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# Kazuma MORITA

Hodge-Tate and de Rham representations in the imperfect residue field case

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

## HODGE-TATE AND DE RHAM REPRESENTATIONS IN THE IMPERFECT RESIDUE FIELD CASE

### BY KAZUMA MORITA

ABSTRACT. – Let K be a p-adic local field with residue field k such that  $[k : k^p] = p^e < +\infty$  and V be a p-adic representation of  $\operatorname{Gal}(\overline{K}/K)$ . Then, by using the theory of p-adic differential modules, we show that V is a Hodge-Tate (resp. de Rham) representation of  $\operatorname{Gal}(\overline{K}/K)$  if and only if V is a Hodge-Tate (resp. de Rham) representation of  $\operatorname{Gal}(\overline{K}/K)$  if and only if V is a Hodge-Tate (resp. de Rham) representation of  $\operatorname{Gal}(\overline{K}/K)$  is a certain p-adic local field with residue field the smallest perfect field  $k^{\text{pf}}$  containing k.

RÉSUMÉ. – Soit K un corps local p-adique de corps résiduel k tel que  $[k : k^p] = p^e < +\infty$  et soit V une représentation p-adique de Gal $(\overline{K}/K)$ . Nous utilisons la théorie des modules différentiels p-adiques pour montrer que V est une représentation de Hodge-Tate (resp. de Rham) de Gal $(\overline{K}/K)$  si et seulement si V est une représentation de Hodge-Tate (resp. de Rham) de Gal $(\overline{K}/K)$  où  $K^{\text{pf}}/K$  est un certain corps local p-adique de corps résiduel le plus petit corps parfait  $k^{\text{pf}}$  contenant k.

#### 1. Introduction

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic p > 0 such that  $[k : k^p] = p^e < +\infty$ . Choose an algebraic closure  $\overline{K}$  of K and put  $G_K = \text{Gal}(\overline{K}/K)$ . By a p-adic representation of  $G_K$ , we mean a finite dimensional vector space V over  $\mathbb{Q}_p$  endowed with a continuous action of  $G_K$ . In the case e = 0 (i.e. k is perfect), following Fontaine, we can classify p-adic representations of  $G_K$  by using the p-adic periods rings  $B_{\text{HT}}$ ,  $B_{\text{dR}}$ ,  $B_{\text{st}}$  and  $B_{\text{cris}}$  (Hodge-Tate, de Rham, semi-stable and crystalline representations). In the general case (i.e. k is not necessarily perfect), Hyodo constructed the imperfect residue field version of the ring  $B_{\text{HT}}$  and Tsuzuki and several authors constructed that of the ring  $B_{\text{dR}}$ . By using these rings, we can define the imperfect residue field version of Hodge-Tate and de Rham representations of  $G_K$  in the evident way ([3], [7], [8], [9], [12]).

Now, we shall state the main result of this article. Let us fix some notations. Fix a lifting  $(b_i)_{1 \le i \le e}$  of a *p*-basis of *k* in  $\mathcal{O}_K$  (the ring of integers of *K*) and for each  $m \ge 1$ , fix a  $p^m$  -th root  $b_i^{1/p^m}$  of  $b_i$  in  $\overline{K}$  satisfying  $(b_i^{1/p^m+1})^p = b_i^{1/p^m}$ . Put  $K^{(\text{pf})} = \bigcup_{m \ge 1} K(b_i^{1/p^m}, 1 \le i \le e)$  and  $K^{\text{pf}}$  = the *p*-adic completion of  $K^{(\text{pf})}$ . These fields depend on the choice of a lifting

of a *p*-basis of *k* in  $\mathscr{O}_K$ . Since  $K^{\text{pf}}$  becomes a complete discrete valuation field with perfect residue field, we can apply theories in the perfect residue field case to *p*-adic representations of  $G_{K^{\text{pf}}} = \text{Gal}(\overline{K^{\text{pf}}}/K^{\text{pf}})$  where we choose an algebraic closure  $\overline{K^{\text{pf}}}$  of  $K^{\text{pf}}$  containing  $\overline{K}$ . Note that, if *V* is a *p*-adic representation of  $G_K$ , it can be also regarded as a *p*-adic representation of  $G_{K^{\text{pf}}}$  (see Section 2.2 for details). Our main result is the following.

THEOREM 1.1. – Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic p > 0 such that  $[k : k^p] = p^e < +\infty$  and V be a p-adic representation of  $G_K$ . Let  $K^{pf}$  be the field extension of K defined as above. Then, we have the following equivalences

- 1. *V* is a Hodge-Tate representation of  $G_K$  if and only if *V* is a Hodge-Tate representation of  $G_{K^{\text{pf}}}$ ,
- 2. *V* is a de Rham representation of  $G_K$  if and only if *V* is a de Rham representation of  $G_{K^{\text{pf}}}$ .

In the case of Hodge-Tate representations, Tsuji [11] had proved a more refined theorem based on this article. This paper is organized as follows. In Section 2, we shall review the definitions and basic known facts on Hodge-Tate and de Rham representations, first in the perfect residue field case and then in the imperfect residue field case. In Section 3, we shall review the theory of *p*-adic differential modules which play a central role in this article. In Section 4, by using the theory of *p*-adic differential modules, we shall prove the main theorem, first for Hodge-Tate representations and then for de Rham representations.

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#### 2. Preliminaries on Hodge-Tate and de Rham representations

#### 2.1. Hodge-Tate and de Rham representations in the perfect residue field case

(See [4] and [5] for details.) Let K be a complete discrete valuation field of characteristic 0 with perfect residue field k of characteristic p > 0. Choose an algebraic closure  $\overline{K}$  of K and consider its p-adic completion  $\mathbb{C}_p$ . Put

$$\widetilde{\mathbb{E}} = \lim_{x \mapsto x^p} \mathbb{C}_p = \{ (x^{(0)}, x^{(1)}, \dots) \mid (x^{(i+1)})^p = x^{(i)}, x^{(i)} \in \mathbb{C}_p \}$$

and let  $\widetilde{\mathbb{E}}^+$  denote the set of  $x = (x^{(i)}) \in \widetilde{\mathbb{E}}$  such that  $x^{(0)} \in \mathscr{O}_{\mathbb{C}_p}$  where  $\mathscr{O}_{\mathbb{C}_p}$  denotes the ring of integers of  $\mathbb{C}_p$ . For two elements  $x = (x^{(i)})$  and  $y = (y^{(i)})$  of  $\widetilde{\mathbb{E}}$ , their sum and

product are defined by  $(x + y)^{(i)} = \lim_{j \to +\infty} (x^{(i+j)} + y^{(i+j)})^{p^j}$  and  $(xy)^{(i)} = x^{(i)}y^{(i)}$ . These sum and product make  $\widetilde{\mathbb{E}}$  a perfect field of characteristic p > 0 ( $\widetilde{\mathbb{E}}^+$  is a subring of  $\widetilde{\mathbb{E}}$ ). Let  $\epsilon = (\epsilon^{(n)})$  be an element of  $\widetilde{\mathbb{E}}$  such that  $\epsilon^{(0)} = 1$  and  $\epsilon^{(1)} \neq 1$ . Then,  $\widetilde{\mathbb{E}}$  is the completion of an algebraic closure of  $k((\epsilon - 1))$  for the valuation defined by  $v_{\mathbb{E}}(x) = v_p(x^{(0)})$  where  $v_p$  denotes the *p*-adic valuation of  $\mathbb{C}_p$  normalized by  $v_p(p) = 1$ . The field  $\widetilde{\mathbb{E}}$  is equipped with a continuous action of the Galois group  $G_K = \operatorname{Gal}(\overline{K}/K)$  with respect to the topology defined by the valuation  $v_{\mathbb{E}}$ . Put  $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$  (the ring of Witt vectors with coefficients in  $\widetilde{\mathbb{E}}^+$ ) and  $\widetilde{\mathbb{B}}^+ = \widetilde{\mathbb{A}}^+[1/p] = \{\sum_{k\gg -\infty} p^k[x_k] \mid x_k \in \widetilde{\mathbb{E}}^+\}$  where [\*] denotes the Teichmüller lift of  $* \in \widetilde{\mathbb{E}}^+$ . This ring  $\widetilde{\mathbb{B}}^+$  is equipped with a surjective homomorphism

$$\theta: \widetilde{\mathbb{B}}^+ \twoheadrightarrow \mathbb{C}_p: \sum p^k[x_k] \mapsto \sum p^k x_k^{(0)}.$$

