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NOTE RECTIFICATIVE D'Y. GUIVARCH ET Y. LE JAN CONCERNANT L'ARTICLE

ASYMPTOTIC WINDING OF THE GEODESIC FLOW ON MODULAR SURFACES AND CONTINUOUS FRACTIONS

(paru dans le tome 26, fascicule 1, 1993, pp. 23-50)

The object of this note is to complete the proof of theorem 2.1 in [G-L]. The main steps of the proof are valid, but the replacement of the discrete coding time used in Proposition 4-3 by the time of the flow should have been done more carefully. We use the notations of [G-L].

We recall that the geodesic flow U_t on Γ/G has been presented as a special flow over $\hat{S} = [0, 1]^2 \times \mathbb{Z}/2\mathbb{Z} \times \Gamma_0/\Gamma$. The transformation $\hat{\theta}$ on $\hat{S}$ is an extension of the continuous fraction transformation and the height function Φ is explicit. We denote by $\hat{g}$ the projection of $g \in G$ on Γ/G and by $t_{m_k}(\hat{g}) = t_{m_k}(g)$ the sequence of return times to the section $\hat{S}$. The following points are essential in the proof.

a) Set $N_t(g) = \text{Inf}\{k \in \mathbb{N};\ t_{m_k}(\hat{g}) > t\}$, $\quad A(\hat{g}) = \int_{\hat{\gamma}_{t_{m_1}}(g)} \omega$.

One has $\int_{\hat{\gamma}^t(g)} \omega - \sum\limits_{1}^{N_t[g]-1} \psi\,[\hat{\theta}^n\,\hat{p}\,(g)] = A\,(\hat{g}) - A\,(\hat{g}\,U_t)$.

Because the Liouville measure is U_t-invariant, the laws of $A\,(\hat{g})$ and $A\,(\hat{g}\,U_t)$ are the same; hence $\frac{A\,(\hat{g})}{t}$ and $\frac{A\,(\hat{g}\,U_t)}{t}$ converge to zero in probability. The case of ω^c can be treated in the same way.

b) To reduce Theorem 2.1 to Proposition 4-3, we will use, in addition to Lemma 4.1 (Lemma 4.2 is incorrect) the tightness of $\frac{N_t\,(g) - \pi^2\,t/6\,\text{Log}\,2}{\sqrt{t}}$, as $t \uparrow \infty$.

To show this result, set $\ell = \pi^2/6\,\text{Log}\,2$ and note that:

$$N_t\,(g) - [\ell^{-1}\,t] \geq A\sqrt{t} \qquad \text{iff} \quad \tau_{[\ell^{-1}\,t]+[A\sqrt{t}]}\,(g) + T\,(g) \leq t$$

and that:

$$N_t\,(g) - [\ell^{-1}\,t] \leq -A\sqrt{t} \qquad \text{iff} \quad \tau_{[\ell^{-1}\,t]-[A\sqrt{t}]}\,(g) + T\,(g) \geq t.$$

Hence the result will follow from the tightness of the sequence $\frac{\tau_n - n\ell}{\sqrt{n}}$. Set $(\chi_-^{(n)}, -\chi_+^{(n)}) = \theta^n(\chi_-, \chi_+)$ (N.B. $\theta(\chi_-, \chi_+) = ((\chi_- + [\chi_+^{-1}])^{-1}, \chi_+^{-1} - [\chi_+^{-1}])$: a parenthesis was dropped in [G-L].) From the expression of Φ, it is enough to prove a central limit theorem for the sequences $\sum_1^n \mathrm{Log}(\chi_-^{(m)})$ and $\sum_1^n \mathrm{Log}(\chi_+^{(m)})$ under ν.

The law of $\{\chi_-^{(m)}, m \in \mathbb{N}\}$ is the law of the stationary Markov chain x_n studied in §5. The law of $(\chi_+^{(n)}, \ldots, \chi_+^{(1)}, \chi_+)$ given that $\chi_+^{(n)} = x$ coincides with the law of $(\chi_-, \chi_-^{(1)}, \ldots, \chi_-^{(n)})$ given that $\chi_- = x$.

Henceforth, it is sufficient to prove the central limit theorem for $\sum_1^n \mathrm{Log}(x_m)$. This can be done easily by following the steps given in §5, using the perturbed transfer operator Q'_λ defined by $Q'_\lambda u(x) = \sum_1^\infty p(x, k)\, e^{i\lambda \mathrm{Log}(k\cdot x)}\, u(k\cdot x)$ if we take into account the following remarks (*see* also [G-H]):

The function $\mathrm{Log}\, x$ on $[0, 1]$ is not Lipchitz as in [G-H] but the key properties of the operators are the same, namely.

1) Q'_λ acts on the space of Lipchitz functions on $[0, 1]$ and satisfies the basic inequality

$$\|Q'_\lambda u\| \le \frac{3}{8}\|u\| + C|u|_\infty$$

for some constant $C > 0$.

2) Q'_λ is an analytic family of operators ($|\lambda| < 1$). In particular, if we set

$$D' u(x) = \sum_1^\infty p(x, k)\, e^{i\lambda \mathrm{Log}(k\cdot x)}\, u(k\cdot x)$$

there exist a constant $C' > 0$ such that $\|Q'_\lambda - Q' - i\lambda D'\| \le C'|\lambda|^2$.

