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THE HOCHSCHILD COHOMOLOGY RING OF REGULAR MAXIMAL PRIMITIVE QUOTIENTS OF ENVELOPING ALGEBRAS OF SEMISIMPLE LIE ALGEBRAS

BY WOLFGANG SOERGEL

ABSTRACT. – Let U be the enveloping algebra of a semisimple complex Lie algebra, χ a regular maximal ideal of its center. We show that the Hochschild cohomology ring of $U/\chi U$ is just the coinvariant algebra of the Weyl group. This turns out to be almost immediate if one uses localization.

Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{h} a Cartan subalgebra, $U = U(\mathfrak{g}) \supset Z$ the enveloping algebra with its center. Let $S = S(\mathfrak{h}^*)$ be the regular functions on \mathfrak{h} and $S^+ \subset S$ the maximal ideal of all functions vanishing at $0 \in \mathfrak{h}$.

The Weyl group \mathcal{W} acts on S and we consider the coinvariant algebra $C = S/(S^+)^{\mathcal{W}}S$, the quotient of S by the ideal generated by all \mathcal{W} -invariant functions vanishing at zero.

THEOREM 1. – Let $\chi \subset Z$ be a regular maximal ideal, i.e. the kernel of a regular central character. Then the Hochschild cohomology ring of $U/\chi U$ is the coinvariant algebra with its obvious grading doubled,

$$HH^\bullet(U/\chi U) = C^{2\bullet}.$$

Remark. – I thank Patrick Polo for telling me this problem along with the expected answer, and for his helpful comments on a first version.

Proof. – By the Hochschild cohomology ring of an associative k -algebra A we mean just the ring $HH^\bullet(A) = \text{Ext}_{A \otimes_k A^{\text{opp}}}^\bullet(A, A)$ of self-extensions of A considered as an $A \otimes_k A^{\text{opp}}$ -module.

I now give an outline of the proof. Let X be the variety of all Borel subalgebras of \mathfrak{g} . Under localization [BB81] the bimodule $U/\chi U$ becomes a \mathcal{D} -module \mathcal{M} for some

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sheaf of twisted differential operators \mathcal{D} on $X \times X$ such that $X \times X$ is \mathcal{D} -affine, and we just need to compute the selfextensions of this \mathcal{D} -module \mathcal{M} . Now choosing \mathcal{D} correctly we can assume that

1. the restriction of \mathcal{D} to the open orbit $Y \subset X \times X$ for the diagonal action of G is just the standard sheaf of differential operators \mathcal{D}_Y on Y and
2. the \mathcal{D} -module \mathcal{M} is just the standard module (with unique simple quotient) $i_! \mathcal{O}_Y$, where i is the inclusion of Y into $X \times X$.

Once we know that, we find quickly

$$\mathrm{Ext}_{\mathcal{D}}^{\bullet}(\mathcal{M}, \mathcal{M}) = \mathrm{Ext}_{\mathcal{D}_Y}^{\bullet}(\mathcal{O}_Y, \mathcal{O}_Y) = H^{\bullet}(Y, \mathbb{C}),$$

the first step since $i_!$ is fully faithful even on derived categories, the last step by the Riemann-Hilbert-correspondence. Since Y is fibered over X with fibers just affine spaces, the cohomology ring of Y coincides with the cohomology ring of the flag manifold X which is known to be the ring C of coinvariants (with its obvious grading doubled).

Certainly 1 and 2 are known in much greater generality, they are just a special case of the correspondence between the classifications of Beilinson-Bernstein and Langlands. However, I want to avoid these heavy arguments and provide a more simple minded approach. So let us now start the work and find the sheaf of twisted differential operators (tdo for short) \mathcal{D} . We fix some notations concerning homogeneous twisted differential operators. I will follow [HMSW87], Appendix A.