If  $\tilde{p} = (p^{(n)})$  denotes an element of  $\mathbb{\tilde{E}}^+$  such that  $p^{(0)} = p$ , we can show that Ker  $(\theta)$  is the principal ideal generated by  $\omega = [\tilde{p}] - p$ . The ring  $B^+_{dR,K}$  is defined to be the Ker  $(\theta)$ -adic completion of  $\mathbb{\tilde{B}}^+$ 

$$B_{\mathrm{dR},K}^+ = \underline{\lim}_{n \ge 0} \widetilde{\mathbb{B}}^+ / (\mathrm{Ker}\,(\theta)^n).$$

This is a discrete valuation ring and  $t = \log([\epsilon])$  which converges in  $B_{dR,K}^+$  is a generator of the maximal ideal. Put  $B_{dR,K} = B_{dR,K}^+[1/t]$ . This ring  $B_{dR,K}$  becomes a field and is equipped with an action of the Galois group  $G_K$  and a filtration defined by  $\operatorname{Fil}^i B_{dR,K} = t^i B_{dR,K}^+$   $(i \in \mathbb{Z})$ . Then,  $(B_{dR,K})^{G_K}$  is canonically isomorphic to K. Thus, for a p-adic representation V of  $G_K$ ,  $D_{dR,K}(V) = (B_{dR,K} \otimes_{\mathbb{Q}_p} V)^{G_K}$  is naturally a K-vector space. We say that a p-adic representation V of  $G_K$  is a de Rham representation of  $G_K$  if we have

$$\dim_{\mathbb{Q}_n} V = \dim_K D_{\mathrm{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_n} V \ge \dim_K D_{\mathrm{dR},K}(V)).$$

Furthermore, we say that a *p*-adic representation V of  $G_K$  is a potentially de Rham representation of  $G_K$  if there exists a finite field extension L/K in  $\overline{K}$  such that V is a de Rham representation of  $G_L$ . It is known that a potentially de Rham representation V of  $G_K$  is a de Rham representation of  $G_K$  (see [5, 3.9]).

Define  $B_{\text{HT},K}$  to be the associated graded algebra to the filtration  $\text{Fil}^{i}B_{dR,K}$ . The quotient  $\text{gr}^{i}B_{\text{HT},K} = \text{Fil}^{i}B_{dR,K}/\text{Fil}^{i+1}B_{dR,K}$   $(i \in \mathbb{Z})$  is a one-dimensional  $\mathbb{C}_{p}$ -vector space spanned by the image of  $t^{i}$ . Thus, we obtain the presentation

$$B_{\mathrm{HT},K} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$$

where  $\mathbb{C}_p(i) = \mathbb{C}_p \otimes \mathbb{Z}_p(i)$  is the Tate twist. Then,  $(B_{\mathrm{HT},K})^{G_K}$  is canonically isomorphic to K. Thus, for a p-adic representation V of  $G_K$ ,  $D_{\mathrm{HT},K}(V) = (B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$  is naturally a K-vector space. We say that a p-adic representation V of  $G_K$  is a Hodge-Tate representation of  $G_K$  if we have

 $\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \ge \dim_K D_{\mathrm{HT},K}(V)).$ 

Furthermore, we say that a *p*-adic representation V of  $G_K$  is a potentially Hodge-Tate representation of  $G_K$  if there exists a finite field extension L/K in  $\overline{K}$  such that V is a Hodge-Tate representation of  $G_L$ . It is known that a potentially Hodge-Tate representation V of  $G_K$  is a Hodge-Tate representation of  $G_K$  (see [5, 3.9]). Since we have  $\operatorname{gr} B_{dR,K} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i)$ ,

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if V is a de Rham representation of  $G_K$ , there exists a  $G_K$ -equivariant isomorphism  $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} \mathbb{C}_p(n_j) \ (n_j \in \mathbb{Z})$ . Thus, it follows that a de Rham representation V of  $G_K$  is a Hodge-Tate representation of  $G_K$ .

#### 2.2. Hodge-Tate and de Rham representations in the imperfect residue field case

Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic p > 0 such that  $[k : k^p] = p^e < +\infty$ . Choose an algebraic closure  $\overline{K}$  of K and put  $G_K = \text{Gal}(\overline{K}/K)$ . As in the introduction, fix a lifting  $(b_i)_{1 \le i \le e}$  of a p-basis of k in  $\mathcal{O}_K$  (the ring of integers of K) and for each  $m \ge 1$ , fix a  $p^m$  -th root  $b_i^{1/p^m}$  of  $b_i$  in  $\overline{K}$  satisfying  $(b_i^{1/p^{m+1}})^p = b_i^{1/p^m}$ . Put

$$K^{(\mathrm{pf})} = \cup_{m \ge 0} K(b_i^{1/p^m}, 1 \le i \le e)$$
 and  $K^{\mathrm{pf}} = \mathrm{the} \ p\mathrm{-adic} \ \mathrm{completion} \ \mathrm{of} \ K^{(\mathrm{pf})}.$ 

These fields depend on the choice of a lifting of a *p*-basis of k in  $\mathcal{O}_K$ . Since  $K^{(\text{pf})}$  is a Henselian discrete valuation field, we have an isomorphism  $G_{K^{\text{pf}}} = \text{Gal}(\overline{K^{\text{pf}}}/K^{\text{pf}}) \simeq G_{K^{(\text{pf})}} = \text{Gal}(\overline{K}/K^{(\text{pf})}) (\subset G_K)$  where we choose an algebraic closure  $\overline{K^{\text{pf}}}$  of  $K^{\text{pf}}$  containing  $\overline{K}$ . With this isomorphism, we identify  $G_{K^{\text{pf}}}$  with a subgroup of  $G_K$ . We have a bijective map from the set of finite extensions of  $K^{(\text{pf})}$  contained in  $\overline{K}$  to the set of finite extensions of  $K^{\text{pf}}$ . Furthermore,  $LK^{\text{pf}}$  is the *p*-adic completion of *L*. Hence, we have an isomorphism of rings

$$\mathscr{O}_{\overline{K}}/p^n\mathscr{O}_{\overline{K}}\simeq \mathscr{O}_{\overline{K^{\mathrm{pf}}}}/p^n\mathscr{O}_{\overline{K^{\mathrm{pf}}}}$$

