomial standard theory (V. Lakshmibai, C. S. Seshadri [LS]). The relationship between good filtrations and PVR components is simple: the PVR component appears as the first term of some good filtrations of $M(\Sigma)$ (see the proof of Theorem 3 in [M2]).

2. Frobenius splittings

In this section we have put the results involving Frobenius splittings. Recall that the notion of Frobenius splittings was introduced by V. B. Mehta, S. Ramanan and A. Ramanathan in order to prove, among other things, the Demazure character formula [D] (see [MR1, 2], [RR], [R1, 2], see also [A2], [LS]). Note also that a closely related

idea appeared previously in the nice proof by H. H. Andersen [A1] and W. Haboush [H] of Kempf Theorem [Ke]. The only new result of the section is Lemma 2.4.

Let X be a scheme, let Y be a subscheme defined by the ideal $\mathcal{I}$, and let R be a not necessary unitary commutative k-algebra. A Frobenius morphism of X (respectively of R) is a morphism of sheaves $\varphi : \mathcal{O}_X \to \mathcal{O}_X$ (respectively $\varphi : R \to R$) such that $\varphi(a^p b) = a \varphi(b)$ for every sections a, b of $\mathcal{O}_X$ (respectively for every a, $b \in R$). Moreover when it satisfies $\varphi(a^p) = a$ for every section a of $\mathcal{O}_X$ (respectively for every $a \in R$), the Frobenius morphism φ is called a Frobenius splitting. When φ satisfies: $\varphi(\mathcal{I}) \subseteq \mathcal{I}$ one says that the Frobenius morphism φ of X is compatible with Y (such a Frobenius morphism induces a Frobenius morphism of Y).

LEMMA 2.1. — (*Ramanathan lemma* [R1]) *Let* R *be a* k*-algebra (respectively: let* X *be a scheme). If* R *(respectively* X*) admits a Frobenius splitting, then the algebra* R *(respectively the scheme* X*) is reduced.*

Proof. — If R (respectively X) admits a Frobenius splitting, then the map $a \mapsto a^p$ is injective (*see* [R1], section 2.1).

Remark. — Let X be a scheme, let Y_α be a family of subchemes, and let σ be a Frobenius splitting of X compatible with all the subschemes Y_α. Since σ induces a Frobenius splitting of the scheme-theoric intersections $Y_\alpha \cap Y_\beta$, these intersections are reduced. In the paper, we will work mostly with Frobenius splittable schemes and compatibly splittable subschemes. Also there will be no differences between set-theoretical intersections and scheme-theoretical intersections. That explains our convention of assuming that every scheme is reduced.

Let X be a scheme and Y be a subscheme of X. We will denote by SF (X, Y) the group of Frobenius morphisms of X which are compatible with Y, and let $\mathcal{SF}(\mathcal{X}, \mathcal{Y})$ be the corresponding sheaf.

LEMMA 2.2. — (*Mehta-Ramanathan theorem*)

1) *We assume that* X *is smooth and that* Y *is purely of codimension* 1 *in* X. *Then there is a canonical isomorphism of scheaves:* $\mathcal{SF}(\mathcal{X}, \mathcal{Y}) \simeq \omega_X(Y)^{\otimes 1-p}$. *In particular* $\mathcal{SF}(\mathcal{X}, \mathcal{Y})$ *is an* $\mathcal{O}_X$*-invertible sheaf [for the right action defined as follows: if* a, φ *are sections of* $\mathcal{O}_X$, $\mathcal{SF}(\mathcal{X}, \mathcal{Y})$, *then* $a . \varphi$ *is the Frobenius morphism* $b \mapsto \varphi(ab)$].

2) *Let* N *be a closed point in* X, *and let* $\varphi \in \mathcal{SF}(\mathcal{X}, \mathcal{Y})$. *We assume that in a neighbourhood of* N, Y *is a divisor with normal crossings such that the intersection of its irreducible components is reduced to* N, *and that* φ *generates* $\mathcal{SF}(\mathcal{X}, \mathcal{Y})$. *Set* $\lambda = \varphi(1)$. *Then we have* $\lambda(N) \neq 0$. *In particular, if* X *is projective,* φ *is a nonzero constant and* $\lambda^{-1} \varphi$ *is a Frobenius splitting.*

Proof. — The proof of the lemma can be found in [MR1] as follows: the formula $\mathcal{SF}(\mathcal{X}, \varnothing) \simeq \omega_X^{\otimes 1-p}$ is stated in Proposition 5. The formula with Y not necessarily empty follows the proof of the Proposition 6. Let m be the dimension of X. The hypotheses of point 2 imply the existence of a local system of parameters $z_1, \ldots, z_m$ in a neighbourhood of N such that Y is locally defined by the equation $z_1 \ldots z_m = 0$. So the lemma results from Proposition 7, 8 of [MR1].

Let X be a scheme, let Y be a subscheme of X defined by the ideal $\mathcal{I}$, let σ be a Frobenius splitting of X compatible with Y, and let $\mathcal{L}$ be an invertible sheaf on X. Then σ induces a Frobenius splitting of the $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{n > 0} \mathcal{L}^{\otimes n} \otimes \mathcal{I}$. The splitting (still denoted σ) is defined by the following two rules:

R_1: For every m not divisible by p, $\sigma(\mathcal{L}^{\otimes m} \otimes \mathcal{I}) = 0$.

R_2: Let $m > 0$ be an integer, let y be a section of $\mathcal{L}^{\otimes m}$ of degree m, and let a be a section of $\mathcal{I}$. Then we have: $\sigma(y^{\otimes p} \otimes a) = y \otimes \sigma(a)$.

Note that the rules R_1 and R_2 are actually a good definition of a Frobenius splitting of $\mathcal{A}$: by localization, it suffices to check it when the invertible sheaf $\mathcal{L}$ is trivial. In particular σ induces a Frobenius splitting of the not necessary unitary commutative algebra $\Gamma(X, \mathcal{A})$.

