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$$
\begin{aligned}
& \text { ANNALES } \\
& \text { SCIENTIFIQUES } \\
& \text { de } \\
& \text { L'ECOLE } \\
& \text { NORMALE } \\
& \text { SUPÉRIEURE }
\end{aligned}
$$

Kazuma MORITA
Hodge-Tate and de Rbam representations
in the imperfect residue field case

# HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE 

by Kazuma MORITA


#### Abstract

Let $K$ be a $p$-adic local field with residue field $k$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$ and $V$ be a $p$-adic representation of $\operatorname{Gal}(\bar{K} / K)$. Then, by using the theory of $p$-adic differential modules, we show that $V$ is a Hodge-Tate (resp. de Rham) representation of $\operatorname{Gal}(\bar{K} / K)$ if and only if $V$ is a Hodge-Tate (resp. de Rham) representation of $\operatorname{Gal}\left(\overline{K^{\mathrm{pf}}} / K^{\mathrm{pf}}\right)$ where $K^{\mathrm{pf}} / K$ is a certain $p$-adic local field with residue field the smallest perfect field $k^{\mathrm{pf}}$ containing $k$.

Résumé. - Soit $K$ un corps local $p$-adique de corps résiduel $k$ tel que $\left[k: k^{p}\right]=p^{e}<+\infty$ et soit $V$ une représentation $p$-adique de $\operatorname{Gal}(\bar{K} / K)$. Nous utilisons la théorie des modules différentiels $p$-adiques pour montrer que $V$ est une représentation de Hodge-Tate (resp. de Rham) de $\operatorname{Gal}(\bar{K} / K)$ si et seulement si $V$ est une représentation de Hodge-Tate (resp. de Rham) de Gal $\left(\overline{K^{\mathrm{pf}}} / K^{\mathrm{pf}}\right)$ où $K^{\mathrm{pf}} / K$ est un certain corps local $p$-adique de corps résiduel le plus petit corps parfait $k^{\mathrm{pf}}$ contenant $k$.


## 1. Introduction

Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$. Choose an algebraic closure $\bar{K}$ of $K$ and put $G_{K}=\operatorname{Gal}(\bar{K} / K)$. By a $p$-adic representation of $G_{K}$, we mean a finite dimensional vector space $V$ over $\mathbb{Q}_{p}$ endowed with a continuous action of $G_{K}$. In the case $e=0$ (i.e. $k$ is perfect), following Fontaine, we can classify $p$-adic representations of $G_{K}$ by using the $p$-adic periods rings $B_{\mathrm{HT}}, B_{\mathrm{dR}}, B_{\mathrm{st}}$ and $B_{\text {cris }}$ (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. $k$ is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring $B_{\mathrm{HT}}$ and Tsuzuki and several authors constructed that of the ring $B_{\mathrm{dR}}$. By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of $G_{K}$ in the evident way ([3], [7], [8], [9], [12]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting $\left(b_{i}\right)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathscr{O}_{K}$ (the ring of integers of $K$ ) and for each $m \geq 1$, fix a $p^{m}$-th root $b_{i}^{1 / p^{m}}$ of $b_{i}$ in $\bar{K}$ satisfying $\left(b_{i}^{1 / p^{m}+1}\right)^{p}=b_{i}^{1 / p^{m}}$. Put $K^{(\mathrm{pf})}=\cup_{m \geq 1} K\left(b_{i}^{1 / p^{m}}, 1 \leq i \leq e\right)$ and $K^{\mathrm{pf}}=$ the $p$-adic completion of $K^{(\mathrm{pf})}$. These fields depend on the choice of a lifting
of a $p$-basis of $k$ in $\mathscr{O}_{K}$. Since $K^{\text {pf }}$ becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to $p$-adic representations of $G_{K^{\mathrm{pf}}}=\operatorname{Gal}\left(\overline{K^{\mathrm{pf}}} / K^{\mathrm{pf}}\right)$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of $K^{\mathrm{pf}}$ containing $\bar{K}$. Note that, if $V$ is a $p$-adic representation of $G_{K}$, it can be also regarded as a $p$-adic representation of $G_{K^{\text {pf }}}$ (see Section 2.2 for details). Our main result is the following.

Theorem 1.1. - Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$ and $V$ be a p-adic representation of $G_{K}$. Let $K^{\mathrm{pf}}$ be the field extension of $K$ defined as above. Then, we have the following equivalences

1. $V$ is a Hodge-Tate representation of $G_{K}$ if and only if $V$ is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$,
2. $V$ is a de Rham representation of $G_{K}$ if and only if $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$.

In the case of Hodge-Tate representations, Tsuji [11] had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of $p$-adic differential modules which play a central role in this article. In Section 4, by using the theory of $p$-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

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## 2. Preliminaries on Hodge-Tate and de Rham representations

### 2.1. Hodge-Tate and de Rham representations in the perfect residue field case

(See [4] and [5] for details.) Let $K$ be a complete discrete valuation field of characteristic 0 with perfect residue field $k$ of characteristic $p>0$. Choose an algebraic closure $\bar{K}$ of $K$ and consider its $p$-adic completion $\mathbb{C}_{p}$. Put

$$
\widetilde{\mathbb{E}}=\lim _{x \mapsto x^{p}} \mathbb{C}_{p}=\left\{\left(x^{(0)}, x^{(1)}, \ldots\right) \mid\left(x^{(i+1)}\right)^{p}=x^{(i)}, x^{(i)} \in \mathbb{C}_{p}\right\}
$$

and let $\widetilde{\mathbb{E}}^{+}$denote the set of $x=\left(x^{(i)}\right) \in \widetilde{\mathbb{E}}$ such that $x^{(0)} \in \mathscr{O}_{\mathbb{C}_{p}}$ where $\mathscr{O}_{\mathbb{C}_{p}}$ denotes the ring of integers of $\mathbb{C}_{p}$. For two elements $x=\left(x^{(i)}\right)$ and $y=\left(y^{(i)}\right)$ of $\widetilde{\mathbb{E}}$, their sum and
$4^{\mathrm{e}}$ SÉRIE - TOME 43 - 2010 - $\mathrm{N}^{\mathrm{o}} 2$
product are defined by $(x+y)^{(i)}=\lim _{j \rightarrow+\infty}\left(x^{(i+j)}+y^{(i+j)}\right)^{p^{j}}$ and $(x y)^{(i)}=x^{(i)} y^{(i)}$. These sum and product make $\widetilde{\mathbb{E}}$ a perfect field of characteristic $p>0(\widetilde{\mathbb{E}}+$ is a subring of $\widetilde{\mathbb{E}})$. Let $\epsilon=\left(\epsilon^{(n)}\right)$ be an element of $\widetilde{\mathbb{E}}$ such that $\epsilon^{(0)}=1$ and $\epsilon^{(1)} \neq 1$. Then, $\widetilde{\mathbb{E}}$ is the completion of an algebraic closure of $k((\epsilon-1))$ for the valuation defined by $v_{\mathbb{E}}(x)=v_{p}\left(x^{(0)}\right)$ where $v_{p}$ denotes the $p$-adic valuation of $\mathbb{C}_{p}$ normalized by $v_{p}(p)=1$. The field $\widetilde{\mathbb{E}}$ is equipped with a continuous action of the Galois group $G_{K}=\operatorname{Gal}(\bar{K} / K)$ with respect to the topology defined by the valuation $v_{\mathbb{E}}$. Put $\widetilde{\mathbb{A}}^{+}=W\left(\widetilde{\mathbb{E}}^{+}\right)$(the ring of Witt vectors with coefficients in $\widetilde{\mathbb{E}}^{+}$) and $\widetilde{\mathbb{B}}^{+}=\widetilde{\mathbb{A}}^{+}[1 / p]=\left\{\sum_{k \gg-\infty} p^{k}\left[x_{k}\right] \mid x_{k} \in \widetilde{\mathbb{E}}^{+}\right\}$where $[*]$ denotes the Teichmüller lift of $* \in \widetilde{\mathbb{E}}^{+}$. This ring $\widetilde{\mathbb{B}}^{+}$is equipped with a surjective homomorphism

$$
\theta: \widetilde{\mathbb{B}}^{+} \rightarrow \mathbb{C}_{p}: \sum p^{k}\left[x_{k}\right] \mapsto \sum p^{k} x_{k}^{(0)} .
$$

If $\tilde{p}=\left(p^{(n)}\right)$ denotes an element of $\widetilde{\mathbb{E}}^{+}$such that $p^{(0)}=p$, we can show that $\operatorname{Ker}(\theta)$ is the principal ideal generated by $\omega=[\tilde{p}]-p$. The ring $B_{\mathrm{dR}, K}^{+}$is defined to be the $\operatorname{Ker}(\theta)$-adic completion of $\widetilde{\mathbb{B}}^{+}$

$$
B_{\mathrm{dR}, K}^{+}=\varliminf_{\leftrightarrows}^{\lim _{n \geq 0} \widetilde{\mathbb{B}}^{+} /\left(\operatorname{Ker}(\theta)^{n}\right) . ~}
$$

This is a discrete valuation ring and $t=\log ([\epsilon])$ which converges in $B_{\mathrm{dR}, K}^{+}$is a generator of the maximal ideal. Put $B_{\mathrm{dR}, K}=B_{\mathrm{dR}, K}^{+}[1 / t]$. This ring $B_{\mathrm{dR}, K}$ becomes a field and is equipped with an action of the Galois group $G_{K}$ and a filtration defined by Fil $^{i} B_{\mathrm{dR}, K}=t^{i} B_{\mathrm{dR}, K}^{+}(i \in \mathbb{Z})$. Then, $\left(B_{\mathrm{dR}, K}\right)^{G_{K}}$ is canonically isomorphic to $K$. Thus, for a $p$-adic representation $V$ of $G_{K}, D_{\mathrm{dR}, K}(V)=\left(B_{\mathrm{dR}, K} \otimes_{\mathbb{Q}_{p}} V\right)^{G_{K}}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_{K}$ is a de Rham representation of $G_{K}$ if we have

$$
\operatorname{dim}_{\mathbb{Q}_{p}} V=\operatorname{dim}_{K} D_{\mathrm{dR}, K}(V) \quad \text { (we always have } \operatorname{dim}_{\mathbb{Q}_{p}} V \geq \operatorname{dim}_{K} D_{\mathrm{dR}, K}(V) \text { ). }
$$

Furthermore, we say that a $p$-adic representation $V$ of $G_{K}$ is a potentially de Rham representation of $G_{K}$ if there exists a finite field extension $L / K$ in $\bar{K}$ such that $V$ is a de Rham representation of $G_{L}$. It is known that a potentially de Rham representation $V$ of $G_{K}$ is a de Rham representation of $G_{K}$ (see [5, 3.9]).