where  $\mathscr{O}_{\overline{K}}$  and  $\mathscr{O}_{\overline{K^{pf}}}$  denote the rings of integers of  $\overline{K}$  and  $\overline{K^{pf}}$ . Thus, the *p*-adic completion of  $\overline{K}$  is isomorphic to the *p*-adic completion of  $\overline{K^{pf}}$ , which we will write  $\mathbb{C}_p$ . As in Subsection 2.1, construct the rings  $\widetilde{\mathbb{E}}^+$  and  $\widetilde{\mathbb{A}}^+ = W(\widetilde{\mathbb{E}}^+)$  from this  $\mathbb{C}_p$ . Let  $k^{pf}$  denote the perfect residue field of  $K^{pf}$  and put  $\mathscr{O}_{K_0} = \mathscr{O}_K \cap W(k^{pf})$ . Let  $\alpha : \mathscr{O}_K \otimes_{\mathscr{O}_{K_0}} \widetilde{\mathbb{A}}^+ \twoheadrightarrow \mathscr{O}_{\overline{K}}/p\mathscr{O}_{\overline{K}}$  be the natural surjection and define  $\widetilde{\mathbb{A}}^+_{(K)}$  to be  $\widetilde{\mathbb{A}}^+_{(K)} = \varprojlim_{n\geq 0}(\mathscr{O}_K \otimes_{\mathscr{O}_{K_0}} \widetilde{\mathbb{A}}^+)/(\operatorname{Ker}(\alpha))^n$ . Let  $\theta_K : \widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \twoheadrightarrow \mathbb{C}_p$  be the natural extension of  $\theta : \widetilde{\mathbb{A}}^+[1/p] \twoheadrightarrow \mathbb{C}_p$ . Define  $B^+_{dR,K}$  to be the Ker  $(\theta_K)$ -adic completion of  $\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ 

$$B^+_{\mathrm{dR},K} = \varprojlim_{n \ge 0} (\widetilde{\mathbb{A}}^+_{(K)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) / (\mathrm{Ker}\,(\theta_K)^n).$$

This is a K-algebra equipped with an action of the Galois group  $G_K$ . Let  $\tilde{b}_i$  denote  $(b_i^{(n)}) \in \mathbb{E}^+$  such that  $b_i^{(0)} = b_i$  and then the series which defines  $\log([\tilde{b}_i]/b_i)$  converges to an element  $t_i$  in  $B_{dR,K}^+$ . Then, the ring  $B_{dR,K}^+$  becomes a local ring with the maximal ideal  $m_{dR} = (t, t_1, \ldots, t_e)$ . Define a filtration on  $B_{dR,K}^+$  by fil<sup>*i*</sup> $B_{dR,K}^+ = m_{dR}^i$ . Then, the homomorphism

$$f: B^+_{\mathrm{dR}, K^{\mathrm{pf}}}[[t_1, \dots, t_e]] \to B^+_{\mathrm{dR}, K}$$

is an isomorphism of filtered algebras (see [3, Proposition 2.9]). From this isomorphism, it follows easily that

$$i: B^+_{\mathrm{dR},K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR},K}$$
 and  $p: B^+_{\mathrm{dR},K} \twoheadrightarrow B^+_{\mathrm{dR},K^{\mathrm{pf}}}: t_i \mapsto 0$ 

are  $G_{K^{pf}}$ -equivariant homomorphisms and the composition

$$p \circ i : B^+_{\mathrm{dR}, K^{\mathrm{pf}}} \hookrightarrow B^+_{\mathrm{dR}, K} \twoheadrightarrow B^+_{\mathrm{dR}, K^{\mathrm{pf}}}$$

is an identity. Put  $B_{dR,K} = B^+_{dR,K}[1/t]$ . Then, K is canonically embedded in  $B_{dR,K}$  and we have a canonical isomorphism  $(B_{dR,K})^{G_K} = K$ . Thus, for a p-adic representation V of  $G_K$ ,  $D_{dR,K}(V) = (B_{dR,K} \otimes_{\mathbb{Q}_p} V)^{G_K}$  is naturally a K-vector space. We say that a p-adic representation V of  $G_K$  is a de Rham representation of  $G_K$  if we have

 $\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{dR},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \ge \dim_K D_{\mathrm{dR},K}(V)).$ 

Furthermore, we say that a *p*-adic representation V of  $G_K$  is a potentially de Rham representation of  $G_K$  if there exists a finite field extension L/K in  $\overline{K}$  such that V is a de Rham representation of  $G_L$ . We can show that a potentially de Rham representation V of  $G_K$  is a de Rham representation of  $G_K$  in the same way as in the perfect residue field case.

Define a filtration on  $B_{dR,K}$  to be

$$\operatorname{Fil}^{0}B_{\mathrm{dR},K} = \sum_{n=0}^{\infty} t^{-n} \operatorname{fil}^{n}B_{\mathrm{dR},K}^{+} = B_{\mathrm{dR},K}^{+} [\frac{t_{1}}{t}, \dots, \frac{t_{e}}{t}],$$
  
$$\operatorname{Fil}^{i}B_{\mathrm{dR},K} = t^{i} \operatorname{Fil}^{0}B_{\mathrm{dR},K} \ (i \in \mathbb{Z}).$$

Define  $B_{\text{HT},K}$  to be the associated graded algebra to this filtration. Since the quotient  $\operatorname{gr}^{i}B_{\text{HT},K} = \operatorname{Fil}^{i}B_{dR,K}/\operatorname{Fil}^{i+1}B_{dR,K}$   $(i \in \mathbb{Z})$  is given by  $\operatorname{gr}^{i}B_{\text{HT},K} = t^{i}\mathbb{C}_{p}[\frac{t_{1}}{t},\ldots,\frac{t_{e}}{t}]$ , we obtain the presentation

$$B_{\mathrm{HT},K} = \mathbb{C}_p[t,t^{-1},\frac{t_1}{t},\ldots,\frac{t_e}{t}] = B_{\mathrm{HT},K^{\mathrm{pf}}}[\frac{t_1}{t},\ldots,\frac{t_e}{t}].$$

From this presentation, it follows easily that

$$i: B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K}$$
 and  $p: B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}}: t_i/t \mapsto 0$ 

are  $G_{Kpf}$ -equivariant homomorphisms and the composition

$$p \circ i : B_{\mathrm{HT},K^{\mathrm{pf}}} \hookrightarrow B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}}$$

is an identity. The field K is canonically embedded in  $B_{\text{HT},K}$  and we have  $(B_{\text{HT},K})^{G_K} = K$ . Thus, for a p-adic representation V of  $G_K$ ,  $D_{\text{HT},K}(V) = (B_{\text{HT},K} \otimes_{\mathbb{Q}_p} V)^{G_K}$  is naturally a K-vector space. We say that a p-adic representation V of  $G_K$  is a Hodge-Tate representation of  $G_K$  if we have

 $\dim_{\mathbb{Q}_p} V = \dim_K D_{\mathrm{HT},K}(V) \quad (\text{we always have } \dim_{\mathbb{Q}_p} V \ge \dim_K D_{\mathrm{HT},K}(V)).$ 

Furthermore, we say that a *p*-adic representation V of  $G_K$  is a potentially Hodge-Tate representation of  $G_K$  if there exists a finite field extension L/K in  $\overline{K}$  such that V is a Hodge-Tate representation of  $G_L$ . We can show that a potentially Hodge-Tate representation V of  $G_K$  is a Hodge-Tate representation of  $G_K$  in the same way as in the perfect residue field case.