Let n be an integer. Set $X = (G/B)^n$. Let $\Theta: X \to G/B$ be the projection on the last factor. When X is identified with $G \times^B \ldots \times^B G/B$ as in Section 1.5, the following formula holds: $\Theta((g_1, \ldots, g_n) \bmod B^n) = g_1 \ldots g_n \bmod B$. Set $\mathcal{L}' = (\Theta^* \mathcal{L}(\rho))[-\rho]$, $\mathcal{L} = \mathcal{L}'^{\otimes 1-p}$ (see 1.9, 1.12), where ρ is the half sum of positive roots. So $\mathcal{L}$, $\mathcal{L}'$ are B-equivariant invertible sheaves on X, but they are not G-equivariant. Let u be a reduced decomposition of ω (where ω is the maximal element of W). Set $v = \omega^n$. So v is an element of $\overline{W}$ of length $n \cdot l(\omega)$. Let $\pi: D(v) \to X$ be the Demazure desingularization (see 1.7). We will still denote by $\mathcal{L}'$, $\mathcal{L}$ the pullbacks of $\mathcal{L}'$, $\mathcal{L}$ to $D(u)$. Recall that π is proper and birational. Moreover, by smoothness of X, we have: $\pi_* \mathcal{O}_{D(v)} = \mathcal{O}_{G/B}$. Let Z (respectively $Z(v)$) be the union of all codimension 1 generalized Schubert subvarieties in X (respectively: all codimension 1 Demazure subvarieties in $D(v)$), and let $\mathcal{I}$ (respectively $\mathcal{I}_v$) be the corresponding ideal. Recall that $\pi(Z(v)) = Z$. Set N be the point corresponding to the zero-dimensional Demazure subvariety $D(1)$.

Note that the equality $\pi_* \mathcal{O}_{D(v)} = \mathcal{O}_X$ and $\pi(Z(v)) = Z$ implies that π induces a morphism $\pi: SF(D(v), Z(v)) \to SF(X, Z)$.

LEMMA 2.3:

1) *We have* $\omega_X = \mathcal{L}' \otimes \mathcal{I}$ *and* $\omega_{D(v)} = \mathcal{L}' \otimes \mathcal{I}_v$.

2) *We have* $\mathcal{SF}(\mathcal{X}, \mathcal{L}) = \mathcal{L}$ *and* $\mathcal{SF}(D(v), Z(v)) = \mathcal{L}$.

3) *The* B-modules *SF(X, Z) and SF(D(v), Z(v)) are equal and isomorphic to* $F((p-1)\rho)[(p-1)\rho]$. *In particular every integer* m *such that for some* $i \in I$, $m\alpha_i$ *is a weight of SF(X, Z) satisfies:* $0 \leq m \leq p-1$. *Moreover the weight 0 has multiplicity 1 in* SF(X, Z).

4) *Let* $\varphi \in SF(X, Z)$ *and let* φ_0 *be the weight zero component of* φ. *Then we have* $\varphi(1) \neq 0$ *if and only if* $\varphi_0 \neq 0$.

Proof. — The formula for $\omega_{D(v)}$ is due to Ramanathan ([R1] Proposition 2; here we added the shift $-\rho$ to get the B-equivariance). The formula for ω_X is similar, and can be proved by an easy induction (e.g. using [M2] Lemma 8). Then Point 2 is an immediate corollary of Lemma 2.2 (Point 1). Point 3 is a consequence of Point 2. Note that $Z(v)$ is a divisor with normal crossings intersecting at N. We have $\Theta(\pi(N)) = B/B$. Let $\varphi \in SF(X, Z)$, and let $D(\varphi)$ be the divisor of the corresponding section of

$\Gamma(G/B, \mathscr{L}((1-p)\rho)))$. It is clear that $N \notin D(\varphi)$ if and only if $\varphi_0 \neq 0$. Hence when $\varphi_0 \neq 0$, the Frobenius morphism φ satisfies the hypotheses of Point 2 of Lemma 2.2, and we have $\varphi(1) \neq 0$.   Q.E.D.

For every $\varphi \in SF(X, Z)$, we have $\varphi(1) \in \Gamma(X, \mathscr{O}_X)$, hence $\varphi(1)$ is a constant. Let $R : SF(X, Z) \to k$ be the linear form defined by $R(\varphi) = \varphi(1)$. Let F be the subspace of H-invariant vectors in $SF(X, Z)$. By Points 3 and 4 of previous lemma, F has dimension 1 and $R(F) \neq 0$. So there exists a unique H-invariant Frobenius splitting $\sigma \in SF(X, Z)$: this element is called the canonical Frobenius splitting of X. Let $\mathscr{S}$ be the set of generalized Schubert subschemes in X. It is easy to prove that $\mathscr{S}$ is the smallest set of subschemes of X which contains Z and which is stable under the following operations: union, intersections decomposition into irreducible components. Hence σ is compatible with all the generalized Schubert subschemes.

Let Σ be a $\mathscr{S}$-scheme. There exist generalized Schubert schemes S, S' (with $S \supseteq S'$) and a B-equivariant morphism of scheme $\pi : S \to Y$ (with $\pi_* \mathscr{O}_S \mathscr{O}_Y$) such that Σ is isomorphic to $\pi(S')$. Hence σ induces a Frobenius splitting of Σ. This Frobenius splitting is still denoted σ and it is called the canonical Frobenius splitting of Σ. Actually this Frobenius splitting depends on the choices of S, S', Y, π. However for usual $\mathscr{S}$-schemes, i.e. for generalized Schubert schemes or parabolic analogs, the Frobenius splitting σ does not depend on natural possible choices. A typical example is the following: for every $m \geq n$ and every sequence $w \in W^m$ in which 1 occurs $m - n$ times and ω occurs n times, the generalized Schubert variety S_w is naturally isomorphic to X. Hence there are many ways to realize X as a generalized Schubert variety in G/B^m. It is easy to prove that the canonical Frobenius splitting does not depend on w. Note also that the canonical Frobenius splitting of a $\mathscr{S}$-scheme Σ is compatible with all $\mathscr{S}$-subschemes of Σ (more precisely with all the $\mathscr{S}$-subschemes obtained by the same construction as σ).

KEY LEMMA 2.4. $-$ (commutation relations for σ) Let Σ be a $\mathscr{S}$-scheme, let $\mathscr{L}$ be a invertible sheaf on Σ. Set $\mathscr{M} = \bigoplus_{n>0} \mathscr{L}^{\otimes n}$.

1) Let $\lambda \in X(H)$, let $y \in \Gamma(\Sigma, \mathscr{M})$ be a section of weight λ. Then if $\lambda \notin pX(H)$, we have $\sigma(y) = 0$. If $\lambda \in pX(H)$, then $\sigma(y)$ is a weight vector with weight λ/p.

2) Let z be any section of $\mathscr{M}$, let $i \in I$ and let s be a positive integer. Then we have $\sigma(e_i^{(ps)} z) = e_i^{(s)} \sigma(z)$.