Define $B_{\mathrm{HT}, K}$ to be the associated graded algebra to the filtration $\mathrm{Fil}^{i} B_{\mathrm{dR}, K}$. The quotient $\mathrm{gr}^{i} B_{\mathrm{HT}, K}=\mathrm{Fil}^{i} B_{\mathrm{dR}, K} / \mathrm{Fil}^{i+1} B_{\mathrm{dR}, K}(i \in \mathbb{Z})$ is a one-dimensional $\mathbb{C}_{p}$-vector space spanned by the image of $t^{i}$. Thus, we obtain the presentation

$$
B_{\mathrm{HT}, K}=\bigoplus_{i \in \mathbb{Z}} \mathbb{C}_{p}(i)
$$

where $\mathbb{C}_{p}(i)=\mathbb{C}_{p} \otimes \mathbb{Z}_{p}(i)$ is the Tate twist. Then, $\left(B_{\mathrm{HT}, K}\right)^{G_{K}}$ is canonically isomorphic to $K$. Thus, for a $p$-adic representation $V$ of $G_{K}, D_{\mathrm{HT}, K}(V)=\left(B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_{p}} V\right)^{G_{K}}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_{K}$ is a Hodge-Tate representation of $G_{K}$ if we have

$$
\left.\operatorname{dim}_{\mathbb{Q}_{p}} V=\operatorname{dim}_{K} D_{\mathrm{HT}, K}(V) \quad \text { (we always have } \operatorname{dim}_{\mathbb{Q}_{p}} V \geq \operatorname{dim}_{K} D_{\mathrm{HT}, K}(V)\right) .
$$

Furthermore, we say that a $p$-adic representation $V$ of $G_{K}$ is a potentially Hodge-Tate representation of $G_{K}$ if there exists a finite field extension $L / K$ in $\bar{K}$ such that $V$ is a Hodge-Tate representation of $G_{L}$. It is known that a potentially Hodge-Tate representation $V$ of $G_{K}$ is a Hodge-Tate representation of $G_{K}$ (see [5, 3.9]). Since we have $\operatorname{gr} B_{\mathrm{dR}, K} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_{p}(i)$,
if $V$ is a de Rham representation of $G_{K}$, there exists a $G_{K}$-equivariant isomorphism $\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V \simeq \bigoplus_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} \mathbb{C}_{p}\left(n_{j}\right)\left(n_{j} \in \mathbb{Z}\right)$. Thus, it follows that a de Rham representation $V$ of $G_{K}$ is a Hodge-Tate representation of $G_{K}$.

### 2.2. Hodge-Tate and de Rham representations in the imperfect residue field case

Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$. Choose an algebraic closure $\bar{K}$ of $K$ and put $G_{K}=\operatorname{Gal}(\bar{K} / K)$. As in the introduction, fix a lifting $\left(b_{i}\right)_{1 \leq i \leq e}$ of a $p$-basis of $k$ in $\mathscr{O}_{K}$ (the ring of integers of $K$ ) and for each $m \geq 1$, fix a $p^{m}$-th root $b_{i}^{1 / p^{m}}$ of $b_{i}$ in $\bar{K}$ satisfying $\left(b_{i}^{1 / p^{m+1}}\right)^{p}=b_{i}^{1 / p^{m}}$. Put

$$
K^{(\mathrm{pf})}=\cup_{m \geq 0} K\left(b_{i}^{1 / p^{m}}, 1 \leq i \leq e\right) \quad \text { and } \quad K^{\mathrm{pf}}=\text { the } p \text {-adic completion of } K^{(\mathrm{pf})}
$$

These fields depend on the choice of a lifting of a $p$-basis of $k$ in $\mathscr{O}_{K}$. Since $K^{(\mathrm{pf})}$ is a Henselian discrete valuation field, we have an isomorphism $G_{K^{\mathrm{pf}}}=\operatorname{Gal}\left(\overline{K^{\mathrm{pf}}} / K^{\mathrm{pf}}\right) \simeq$ $G_{K^{(\mathrm{pf})}}=\operatorname{Gal}\left(\bar{K} / K^{(\mathrm{pf})}\right)\left(\subset G_{K}\right)$ where we choose an algebraic closure $\overline{K^{\mathrm{pf}}}$ of $K^{\mathrm{pf}}$ containing $\bar{K}$. With this isomorphism, we identify $G_{K^{\text {pf }}}$ with a subgroup of $G_{K}$. We have a bijective map from the set of finite extensions of $K^{(\mathrm{pf})}$ contained in $\bar{K}$ to the set of finite extensions of $K^{\mathrm{pf}}$ contained in $\overline{K^{\mathrm{pf}}}$ defined by $L \rightarrow L K^{\mathrm{pf}}$. Furthermore, $L K^{\mathrm{pf}}$ is the $p$-adic completion of $L$. Hence, we have an isomorphism of rings

$$
\mathscr{O}_{\bar{K}} / p^{n} \mathscr{O}_{\bar{K}} \simeq \mathscr{O}_{\overline{K^{\mathrm{pf}}}} / p^{n} \mathscr{O}_{\overline{K^{\mathrm{pf}}}}
$$

where $\mathscr{O}_{\bar{K}}$ and $\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$ denote the rings of integers of $\bar{K}$ and $\overline{K^{\mathrm{pf}}}$. Thus, the $p$-adic completion of $\bar{K}$ is isomorphic to the $p$-adic completion of $\overline{K^{\mathrm{pf}}}$, which we will write $\mathbb{C}_{p}$. As in Subsection 2.1, construct the rings $\widetilde{\mathbb{E}}^{+}$and $\widetilde{\mathbb{A}}^{+}=W\left(\widetilde{\mathbb{E}}^{+}\right)$from this $\mathbb{C}_{p}$. Let $k^{\text {pf }}$ denote the perfect residue field of $K^{\text {pf }}$ and put $\mathscr{O}_{K_{0}}=\mathscr{O}_{K} \cap W\left(k^{\mathrm{pf}}\right)$. Let $\alpha: \mathscr{O}_{K} \otimes_{\mathscr{O}_{K_{0}}} \widetilde{\mathbb{A}}^{+} \rightarrow \mathscr{O}_{\bar{K}} / p \mathscr{O}_{\bar{K}}$ be the natural surjection and define $\widetilde{\mathbb{A}}_{(K)}^{+}$to be $\widetilde{\mathbb{A}}_{(K)}^{+}=\lim _{n \geq 0}\left(\mathscr{O}_{K} \otimes_{\mathscr{O}_{K_{0}}} \widetilde{\mathbb{A}}^{+}\right) /(\operatorname{Ker}(\alpha))^{n}$. Let $\theta_{K}: \tilde{\mathbb{A}}_{(K)}^{+} \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p} \rightarrow \mathbb{C}_{p}$ be the natural extension of $\theta: \widetilde{\mathbb{A}}^{+}[1 / p] \rightarrow \mathbb{C}_{p}$. Define $B_{\mathrm{dR}, K}^{+}$to be the $\operatorname{Ker}\left(\theta_{K}\right)$-adic completion of $\widetilde{\mathbb{A}}_{(K)}^{+} \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p}$

$$
B_{\mathrm{dR}, K}^{+}=\lim _{n \geq 0}\left(\tilde{\mathbb{A}}_{(K)}^{+} \otimes_{\mathbb{Z}_{p}} \mathbb{Q}_{p}\right) /\left(\operatorname{Ker}\left(\theta_{K}\right)^{n}\right)
$$