#### 3. Preliminaries on *p*-adic differential modules

In this section, we shall review the theory of *p*-adic differential modules which plays an important role in this article. First, let us fix the notations. Let *K* be a complete discrete valuation field of characteristic 0 with residue field *k* of characteristic p > 0 such that  $[k : k^p] = p^e < \infty$  and *V* be a *p*-adic representation of  $G_K$ . Define  $K^{(\text{pf})}$  and  $K^{\text{pf}}$  as in the introduction and in Subsection 2.2. Put  $K_{\infty}^{(\text{pf})} = \bigcup_{m \ge 0} K^{(\text{pf})}(\zeta_{p^m})$  (resp.  $K_{\infty}^{\text{pf}} = \bigcup_{m \ge 0} K^{\text{pf}}(\zeta_{p^m})$ ) where  $\zeta_{p^m}$  denotes a primitive  $p^m$ -th root of unity in  $\overline{K}$  (resp.  $\overline{K^{\text{pf}}}$ ) such

that  $(\zeta_{p^{m+1}})^p = \zeta_{p^m}$ . Let  $\hat{K}^{\text{pf}}_{\infty}$  denote the *p*-adic completion of  $K^{\text{pf}}_{\infty}$ . These fields  $K^{(\text{pf})}_{\infty}$ ,  $K^{\text{pf}}_{\infty}$  and  $\hat{K}^{\text{pf}}_{\infty}$  depend on the choice of a lifting of a *p*-basis of *k* in  $\mathcal{O}_K$ . Then, we have the following inclusions

$$K^{(\mathrm{pf})}_{\infty} \subset K^{\mathrm{pf}}_{\infty} \subset \hat{K}^{\mathrm{pf}}_{\infty}.$$

Let *H* denote the kernel of the cyclotomic character  $\chi : G_{K^{\text{pf}}} \to \mathbb{Z}_p^*$ . Then, the Galois group *H* is isomorphic to the subgroup  $\text{Gal}(\overline{K}/K_{\infty}^{(\text{pf})})$  of  $G_K$ . Define  $\Gamma_K = G_K/H$ . Let  $\Gamma_0$  denote the subgroup  $\text{Gal}(K_{\infty}^{(\text{pf})}/K^{(\text{pf})}) (\simeq G_{K^{\text{pf}}}/H)$  of  $\Gamma_K$ . Let  $\Gamma_i$   $(1 \le i \le e)$  be the subgroup of  $\Gamma_K$  such that actions of  $\beta_i \in \Gamma_i$   $(1 \le i \le e)$  satisfy  $\beta_i(\zeta_{p^m}) = \zeta_{p^m}$  and  $\beta_i(b_j^{1/p^m}) = b_j^{1/p^m}$   $(i \ne j)$  and define the homomorphism  $c_i : \Gamma_i \to \mathbb{Z}_p$  such that we have  $\beta_i(b_i^{1/p^m}) = b_i^{1/p^m} \zeta_{p^m}^{c_i(\beta_i)}$ . Then, the homomorphism  $c_i$  defines an isomorphism  $\Gamma_i \simeq \mathbb{Z}_p$  of profinite groups. With this, we can see that there exist isomorphisms of profinite groups

$$\Gamma_K \simeq \Gamma_0 \ltimes (\bigoplus_{i=1}^e \Gamma_i) \simeq \Gamma_0 \ltimes \mathbb{Z}_p^{\oplus e}$$

#### **3.1.** Definitions of *p*-adic differential modules

We shall review the definitions of p-adic differential modules and have the following diagram, for a p-adic representation V of  $G_K$ ,

$$(B^{+}_{\mathrm{dR},K} \otimes_{\mathbb{Q}_{p}} V)^{H} \xrightarrow{\theta_{K}} (\mathbb{C}_{p} \otimes_{\mathbb{Q}_{p}} V)^{H} \\ \cup \qquad \cup \\ D^{+}_{\mathrm{dif}}(V) \xrightarrow{} D_{\mathrm{Sen}}(V) \\ \cup \qquad \cup \\ D^{+}_{e,\mathrm{dif}}(V) \xrightarrow{} D_{\mathrm{Bri}}(V).$$

3.1.1. The module  $D_{\text{Sen}}(V)$ . – In the article [10], Sen shows that, for a *p*-adic representation V of  $G_{K^{\text{pf}}}$ , the  $\hat{K}^{\text{pf}}_{\infty}$ -vector space  $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$  has dimension  $d = \dim_{\mathbb{Q}_p} V$  and the union of the finite dimensional  $K^{\text{pf}}_{\infty}$ -subspaces of  $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$  stable under  $\Gamma_0$   $(\simeq G_{K^{\text{pf}}}/H)$  is a  $K^{\text{pf}}_{\infty}$ -vector space of dimension d stable under  $\Gamma_0$  (called  $D_{\text{Sen}}(V)$ ). We have  $\mathbb{C}_p \otimes_{K^{\text{pf}}_{\infty}} D_{\text{Sen}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$  and the natural map  $\hat{K}^{\text{pf}}_{\infty} \otimes_{K^{\text{pf}}_{\infty}} D_{\text{Sen}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$  is an isomorphism. Furthermore, if  $\gamma \in \Gamma_0$  is close enough to 1, then the series of operators on  $D_{\text{Sen}}(V)$ 

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{k>1} \frac{(1-\gamma)^k}{k}$$

converges to a  $K^{\text{pf}}_{\infty}$ -linear derivation  $\nabla^{(0)} : D_{\text{Sen}}(V) \to D_{\text{Sen}}(V)$  and does not depend on the choice of  $\gamma$ .

3.1.2. The module  $D_{\text{Bri}}(V)$ . – In the article [2], Brinon generalizes Sen's work above. For a *p*-adic representation V of  $G_K$ , he shows that the union of the finite dimensional  $K_{\infty}^{(\text{pf})}$ subspaces of  $(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$  stable under  $\Gamma_K$  is a  $K_{\infty}^{(\text{pf})}$ -vector space of dimension *d* stable under  $\Gamma_K$  (we call it  $D_{\text{Bri}}(V)$ ). We have  $\mathbb{C}_p \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V) = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$  and the natural map  $\hat{K}_{\infty}^{\text{pf}} \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V) \to (\mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^H$  is an isomorphism. As in the case of  $D_{\text{Sen}}(V)$ , the  $K_{\infty}^{(\text{pf})}$ -vector space  $D_{\text{Bri}}(V)$  is endowed with the action of the  $K_{\infty}^{(\text{pf})}$ -linear derivation

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 $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$  if  $\gamma \in \Gamma_0$  is close enough to 1. In addition to this operator  $\nabla^{(0)}$ , if  $\beta_i \in \Gamma_i$  is close enough to 1, then the series of operators on  $D_{\text{Bri}}(V)$ 

$$\frac{\log(\beta_i)}{c_i(\beta_i)} = -\frac{1}{c_i(\beta_i)} \sum_{k \ge 1} \frac{(1-\beta_i)^k}{k}$$

converges to a  $K_{\infty}^{(\mathrm{pf})}$ -linear derivation  $\nabla^{(i)}: D_{\mathrm{Bri}}(V) \to D_{\mathrm{Bri}}(V)$  and does not depend on the choice of  $\beta_i$ .

3.1.3. The module  $D_{e-\text{dif}}^+(V)$ . – In the article [1], Andreatta and Brinon generalize Fontaine's work [6]. For a *p*-adic representation *V* of  $G_K$ , they show that the union of  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$ -submodules of finite type of  $(B_{dR,K}^+ \otimes_{\mathbb{Q}_p} V)^H$  stable under  $\Gamma_K$  is a free  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$ -module of rank d stable under  $\Gamma_K$  (we call it  $D_{e-\text{dif}}^+(V)$ ). We have  $B_{dR,K}^+ \otimes_{K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]} D_{e-\text{dif}}^+(V) = B_{dR,K}^+ \otimes_{\mathbb{Q}_p} V$  and the natural map

$$(B^+_{\mathsf{dR},K})^H \otimes_{K^{(\mathrm{pf})}_{\infty}[[t,t_1,\ldots,t_e]]} D^+_{e-\mathrm{dif}}(V) \to (B^+_{\mathsf{dR},K} \otimes_{\mathbb{Q}_p} V)^H$$

is an isomorphism. The  $K_{\infty}^{(\mathrm{pf})}[[t, t_1, \ldots, t_e]]$ -module  $D_{e-\mathrm{dif}}^+(V)$  is endowed with the action of the  $K_{\infty}^{(\mathrm{pf})}$ -linear derivations  $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$  if  $\gamma \in \Gamma_0$  is close enough to 1 and  $\nabla^{(i)} = \frac{\log(\beta_i)}{c_i(\beta_i)}$   $(1 \le i \le e)$  if  $\beta_i \in \Gamma_i$  is close enough to 1.