Proof. $-$ Assertion 1 comes easily from the H-invariance of σ. In order to prove Assertion 2, we will prove it first for the following case: $\Sigma = X$, $\mathscr{L} = \mathscr{O}_X$. We will argue by induction on s. Note that for $s = 0$, the assertion is obvious. Let σ' be the endomorphism of $\mathscr{O}_X$ defined by the formula: $\sigma'(a) = e_i^{(s)} \sigma(a) - \sigma(e_i^{(ps)} a)$ for every section a of $\mathscr{O}_X$. We claim that σ' is a Frobenius morphism. Recall that:

$$e_i^{(r)} a^p = 0 \text{ for every integer } r \notin p\mathbb{N},$$

$$e_i^{(pr)} a^p = (e_i^{(r)} a)^p.$$

We have:

$$\sigma'(a^p\, b) = e_i^{(s)}\,\sigma\,(a^p\, b) - \sigma\,(e_i^{(ps)}\,(a^p\, b))$$

$$= e_i^{(s)}\,(a\,\sigma\,(b)) - \sum_{s'+s''s}\sigma\,((e_i^{(s')}\,a)^p\,(e_i^{(ps'')}\,b))$$

$$= \sum_{s'+s''s}(e_i^{(s')}\,a)\,(e_i^{(s'')}\,\sigma\,(b)) - (e_i^{(s')}\,a)\,\sigma\,(e_i^{(ps'')}\,b)).$$

By the induction hypothesis we have: $e_i^{(s'')}\,\sigma\,(b) = \sigma\,(e_i^{(ps'')}\,b)$ whenever $s'' < s$. So we have:

$$\sigma'(a^p\, b) = a e_i^{(s)}\,\sigma\,(b) - a\,\sigma\,(e_i^{(ps)}\,b)$$

$$= a\,\sigma'(b).$$

Note that we have $\sigma'(\mathscr{I}) \subseteq \mathscr{I}$. Hence we have $\sigma' \in SF(X, Z)$. But σ' is a vector of weight $ps\,\alpha_i$. (We should note that the action of H on the image is shifted by the absolute Frobenius F.) As the multiplicity of the weight $ps\,\alpha_i$ in $SF(X, Z)$ is zero (Lemma 2.3, Point 3), we get $\sigma' = 0$. This proves the assertion (2) when $\mathscr{L}$ is the structural sheaf on X. The general case follows easily by application of rules R_1 and R_2.

Remark. — Note as a corallary of the previous lemma that σ maps B-modules on B-modules (see the proof of Lemma 4.3).

3. Filtration of B-algebras

Conventions: We will fix a linear form $E: X(H) \to \mathbb{R}$ *such that* $E(\alpha) > 0$ *for every positive root* α. *For simplicity, we will assume that E is injective (it is easy to show that there exists such a form* E).

Definition of the filtration: Let $M \in \mathscr{C}(B)$ *and let* $x \in \mathbb{R}$. *Let* $\mathscr{F}_x M$ *(respectively* $\mathscr{F}_x^- M$) *be the largest* B-*submodule of* M, *all of whose weights* λ *satisfies* $E(\lambda) \leq x$ *(respectively* $E(\lambda) < x$).

Hence $M \mapsto \mathscr{F}_x M$, $M \mapsto \mathscr{F}_x^- M$ are functors defined on the category $\mathscr{C}(B)$. Set $\mathscr{H}_x M = \mathscr{F}_x M / \mathscr{F}_x^- M$ (the notations agree with 1.11). For every $\lambda \in X(B)$, set $I(\lambda) = \mathrm{Ind}_H^B \lambda$.

LEMMA 3.1. — *Let* M, $N \in \mathscr{C}(B)$ *and let* $x, y \in \mathbb{R}$.

1) *For every one-to-one morphism* $M \to N$, *the associated morphism* $\mathscr{H}_x M \to \mathscr{H}_x N$ *is one-to-one.*

2) *We have* $(\mathscr{F}_x M) \otimes (\mathscr{F}_y N) \subseteq \mathscr{F}_{x+y}(M \otimes N)$. *This induces a functorial morphism* $(\mathscr{H}_x M) \otimes (\mathscr{H}_y N) \to \mathscr{H}_{x+y}(M \otimes N)$.

3) *If* M *is a G-module, then* $\mathscr{F}_x M$, $\mathscr{F}_x^- M$ *and* $\mathscr{H}_x M$ *are G-modules.*

4) *If* $E^{-1}(x) = \varnothing$, *then* $\mathscr{H}_x = 0$. *If* $E^{-1}(x) \neq \varnothing$, *then* $E^{-1}(x)$ *contains exactly one element* λ, *and the socle of the B-module* $\mathscr{H}_x M$ *is isotypical of type* λ.

5) *If* M *is an injective* B-*module, then the* B-*modules* $\mathscr{F}_x M$, $\mathscr{F}_x^- M$ *and* $\mathscr{H}_x M$ *are injective.*

Proof. — Point 1 of the lemma comes from the following equalities: $\mathscr{F}_x N \cap M = \mathscr{F}_x M$, $\mathscr{F}_x^- N \cap M = \mathscr{F}_x^- M$. Point 2 is obvious. Let M be a G-module. Note that the G-module M′ generated by $\mathscr{F}_x M$ is equal to $\mathscr{U}(U^- \mathscr{F}_x M$ (by Poincaré-Birkhoff-Witt Theorem). Hence for every weight μ′ of M′, there exists a weight μ of $\mathscr{F}_x M$ such that μ − μ′ is a linear combination with positive coefficients of positive roots. Hence we have $E(\mu') \leqq E(\mu) \leqq x$, and $M' \subseteq \mathscr{F}_x M$. That proves Point 3. Point 4 is clear, because we have: $\mathscr{F}_x^- \mathscr{H}_x M = 0$. Note that the functors $\mathscr{F}_x^-$, $\mathscr{F}_x$ and $\mathscr{H}_x$ are additive. So it suffices to prove Point 5 for injective indecomposable modules $I(\mu)$. We have:

$$\mathscr{F}_x I(\mu) = I(\mu) \ \textit{if } E(\mu) \leqq x \quad \textit{and} \quad \mathscr{F}_x I(\mu) = 0 \ \textit{if not},$$

$$\mathscr{F}_x^- I(\mu) = I(\mu) \ \textit{if } E(\mu) < x \quad \textit{and} \quad \mathscr{F}_x^- I(\mu) = 0 \ \textit{if not},$$

$$\mathscr{H}_x I(\mu) = I(\mu) \ \textit{if } E(\mu) = x \quad \textit{and} \quad \mathscr{H}_x I(\mu) = 0 \ \textit{if not}.$$

Q.E.D.

Let $x \in \mathbb{R}$. Suppose that $E^{-1}(x)$ contains a (unique) weight λ. We will denote by $\mathscr{C}(B, x)$ the category of B-modules whose socle is isotypical of type λ. When $E^{-1}(x)$ is empty, then we will denote by $\mathscr{C}(B, x)$ the category containing only $\{O\}$. Let $\mathrm{Alg}(B, x)$ be the category of not necessarily unitary graded B-algebras $\mathscr{A} = \bigoplus_{n \geq 0} \mathscr{A}_n$ such that for every $n \in \mathbb{N}$, we have $\mathscr{A}_n \in \mathscr{C}(B, nx)$. Set $\mathscr{C}(G, x) = \mathscr{C}(B, x) \cap \mathscr{C}(G)$, $\mathrm{Alg}(G, x) = \mathrm{Alg}(B, x) \cap \mathscr{C}(G)$.