For a complex algebraic group G and a closed subgroup $B \subset G$ and a B -invariant linear form λ on $\mathrm{Lie} B$ we have a sheaf \mathcal{D}_{λ} of homogeneous twisted differential operators on G/B . Let $\mathfrak{g} \supset \mathfrak{b}$ be the Lie algebras of $G \supset B$. The geometric stalk of \mathcal{D}_{λ} at the point $x = B$ of G/B is given as

$$\mathcal{D}_{\lambda} / \mathcal{D}_{\lambda} \mathfrak{m}_x = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} (\mathbb{C}_{\lambda} \otimes \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b})).$$

The ordinary differential operators are \mathcal{D}_0 . Suppose now $H \subset G$ is a closed subgroup and let $f : H/H \cap B \hookrightarrow G/B$ be the inclusion. Pulling back the tdo \mathcal{D}_{λ} via f we find $\mathcal{D}_{\lambda}^f = \mathcal{D}_{\mu}$ where μ is the restriction of f to $\mathrm{Lie}(H \cap B)$.

Let us return now to our situation. We fix a semisimple connected complex algebraic group G with $\mathrm{Lie} G = \mathfrak{g}$, let $T \subset G$ be the maximal torus with $\mathrm{Lie} T = \mathfrak{h}$ and fix two Borel subgroups $B^+, B^- \subset G$ containing T which are opposite, *i.e.* $B^+ \cap B^- = T$. Their Lie algebras will be denoted \mathfrak{b}^+ and \mathfrak{b}^- respectively. Any $\lambda \in \mathfrak{h}^*$ determines a homogeneous tdo \mathcal{D}_{λ}^+ on $X = G/B^+$ whose stalk at B^+ is the Verma module $M^+(\lambda) = U \otimes_{U(\mathfrak{b}^+)} (\mathbb{C}_{\lambda} \otimes \bigwedge^{\max}(\mathfrak{g}/\mathfrak{b}^+))$. If we put $\xi^+(\lambda) = \mathrm{Ann}_Z M^+(\lambda)$, then $\Gamma(\mathcal{D}_{\lambda}^+) = U/\xi^+(\lambda)U$. The same statements hold with $+$ replaced by $-$, and since the \mathfrak{h} -finite vectors in the (ordinary) dual of $M^+(\lambda)$ have the same \mathfrak{h} -weights as $M^-(-\lambda)$, we see directly that the principal antiautomorphism of \mathfrak{g} induces an isomorphism $U/\xi^+(\lambda)U \rightarrow (U/\xi^-(-\lambda)U)^{\mathrm{opp}}$. This way we associate to $\lambda \in \mathfrak{h}^*$ a tdo $\mathcal{D}_{(\lambda)} = \mathcal{D}_{\lambda}^+ \boxtimes \mathcal{D}_{-\lambda}^-$ on $X \times X$ with global sections $\Gamma(\mathcal{D}_{(\lambda)}) = U/\xi^+(\lambda)U \otimes (U/\xi^+(\lambda)U)^{\mathrm{opp}}$. We leave it to the interested reader to verify that $\mathcal{D}_{(\lambda)}$ and this identification of the global sections depend only on the linear form λ on the “abstract Cartan subgroup”.

The localization theorem [BB81] says that for a homogeneous tdo \mathcal{D} on X the functor of global sections is an equivalence of categories

$$\Gamma : \mathcal{D}\text{-mod} \rightarrow \Gamma(\mathcal{D})\text{-mod}$$

if and only if for one (equivalently any) point $x \in X$ the geometric stalk $\mathcal{D}/\mathcal{D}m_x$ at x is a simple \mathfrak{g} -module with regular central character. In this case X is called \mathcal{D} -affine. It is clear that for any regular maximal ideal $\chi \subset Z$ we can find $\lambda \in \mathfrak{h}^*$ such that $\xi^+(\lambda) = \chi$ and that $X \times X$ is $\mathcal{D}_{(\lambda)}$ -affine. This homogeneous tdo $\mathcal{D}_{(\lambda)}$ is the \mathcal{D} we were looking for.