3.1.4. The module  $D_{dif}^+(V)$ . – For a *p*-adic representation *V* of  $G_K$ , define  $D_{dif}^+(V)$  to be  $\lim_{K \to 0} r(K_{\infty}^{pf}[[t,t_1,\ldots,t_e]] \otimes_{K_{\infty}^{(pf)}[[t,t_1,\ldots,t_e]]} D_{e-dif}^{+,(r)}(V))$  where we put  $D_{e-dif}^{+,(r)}(V) = D_{e-dif}^+(V)/(t,t_1,\ldots,t_e)^r D_{e-dif}^+(V)$ . One can verify that  $D_{dif}^+(V)$  is the union of  $K_{\infty}^{pf}[[t,t_1,\ldots,t_e]]$ -submodules of finite type of  $(B_{dR,K}^+ \otimes_{\mathbb{Q}_p} V)^H$  stable under  $\Gamma_0 (\simeq G_{K^{pf}}/H)$  and is a free  $K_{\infty}^{pf}[[t,t_1,\ldots,t_e]]$ -module of rank *d* stable under  $\Gamma_0$ . Furthermore, we have  $B_{dR,K}^+ \otimes_{K_{\infty}^{pf}}[[t,t_1,\ldots,t_e]] D_{dif}^+(V) = B_{dR,K}^+ \otimes_{\mathbb{Q}_p} V$  and the natural map  $(B_{dR,K}^+)^H \otimes_{K_{\infty}^{pf}}[[t,t_1,\ldots,t_e]] D_{dif}^+(V) \to (B_{dR,K}^+ \otimes_{\mathbb{Q}_p} V)^H$  is an isomorphism. As in the case of  $D_{e-dif}^+(V)$ , the  $K_{\infty}^{pf}[[t,t_1,\ldots,t_e]]$ -module  $D_{dif}^+(V)$  is endowed with the action of the  $K_{\infty}^{pf}$ -linear derivation  $\nabla^{(0)} = \frac{\log(\gamma)}{\log(\chi(\gamma))}$  if  $\gamma \in \Gamma_0$  is close enough to 1.

- REMARK 3.1. 1. The preceding results in Subsection 3.1.1 are obtained when V is a p-adic representation of  $G_L = \text{Gal}(\overline{L}/L)$  where L is a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic p > 0 and we choose an algebraic closure  $\overline{L}$  of L. However, in Subsection 3.1.1, for simplicity, we stated the results in the case  $L = K^{\text{pf}}$ .
- 2. Note that, though many people denote the *p*-adic differential module constructed by Fontaine in [6] by  $D_{dif}^+(V)$ , the module  $D_{dif}^+(V)$  in Subsection 3.1.4 is a little different from this module.

#### 3.2. Some properties of differential operators

We shall describe the action of derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  on  $D_{\text{Bri}}(V)$  and  $D_{e\text{-dif}}^{+}(V)$ . First, by a standard argument, we can show that, if  $x \in D_{\text{Bri}}(V)$  (resp.  $D_{e\text{-dif}}^{+}(V)$ ), we have

$$\nabla^{(0)}(x) = \lim_{\gamma \to 1} \frac{\gamma(x) - x}{\chi(\gamma) - 1} \quad \text{and} \quad \nabla^{(i)}(x) = \lim_{\beta_i \to 1} \frac{\beta_i(x) - x}{c_i(\beta_i)}.$$

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With this, we can easily describe the actions of  $K_{\infty}^{(\text{pf})}$ -linear derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  on  $K_{\infty}^{(\text{pf})}[[t, t_1, \dots, t_e]] = D_{e-\text{dif}}^+(\mathbb{Q}_p)$  where  $\mathbb{Q}_p$  is equipped with the structure of *p*-adic representations of  $G_K$  induced by the trivial action of  $G_K$ .

LEMMA 3.2. – The actions of  $K_{\infty}^{(\text{pf})}$ -linear derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  on  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$  are given by  $\nabla^{(0)} = t \frac{d}{dt}$  and  $\nabla^{(i)} = t \frac{d}{dt_i}$   $(1 \le i \le e)$ .

*Proof.* – Since  $\{\nabla^{(j)}\}_{j=0}^{e}$  are  $K_{\infty}^{(\text{pf})}$ -linear derivations and we can see that we have  $\nabla^{(j)}(t_k) = 0$   $(j \neq k)$  and  $\nabla^{(i)}(t) = 0$   $(i \neq 0)$ , it suffices to show that we have  $\nabla^{(0)}(t) = t$  and  $\nabla^{(i)}(t_i) = t$ . These follow from

$$\nabla^{(0)}(t) = \lim_{\gamma \to 1} \frac{\gamma(t) - t}{\chi(\gamma) - 1} = \lim_{\gamma \to 1} \frac{\chi(\gamma)t - t}{\chi(\gamma) - 1} = t$$
$$\nabla^{(i)}(t_i) = \lim_{\beta_i \to 1} \frac{\beta_i(t_i) - t_i}{c_i(\beta_i)} = \lim_{\beta_i \to 1} \frac{(t_i + c_i(\beta_i)t) - t_i}{c_i(\beta_i)} = t.$$

We extend naturally actions of  $K_{\infty}^{(\text{pf})}$ -linear derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  on  $K_{\infty}^{(\text{pf})}[[t, t_{1}, \ldots, t_{e}]]$  to  $K_{\infty}^{(\text{pf})}[[t, t_{1}, \ldots, t_{e}]][t^{-1}] (\subset B_{dR,K})$  by putting  $\nabla^{(0)}(t^{-1}) = -t^{-1}$  and  $\nabla^{(i)}(t^{-1}) = 0$   $(1 \leq i \leq e)$ . Now, we compute the bracket [, ] of derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  on  $D_{\text{Bri}}(V)$  (resp.  $D_{e-\text{dif}}^{+}(V)$ ).

**PROPOSITION 3.3.** – On the p-adic differential module  $D_{\text{Bri}}(V)$  (resp.  $D_{e-\text{dif}}^+(V)$ ), we have  $[\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)}$   $(i \neq 0)$  and  $[\nabla^{(i)}, \nabla^{(j)}] = 0$   $(i, j \neq 0)$ .