LEMMA 3.2. — 1) *Let* $V \in \mathscr{C}(B, x)$. *The* B-*module* V *has a unique injective envelop* $C_x V$ *(so* $V \mapsto C_x V$ *is a functor). We have* $C_x V \in \mathscr{C}(B, x)$.

2) *Let* $V \in \mathscr{C}(G, x)$ *and let* E *be an injective envelop of the* G-*module* V. *Then the module* $\bar{C}_x V = \mathscr{F}_x E$ *does not depend on a choice for* E *(so* $V \mapsto C_x V$ *is a functor). We have* $\bar{C}_x V \in \mathscr{C}(G, x)$.

3) *Let* $V \in \mathscr{C}(G, x)$. *There exists a unique morphism* $\bar{C}_x V \mapsto C_x V$ *which factors* $V \mapsto C_x V$ *(so we get a morphism of functors* $\bar{C}_x \to C_x$*). Moreover the morphism is one-to-one.*

4) *Let* $y \in \mathbb{R}$, *let* $V \in \mathscr{C}(B, x)$, *let* $V' \in \mathscr{C}(B, y)$. *We then have a natural isomorphism* $\mathscr{H}_{x+y}(C_x V \otimes C_y V') = C_{x+y} \mathscr{H}_{x+y}(V \otimes V')$. *Moreover if* V, V′ *are G-modules we have:* $\mathscr{H}_{x+y}(\bar{C}_x V \otimes \bar{C}_y V') = \bar{C}_{x+y} \mathscr{H}_{x+y}(V \otimes V')$.

Proof. — 1) Let F be an injective envelop of V. The set of injective envelop of V is classified by the group Γ of automorphisms of F which act trivially on V. It is clear that F and V have the same socle. In particular $F \in \mathscr{C}(B, x)$. Let $\gamma \in \Gamma$, and let $F' = (1 - \gamma) F$. We have $F' \subseteq \mathscr{F}_x^- F$, hence we have $F' = 0$ and $\Gamma = 1$. So the injective envelop is unique.

For Point 2, the proof is similar: we have to prove that every isomorphism γ of E which acts trivially on V acts also trivially on $\mathscr{F}_x E$. Point 3 is obvious.

By Lemma 3.1 (Point 2), there exists a functorial morphism

$$\mathcal{H}_{x+y}(V \otimes V') \to \mathcal{H}_{x+y}(C_x V \otimes C_y V').$$

We can suppose that there exist λ, $\mu \in X(B)$ such that $E(\lambda) = x$, $E(\mu) = y$. Note that the natural morphism $I(\lambda) \otimes I(\mu) \to I(\lambda + \mu)$ is onto. So $\mathcal{H}_{x+y}(C_x V \otimes C_y V')$ is a injective B-module. Hence there exists a natural map

$$\pi: C_{x+y} \mathcal{H}_{x+y}(V \otimes V') \to \mathcal{H}_{x+y}(C_x V \otimes C_y V').$$

It is clear that the restriction to socles of π is an isomorphism. But π maps an injective module on another injective module. Hence π is an isomorphism. The same proof holds for G-modules, because the morphism $F(\lambda) \otimes F(\mu) \to F(\lambda + \mu)$ is onto (*see* [RR] Theorem 1; for a classical group G, the result is also consequence of the standard monomial theory [LS]).

LEMMA 3.3. — *Let* $x \in \mathbb{R}$.

1) *Let* $A \in \mathrm{Alg}(B, x)$. *Set* $N_x A = \bigoplus_{n \geq 0} C_{nx} A_n$. *Then there exists a unique structure of B-algebra on* $N_x A$, *extending the algebra structure of* A (*so* $A \mapsto N_x A$ *defines a functor* $N_x: \mathrm{Alg}(B, x) \to \mathrm{Alg}(B, x)$).

2) *Let* $A \in \mathrm{Alg}(G, x)$. *Set* $\bar{N}_x A = \bigoplus_{n \geq 0} \bar{C}_{nx} A_n$. *Then there exists a unique structure of G-algebra on* $\bar{N}_x A$, *extending the algebra structure of* A (*so* $A \mapsto \bar{N}_x A$ *is a functor* $\bar{N}_x: \mathrm{Alg}(G, x) \to \mathrm{Alg}(G, x)$). *The natural morphism* $\bar{N}_x A \to N_x A$ *is a morphism of algebras.*

3) *Let* $A \in \mathrm{Alg}(B, x)$. *If* A *is commutative* (*respectively commutative and reduced*) *then* $N_x A$ *is commutative* (*respectively commutative and reduced*).

Proof. — Let n, m be two integers. Note that $A_n \otimes A_m = \mathcal{F}_{(n+m)x}(A_n \otimes A_m)$. So $\mathcal{H}_{(n+m)x}(A_n \otimes A_m)$ is actually a quotient of $A_n \otimes A_m$. The morphisms $C_{nx} A_n \otimes C_{mx} A_m \to C_{(n+m)x} A_{n+m}$ are defined by the composition of the following natural morphisms (*cf.* lemma 3.1, 3.2)

$$C_{nx} A_n \otimes C_{mx} A_m \to \mathcal{H}_{(n+m)x}(C_{nx} A_n \otimes C_{mx} A_m)$$

$$\mathcal{H}_{(n+m)x}(C_{nx} A_n \otimes C_{mx} A_m) \to C_{(n+m)x} \mathcal{H}_{(n+m)x}(A_n \otimes A_m)$$

$$C_{(n+m)x} \mathcal{H}_{(n+m)x}(A_n \otimes A_m) \to C_{(n+m)x} A_{n+m}.$$

The construction for G-modules is exactly the same. We now prove that the map $\bar{N}_x A \to N_x A$ is a morphism of algebras. Denote by $\bar{\mu}$, μ be the multiplications of these algebras and let n, m be two integers. Set $\varphi: \bar{N}_x A_n \otimes \bar{N}_x A_m \to N_x A_{n+m}$ be the morphism $\varphi = \bar{\mu} - \mu$. Note that φ factorizes by $\mathcal{H}_{(n+m)x}(\bar{N}_x A_n \otimes \bar{N}_x A_m)$ and that φ vanishes on the socle of $\mathcal{H}_{(n+m)x}(\bar{N}_x A_n \otimes \bar{N}_x A_m)$. Hence $\mathrm{Im}\,\varphi \subseteq \mathcal{F}_{(n+m)x}^- N_x A_{n+m}$ and so we have $\varphi = 0$, $\bar{\mu} = \mu$.