In order to get the required inequality for $\|Q'_\lambda u\|$ one needs to control the Lipchitz coefficient. One has:

$$|Q'_\lambda u(x) - Q'_\lambda u(x')| \le \sum_1^\infty p(x, k)\left|\frac{1}{(k+x)^{i\lambda}} - \frac{1}{(k+x')^{i\lambda}}\right| |u|_\infty$$
$$+ \sum_1^\infty |p(x, k) - p(x', k)|\,|u(k\cdot x)| + \sum_1^\infty p(x, k)|u(k\cdot x) - u(k\cdot x')|.$$

Clearly $\left|\frac{1}{(k+x)^{i\lambda}} - \frac{1}{(k+x')^{i\lambda}}\right| \le |\lambda|\,|x - x'|\,\frac{1}{k}$ hence the first term is bounded by $|\lambda|\,|x - x'|\,|u|_\infty \sum_1^\infty p(x, k)\,\frac{1}{k} \le C_1|\lambda|\,|u|_\infty\,|x - x'|$.

The other terms are bounded by $[C_2|u|_\infty + \frac{3}{8}\|u\|]|x - x'|$.

The analyticity of the family Q'_λ follows from the Cauchy formula on a loop ℓ in the unit disk:

$$\int_\ell Q'_z\, u\, dz = \int_\ell \sum_1^\infty p(x, k)\,\frac{1}{(x+k)^z}\, u(k\cdot x)\, dz.$$

If $|z| < 1 - \varepsilon$, one has $|p(x, k) \frac{1}{(x+k)^z}| \leq p(x, k)(k+1)^{1-\varepsilon}$.

Hence, summation and integration can be exchanged and the result follows form the relation $\int_\ell \frac{1}{(x+k)^z} dz = 0$.

c) To show that $\frac{1}{t}(\sum_1^{N_t} \psi \circ \hat{\theta}^n - \sum_1^{[\ell^{-1} t]} \psi \circ \hat{\theta}^n)$ converges to zero in probability, it is enough to show the convergence to 0 in probability of $\frac{1}{n} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m|$ as $n \uparrow \infty$. From the expressions of ψ and $\hat{\nu}$, it is clear that $\int |\psi|^{1-\varepsilon} d\hat{\nu}$ is finite for any ε in $]0, 1[$. Moreover the inequality $(\sum_1^k |a_i|)^\varepsilon \leq \sum_1^k |a_i|^\varepsilon$ $(0 < \varepsilon < 1)$ implies:

$$\left(n^{-1} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m|\right)^{1-\varepsilon} \leq n^{\varepsilon - 1} \sum_1^{A\sqrt{n}} |\psi \circ \hat{\theta}^m|^{1-\varepsilon}.$$

Hence, for any p such that $1 < p < (1-\varepsilon)^{-1}$, from the triangular inequality in L^p

$$\left\| \left(n^{-1} \sum_1^{[A\sqrt{n}]} |\psi \circ \hat{\theta}^m|\right)^{1-\varepsilon} \right\|_{L^p(\hat{\nu})} \leq [A\sqrt{n}]\, n^{\varepsilon-1} \left(\int |\psi|^{p(1-\varepsilon)} d\hat{\nu}\right)^{1/p}$$

which converges to zero as $n \uparrow \infty$ for $\varepsilon < \frac{1}{2}$.

This completes the proof of 2.1 a) (A different approach was also given in [LJ], and developed in [E] for a) and b) using Brownian motion instead of the coding.)

d) To show that $\frac{1}{\sqrt{t}}(\sum_1^{N_t} \eta \circ \hat{\theta}^n - \sum_1^{[\ell^{-1}t]} \eta \circ \hat{\theta}^n)$ converges to zero in probability, it is enough to show the convergence to 0 in probability of $\frac{1}{\sqrt{n}} S^*_{A\sqrt{n}}$ with $S_n = \sum_1^n \eta \circ \hat{\theta}^m$, $S_n^* = \sup_{k \leq n} |S_k|$.

Since η is bounded and of integral zero, an extension of the classical Kolmogorov inequality is required here, in the case of the Markov chain on $[0, 1] \times \mathbb{Z}/2\mathbb{Z} \times \Gamma_0/\Gamma$ defined by Q, and the functional $S_n = \sum_1^n \eta \circ \hat{\theta}^m$: there exists a constant $C > 0$ such that:

$$\tilde{m}\{S_n^* > b\} \leq C \frac{n}{b^2}.$$

This implies $\tilde{m}\{\frac{1}{\sqrt{n}} S^*_{[A\sqrt{n}]} > b\} \leq C \frac{A}{b^2\sqrt{n}}$, hence the convergence to zero in probability of $\frac{1}{\sqrt{n}} S^*_{[A\sqrt{n}]}$.

Such an inequality is proved in [G] p. 451 Cor. 1, on the basis of the Berry-Essen estimate for S_n. In turn, this estimate is obtained in [G-H] p. 80 Th. 2. The operator Q which occurs here satisfies the properties required in [G-H] as follows from Prop. 5.2, p. 36. Such estimates are studied in full detail in [Br] in the context of expanding transformations of the interval, a situation close to the situation considered here.

The proof of 2.1 b) and c) is now completed.

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