*Proof.* – The second equality follows from the commutativity of  $\beta_i$  and  $\beta_j$ . For the first equality, we have the relation  $\gamma \beta_i = \beta_i^{\chi(\gamma)} \gamma$ . Then, since we have

$$\lim_{h \to 0} \frac{a^{h+1} - a}{(h+1) - 1} = a\log(a),$$

we obtain

$$\begin{split} [\nabla^{(0)}, \nabla^{(i)}](*) &= \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1} \lim_{\beta_i \to 1} \frac{\beta_i - 1}{c_i(\beta_i)}(*) - \lim_{\beta_i \to 1} \frac{\beta_i - 1}{c_i(\beta_i)} \lim_{\gamma \to 1} \frac{\gamma - 1}{\chi(\gamma) - 1}(*) \\ &= \lim_{\beta_i \to 1} \lim_{\gamma \to 1} \frac{\gamma \beta_i - \gamma - \beta_i + 1}{(\chi(\gamma) - 1)c_i(\beta_i)}(*) - \lim_{\beta_i \to 1} \lim_{\gamma \to 1} \frac{\beta_i \gamma - \gamma - \beta_i + 1}{(\chi(\gamma) - 1)c_i(\beta_i)}(*) \\ &= \lim_{\beta_i \to 1} \lim_{\gamma \to 1} \frac{\beta_i^{\chi(\gamma)} \gamma - \beta_i \gamma}{(\chi(\gamma) - 1)c_i(\beta_i)}(*) \\ &= \lim_{\beta_i \to 1} \frac{\beta_i \log(\beta_i)}{c_i(\beta_i)}(*) \\ &= \nabla^{(i)}(*). \end{split}$$

**PROPOSITION 3.4.** – The action of the  $K_{\infty}^{(\text{pf})}$ -linear derivation  $\nabla^{(i)}$   $(i \neq 0)$  on  $D_{\text{Bri}}(V)$  is nilpotent.

*Proof.* – From the equality  $\nabla^{(0)}\nabla^{(i)} - \nabla^{(i)}\nabla^{(0)} = \nabla^{(i)}$ , we get  $\nabla^{(0)}(\nabla^{(i)})^r - (\nabla^{(i)})^r\nabla^{(0)} = r(\nabla^{(i)})^r$  and  $\operatorname{tr}(r(\nabla^{(i)})^r) = 0$  for all  $r \in \mathbb{N}$ . Since the characteristic of  $K_{\infty}^{(\mathrm{pf})}$  is 0, we obtain  $\operatorname{tr}((\nabla^{(i)})^r) = 0$  for all  $r \in \mathbb{N}$ . As is well known in linear algebra, this shows that the action of the  $K_{\infty}^{(\mathrm{pf})}$ -linear derivation  $\nabla^{(i)}$   $(i \neq 0)$  on  $D_{\mathrm{Bri}}(V)$  is nilpotent.

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NOTATION . - For simplicity, put

$$R = K_{\infty}^{(\mathrm{pf})}[t, \frac{t_1}{t}, \dots, \frac{t_e}{t}] \quad \text{or} \quad K_{\infty}^{(\mathrm{pf})}[[t, t_1, \dots, t_e]].$$

**PROPOSITION 3.5.** – Let M be a finitely generated free R[1/t]-module endowed with  $K_{\infty}^{(\text{pf})}$ -linear derivations  $\{\nabla^{(i)}\}_{i=0}^{e}$  which satisfy the same properties in Lemma 3.2 and Proposition 3.3. Assume that we can choose a basis  $\{g_j\}_{j=1}^{d}$  of M over R[1/t] such that  $\nabla^{(0)}(g_j) = 0$ . Then, the action of  $\nabla^{(i)}$   $(i \neq 0)$  on this basis is given by  $\nabla^{(i)}(g_j) = t \sum_{k=1}^{d} c_k g_k$  where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ .

*Proof.* – Since  $\{g_j\}_{j=1}^d$  forms a basis of M over R[1/t], we can write, for  $i \neq 0$ ,

(3.1) 
$$\nabla^{(i)}(g_j) = \sum_{k=1}^d a_k g_k \quad (a_k \in R[1/t]).$$

Then, the relation  $[\nabla^{(0)}, \nabla^{(i)}] = \nabla^{(i)}$   $(i \neq 0)$  of Proposition 3.3 says that we have  $\sum_{k=1}^{d} \nabla^{(0)}(a_k)g_k = \sum_{k=1}^{d} a_k g_k$ . Note that we have  $\nabla^{(0)}(g_j) = 0$  by hypothesis. Hence, we obtain the differential equation  $\nabla^{(0)}(a_k) = a_k$ . Define an element  $c_k$  of R[1/t] to be  $a_k/t$ . Then, we can see that  $c_k$  satisfies  $\nabla^{(0)}(c_k) = a_k/t - a_k/t = 0$  and that  $c_k$  is contained in R. Thus, the solution of the differential equation  $\nabla^{(0)}(a_k) = a_k$  in R[1/t] has the following form

where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ . Hence, from (3.1) and (3.2), we obtain, for  $i \neq 0$ ,  $\nabla^{(i)}(g_j) = t \sum_{k=1}^d c_k g_k$  where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ .  $\Box$ 

COROLLARY 3.6. – With notations as in Proposition 3.5 above, we have the following presentation

$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e}(g_j) = t^{k_1 + \dots + k_e} \sum_{k=1}^d c_k g_k$$

where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ .

#### 4. Proof of the main theorem

In this section, we keep the notation and the assumption of Section 3.

#### 4.1. Main theorem for Hodge-Tate representations

**PROPOSITION 4.1 ([10, Section (2.3)]).** – If V is a Hodge-Tate representation of  $G_{K^{\text{pf}}}$ , there exists a  $\Gamma_0$ -equivariant isomorphism of  $K_{\infty}^{\text{pf}}$ -vector spaces

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K^{\mathrm{pf}}_{\infty}(n_j) \quad (n_j \in \mathbb{Z}).$$

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REMARK 4.2. – In general, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic p > 0 and V is a Hodge-Tate representation of  $G_L = \text{Gal}(\overline{L}/L)$  where we choose an algebraic closure  $\overline{L}$  of L, Sen shows that there exists a  $G_L/H$ -equivariant isomorphism of  $L_{\infty}(= \bigcup_{m \ge 1} L(\zeta_{p^m}))$ -vector spaces ([10, Section (2.3)])

$$D_{\mathrm{Sen}}(V) \simeq \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p}V} L_{\infty}(n_j) \quad (n_j \in \mathbb{Z}).$$

COROLLARY 4.3. – For a p-adic representation V of  $G_K$ , assume that V is a Hodge-Tate representation of  $G_{K^{\text{pf}}}$ . Then, there exists a  $\nabla^{(0)}$ - equivariant isomorphism of  $K_{\infty}^{(\text{pf})}$ -vector spaces

$$D_{\mathrm{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_{\infty}^{(\mathrm{pf})}(n_j) \quad (n_j \in \mathbb{Z}).$$

Here,  $\simeq_{\nabla^{(0)}}$  denotes a  $\nabla^{(0)}$ -equivariant isomorphism. Furthermore, the multiplicity of  $\{n_j\}_{j=1}^d$  is the same as that of  $\{n_j\}_{j=1}^d$  in Proposition 4.1.