This is a $K$-algebra equipped with an action of the Galois group $G_{K}$. Let $\widetilde{b_{i}}$ denote $\left(b_{i}^{(n)}\right) \in \widetilde{\mathbb{E}}^{+}$such that $b_{i}^{(0)}=b_{i}$ and then the series which defines $\log \left(\left[\widetilde{b}_{i}\right] / b_{i}\right)$ converges to an element $t_{i}$ in $B_{\mathrm{dR}, K}^{+}$. Then, the ring $B_{\mathrm{dR}, K}^{+}$becomes a local ring with the maximal ideal $m_{\mathrm{dR}}=\left(t, t_{1}, \ldots, t_{e}\right)$. Define a filtration on $B_{\mathrm{dR}, K}^{+}$by fil ${ }^{i} B_{\mathrm{dR}, K}^{+}=m_{\mathrm{dR}}^{i}$. Then, the homomorphism

$$
f: B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+}\left[\left[t_{1}, \ldots, t_{e}\right]\right] \rightarrow B_{\mathrm{dR}, K}^{+}
$$

is an isomorphism of filtered algebras (see [3, Proposition 2.9]). From this isomorphism, it follows easily that

$$
i: B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+} \hookrightarrow B_{\mathrm{dR}, K}^{+} \quad \text { and } \quad p: B_{\mathrm{dR}, K}^{+} \rightarrow B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+}: t_{i} \mapsto 0
$$

are $G_{K^{\mathrm{pf}}}$-equivariant homomorphisms and the composition

$$
p \circ i: B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+} \hookrightarrow B_{\mathrm{dR}, K}^{+} \rightarrow B_{\mathrm{dR}, K_{\mathrm{pf}}}^{+}
$$

is an identity. Put $B_{\mathrm{dR}, K}=B_{\mathrm{dR}, K}^{+}[1 / t]$. Then, $K$ is canonically embedded in $B_{\mathrm{dR}, K}$ and we have a canonical isomorphism $\left(B_{\mathrm{dR}, K}\right)^{G_{K}}=K$. Thus, for a $p$-adic representation $V$ of $G_{K}, D_{\mathrm{dR}, K}(V)=\left(B_{\mathrm{dR}, K} \otimes \otimes_{\mathbb{Q}_{p}} V\right)^{G_{K}}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_{K}$ is a de Rham representation of $G_{K}$ if we have

$$
\operatorname{dim}_{\mathbb{Q}_{p}} V=\operatorname{dim}_{K} D_{\mathrm{dR}, K}(V) \quad \text { (we always have } \operatorname{dim}_{\mathbb{Q}_{p}} V \geq \operatorname{dim}_{K} D_{\mathrm{dR}, K}(V) \text { ). }
$$

Furthermore, we say that a $p$-adic representation $V$ of $G_{K}$ is a potentially de Rham representation of $G_{K}$ if there exists a finite field extension $L / K$ in $\bar{K}$ such that $V$ is a de Rham representation of $G_{L}$. We can show that a potentially de Rham representation $V$ of $G_{K}$ is a de Rham representation of $G_{K}$ in the same way as in the perfect residue field case.

Define a filtration on $B_{\mathrm{dR}, K}$ to be

$$
\begin{aligned}
& \mathrm{Fil}^{0} B_{\mathrm{dR}, K}=\sum_{n=0}^{\infty} t^{-n} \mathrm{fil}^{n} B_{\mathrm{dR}, K}^{+}=B_{\mathrm{dR}, K}^{+}\left[\frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right], \\
& \mathrm{Fil}^{i} B_{\mathrm{dR}, K}=t^{i} \mathrm{Fil}^{0} B_{\mathrm{dR}, K}(i \in \mathbb{Z}) .
\end{aligned}
$$

Define $B_{\mathrm{HT}, K}$ to be the associated graded algebra to this filtration. Since the quotient $\operatorname{gr}^{i} B_{\mathrm{HT}, K}=\mathrm{Fil}^{i} B_{\mathrm{dR}, K} / \mathrm{Fil}^{i+1} B_{\mathrm{dR}, K}(i \in \mathbb{Z})$ is given by $\mathrm{gr}^{i} B_{\mathrm{HT}, K}=t^{i} \mathbb{C}_{p}\left[\frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right]$, we obtain the presentation

$$
B_{\mathrm{HT}, K}=\mathbb{C}_{p}\left[t, t^{-1}, \frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right]=B_{\mathrm{HT}, K^{\mathrm{pr}}}\left[\frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right] .
$$

From this presentation, it follows easily that

$$
i: B_{\mathrm{HT}, K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT}, K} \quad \text { and } \quad p: B_{\mathrm{HT}, K} \rightarrow B_{\mathrm{HT}, K^{\mathrm{pf}}}: t_{i} / t \mapsto 0
$$

are $G_{K^{\mathrm{p}}}$-equivariant homomorphisms and the composition

$$
p \circ i: B_{\mathrm{HT}, K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT}, K} \rightarrow B_{\mathrm{HT}, K^{\mathrm{pf}}}
$$

is an identity. The field $K$ is canonically embedded in $B_{\mathrm{HT}, K}$ and we have $\left(B_{\mathrm{HT}, K}\right)^{G_{K}}=K$. Thus, for a $p$-adic representation $V$ of $G_{K}, D_{\mathrm{HT}, K}(V)=\left(B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_{p}} V\right)^{G_{K}}$ is naturally a $K$-vector space. We say that a $p$-adic representation $V$ of $G_{K}$ is a Hodge-Tate representation of $G_{K}$ if we have

$$
\left.\operatorname{dim}_{\mathbb{Q}_{p}} V=\operatorname{dim}_{K} D_{\mathrm{HT}, K}(V) \quad \text { (we always have } \operatorname{dim}_{\mathbb{Q}_{p}} V \geq \operatorname{dim}_{K} D_{\mathrm{HT}, K}(V)\right) .
$$

Furthermore, we say that a $p$-adic representation $V$ of $G_{K}$ is a potentially Hodge-Tate representation of $G_{K}$ if there exists a finite field extension $L / K$ in $\bar{K}$ such that $V$ is a HodgeTate representation of $G_{L}$. We can show that a potentially Hodge-Tate representation $V$ of $G_{K}$ is a Hodge-Tate representation of $G_{K}$ in the same way as in the perfect residue field case.

## 3. Preliminaries on $p$-adic differential modules

In this section, we shall review the theory of $p$-adic differential modules which plays an important role in this article. First, let us fix the notations. Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<\infty$ and $V$ be a $p$-adic representation of $G_{K}$. Define $K^{(\mathrm{pf})}$ and $K^{\mathrm{pf}}$ as in the introduction and in Subsection 2.2. Put $K_{\infty}^{(\mathrm{pf})}=\cup_{m \geq 0} K^{(\mathrm{pf})}\left(\zeta_{p^{m}}\right)$ (resp. $K_{\infty}^{\mathrm{pf}}=$ $\cup_{m \geq 0} K^{\mathrm{pf}}\left(\zeta_{p^{m}}\right)$ ) where $\zeta_{p^{m}}$ denotes a primitive $p^{m}$-th root of unity in $\bar{K}$ (resp. $\overline{K^{\mathrm{pf}}}$ ) such
that $\left(\zeta_{p^{m+1}}\right)^{p}=\zeta_{p^{m}}$. Let $\hat{K}_{\infty}^{\mathrm{pf}}$ denote the $p$-adic completion of $K_{\infty}^{\mathrm{pf}}$. These fields $K_{\infty}^{(\mathrm{pf})}, K_{\infty}^{\mathrm{pf}}$ and $\hat{K}_{\infty}^{\text {pf }}$ depend on the choice of a lifting of a $p$-basis of $k$ in $\mathscr{O}_{K}$. Then, we have the following inclusions

$$
K_{\infty}^{(\mathrm{pf})} \subset K_{\infty}^{\mathrm{pf}} \subset \hat{K}_{\infty}^{\mathrm{pf}}
$$

Let $H$ denote the kernel of the cyclotomic character $\chi: G_{K^{\mathrm{pf}}} \rightarrow \mathbb{Z}_{p}^{*}$. Then, the Galois group $H$ is isomorphic to the subgroup $\operatorname{Gal}\left(\bar{K} / K_{\infty}^{(\text {pf })}\right)$ of $G_{K}$. Define $\Gamma_{K}=G_{K} / H$. Let $\Gamma_{0}$ denote the subgroup $\operatorname{Gal}\left(K_{\infty}^{(\mathrm{pf})} / K^{(\mathrm{pf})}\right)\left(\simeq G_{K^{\mathrm{pf}}} / H\right)$ of $\Gamma_{K}$. Let $\Gamma_{i}(1 \leq i \leq e)$ be the subgroup of $\Gamma_{K}$ such that actions of $\beta_{i} \in \Gamma_{i}(1 \leq i \leq e)$ satisfy $\beta_{i}\left(\zeta_{p^{m}}\right)=\zeta_{p^{m}}$ and $\beta_{i}\left(b_{j}^{1 / p^{m}}\right)=b_{j}^{1 / p^{m}}$ $(i \neq j)$ and define the homomorphism $c_{i}: \Gamma_{i} \rightarrow \mathbb{Z}_{p}$ such that we have $\beta_{i}\left(b_{i}^{1 / p^{m}}\right)=b_{i}^{1 / p^{m}} \zeta_{p^{m}}^{c_{i}\left(\beta_{i}\right)}$. Then, the homomorphism $c_{i}$ defines an isomorphism $\Gamma_{i} \simeq \mathbb{Z}_{p}$ of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

$$
\Gamma_{K} \simeq \Gamma_{0} \ltimes\left(\oplus_{i=1}^{e} \Gamma_{i}\right) \simeq \Gamma_{0} \ltimes \mathbb{Z}_{p}^{\oplus e}
$$