The proof of Point 3 is similar.

KEY LEMMA 3.4. — (criterion for good filtrations) *Let* A *be a commutative and reduced algebra in* Alg (G, *x*). *Suppose that the morphism* A → $\bar{N}_x$ A *is not an isomorphism. Then there exists an integer* $m > 0$, *and* $u \in \bar{N}_x A_m$ *such that* $u \notin A$ *and* $u^p \in A$.

Proof. — We first state a particular case of Lemma 13 of [M1]:

Let $\mathscr{A}_1 \supseteq \mathscr{A}_2$ *be finitely generated unitary graded commutative algebras over* k. *Suppose that the morphism* Spec $\mathscr{A}_1$ → Spec $\mathscr{A}_2$ *is a finite homeomorphism, but that* $\mathscr{A}_1 \neq \mathscr{A}_2$. *Then there exists an homogeneous element* u *with* $u \in \mathscr{A}_1$, $u \notin \mathscr{A}_2$ *and* $u^p \in \mathscr{A}_2$ (set: $\tilde{R} = R = S = \mathscr{A}_2$ *and* $\tilde{S} = \mathscr{A}_3$ *in Lemma 13 of* [M1]).

We can suppose that $A_0 = k$ and $A_1 \neq \bar{C}_x A_1$. Let $\lambda \in X(H)$ be the weight such that $E(\lambda) = x$. There exists a G-submodule $X \subseteq \bar{C}_x A_1$ such that X is isomorphic to $F(\lambda)$ and X is not contained in A_1. Let Y be the socle of the G-module X. Let $\mathscr{A}_1$ (respectively $\mathscr{A}_3$) be the subalgebra generated by X (respectively by Y). Set $\mathscr{A}_2 = \mathscr{A}_1 \cap A$. By construction, we have $\mathscr{A}_1 \supseteq \mathscr{A}_2 \supseteq \mathscr{A}_3$.

It is clear that $\mathscr{A}_1$ is isomorphic to the Cartan algebra $\underset{n \geq 0}{\oplus} F(m\lambda)$. Let Σ be the orbit in X* of the highest weight line, and let L be the kernel of $\pi : X^* \to Y^*$. Set $\Sigma' = \pi(\Sigma)$. We have $L \cap \Sigma = 0$ and Σ, Σ' are cones. Hence the induced morphism $\chi : \Sigma \to \Sigma'$ is finite. Moreover χ is an homeomorphism. Note that Spec $\mathscr{A}_1 = \Sigma$, Spec $\mathscr{A}_3 = \Sigma'$. Since Spec $\mathscr{A}_1$ → Spec $\mathscr{A}_3$ is a finite homeomorphism, the morphism Spec $\mathscr{A}_1$ → Spec $\mathscr{A}_2$ is a finite homeomorphism. Hence by the previous assertion, we get an integer $m > 0$, and $u \in \bar{N}_x A_m$ such that $u \notin A$ and $u^p \in A$.

Remark. — Let M be a finite dimensional G-module. Note that for all but finite many $x \in \mathbb{R}$, we have $\mathscr{H}_x M = 0$. Hence the finite family $\mathscr{F}_x M$ is a filtration of M whose successive quotients are $\mathscr{H}_x M$. Hence M has a god filtration whenever $\bar{C}_x \mathscr{H}_x M = \mathscr{H}_x M$. The converse (not used in what follows), is also true, and we can see that the previous lemma give a criterion for good filtrations for the homogeneous components of graded G-algebras.

4. Proof of the theorem

Let G′ be a semisimple subgroup of G corresponding to a Dynkin subdiagram, let Q = G′ B be the corresponding parabolic group (so G′ is the Levi component of Q), let B′ = G′ ∩ B be the standard Borel subgroup of G′, let S ⊇ S′ be two Q-invariant $\mathscr{S}$-schemes, and let $\mathscr{L}$ be a Q-equivariant invertible sheaf over S. Let $\mathscr{I}$ be the ideal defining S′ in S. In this section we will prove the following result:

THEOREM 3. — *The* G′-*module* M = Γ(S, $\mathscr{L} \otimes \mathscr{I}$) *has a good filtration.*

Remarks. — 1) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of G′-modules. Recall that if M′ and M have good filtrations, then M″ has a good filtration (Wang [W] Proposition 3.3). Let Z be a G′-invariant $\mathscr{S}$-subscheme of S. By Wang's Lemma, the G′-module image of Γ(S, $\mathscr{L} \otimes \mathscr{I}$) → Γ(Z, $\mathscr{L} \otimes \mathscr{I}$) has a good filtration. Hence Theorem 3 implies Theorem 2 stated in the introduction.

2) It is possible to show a similar result for the sections of $\mathscr{L} \otimes \mathscr{I}$ over any Q-invariant open subset of X.

Let ω' be the maximal element of the Weyl group of G', let $\Omega = B \omega' B$ be the big cell of Q and let $H' = B' \cap \mathrm{Ad}(\omega')(B')$ be the Cartan group of G'. In order to prove the theorem, we will first reduce to a simpler case:

LEMMA 4.1. — *We can assume that there exist $\mathscr{S}$-schemes $\Sigma \supseteq \Sigma'$ such that $S = Q \times^B \Sigma$, $S' = Q \times^B \Sigma'$.*

Proof. — Let Z be an arbitrary B-scheme and let j be the canonical morphism $j: Z \to Q \times^B Z$. The inverse image functor j^* induces an equivalence of categories between the category of B-equivariant quasicoherent sheaves on Z and the category of Q-equivariant quasicoherent sheaves on $Q \times^B Z$. Let $\mathscr{D}'$ be its inverse functor and let D' be the induction fuctor from B to Q. Note that $X = Q \times^B S$ and $X' = Q \times^B S'$ are $\mathscr{S}$-schemes, note that $\mathscr{I}' = \mathscr{D}' \mathscr{I}$ is the ideal defining X' in X and note that $\mathscr{L}' = \mathscr{D}' \mathscr{L}$ is an invertible sheaf of X. We have $D' \Gamma(S, \mathscr{L} \otimes \mathscr{I}) = \Gamma(X, \mathscr{L}' \otimes \mathscr{I}')$. Since M is already a Q-module, we have $M = D' M$. Using X, X' instead of S, S', we reduce the proof of the theorem to the case stated in the lemma.

Hence the lemma is proved and we will suppose from now on that the assumption of lemma 4.1 holds. Set $V = \Omega \times^B \Sigma$. For every integer $m > 0$, set $A_m = \Gamma(V, \mathscr{I} \otimes \mathscr{L}^{\otimes m})$ and set $\bar{A}_m = \Gamma(S, \mathscr{I} \otimes \mathscr{L}^{\otimes m})$. Consequently, the groups $A = \underset{m > 0}{\oplus} A_m$ and $\bar{A} = \underset{m > 0}{\oplus} \bar{A}_m$ are non unitary algebras.