*Proof.* – From the presentation of Proposition 4.1, the action of the  $K^{\text{pf}}_{\infty}$ -linear derivation  $\nabla^{(0)}$  on  $D_{\text{Sen}}(V)$  is semi-simple and its eigenvalues are integers. Thus, the action of the  $K^{(\text{pf})}_{\infty}$ -linear derivation  $\nabla^{(0)}$  on the subspace  $D_{\text{Bri}}(V)$  of  $D_{\text{Sen}}(V)$  is also semi-simple and its eigenvalues are the same. Therefore, we obtain a  $\nabla^{(0)}$ -equivariant isomorphism  $D_{\text{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K^{(\text{pf})}_{\infty}(n_j) \ (n_j \in \mathbb{Z})$ . By tensoring  $K^{\text{pf}}_{\infty} \otimes_{K^{(\text{pf})}_{\infty}}$  over both sides, we obtain  $K^{\text{pf}}_{\infty} \otimes_{K^{(\text{pf})}_{\infty}} D_{\text{Bri}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d} K^{\text{pf}}_{\infty}(n_j) \ (n_j \in \mathbb{Z})$ . Furthermore, since we have  $K^{\text{pf}}_{\infty} \otimes_{K^{(\text{pf})}_{\infty}} D_{\text{Bri}}(V) \hookrightarrow D_{\text{Sen}}(V)$  by definition and both sides have the same dimension d over  $K^{\text{pf}}_{\infty}$ , we obtain  $K^{\text{pf}}_{\infty} \otimes_{K^{(\text{pf})}_{\infty}} D_{\text{Bri}}(V) = D_{\text{Sen}}(V)$  and can see that the multiplicity of  $\{n_j\}_{j=1}^d$  is the same as that of  $\{n_j\}_{j=1}^d$  in Proposition 4.1. □

THEOREM 4.4. – Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic p > 0 such that  $[k : k^p] = p^e < +\infty$  and V be a p-adic representation of  $G_K$ . Let  $K^{pf}$  be the field extension of K defined as before. Then, V is a Hodge-Tate representation of  $G_K$  if and only if V is a Hodge-Tate representation of  $G_{K^{pf}}$ .

*Proof.* – We shall prove the main theorem in two parts.

(1) V: HT rep. of  $G_K \Rightarrow V$ : HT rep. of  $G_{K^{pf}}$ . – Since V is a Hodge-Tate representation of  $G_K$ , there exists a  $G_K$ -equivariant isomorphism of  $B_{HT,K}$ -modules

$$(4.1) B_{\mathrm{HT},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K})^{d=\dim_{\mathbb{Q}_p} V}.$$

Now, by tensoring  $B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{B_{\mathrm{HT},K}}$  (which is induced by the  $G_{K^{\mathrm{pf}}}$ -equivariant surjection  $p: B_{\mathrm{HT},K} \twoheadrightarrow B_{\mathrm{HT},K^{\mathrm{pf}}}: t_i/t \mapsto 0$ ) over (4.1), we obtain a  $G_{K^{\mathrm{pf}}}$ -equivariant isomorphism of  $B_{\mathrm{HT},K^{\mathrm{pf}}}$ -modules

$$B_{\mathrm{HT},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{HT},K^{\mathrm{pf}}})^d.$$

This means that V is a Hodge-Tate representation of  $G_{K^{\text{pf}}}$ .

(2) V: HT rep. of  $G_{K^{\text{pf}}} \Rightarrow V$ : HT rep. of  $G_K$ . – For simplicity, put  $R = K_{\infty}^{(\text{pf})}[t, \frac{t_1}{t}, \dots, \frac{t_e}{t}]$ . We shall construct the  $K_{\infty}^{(\text{pf})}$ -linearly independent elements  $\{f_j^{(*)}\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$  of  $R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V) \ (\subset B_{\text{HT},K} \otimes_{\mathbb{Q}_p} V)$  such that  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $0 \le i \le e$  and  $1 \le j \le d$ .

(A) Construction of  $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V)$ . – From the presentation of Corollary 4.3 above, if we twist by some powers of t, we obtain a basis  $\{f_j\}_{j=1}^d$  of  $R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V)$  over R[1/t] such that  $\nabla^{(0)}(f_j) = 0$  for all  $1 \leq j \leq d$ . Thus, by applying Corollary 3.6 to the R[1/t]-module  $R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V)$  generated by  $\{f_j\}_{j=1}^d$ , we can deduce

(4.2) 
$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (f_j) = t^{k_1 + \dots + k_e} \sum_{k=1}^d c_k f_k$$

where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ . Furthermore, since the action of  $K_{\infty}^{(\mathrm{pf})}$ -linear derivation  $\nabla^{(i)}$   $(i \neq 0)$  on  $D_{\mathrm{Bri}}(V)$  is nilpotent by Proposition 3.4, if we take  $n \in \mathbb{N}$  large enough, we obtain

(4.3) 
$$(\nabla^{(i)})^n(f_j) = 0 \quad \text{for all } 1 \le j \le d \text{ and } 1 \le i \le e.$$

Define an element  $f_j^{(*)}$  of  $R[1/t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$  by

$$f_j^{(*)} = \sum_{0 \le k_1, \dots, k_e} (-1)^{k_1 + \dots + k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1 + \dots + k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (f_j).$$

Note that this series is a finite sum by (4.3) and thus  $f_j^{(*)}$  actually defines an element of  $R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V)$ . Then, it follows easily that we have  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $1 \le i \le e$  and  $1 \le j \le d$  by using the Leibniz rule. Furthermore, by using (4.2) and the fact  $\nabla^{(0)}(f_j) = 0$ , we can deduce that we have  $\nabla^{(0)}(f_j^{(*)}) = 0$  for all  $1 \le j \le d$ .

(B)  $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_{K_{\infty}^{(\text{pf})}} D_{\text{Bri}}(V)$  is linearly independent over  $K_{\infty}^{(\text{pf})}$ . – By the presentation of  $f_i^{(*)}$ , we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (\frac{t_1}{t}, \dots, \frac{t_e}{t})(B_{\mathrm{HT}, K} \otimes_{\mathbb{Q}_p} V)).$$

Since  $\{f_j\}_{j=1}^d$  forms a basis of  $R[1/t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$  over R[1/t], it is, in particular, linearly independent over  $K_{\infty}^{(\mathrm{pf})}$  ( $\subset R[1/t]$ ). Thus,  $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$  (– denotes the reduction modulo  $(t_1, \ldots, t_e)$ ) is linearly independent over  $K_{\infty}^{(\mathrm{pf})}$  and we can see that  $\{f_j^{(*)}\}_{j=1}^d$  is linearly independent over  $K_{\infty}^{(\mathrm{pf})}$  in  $R[1/t] \otimes_{K_{\infty}^{(\mathrm{pf})}} D_{\mathrm{Bri}}(V)$ .

(C) Conclusion. – Therefore, on the K-vector space generated by  $\{f_j^{(*)}\}_{j=1}^d$ ,  $\log(\gamma)$  and  $\{\log(\beta_i)\}_{i=1}^e$  act trivially ( $\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$  and  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $1 \le i \le e$  and  $1 \le j \le d$ ). Thus, this means that  $\Gamma_K$  acts on this K-vector space via finite quotient and there exists a finite field extension L/K in  $K_{\infty}^{(\text{pf})}$  such that  $\{f^{(*)}\}_{j=1}^d$  forms a basis of  $D_{\text{HT},L}(V)$ 

over L. Since a potentially Hodge-Tate representation of  $G_K$  is a Hodge-Tate representation of  $G_K$ , this completes the proof.

#### 4.2. Main theorem for de Rham representations

LEMMA 4.5. – For a p-adic representation V of  $G_K$ , assume that V is a de Rham representation of  $G_{K^{pf}}$ . Then, we can choose a basis  $\{h_j\}_{j=1}^{d=\dim_{\mathbb{Q}_p}V}$  of  $D^+_{dif}(V)[1/t]$  over  $K^{pf}_{\infty}[[t, t_1, \ldots, t_e]][1/t]$  such that the action of  $\Gamma_0$  on  $\{h_j\}_{j=1}^d$  is trivial.