### 3.1. Definitions of $p$-adic differential modules

We shall review the definitions of $p$-adic differential modules and have the following diagram, for a $p$-adic representation $V$ of $G_{K}$,

$$
\begin{array}{ccc}
\left(B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V\right)^{H} & \xrightarrow{\theta_{K}}\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H} \\
\cup & & \cup \\
D_{\mathrm{dif}}^{+}(V) & \rightarrow & D_{\mathrm{Sen}}(V) \\
\cup & & \cup \\
D_{e-\mathrm{dif}}^{+}(V) & \rightarrow & D_{\mathrm{Bri}}(V) .
\end{array}
$$

3.1.1. The module $D_{\text {Sen }}(V)$. - In the article [10], Sen shows that, for a $p$-adic representation $V$ of $G_{K^{\mathrm{pf}}}$, the $\hat{K}_{\infty}^{\mathrm{pf}}$-vector space $\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ has dimension $d=\operatorname{dim}_{\mathbb{Q}_{p}} V$ and the union of the finite dimensional $K_{\infty}^{\mathrm{pf}}$-subspaces of $\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ stable under $\Gamma_{0}$ $\left(\simeq G_{K^{\mathrm{pf}}} / H\right)$ is a $K_{\infty}^{\mathrm{pf}}$-vector space of dimension $d$ stable under $\Gamma_{0}$ (called $D_{\text {Sen }}(V)$ ). We have $\mathbb{C}_{p} \otimes_{K_{\infty}^{\mathrm{pf}}} D_{\operatorname{Sen}}(V)=\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V$ and the natural map $\hat{K}_{\infty}^{\mathrm{pf}} \otimes_{K_{\infty}^{\mathrm{pf}}} D_{\operatorname{Sen}}(V) \rightarrow\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ is an isomorphism. Furthermore, if $\gamma \in \Gamma_{0}$ is close enough to 1 , then the series of operators on $D_{\text {Sen }}(V)$

$$
\frac{\log (\gamma)}{\log (\chi(\gamma))}=-\frac{1}{\log (\chi(\gamma))} \sum_{k \geq 1} \frac{(1-\gamma)^{k}}{k}
$$

converges to a $K_{\infty}^{\mathrm{pf}}$-linear derivation $\nabla^{(0)}: D_{\text {Sen }}(V) \rightarrow D_{\operatorname{Sen}}(V)$ and does not depend on the choice of $\gamma$.
3.1.2. The module $D_{\mathrm{Bri}}(V)$. - In the article [2], Brinon generalizes Sen's work above. For a $p$-adic representation $V$ of $G_{K}$, he shows that the union of the finite dimensional $K_{\infty}^{(\mathrm{pf})}$ subspaces of $\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ stable under $\Gamma_{K}$ is a $K_{\infty}^{(\mathrm{pf})}$-vector space of dimension $d$ stable under $\Gamma_{K}$ (we call it $D_{\text {Bri }}(V)$ ). We have $\mathbb{C}_{p} \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)=\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V$ and the natural $\operatorname{map} \hat{K}_{\infty}^{\mathrm{pf}} \otimes_{K_{\mathrm{pf}}^{(\mathrm{pf}}} D_{\mathrm{Bri}}(V) \rightarrow\left(\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ is an isomorphism. As in the case of $D_{\mathrm{Sen}}(V)$, the $K_{\infty}^{(\mathrm{pf})}$-vector space $D_{\mathrm{Bri}}(V)$ is endowed with the action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivation
$\nabla^{(0)}=\frac{\log (\gamma)}{\log (\chi(\gamma))}$ if $\gamma \in \Gamma_{0}$ is close enough to 1 . In addition to this operator $\nabla^{(0)}$, if $\beta_{i} \in \Gamma_{i}$ is close enough to 1 , then the series of operators on $D_{\operatorname{Bri}}(V)$

$$
\frac{\log \left(\beta_{i}\right)}{c_{i}\left(\beta_{i}\right)}=-\frac{1}{c_{i}\left(\beta_{i}\right)} \sum_{k \geq 1} \frac{\left(1-\beta_{i}\right)^{k}}{k}
$$

converges to a $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}: D_{\mathrm{Bri}}(V) \rightarrow D_{\mathrm{Bri}}(V)$ and does not depend on the choice of $\beta_{i}$.
3.1.3. The module $D_{e \text {-dif }}^{+}(V)$. - In the article [1], Andreatta and Brinon generalize Fontaine's work [6]. For a $p$-adic representation $V$ of $G_{K}$, they show that the union of $K_{\infty}^{(\text {pf) }}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-submodules of finite type of $\left(B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ stable under $\Gamma_{K}$ is a free $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-module of rank d stable under $\Gamma_{K}$ (we call it $D_{e \text {-dif }}^{+}(V)$ ). We have $B_{\mathrm{dR}, K}^{+} \otimes_{K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]} D_{e-\mathrm{dif}}^{+}(V)=B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V$ and the natural map

$$
\left(B_{\mathrm{dR}, K}^{+}\right)^{H} \otimes_{K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]} D_{e-\mathrm{dif}}^{+}(V) \rightarrow\left(B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V\right)^{H}
$$

is an isomorphism. The $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-module $D_{e \text {-dif }}^{+}(V)$ is endowed with the action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivations $\nabla^{(0)}=\frac{\log (\gamma)}{\log (\chi(\gamma))}$ if $\gamma \in \Gamma_{0}$ is close enough to 1 and $\nabla^{(i)}=\frac{\log \left(\beta_{i}\right)}{c_{i}\left(\beta_{i}\right)}(1 \leq i \leq e)$ if $\beta_{i} \in \Gamma_{i}$ is close enough to 1.
3.1.4. The module $D_{\text {dif }}^{+}(V)$. - For a $p$-adic representation $V$ of $G_{K}$, define $D_{\text {dif }}^{+}(V)$ to be $\lim _{r}\left(K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right] \otimes_{K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]} D_{e \text {-dif }}^{+,(r)}(V)\right)$ where we put $D_{e \text {-dif }}^{+,(r)}(V)=$ $D_{e \text {-dif }}^{+}(V) /\left(t, t_{1}, \ldots, t_{e}\right)^{r} D_{e-d i f}^{+}(V)$. One can verify that $D_{\text {dif }}^{+}(V)$ is the union of $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-submodules of finite type of $\left(B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ stable under $\Gamma_{0}\left(\simeq G_{K^{\mathrm{pf}}} / H\right)$ and is a free $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-module of rank $d$ stable under $\Gamma_{0}$. Furthermore, we have $B_{\mathrm{dR}, K}^{+} \otimes_{K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]} D_{\mathrm{dif}}^{+}(V)=B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V$ and the natural map $\left(B_{\mathrm{dR}, K}^{+}\right)^{H}$ $\otimes_{K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]} D_{\mathrm{dif}}^{+}(V) \rightarrow\left(B_{\mathrm{dR}, K}^{+} \otimes_{\mathbb{Q}_{p}} V\right)^{H}$ is an isomorphism. As in the case of $D_{e \text {-dif }}^{+}(V)$, the $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-module $D_{\mathrm{dif}}^{+}(V)$ is endowed with the action of the $K_{\infty}^{\mathrm{pf}}$-linear derivation $\nabla^{(0)}=\frac{\log (\gamma)}{\log (\chi(\gamma))}$ if $\gamma \in \Gamma_{0}$ is close enough to 1.

REMARK 3.1. - 1. The preceding results in Subsection 3.1.1 are obtained when $V$ is a $p$-adic representation of $G_{L}=\operatorname{Gal}(\bar{L} / L)$ where $L$ is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p>0$ and we choose an algebraic closure $\bar{L}$ of $L$. However, in Subsection 3.1.1, for simplicity, we stated the results in the case $L=K^{\mathrm{pf}}$.
2. Note that, though many people denote the $p$-adic differential module constructed by Fontaine in [6] by $D_{\text {dif }}^{+}(V)$, the module $D_{\text {dif }}^{+}(V)$ in Subsection 3.1.4 is a little different from this module.

### 3.2. Some properties of differential operators

We shall describe the action of derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ on $D_{\mathrm{Bri}}(V)$ and $D_{e-\mathrm{dif}}^{+}(V)$. First, by a standard argument, we can show that, if $x \in D_{\mathrm{Bri}}(V)$ (resp. $D_{e \text {-dif }}^{+}(V)$ ), we have

$$
\nabla^{(0)}(x)=\lim _{\gamma \rightarrow 1} \frac{\gamma(x)-x}{\chi(\gamma)-1} \quad \text { and } \quad \nabla^{(i)}(x)=\lim _{\beta_{i} \rightarrow 1} \frac{\beta_{i}(x)-x}{c_{i}\left(\beta_{i}\right)}
$$

With this, we can easily describe the actions of $K_{\infty}^{(\mathrm{pf})}$-linear derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ on $K_{\infty}^{(\text {pf })}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]=D_{e-\text { dif }}^{+}\left(\mathbb{Q}_{p}\right)$ where $\mathbb{Q}_{p}$ is equipped with the structure of $p$-adic representations of $G_{K}$ induced by the trivial action of $G_{K}$.

Lemma 3.2. - The actions of $K_{\infty}^{(\mathrm{pf})}$-linear derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ on $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$ are given by $\nabla^{(0)}=t \frac{d}{d t}$ and $\nabla^{(i)}=t \frac{d}{d t_{i}}(1 \leq i \leq e)$.