LEMMA 4.2. — *The B'-module A is injective.*

Proof. — Let $g: \Sigma \to V$ be the morphism given by the formula $g(\sigma) = \omega' j(\sigma)$. We have $V \simeq B' \times^{H'} g(\Sigma)$. Hence we have $A_m = \mathrm{Ind}_{H'}^{B'} \Gamma(g(\Sigma), g^*(\mathscr{I} \otimes \mathscr{L}^{\otimes m}))$. So each B'-module A_m is injective, and A is an injective B'-module. Q.E.D.

Choose a linear form $E': X(H') \to \mathbb{R}$, as in section 3. We will apply the notations and the results of Section 3 to the semi-simple group G' and to its Borel subgroup B'. Let $x \in \mathbb{R}$. Set:

$$A(x) = \underset{m > 0}{\oplus} \mathscr{F}_{mx} A_m,$$

$$\bar{A}(x) = \underset{m > 0}{\oplus} \mathscr{F}_{mx} \bar{A}_m,$$

$$A^-(x) = \underset{m > 0}{\oplus} \mathscr{F}_{mx}^- A_m$$

$$\bar{A}^-(x) = \underset{m > 0}{\oplus} \mathscr{F}_{mx}^- \bar{A}_m.$$

By Lemma 3.1 (Point 2), $A(x)$ and $\bar{A}(x)$ are non unitary subalgebras of A, and $A^-(x)$ [respectively $\bar{A}^-(x)$] is an ideal of $A(x)$ [respectively of $A(x)$]. Set $\mathscr{A}(x) = A(x)/A^-(x)$ and $\bar{\mathscr{A}}(x) = \bar{A}(x)/\bar{A}^-(x)$. The algebras $\mathscr{A}(x)$, $\bar{\mathscr{A}}(x)$ are endowed with a natural grading. For every integer $m > 0$, let $\mathscr{A}_m(x)$, $\bar{\mathscr{A}}_m(x)$ be their homogeneous components of degree m.

Key Lemma 4.3. — *(compatibility of the splitting σ with the filtration) For every $x \in \mathbb{R}$, we have $\sigma(A(x)) \subseteq A(x)$ and $\sigma(\bar{A}(x)) \subseteq \bar{A}(x)$.*

Proof. — We will prove that $\sigma(A(x)) \subseteq A(x)$ (the second assertion is proved similarly). Note that for every integer m not divisible by p, we have $\sigma(A_m) = 0$. Hence it suffices to prove that $\sigma(\mathscr{F}_{pmx} A_{pm}) \subseteq \mathscr{F}_{mx} A_m$, for every $m > 0$. Set $N = \mathscr{F}_{pmx} A_{pm}$. Let I' be the Dynkin subdiagram corresponding to G' and let $U' = (B', B')$. By the key Lemma 2.4, we have:

$$e_i^{(s)} \cdot \sigma(n) = \sigma(e_i^{(ps)} \cdot n), \qquad \text{for every} \quad i \in I', s \in \mathbb{N}, n \in N.$$

Hence $\sigma(N)$ is a U'-module (lemma 1.1). Moreover, for every weight $\lambda \in X(H')$, for every $n \in N_\lambda$ we have $\sigma(n) = 0$ if $\lambda \notin p\,X(H')$ and $\sigma(n)$ has weight λ/p whenever $\lambda \in p\,X(H')$. Hence $\sigma(N)$ is a B'-module, for which every weight μ satisfies $E(\mu) \leq mx$. So we have $\sigma(A(x)) \subseteq A(x)$.

Lemma 4.4. — *Let $x \in \mathbb{R}$.*

1) *The algebra morphism $\bar{\mathscr{A}}(x) \to \mathscr{A}(x)$ is one-to-one.*

2) *The B'-module $\mathscr{A}(x)$ is injective.*

3) *We have $\bar{\mathscr{A}}(x) \in \mathrm{Alg}(G', x)$ and $\mathscr{A}(x) \in \mathrm{Alg}(B', x)$.*

4) *The Frobenius splitting σ of A induces a Frobenius splitting of $\mathscr{A}(x)$ and of its subalgebra $\bar{\mathscr{A}}(x)$.*

Proof. — The points 1, 2 arise from Lemma 3.1 (Point 1 and 5 respectively). Point 3 of the lemma results from points 2 and 3 of Lemma 3.1. By the key Lemma 4.3, we have $\sigma(A(x)) \subseteq A(x)$ and $\sigma(A^-(x)) \subseteq A^-(x)$. Hence the Frobenius splitting σ acts on the quotient algebra $\mathscr{A}(x)$. The assertion for the algebra $\bar{\mathscr{A}}(x)$ is similar.

Proof of the theorem. — To prove the theorem, it suffices to prove that for every $x \in \mathbb{R}$, we have $\bar{C}_x \mathscr{H}_x M = \mathscr{H}_x M$. Note that $\mathscr{H}_x M = \bar{\mathscr{A}}_1(x)$. We also have $\bar{\mathscr{A}}(x) \in \mathrm{Alg}(G', x)$ (lemma 4.4). Hence it suffices to prove the more general assertion: $\bar{N}_x \bar{\mathscr{A}}(x) = \bar{\mathscr{A}}(x)$.

We first prove that there exists a natural immersion $\bar{N}_x \bar{\mathscr{A}}(x) \subseteq \mathscr{A}(x)$. We have $\mathscr{A}(x) \in \mathrm{Alg}(B', x)$ (Lemma 4.4) and the B'-module $\mathscr{A}(x)$ is injective (Lemma 4.4). The composition of natural morphisms involved in Lemma 3.3: $\bar{N}_x \bar{\mathscr{A}}(x) \hookrightarrow N_x \bar{\mathscr{A}}(x)$ and $N_x \bar{\mathscr{A}}(x) \to N_x \mathscr{A}(x)$ is a morphism $h: \bar{N}_x \bar{\mathscr{A}}(x) \to N_x \mathscr{A}(x)$. By Lemma 3.2 (Point 3) and Lemma 4.4 (Point 1), the morphism h is one-to-one. The injectivity of the module $\mathscr{A}(x)$ implies that we have $N_x \mathscr{A}(x) = \mathscr{A}(x)$. So we get the required immersion, and we have $\bar{\mathscr{A}}(x) \subseteq \bar{N}_x \bar{\mathscr{A}}(x) \subseteq \mathscr{A}(x)$.