To see this, remark first that by definition of λ the functor

$$\Gamma : \mathcal{D}_{(\lambda)}\text{-mod} \rightarrow U/\chi U\text{-mod-}U/\chi U$$

is an equivalence of categories. Furthermore, if $i : Y \hookrightarrow X \times X$ denotes the inclusion of the open G -orbit $Y = G(B^+, B^-) = G/T$, it is plain that $\mathcal{D}_{(\lambda)}$ pulls back to the ordinary differential operators $\mathcal{D}_{(\lambda)}^i = \mathcal{D}_Y$ on Y .

We have to check that $\Gamma(i_! \mathcal{O}_Y) = U/\chi U$. Remark first that since i is an open embedding, the direct and inverse images in the category of sheaves, \mathcal{O} -modules and \mathcal{D} -modules coincide. We thus have an adjoint pair of functors (i^*, i_*) between sheaves, \mathcal{O} -modules and \mathcal{D} -modules for any tdo \mathcal{D} on $X \times X$. Certainly i^* is exact, and since i is affine i_* is exact, too. Now $\Gamma(i_* \mathcal{O}_Y)$ coincides with $\Gamma(\mathcal{O}_Y)$ and it is clear that every nonzero $\mathcal{D}_{(\lambda)}$ -submodule of $i_* \mathcal{O}_Y$ contains the global section 1_Y on Y . Thus $i_* \mathcal{O}_Y$ has a simple socle. Let us define the adjoint \mathfrak{g} -action on a U -bimodule by the formula

$$(\text{ad}A)(m) = Am - mA.$$

It is then clear that the socle of the U -bimodule $\Gamma(i_* \mathcal{O}_Y)$ is simple and generated by an $(\text{ad}\mathfrak{g})$ -invariant line.

Now we study $i_! \mathcal{O}_Y$. Recall its definition from [Bei83]. If \mathcal{D} is a tdo on a smooth variety of dimension n and \mathcal{M} a holonomic \mathcal{D} -module, one defines its dual $*\mathcal{M} = \mathcal{R}Hom_{\mathcal{D}}^n(\mathcal{M}, \mathcal{D})$. This is a holonomic \mathcal{D}^{opp} -module, and $**\mathcal{M} = \mathcal{M}$ naturally. By definition $i_! \mathcal{O}_Y = *i_* \mathcal{O}_Y$. It is clear that in our case (i an open immersion) the duality commutes with i^* , whence an adjoint pair $(i_!, i^*)$ of functors between holonomic modules. It is known in general (and clear in our case) that $i_! \mathcal{O}_Y$ has as its unique simple quotient the socle of $i_* \mathcal{O}_Y$. Passing to global sections we see that $\Gamma(i_! \mathcal{O}_Y)$ is generated by an $(\text{ad}\mathfrak{g})$ -invariant line. From this we already find a surjection of bimodules

$$U/\chi U \twoheadrightarrow \Gamma(i_! \mathcal{O}_Y).$$

Now let us go to the category ${}_{\chi}\mathcal{H}_{\chi}$ of all locally $(\text{ad}\mathfrak{g})$ -finite $U/\chi U$ -bimodules. Under localization this corresponds to the category of all G -equivariant $\mathcal{D}_{(\lambda)}$ -modules. All objects of this category are mapped under i^* to a finite direct sum of some copies of \mathcal{O}_Y . Hence $i_! \mathcal{O}_Y$ is a projective object in our category, whence $\Gamma(i_! \mathcal{O}_Y)$ is a projective object of ${}_{\chi}\mathcal{H}_{\chi}$. But this means that the surjection $U/\chi U \twoheadrightarrow \Gamma(i_! \mathcal{O}_Y)$ is split, and since $U/\chi U$ is indecomposable it has to be an isomorphism. \square

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