Proof. - Since $\left\{\nabla^{(j)}\right\}_{j=0}^{e}$ are $K_{\infty}^{(\mathrm{pf})}$-linear derivations and we can see that we have $\nabla^{(j)}\left(t_{k}\right)=0(j \neq k)$ and $\nabla^{(i)}(t)=0(i \neq 0)$, it suffices to show that we have $\nabla^{(0)}(t)=t$ and $\nabla^{(i)}\left(t_{i}\right)=t$. These follow from

$$
\begin{aligned}
\nabla^{(0)}(t) & =\lim _{\gamma \rightarrow 1} \frac{\gamma(t)-t}{\chi(\gamma)-1}=\lim _{\gamma \rightarrow 1} \frac{\chi(\gamma) t-t}{\chi(\gamma)-1}=t \\
\nabla^{(i)}\left(t_{i}\right) & =\lim _{\beta_{i} \rightarrow 1} \frac{\beta_{i}\left(t_{i}\right)-t_{i}}{c_{i}\left(\beta_{i}\right)}=\lim _{\beta_{i} \rightarrow 1} \frac{\left(t_{i}+c_{i}\left(\beta_{i}\right) t\right)-t_{i}}{c_{i}\left(\beta_{i}\right)}=t .
\end{aligned}
$$

We extend naturally actions of $K_{\infty}^{(\mathrm{pf})}$-linear derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ on $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$ to $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\left[t^{-1}\right]\left(\subset B_{\mathrm{dR}, K}\right)\right.$ by putting $\nabla^{(0)}\left(t^{-1}\right)=-t^{-1}$ and $\nabla^{(i)}\left(t^{-1}\right)=0$ $(1 \leq i \leq e)$. Now, we compute the bracket [, ] of derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ on $D_{\mathrm{Bri}}(V)$ (resp. $\left.D_{e-\mathrm{dif}}^{+}(V)\right)$.

Proposition 3.3. - On the p-adic differential module $D_{\mathrm{Bri}}(V)$ (resp. $D_{e-\mathrm{dif}}^{+}(V)$ ), we have $\left[\nabla^{(0)}, \nabla^{(i)}\right]=\nabla^{(i)}(i \neq 0)$ and $\left[\nabla^{(i)}, \nabla^{(j)}\right]=0(i, j \neq 0)$.

Proof. - The second equality follows from the commutativity of $\beta_{i}$ and $\beta_{j}$. For the first equality, we have the relation $\gamma \beta_{i}=\beta_{i}^{\chi(\gamma)} \gamma$. Then, since we have

$$
\lim _{h \rightarrow 0} \frac{a^{h+1}-a}{(h+1)-1}=a \log (a)
$$

we obtain

$$
\begin{aligned}
{\left[\nabla^{(0)}, \nabla^{(i)}\right](*) } & =\lim _{\gamma \rightarrow 1} \frac{\gamma-1}{\chi(\gamma)-1} \lim _{\beta_{i} \rightarrow 1} \frac{\beta_{i}-1}{c_{i}\left(\beta_{i}\right)}(*)-\lim _{\beta_{i} \rightarrow 1} \frac{\beta_{i}-1}{c_{i}\left(\beta_{i}\right)} \lim _{\gamma \rightarrow 1} \frac{\gamma-1}{\chi(\gamma)-1}(*) \\
& =\lim _{\beta_{i} \rightarrow 1} \lim _{\gamma \rightarrow 1} \frac{\gamma \beta_{i}-\gamma-\beta_{i}+1}{(\chi(\gamma)-1) c_{i}\left(\beta_{i}\right)}(*)-\lim _{\beta_{i} \rightarrow 1} \lim _{\gamma \rightarrow 1} \frac{\beta_{i} \gamma-\gamma-\beta_{i}+1}{(\chi(\gamma)-1) c_{i}\left(\beta_{i}\right)}(*) \\
& =\lim _{\beta_{i} \rightarrow 1} \lim _{\gamma \rightarrow 1} \frac{\beta_{i}^{\chi(\gamma)} \gamma-\beta_{i} \gamma}{(\chi(\gamma)-1) c_{i}\left(\beta_{i}\right)}(*) \\
& =\lim _{\beta_{i} \rightarrow 1} \frac{\beta_{i} \log \left(\beta_{i}\right)}{c_{i}\left(\beta_{i}\right)}(*) \\
& =\nabla^{(i)}(*) .
\end{aligned}
$$

Proposition 3.4. - The action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent.

Proof. - From the equality $\nabla^{(0)} \nabla^{(i)}-\nabla^{(i)} \nabla^{(0)}=\nabla^{(i)}$, we get $\nabla^{(0)}\left(\nabla^{(i)}\right)^{r}-\left(\nabla^{(i)}\right)^{r} \nabla^{(0)}$ $=r\left(\nabla^{(i)}\right)^{r}$ and $\operatorname{tr}\left(r\left(\nabla^{(i)}\right)^{r}\right)=0$ for all $r \in \mathbb{N}$. Since the characteristic of $K_{\infty}^{(\mathrm{pf})}$ is 0 , we obtain $\operatorname{tr}\left(\left(\nabla^{(i)}\right)^{r}\right)=0$ for all $r \in \mathbb{N}$. As is well known in linear algebra, this shows that the action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent.

Notation. - For simplicity, put

$$
R=K_{\infty}^{(\mathrm{pf})}\left[t, \frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right] \quad \text { or } \quad K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right] .
$$

Proposition 3.5. - Let $M$ be a finitely generated free $R[1 / t]$-module endowed with $K_{\infty}^{(\mathrm{pf})}$-linear derivations $\left\{\nabla^{(i)}\right\}_{i=0}^{e}$ which satisfy the same properties in Lemma 3.2 and Proposition 3.3. Assume that we can choose a basis $\left\{g_{j}\right\}_{j=1}^{d}$ of $M$ over $R[1 / t]$ such that $\nabla^{(0)}\left(g_{j}\right)=0$. Then, the action of $\nabla^{(i)}(i \neq 0)$ on this basis is given by $\nabla^{(i)}\left(g_{j}\right)=t \sum_{k=1}^{d} c_{k} g_{k}$ where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$.

Proof. - Since $\left\{g_{j}\right\}_{j=1}^{d}$ forms a basis of $M$ over $R[1 / t]$, we can write, for $i \neq 0$,

$$
\begin{equation*}
\nabla^{(i)}\left(g_{j}\right)=\sum_{k=1}^{d} a_{k} g_{k} \quad\left(a_{k} \in R[1 / t]\right) \tag{3.1}
\end{equation*}
$$

Then, the relation $\left[\nabla^{(0)}, \nabla^{(i)}\right]=\nabla^{(i)}(i \neq 0)$ of Proposition 3.3 says that we have $\sum_{k=1}^{d} \nabla^{(0)}\left(a_{k}\right) g_{k}=\sum_{k=1}^{d} a_{k} g_{k}$. Note that we have $\nabla^{(0)}\left(g_{j}\right)=0$ by hypothesis. Hence, we obtain the differential equation $\nabla^{(0)}\left(a_{k}\right)=a_{k}$. Define an element $c_{k}$ of $R[1 / t]$ to be $a_{k} / t$. Then, we can see that $c_{k}$ satisfies $\nabla^{(0)}\left(c_{k}\right)=a_{k} / t-a_{k} / t=0$ and that $c_{k}$ is contained in $R$. Thus, the solution of the differential equation $\nabla^{(0)}\left(a_{k}\right)=a_{k}$ in $\mathrm{R}[1 / t]$ has the following form

$$
\begin{equation*}
a_{k}=c_{k} t \tag{3.2}
\end{equation*}
$$

where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$. Hence, from (3.1) and (3.2), we obtain, for $i \neq 0, \nabla^{(i)}\left(g_{j}\right)=t \sum_{k=1}^{d} c_{k} g_{k}$ where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$.

Corollary 3.6. - With notations as in Proposition 3.5 above, we have the following presentation

$$
\left(\nabla^{(1)}\right)^{k_{1}} \cdots\left(\nabla^{(e)}\right)^{k_{e}}\left(g_{j}\right)=t^{k_{1}+\cdots+k_{e}} \sum_{k=1}^{d} c_{k} g_{k}
$$

where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$.

## 4. Proof of the main theorem

In this section, we keep the notation and the assumption of Section 3.

### 4.1. Main theorem for Hodge-Tate representations

Proposition 4.1 ([10, Section (2.3)]). - If $V$ is a Hodge-Tate representation of $G_{K^{\mathrm{pr}}}$, there exists a $\Gamma_{0}$-equivariant isomorphism of $K_{\infty}^{\mathrm{pf}}$-vector spaces

$$
D_{\operatorname{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\operatorname{dim}_{Q_{p}} V} K_{\infty}^{\mathrm{pf}}\left(n_{j}\right) \quad\left(n_{j} \in \mathbb{Z}\right) .
$$

REmark 4.2. - In general, if $L$ denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p>0$ and $V$ is a Hodge-Tate representation of $G_{L}=\operatorname{Gal}(\bar{L} / L)$ where we choose an algebraic closure $\bar{L}$ of $L$, Sen shows that there exists a $G_{L} / H$-equivariant isomorphism of $L_{\infty}\left(=\cup_{m \geq 1} L\left(\zeta_{p^{m}}\right)\right)$-vector spaces ([10, Section (2.3)])

$$
D_{\operatorname{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} L_{\infty}\left(n_{j}\right) \quad\left(n_{j} \in \mathbb{Z}\right)
$$

Corollary 4.3. - For a p-adic representation $V$ of $G_{K}$, assume that $V$ is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$ - equivariant isomorphism of $K_{\infty}^{(\mathrm{pf})}$-vector spaces

$$
D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} K_{\infty}^{(\mathrm{pf})}\left(n_{j}\right) \quad\left(n_{j} \in \mathbb{Z}\right)
$$

Here, $\simeq_{\nabla^{(0)}}$ denotes $a \nabla^{(0)}$-equivariant isomorphism. Furthermore, the multiplicity of $\left\{n_{j}\right\}_{j=1}^{d}$ is the same as that of $\left\{n_{j}\right\}_{j=1}^{d}$ in Proposition 4.1.