The algebra $\mathscr{A}(x)$ and its subalgebra $\bar{\mathscr{A}}(x)$ have a Frobenius splitting σ (Lemma 4.4, Point 4). So by Ramanathan's Lemma 2.1, these algebras are reduced. Let u be a homogeneous element of $\bar{N}_x \bar{\mathscr{A}}(x)$ such that u^p belongs to $\bar{\mathscr{A}}(x)$. In particular, we have $u \in \mathscr{A}(x)$ and $u^p \in \bar{\mathscr{A}}(x)$. So we get $u = \sigma(u^p) \in \bar{\mathscr{A}}(x)$. Hence by the key Lemma 3.4, we have $\bar{N}_x \bar{\mathscr{A}}(x) = \bar{\mathscr{A}}(x)$.

5. Filtrations of B-modules

For B-modules, there are four generalizations of the notion of good filtrations. Let $x, y \in W$ with $x \geq y$, let $\lambda \in X^{+}(B)$ and let S be a Schubert scheme of G/B. Let δS_x be the union of codimension 1 Schubert subvarieties in S_x. Set:

$$F_x(\lambda) = \Gamma(S_x, \mathscr{L}(-\lambda))$$

$$F_S(\lambda) = \Gamma(S, \mathscr{L}(-\lambda))$$

$$R_x(\lambda) = \text{the kernel of } F_x(\lambda) \to F_{\delta S_x}(-\lambda)$$

$$K_{x,y}(\lambda) = \text{the kernel of } F_x(\lambda) \to F_y(\lambda).$$

We define the category $\mathscr{S} \subset \mathscr{C}(B)$ (respectively $\mathscr{W}, \mathscr{H}, \mathscr{K}$) to be the category of B-modules which have a filtration whose subquotients are some $F_x(\lambda)$ (respectively: $R_x(\lambda)$, $F_S(\lambda)$, $K_{x,y}(\lambda)$). The class $\mathscr{S}$ (respectively $\mathscr{W}$, the classes $\mathscr{H}$, $\mathscr{K}$) was introduced by A. Joseph [J] (respectively by W. van der Kallen [vdK1], by P. Polo [P1], [P2]). The modules of the category $\mathscr{S}$, respectively $\mathscr{W}$, are called strong, respectively weak (in the literature, strong modules are also called excellent or Joseph modules). Following [vdK1], we have $\mathscr{S} \subset \mathscr{H}$, $\mathscr{K} \subset \mathscr{W}$. By induction from B to G, it is easy to show that any G-module $M \in \mathscr{W}$ has a good filtration. Conversely any G-module which has a good filtration is automatically strong. Hence the categories of B-modules $\mathscr{S}, \mathscr{W}, \mathscr{H}, \mathscr{K}$ are generalizations of the notion of good filtration.

COROLLARY 1. — *Let* $M \in \mathscr{C}(B)$, *and let* $\lambda \in X^{+}(B)$. *If we have:*

$M \in \mathscr{S}$ [*respectively:* (2) $M \in \mathscr{W}$, (3) $M \in \mathscr{H}$, (4) $M \in \mathscr{K}$], *then we have:*

$M[-\lambda] \in \mathscr{S}$ (*respectively:* (2) $M[-\lambda] \in \mathscr{W}$, (3) $M[-\lambda] \in \mathscr{H}$, (4) $M[-\lambda] \in \mathscr{K}$).

COROLLARY 2. — *Let* M *be a weak* B-*module and let* λ *be a dominant and regular weight. Then the* B-*module* $M[-\lambda]$ *is strong.*

Remark. — As for Theorem 1, the corollaries were already known for almost all cases. The point 1 of corollary 1 was first proved for $SL(n)$ by P. Polo ([P1], corollary 4.11), and it was conjectured by him for any group G (the special case: $F(\lambda)[-\mu] \in \mathscr{S}$ was already conjectured by A. Joseph [J]). Later, both corollaries were proved for any group G when p is large (*see* [M2], Theorem 1, [M3], Theorem 1 and its remark) and they were announced for any p when G is a group without components of type F_4, E_7, E_8 [P3]. As pointed out by P. Polo, Corollary 1 is actually a generalization of Theorem 1, because inducing a strong filtration gives a good filtration.

We will recall two lemmas in order to prove the corollaries.

LEMMA 5.1. — *The statement of the corollary 1 is actually equivalent to the following one: For every* $\lambda, \mu, \nu \in X^{+}(H)$, *the* G-*module* $F(\lambda) \otimes F(\mu) \otimes F(\nu)$ *has a good filtration compatible with any special double scheme.*

Proof. — By van der Kallen Criteria for strong and for weak B-modules (Theorem 3.1 and 3.5 of [vdK1]), Assertions (1) and (2) of Corollary 1 are equivalent. Similarly, by Polo Criteria for classes $\mathscr{H}$ and $\mathscr{K}$ ([P2], [P3]) Assertions (3) and (4) of Corollary 1 are

equivalent. Moreover the statement in Lemma 5.1 is equivalent to Assertions (1) and (3) of the corollary (Proposition 4 of [M4]). Hence the lemma is proved.

LEMMA 5.2. − *Any double Schubert scheme Σ is a G-invariant generalized Schubert scheme in $(G/B)^3$.*

Proof. − It suffices to prove the lemma when Σ is irreducible. Let x, y be the elements of W such that $\Sigma = G \times^B (S_x \times S_y)$. We have a natural isomorphism $\Sigma \simeq S_{\omega, x-1, y}$.

<div align="right">Q.E.D.</div>

Proof of Corollary 1. − By theorem 3, the modules $F(\lambda) \otimes F(\mu) \otimes F(\nu)$ have good filtrations compatible with G-invariant generalized Schubert schemes. Hence Lemmas 5.1 and 5.2 imply Corollary 1.

Proof of Corollary 2. − The proof is similar, but we need to work more because the analog of Lemma 5.1 is not stated in [M2]. Let D be the induction functor from B to G. By van der Kallen Criterion for strong modules (Theorem 3.1 and 3.5 of [vdK1]), it suffices to prove that for every $x, y \in W$, $\mu, \nu \in X^+(B)$, we have

$$H^1(B, R_x(\mu) \otimes R_y(\nu)[-\lambda]) = 0.$$

Set $N = R_x(\mu) \otimes R_y(\nu)[-\lambda]$. By [M2] (Lemma 16), we have $R^1 DN = 0$. Hence it suffices to prove that $H^1(G, DN) = 0$. Set $\Sigma = G \times^B (S_x \times S_y)$, set $\Sigma' = G \times^B ((S_x \times \delta S_y) \cup (\delta S_x \times S_y))$ and set $\mathscr{L} = \mathscr{L}(-\lambda, -\mu, -\nu)$. Moreover, by [M2] (lemma 16), the G-module DN can be identified with the kernel of the restriction map $\Gamma(\Sigma, \mathscr{L}) \to \Gamma(\Sigma', \mathscr{L})$. By Lemma 5.2, the schemes Σ, Σ' are generalized Schubert schemes. Hence by Theorem 3, the module DN has a good filtration. Hence we have $H^1(G, DN) = 0$.