Proof. - From the presentation of Proposition 4.1, the action of the $K_{\infty}^{\mathrm{pf}}$-linear derivation $\nabla^{(0)}$ on $D_{\text {Sen }}(V)$ is semi-simple and its eigenvalues are integers. Thus, the action of the $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(0)}$ on the subspace $D_{\mathrm{Bri}}(V)$ of $D_{\operatorname{Sen}}(V)$ is also semi-simple and its eigenvalues are the same. Therefore, we obtain a $\nabla^{(0)}$-equivariant isomorphism $D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K_{\infty}^{(\mathrm{pf})}\left(n_{j}\right)\left(n_{j} \in \mathbb{Z}\right)$. By tensoring $K_{\infty}^{\mathrm{pf}} \otimes_{K_{\infty}^{(\mathrm{pf})}}$ over both sides, we obtain $K_{\infty}^{\mathrm{pf}} \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K_{\infty}^{\mathrm{pf}}\left(n_{j}\right)\left(n_{j} \in \mathbb{Z}\right)$. Furthermore, since we have $K_{\infty}^{\mathrm{pf}} \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V) \hookrightarrow D_{\mathrm{Sen}}(V)$ by definition and both sides have the same dimension $d$ over $K_{\infty}^{\mathrm{pf}}$, we obtain $K_{\infty}^{\mathrm{pf}} \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)=D_{\mathrm{Sen}}(V)$ and can see that the multiplicity of $\left\{n_{j}\right\}_{j=1}^{d}$ is the same as that of $\left\{n_{j}\right\}_{j=1}^{d}$ in Proposition 4.1.

Theorem 4.4. - Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$ and $V$ be a p-adic representation of $G_{K}$. Let $K^{\mathrm{pf}}$ be the field extension of $K$ defined as before. Then, $V$ is a Hodge-Tate representation of $G_{K}$ if and only if $V$ is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.

Proof. - We shall prove the main theorem in two parts.
(1) $V$ : HT rep. of $G_{K} \Rightarrow V$ : HT rep. of $G_{K^{\mathrm{pf}}}$. - Since $V$ is a Hodge-Tate representation of $G_{K}$, there exists a $G_{K}$-equivariant isomorphism of $B_{\mathrm{HT}, K}$-modules

$$
\begin{equation*}
B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_{p}} V \simeq\left(B_{\mathrm{HT}, K}\right)^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} \tag{4.1}
\end{equation*}
$$

Now, by tensoring $B_{\mathrm{HT}, K^{\mathrm{pf}}} \otimes_{B_{\mathrm{HT}, K}}$ (which is induced by the $G_{K^{\mathrm{pf}}}$-equivariant surjection $p: B_{\mathrm{HT}, K} \rightarrow B_{\mathrm{HT}, K^{\mathrm{pf}}}: t_{i} / t \mapsto 0$ ) over (4.1), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{HT}, K^{\mathrm{pf}}}$-modules

$$
B_{\mathrm{HT}, K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_{p}} V \simeq\left(B_{\mathrm{HT}, K^{\mathrm{pf}}}\right)^{d}
$$

This means that $V$ is a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$.
$4^{\mathrm{e}}$ SÉRIE - TOME 43 - $2010-\mathrm{N}^{\mathrm{o}} 2$
(2) $V: \quad H T$ rep. of $G_{K^{\mathrm{pf}}} \Rightarrow V$ : HT rep. of $G_{K}$. For simplicity, put $R=K_{\infty}^{(\mathrm{pf})}\left[t, \frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right]$. We shall construct the $K_{\infty}^{(\mathrm{pf})}$-linearly independent elements $\left\{f_{j}^{(*)}\right\}_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V}$ of $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)\left(\subset B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_{p}} V\right)$ such that $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.
(A) Construction of $\left\{f_{j}^{(*)}\right\}_{j=1}^{d} \in R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$. - From the presentation of Corollary 4.3 above, if we twist by some powers of $t$, we obtain a basis $\left\{f_{j}\right\}_{j=1}^{d}$ of $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1 / t]$ such that $\nabla^{(0)}\left(f_{j}\right)=0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1 / t]$-module $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\text {Bri }}(V)$ generated by $\left\{f_{j}\right\}_{j=1}^{d}$, we can deduce

$$
\begin{equation*}
\left(\nabla^{(1)}\right)^{k_{1}} \cdots\left(\nabla^{(e)}\right)^{k_{e}}\left(f_{j}\right)=t^{k_{1}+\cdots+k_{e}} \sum_{k=1}^{d} c_{k} f_{k} \tag{4.2}
\end{equation*}
$$

where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$. Furthermore, since the action of $K_{\infty}^{(\mathrm{pf})}$-linear derivation $\nabla^{(i)}(i \neq 0)$ on $D_{\mathrm{Bri}}(V)$ is nilpotent by Proposition 3.4, if we take $n \in \mathbb{N}$ large enough, we obtain

$$
\begin{equation*}
\left(\nabla^{(i)}\right)^{n}\left(f_{j}\right)=0 \quad \text { for all } 1 \leq j \leq d \text { and } 1 \leq i \leq e \tag{4.3}
\end{equation*}
$$

Define an element $f_{j}^{(*)}$ of $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ by

$$
f_{j}^{(*)}=\sum_{0 \leq k_{1}, \ldots, k_{e}}(-1)^{k_{1}+\cdots+k_{e}} \frac{t_{1}^{k_{1}} \cdots t_{e}^{k_{e}}}{k_{1}!\cdots k_{e}!t^{k_{1}+\cdots+k_{e}}}\left(\nabla^{(1)}\right)^{k_{1}} \cdots\left(\nabla^{(e)}\right)^{k_{e}}\left(f_{j}\right)
$$

Note that this series is a finite sum by (4.3) and thus $f_{j}^{(*)}$ actually defines an element of $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$. Then, it follows easily that we have $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.2) and the fact $\nabla^{(0)}\left(f_{j}\right)=0$, we can deduce that we have $\nabla^{(0)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq j \leq d$.
(B) $\left\{f_{j}^{(*)}\right\}_{j=1}^{d} \in R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ is linearly independent over $K_{\infty}^{(\mathrm{pf})}$. - By the presentation of $f_{j}^{(*)}$, we have

$$
f_{j}^{(*)}=f_{j}+g_{j} \quad\left(g_{j} \in\left(\frac{t_{1}}{t}, \ldots, \frac{t_{e}}{t}\right)\left(B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_{p}} V\right)\right)
$$

Since $\left\{f_{j}\right\}_{j=1}^{d}$ forms a basis of $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ over $R[1 / t]$, it is, in particular, linearly independent over $K_{\infty}^{(\text {pf })}(\subset R[1 / t])$. Thus, $\left\{\overline{f_{j}}={\overline{f_{j}}}^{(*)}\right\}_{j=1}^{d}(-$ denotes the reduction modulo $\left(t_{1}, \ldots, t_{e}\right)$ ) is linearly independent over $K_{\infty}^{(\mathrm{pf})}$ and we can see that $\left\{f_{j}^{(*)}\right\}_{j=1}^{d}$ is linearly independent over $K_{\infty}^{(\mathrm{pf})}$ in $R[1 / t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$.
(C) Conclusion. - Therefore, on the $K$-vector space generated by $\left\{f_{j}^{(*)}\right\}_{j=1}^{d}, \log (\gamma)$ and $\left\{\log \left(\beta_{i}\right)\right\}_{i=1}^{e}$ act trivially $\left(\Leftrightarrow \nabla^{(0)}\left(f_{j}^{(*)}\right)=0\right.$ and $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d)$. Thus, this means that $\Gamma_{K}$ acts on this $K$-vector space via finite quotient and there exists a finite field extension $L / K$ in $K_{\infty}^{(\mathrm{pf})}$ such that $\left\{f^{(*)}\right\}_{j=1}^{d}$ forms a basis of $D_{\mathrm{HT}, L}(V)$
over $L$. Since a potentially Hodge-Tate representation of $G_{K}$ is a Hodge-Tate representation of $G_{K}$, this completes the proof.

### 4.2. Main theorem for de Rham representations

Lemma 4.5. - For a p-adic representation $V$ of $G_{K}$, assume that $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, we can choose a basis $\left\{h_{j}\right\}_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V}$ of $D_{\mathrm{dif}}^{+}(V)[1 / t]$ over $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right][1 / t]$ such that the action of $\Gamma_{0}$ on $\left\{h_{j}\right\}_{j=1}^{d}$ is trivial.