Conclusion

We would like to mention that there are a few statements in the literature which were proved to be a consequence of Theorem 1 or of Corollary 1. Such statements are for example van der Kallen Conjectures 3.12 on vanishings of functors O_π and Long [vdK1] (*see* also the last chapter of [D1] and Propositions 1 and 2 of [M3]). It is now clear that Theorem 2 of [M3] means some compatibility of good filtrations (and of their generalizations for B-modules) with some Frobenius splittings. We used Kac-Moody groups in its proof. Actually it is likely that methods of the present paper can give a simpler proof.

REFERENCES

[A1]   H. H. ANDERSEN, *The Frobenius morphism on the cohomology of homogeneous vector bundles on G/B*. Ann. Math., 112, 1980, pp. 113-121.

[A2]   H. H. ANDERSEN, *Schubert varieties and Demazure's character formula*, Invent. Math., 79, 1985, pp. 611-618.

[A3]   H. H. ANDERSEN, *The strong linkage principle*. J. Reine Angew Math., 315, 1980, pp. 53-59.

[B]   N. BOURBAKI, *Groupes et algèbres de Lie* (ch. 7 et 8), Diffusion C.C.L.S., Paris, 1975.

[D]   M. DEMAZURE, *Désingularisation des variétés de Schubert généralisées*. Ann. Scient. Ec. Norm. Sup., 7, 1974, pp. 53-88.

[D1]   S. DONKIN, *Rational representations of algebraic groups*. Springer Verlag, Lect. Notes Math., 1140, 1985.

[D2]   S. DONKIN, *A filtration for rational modules*, Math. Z., 117, 1981, pp. 1-8.

[D3]   S. DONKIN, *Good filtrations of rational modules for reductive groups*, Proc. Symp. Pure Math. A.M.S., 177, part 1, 1987, pp. 69-80.

[D4]   S. DONKIN, *Invariants of Unipotent Radicals*. Math. Z., 198, 1988, pp. 117-125.

[D5]   S. DONKIN, *On conjugating representations and adjoint representations of semi-simple groups*, Inv. Math., 91, 1988, pp. 137-145.

[F]   E. FRIEDLANDER, *A canonical filtration for certain rational modules*. Math. Z, 188, 1985, pp. 433-438.

[H]   W. HABOUSH, *A short proof of the Kempf vanishing theorem*. Invent. Math., 56, 1980, pp. 109-112.

[H1]   J. HUMPHREYS, *Ordinary and modular representations of Chevalley groups II*, unpublished, 1977.

[H2]   J. HUMPHREYS, *Linear algebraic groups*, G.T.M. 21, Springer-Verlag, Berlin, 1975.

[J1]   J. JANTZEN, *Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne*. J. Reine Angew. Math., 290, 1980, pp. 157-199.

[J2]   J. JANTZEN, *Representations of algebraic groups*, Academic Press, Orlando, 1987.

[J]   A. JOSEPH, *On the Demazure character formula*. Ann. Scient. Ec. Norm. Sup., 18, 1985, pp. 389-419.

[Ke]   G. KDMPF, *Linear systems on homogeneous spaces*, Ann. Math., 103, 1980, pp. 557-591.

[Ko]   B. KOSTANT, *Groups over $\mathbb{Z}$*. Proc. Symp. Pure Math. A.M.S., 9, 1966, pp. 90-98.

[Ku]   S. KUMAR, *Proof of the Parthasaraty-Ranga-Rao-Varadarajan conjecture*. Invent. Math., 93, 1988, pp. 117-130.

[LS]   V. LAKSHMIBAI, C. S. SESHADRI, *Geometry of G/P-5*. J. Alg., 100, 1986, pp. 462-557.

[L]   P. LITTELMANN, *A generalization of the Littlewood-Richardson rule* (to appear in J. Algebra).

[M1]   O. MATHIEU, *Construction d'un groupe de Kac-Moody et applications*, Comp. Math., 69, 1989, pp. 37-60.

[M2]   O. MATHIEU, *Filtrations of B-modules*. Duke Math. J., 59-2, 1989, pp. 421-442.

[M3]   O. MATHIEU, *Frobenius action on the B-cohomology*, Proc. of "Infinite dimensional Lie algebras and groups" (Luminy 1988), ed. V. Kac, World Scientific, Adv. Ser. Math. Phys., 7, 1989, pp. 39-51.

[M4]   O. MATHIEU, *Filtrations of G-modules*. C. R. Acad. Sci., Paris, 309, 1989, pp. 273-276.

[MG]   M. MCGOVERN, *On a theorem of Mathieu and Kumar* (to appear).

[MR1]   V. MEHTA, A. RAMANATHAN, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. Math., 122, 1985, pp. 27-40.

[MR2]   V. MEHTA, A. RAMANATHAN, *Schubert varieties in G/B × G/B*. Comp. Math., 67, 1988, pp. 355-358.

[P1]   P. POLO, *Variétés de Schubert et excellentes filtrations*. Proc. of "Orbites unipotentes et représentations" (Paris 1987), Astérisque (to appear).

[P2]   P. POLO, *Un critère cohomologique d'existence d'une filtration de Schubert*. C. R. Acad. Sci., Paris, 307, 1988, pp. 791-794.

[P3]   P. POLO, *Modules associés aux variétés de Schubert*. C. R. Acad. Sci., Paris, 308, 1989, pp. 37-60.

[RR]   S. RAMANAN, A. RAMANATHAN, *Projective normality of flag varieties and Schubert varieties*. Invent. Math., 79, 1985, pp. 217-224.

[R1]   A. RAMANATHAN, *Schubert varieties are arithmetically Cohen-Macaulay*. Invent. Math., 80, 1985, pp. 283-294.

[R2]   A. RAMANATHAN, *Equations defining Schubert varieties and Frobenius splitting of diagonals*. Publ. I.H.E.S., 65, 1987, pp. 61-90.

[U]   B. S. UPADHYADA, *Filtrations with Weyl module quotients of the principal indecomposable modules for the group* SL(2, *q*). Comm. Alg., 7, 1979, pp. 1469-1488.

[vdK1]   W. VAN DER KALLEN, *Longest weight vectors and excellent filtrations* (to appear).

[vdK2]   W. VAN DER KALLEN, *Letter to the author* (June 1988).

[W]   WANG JIAN-PIAN, *Sheaf cohomology on* G/B *and tensor products of Weyl modules*, J. Alg., 77, 1982, pp. 162-185.

O. MATHIEU,
U.A. 748 du C.N.R.S.,
28, boulevard de l'Hôpital,
75005 Paris, France.