Proof. - Since $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$, there exists a basis $\left\{h_{j}\right\}_{j=1}^{d}$ of $B_{\mathrm{dR}, K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_{p}} V$ over $B_{\mathrm{dR}, K^{\mathrm{pf}}}$ such that the action of $G_{K^{\mathrm{pf}}}$ on $\left\{h_{j}\right\}_{j=1}^{d}$ is trivial. We can see that these elements $\left\{h_{j}\right\}_{j=1}^{d}$ are contained in $D_{\text {dif }}^{+}(V)[1 / t]$ by definition. For each $j$, if we twist $h_{j}$ by some power of $t$, we obtain an element $g_{j}$ of $B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+} \otimes_{\mathbb{Q}_{p}} V$ such that $g_{j} \notin t B_{\mathrm{dR}, K^{\mathrm{pf}}}^{+} \otimes_{\mathbb{Q}_{p}} V$. Then, it follows that $g_{j}$ is contained in $D_{\text {dif }}^{+}(V)$ and satisfies $\overline{g_{j}} \neq 0(-$ denotes the reduction modulo $\left.\left(t, t_{1}, \ldots, t_{e}\right) D_{\text {dif }}^{+}(V)\right)$. Since $D_{\text {dif }}^{+}(V)$ is a free module of rank $d$ over the local ring $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$ and $\left\{\overline{g_{j}}\right\}_{j=1}^{d}$ forms a basis of $D_{\operatorname{Sen}}(V)$ over $K_{\infty}^{\mathrm{pf}}$, the lifting $\left\{g_{j}\right\}_{j=1}^{d}$ of $\left\{\overline{g_{j}}\right\}_{j=1}^{d}$ in $D_{\text {dif }}^{+}(V)$ forms a basis of $D_{\text {dif }}^{+}(V)$ over $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$. Thus, it follows that $\left\{h_{j}\right\}_{j=1}^{d}$ forms a basis of $D_{\text {dif }}^{+}(V)[1 / t]$ over $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right][1 / t]$.

With notations as above, note that, since we have the inclusion $D_{e \text {-dif }}^{+}(V) \hookrightarrow D_{\text {dif }}^{+}(V)[1 / t]$ by definition, any element $g$ of $D_{e-\text { dif }}^{+}(V)$ can be written as $g=\sum_{k=l}^{+\infty}\left(\sum_{j=1}^{d} a_{j k} h_{j}\right) t^{k}$ $\left(a_{j k} \in K_{\infty}^{\mathrm{pf}}\left[\left[t_{1}, \ldots, t_{e}\right]\right]\right)$.

Remark 4.6. - Keep the notation as in Lemma 4.5. Since we assume that $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\left\{v_{j}\right\}_{j=1}^{d}$ of $D_{\mathrm{Bri}}(V)$ over $K_{\infty}^{(\mathrm{pf})}$ such that $\nabla^{(0)}\left(v_{j}\right)=n_{j} v_{j}$. Put $M=\operatorname{Max}\left(n_{j}\right)_{j=1}^{d}$. Then, for an element $g \in D_{e-\text { dif }}^{+}(V)$, there exists an element $\sum_{k=n}^{+\infty}\left(\sum_{j=1}^{d} c_{j k} h_{j}\right) t^{k}$ of $\left(t, t_{1}, \ldots, t_{e}\right) D_{e \text {-dif }}^{+}(V)$ such that we can write

$$
g=\sum_{k=m}^{M}\left(\sum_{j=1}^{d} b_{j k} h_{j}\right) t^{k}+\sum_{k=n}^{+\infty}\left(\sum_{j=1}^{d} c_{j k} h_{j}\right) t^{k} \quad\left(b_{j k}, c_{j k} \in K_{\infty}^{\mathrm{pf}}\left[\left[t_{1}, \ldots, t_{e}\right]\right]\right)
$$

Thus, $g^{\prime}=\sum_{k=m}^{M}\left(\sum_{j=1}^{d} b_{j k} h_{j}\right) t^{k}$ defines an element of $D_{e \text {-dif }}^{+}(V)$.
Lemma 4.7. - With notations as above, for an element $g^{\prime}=\sum_{k=m}^{M}\left(\sum_{j=1}^{d} b_{j k} h_{j}\right) t^{k}$ of $D_{e-\text { dif }}^{+}(V)$, each $\left(\sum_{j=1}^{d} b_{j k} h_{j}\right) t^{k}$ is contained in $D_{e \text {-dif }}^{+}(V)$.

Proof. - We shall prove this lemma by induction on the smallest degree of $g^{\prime}$ with respect to $t$. Since we have $g^{\prime}-\left(\sum_{j=1}^{d} b_{j m} h_{j}\right) t^{m} \in D_{e \text {-dif }}^{+}(V)$ if $\left(\sum_{j=1}^{d} b_{j m} h_{j}\right) t^{m}$ is contained in $D_{e \text {-dif }}^{+}(V)$, it suffices to show that $\left(\sum_{j=1}^{d} b_{j m} h_{j}\right) t^{m}$ is contained in $D_{e \text {-dif }}^{+}(V)$. Since the $K_{\infty}^{\mathrm{pf}}\left[\left[t_{1}, \ldots, t_{e}\right]\right]$-linear derivation $\nabla^{(0)}$ acts trivially on $\left\{h_{j}\right\}_{j=1}^{d}$, we have

$$
\prod_{k=m+1}^{M}\left(\nabla^{(0)}-k\right)\left(g^{\prime}\right)=\left(\prod_{k=m+1}^{M}(m-k)\right)\left(\sum_{j=1}^{d} b_{j m} h_{j}\right) t^{m}
$$

It follows that $\left(\sum_{j=1}^{d} b_{j m} h_{j}\right) t^{m}$ is contained in $D_{e \text {-dif }}^{+}(V)$ since the action of $\nabla^{(0)}$ on $D_{e-\mathrm{dif}}^{+}(V)$ is stable. Thus, this completes the proof.

Proposition 4.8. - For a p-adic representation $V$ of $G_{K}$, assume that $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$. Then, there exists a $\nabla^{(0)}$-equivariant isomorphism of $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$-modules

$$
D_{e-\text { dif }}^{+}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\operatorname{dim}_{Q_{p}} V} K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]\left(n_{j}\right) \quad\left(n_{j} \in \mathbb{Z}\right)
$$

Proof. - Since $V$ is also a Hodge-Tate representation of $G_{K^{\mathrm{pf}}}$, by Corollary 4.3, there exists a basis $\left\{v_{j}\right\}_{j=1}^{d}$ of $D_{e-\text { dif }}^{+}(V) /\left(t, t_{1}, \ldots, t_{e}\right) D_{e \text {-dif }}^{+}(V) \simeq D_{\mathrm{Bri}}(V)$ over $K_{\infty}^{(\mathrm{pf})}$ such that it gives a $\nabla^{(0)}$-equivariant isomorphism of $K_{\infty}^{(\mathrm{pf})}$-vector spaces

$$
D_{e-\text { dif }}^{+}(V) /\left(t, t_{1}, \ldots, t_{e}\right) D_{e-\mathrm{dif}}^{+}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K_{\infty}^{(\mathrm{pf})}\left(n_{j}\right): v_{j} \mapsto t^{n_{j}}
$$

Since $D_{e-\text { dif }}^{+}(V)$ is a free module of rank $d$ over the local ring $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$, any lifting $\left\{g_{j}\right\}_{j=1}^{d}$ of $\left\{v_{j}\right\}_{j=1}^{d}$ in $D_{e-\text { dif }}^{+}(V)$ forms a basis of $D_{e-\text { dif }}^{+}(V)$ over $K_{\infty}^{(\mathrm{pff})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$. Let $\left\{h_{j}\right\}_{j=1}^{d}$ denote a basis of $D_{\text {dif }}^{+}(V)[1 / t]$ over $K_{\infty}^{\mathrm{pf}}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right][1 / t]$ such that $\nabla^{(0)}\left(h_{j}\right)=0$ obtained in Lemma 4.5. Then, we may assume that each $g_{j}$ is written as $g_{j}=\sum_{k=m}^{M}\left(\sum_{l=1}^{d} b_{k l} h_{l}\right) t^{k}\left(b_{k l} \in K_{\infty}^{\mathrm{pf}}\left[\left[t_{1}, \ldots, t_{e}\right]\right]\right)$ where we take $M \in \mathbb{N}$ as in Remark 4.6. Now, define an element $f_{j}$ of $D_{e \text {-dif }}^{+}(V)$ (Lemma 4.7 above) by

$$
f_{j}=\left(\sum_{l=1}^{d} b_{n_{j}} h_{l}\right) t^{n_{j}} .
$$

It is easy to see $\nabla^{(0)}\left(f_{j}\right)=n_{j} f_{j}$. Therefore, the rest is to show that $\left\{f_{j}\right\}_{j=1}^{d}$ forms a basis of $D_{e-\text { dif }}^{+}(V)$ over $K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$. To prove that $\left\{f_{j}\right\}_{j=1}^{d}$ is a lifting of $\left\{v_{j}\right\}_{j=1}^{d}$, it suffices to show $g_{j}-f_{j} \in\left(t, t_{1}, \ldots, t_{e}\right) D_{e \text { dif }}^{+}(V)$. For each $g_{j}$, put $s_{k}=\left(\sum_{l=1}^{d} b_{k l} h_{l}\right) t^{k} \in D_{e \text {-dif }}^{+}(V)$ (Lemma 4.7 above). Since we have $\nabla^{(0)}\left(\overline{s_{k}}\right)=k \overline{s_{k}}$ ( - denotes the reduction modulo $\left(t, t_{1}, \ldots, t_{e}\right)$ ) and this means that $\overline{s_{k}}$ is an eigenvector of $\nabla^{(0)}$, it follows that the elements $\left\{v_{j}, \overline{s_{k}} \neq 0\right\}_{k \neq n_{j}}$ are linearly independent over $K_{\infty}^{(\mathrm{pf})}$ in $D_{\mathrm{Bri}}(V)$. Since we have $v_{j}=\sum_{k=m}^{M} \overline{s_{k}}$ by definition, it follows that we obtain $\overline{s_{k}}=0$ for $k \neq n_{j}$. This means that we have $s_{k} \in\left(t, t_{1}, \ldots, t_{e}\right) D_{e-\text { dif }}^{+}(V)\left(k \neq n_{j}\right)$ and $g_{j}-f_{j} \in\left(t, t_{1}, \ldots, t_{e}\right) D_{e \text {-dif }}^{+}(V)$. Thus, this completes the proof.

Remark 4.9. - In general, it is evident from the proof that, if $L$ denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic $p>0$ and $V$ is a de Rham representation of $G_{L}=\operatorname{Gal}(\bar{L} / L)$ where we choose an algebraic closure $\bar{L}$ of $L$, we have a $\nabla^{(0)}$-equivariant isomorphism of $L_{\infty}[[t]]$-modules

$$
D_{\mathrm{dif}}^{+}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} L_{\infty}[[t]]\left(n_{j}\right) \quad\left(n_{j} \in \mathbb{Z}\right) .
$$

Theorem 4.10. - Let $K$ be a complete discrete valuation field of characteristic 0 with residue field $k$ of characteristic $p>0$ such that $\left[k: k^{p}\right]=p^{e}<+\infty$ and $V$ be a $p$-adic representation of $G_{K}$. Let $K^{\text {pf }}$ be the field extension of $K$ defined as before. Then, $V$ is a de Rham representation of $G_{K}$ if and only if $V$ is a de Rham representation of $G_{K^{\mathrm{pr}}}$.

Proof. - We shall prove the main theorem in two parts.
(1) $V: d R$ rep. of $G_{K} \Rightarrow V: d R$ rep. of $G_{K^{\mathrm{pf}}}$. - Since $V$ is a de Rham representation of $G_{K}$, there exists a $G_{K}$-equivariant isomorphism of $B_{\mathrm{dR}, K^{\prime}}$-modules

$$
\begin{equation*}
B_{\mathrm{dR}, K} \otimes_{\mathbb{Q}_{p}} V \simeq\left(B_{\mathrm{dR}, K}\right)^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V} \tag{4.4}
\end{equation*}
$$

Now, by tensoring $B_{\mathrm{dR}, K^{\mathrm{pf}}} \otimes_{B_{\mathrm{dR}, K}}$ (which is induced by the $G_{K^{\mathrm{pf}} \text {-equivariant surjection }}$ $p: B_{\mathrm{dR}, K} \rightarrow B_{\mathrm{dR}, K^{\mathrm{pf}}}: t_{i} \mapsto 0$ ) over (4.4), we obtain a $G_{K^{\mathrm{pf}}}$-equivariant isomorphism of $B_{\mathrm{dR}, K^{\mathrm{pf}}-\mathrm{modules}}$

$$
B_{\mathrm{dR}, K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_{p}} V \simeq\left(B_{\mathrm{dR}, K^{\mathrm{pf}}}\right)^{d}
$$

This means that $V$ is a de Rham representation of $G_{K^{\mathrm{pf}}}$.
(2) $V$ : dR rep. of $G_{K^{\mathrm{pf}}} \Rightarrow V: d R$ rep. of $G_{K}$. - For simplicity, put $R=K_{\infty}^{(\mathrm{pf})}\left[\left[t, t_{1}, \ldots, t_{e}\right]\right]$. We shall construct the $K_{\infty}^{(\mathrm{pf})}$-linearly independent elements $\left\{f_{j}^{(*)}\right\}_{j=1}^{d=\operatorname{dim}_{\mathbb{Q}_{p}} V}$ of $R[1 / t] \otimes_{R} D_{e-\text { dif }}^{+}(V)\left(\subset B_{\mathrm{dR}, K} \otimes_{\mathbb{Q}_{p}} V\right)$ such that $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $0 \leq i \leq e$ and $1 \leq j \leq d$.
(A) Construction of $\left\{f_{j}^{(*)}\right\}_{j=1}^{d} \in R[1 / t] \otimes_{R} D_{e \text {-dif }}^{+}(V)$. - From the presentation of Proposition 4.8 above, if we twist by some powers of $t$, we obtain a basis $\left\{f_{j}\right\}_{j=1}^{d}$ of $R[1 / t] \otimes_{R} D_{e \text {-dif }}^{+}(V)$ over $R[1 / t]$ such that $\nabla^{(0)}\left(f_{j}\right)=0$ for all $1 \leq j \leq d$. Thus, by applying Corollary 3.6 to the $R[1 / t]$-module $R[1 / t] \otimes_{R} D_{e-\text { dif }}^{+}(V)$ generated by $\left\{f_{j}\right\}_{j=1}^{d}$, we can deduce

$$
\begin{equation*}
\left(\nabla^{(1)}\right)^{k_{1}} \cdots\left(\nabla^{(e)}\right)^{k_{e}}\left(f_{j}\right)=t^{k_{1}+\cdots+k_{e}} \sum_{k=1}^{d} c_{k} f_{k} \tag{4.5}
\end{equation*}
$$

where $c_{k}$ is an element of $R$ such that $\nabla^{(0)}\left(c_{k}\right)=0$. Define an element $f_{j}^{(*)}$ of $R[1 / t] \otimes_{R} D_{e-\text { dif }}^{+}(V)$ by

$$
f_{j}^{(*)}=\sum_{0 \leq k_{1}, \ldots, k_{e}}(-1)^{k_{1}+\cdots+k_{e}} \frac{t_{1}^{k_{1}} \cdots t_{e}^{k_{e}}}{k_{1}!\cdots k_{e}!t^{k_{1}+\cdots+k_{e}}}\left(\nabla^{(1)}\right)^{k_{1}} \cdots\left(\nabla^{(e)}\right)^{k_{e}}\left(f_{j}\right)
$$

Note that this series converges in $R[1 / t] \otimes_{R} D_{e-\mathrm{dif}}^{+}(V)$ for $\left(t_{1}, \ldots, t_{e}\right)$-adic topology by (4.5) and thus $f_{j}^{(*)}$ actually defines an element of $R[1 / t] \otimes_{R} D_{e-\text { dif }}^{+}(V)$. Then, it follows easily that we have $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d$ by using the Leibniz rule. Furthermore, by using (4.5) and the fact $\nabla^{(0)}\left(f_{j}\right)=0$, we can deduce that we have $\nabla^{(0)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq j \leq d$.
(B) $\left\{f_{j}^{(*)}\right\}_{j=1}^{d} \in R[1 / t] \otimes_{R} D_{e-\operatorname{dif}}^{+}(V)$ is linearly independent over $K_{\infty}^{(\mathrm{pf})}$. - By the presentation of $f_{j}^{(*)}$, we have

$$
f_{j}^{(*)}=f_{j}+g_{j} \quad\left(g_{j} \in\left(t_{1}, \ldots, t_{e}\right)\left(B_{\mathrm{dR}, K} \otimes_{\mathbb{Q}_{p}} V\right)\right)
$$

Since $\left\{f_{j}\right\}_{j=1}^{d}$ forms a basis of $R[1 / t] \otimes_{R} D_{e-\mathrm{dif}}^{+}(V)$ over $R[1 / t]$, it is, in particular, linearly independent over $K_{\infty}^{(\text {pf })}(\subset R[1 / t])$. Thus, $\left\{\overline{f_{j}}={\overline{f_{j}}}^{(*)}\right\}_{j=1}^{d}$ ( - denotes the reduction modulo $\left(t_{1}, \ldots, t_{e}\right)$ ) is linearly independent over $K_{\infty}^{(\text {pf })}$ and we can see that $\left\{f_{j}^{(*)}\right\}_{j=1}^{d}$ is linearly independent over $K_{\infty}^{(\mathrm{pf})}$ in $R[1 / t] \otimes_{R} D_{e \text {-dif }}^{+}(V)$.
(C) Conclusion. - Therefore, on the $K$-vector space generated by $\left\{f_{j}^{(*)}\right\}_{j=1}^{d}, \log (\gamma)$ and $\left\{\log \left(\beta_{i}\right)\right\}_{i=1}^{e}$ act trivially $\left(\Leftrightarrow \nabla^{(0)}\left(f_{j}^{(*)}\right)=0\right.$ and $\nabla^{(i)}\left(f_{j}^{(*)}\right)=0$ for all $1 \leq i \leq e$ and $1 \leq j \leq d)$. Thus, this means that $\Gamma_{K}$ acts on this $K$-vector space via finite quotient and there exists a finite field extension $L / K$ in $K_{\infty}^{(\text {pf })}$ such that $\left\{f^{(*)}\right\}_{j=1}^{d}$ forms a basis of $D_{\mathrm{dR}, L}(V)$ over $L$. Since a potentially de Rham representation of $G_{K}$ is a de Rham representation of $G_{K}$, this completes the proof.

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