*Proof.* – Since V is a de Rham representation of  $G_{K^{\text{pf}}}$ , there exists a basis  $\{h_j\}_{j=1}^d$  of  $B_{dR,K^{\text{pf}}}\otimes_{\mathbb{Q}_p}V$  over  $B_{dR,K^{\text{pf}}}$  such that the action of  $G_{K^{\text{pf}}}$  on  $\{h_j\}_{j=1}^d$  is trivial. We can see that these elements  $\{h_j\}_{j=1}^d$  are contained in  $D_{\text{dif}}^+(V)[1/t]$  by definition. For each j, if we twist  $h_j$  by some power of t, we obtain an element  $g_j$  of  $B_{dR,K^{\text{pf}}}^+\otimes_{\mathbb{Q}_p}V$  such that  $g_j \notin tB_{dR,K^{\text{pf}}}^+\otimes_{\mathbb{Q}_p}V$ . Then, it follows that  $g_j$  is contained in  $D_{\text{dif}}^+(V)$  and satisfies  $\overline{g_j} \neq 0$  (− denotes the reduction modulo  $(t, t_1, \ldots, t_e)D_{\text{dif}}^+(V)$ ). Since  $D_{\text{dif}}^+(V)$  is a free module of rank d over the local ring  $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$  and  $\{\overline{g_j}\}_{j=1}^d$  forms a basis of  $D_{\text{sen}}(V)$  over  $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$ . Thus, it follows that  $\{h_j\}_{j=1}^d$  forms a basis of  $D_{\text{dif}}^+(V)$  over  $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$ . Thus, it follows that  $\{h_j\}_{j=1}^d$  forms a basis of  $D_{\text{dif}}^+(V)$  is  $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]]$ . Thus, it follows that  $\{h_j\}_{j=1}^d$  forms a basis of  $D_{\text{dif}}^+(V)$  forms a basis of  $D_{\text{dif}}^+(t, 1, \ldots, t_e]$ ] [1/t].

With notations as above, note that, since we have the inclusion  $D_{e\text{-dif}}^+(V) \hookrightarrow D_{\text{dif}}^+(V)[1/t]$ by definition, any element g of  $D_{e\text{-dif}}^+(V)$  can be written as  $g = \sum_{k=l}^{+\infty} (\sum_{j=1}^d a_{jk}h_j)t^k$  $(a_{jk} \in K_{\infty}^{\text{pf}}[[t_1, \ldots, t_e]]).$ 

REMARK 4.6. – Keep the notation as in Lemma 4.5. Since we assume that V is a de Rham representation of  $G_{K^{\text{pf}}}$ , by Corollary 4.3, there exists a basis  $\{v_j\}_{j=1}^d$  of  $D_{\text{Bri}}(V)$ over  $K_{\alpha}^{(\text{pf})}$  such that  $\nabla^{(0)}(v_j) = n_j v_j$ . Put  $M = \text{Max}(n_j)_{j=1}^d$ . Then, for an element  $g \in D_{e-\text{dif}}^+(V)$ , there exists an element  $\sum_{k=n}^{+\infty} (\sum_{j=1}^d c_{jk}h_j)t^k$  of  $(t, t_1, \ldots, t_e)D_{e-\text{dif}}^+(V)$  such that we can write

$$g = \sum_{k=m}^{M} \left( \sum_{j=1}^{d} b_{jk} h_j \right) t^k + \sum_{k=n}^{+\infty} \left( \sum_{j=1}^{d} c_{jk} h_j \right) t^k \quad (b_{jk}, c_{jk} \in K_{\infty}^{\text{pf}}[[t_1, \dots, t_e]]).$$

Thus,  $g' = \sum_{k=m}^{M} (\sum_{j=1}^{d} b_{jk} h_j) t^k$  defines an element of  $D_{e-\text{dif}}^+(V)$ .

LEMMA 4.7. – With notations as above, for an element  $g' = \sum_{k=m}^{M} (\sum_{j=1}^{d} b_{jk} h_j) t^k$  of  $D_{e-\text{dif}}^+(V)$ , each  $(\sum_{j=1}^{d} b_{jk} h_j) t^k$  is contained in  $D_{e-\text{dif}}^+(V)$ .

*Proof.* – We shall prove this lemma by induction on the smallest degree of g' with respect to t. Since we have  $g' - (\sum_{j=1}^{d} b_{jm}h_j)t^m \in D^+_{e-\text{dif}}(V)$  if  $(\sum_{j=1}^{d} b_{jm}h_j)t^m$  is contained in  $D^+_{e-\text{dif}}(V)$ , it suffices to show that  $(\sum_{j=1}^{d} b_{jm}h_j)t^m$  is contained in  $D^+_{e-\text{dif}}(V)$ . Since the  $K^{\text{pf}}_{\infty}[[t_1, \ldots, t_e]]$ -linear derivation  $\nabla^{(0)}$  acts trivially on  $\{h_j\}_{j=1}^{d}$ , we have

$$\prod_{k=m+1}^{M} (\nabla^{(0)} - k)(g') = (\prod_{k=m+1}^{M} (m-k))(\sum_{j=1}^{d} b_{jm}h_j)t^m$$

It follows that  $(\sum_{j=1}^{d} b_{jm} h_j) t^m$  is contained in  $D_{e-\text{dif}}^+(V)$  since the action of  $\nabla^{(0)}$  on  $D_{e-\text{dif}}^+(V)$  is stable. Thus, this completes the proof.

**PROPOSITION 4.8.** – For a p-adic representation V of  $G_K$ , assume that V is a de Rham representation of  $G_{K^{\text{pf}}}$ . Then, there exists a  $\nabla^{(0)}$ -equivariant isomorphism of  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$ -modules

$$D_{e\text{-dif}}^+(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\dim_{\mathbb{Q}_p} V} K_{\infty}^{(\text{pf})}[[t, t_1, \dots, t_e]](n_j) \quad (n_j \in \mathbb{Z}).$$

*Proof.* – Since V is also a Hodge-Tate representation of  $G_{K^{\text{pf}}}$ , by Corollary 4.3, there exists a basis  $\{v_j\}_{j=1}^d$  of  $D_{e\text{-dif}}^+(V)/(t, t_1, \ldots, t_e)D_{e\text{-dif}}^+(V) \simeq D_{\text{Bri}}(V)$  over  $K_{\infty}^{(\text{pf})}$  such that it gives a  $\nabla^{(0)}$ -equivariant isomorphism of  $K_{\infty}^{(\text{pf})}$ -vector spaces

$$D_{e-\mathrm{dif}}^+(V)/(t,t_1,\ldots,t_e)D_{e-\mathrm{dif}}^+(V)\simeq_{\nabla^{(0)}} \bigoplus_{j=1}^d K_\infty^{(\mathrm{pf})}(n_j): v_j\mapsto t^{n_j}.$$

Since  $D_{e-\text{dif}}^+(V)$  is a free module of rank d over the local ring  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$ , any lifting  $\{g_j\}_{j=1}^d$  of  $\{v_j\}_{j=1}^d$  in  $D_{e-\text{dif}}^+(V)$  forms a basis of  $D_{e-\text{dif}}^+(V)$  over  $K_{\infty}^{(\text{pf})}[[t, t_1, \ldots, t_e]]$ . Let  $\{h_j\}_{j=1}^d$  denote a basis of  $D_{\text{dif}}^+(V)[1/t]$  over  $K_{\infty}^{\text{pf}}[[t, t_1, \ldots, t_e]][1/t]$  such that  $\nabla^{(0)}(h_j) = 0$  obtained in Lemma 4.5. Then, we may assume that each  $g_j$  is written as  $g_j = \sum_{k=m}^M (\sum_{l=1}^d b_{kl}h_l) t^k (b_{kl} \in K_{\infty}^{\text{pf}}[[t_1, \ldots, t_e]])$  where we take  $M \in \mathbb{N}$  as in Remark 4.6. Now, define an element  $f_j$  of  $D_{e-\text{dif}}^+(V)$  (Lemma 4.7 above) by

$$f_j = \left(\sum_{l=1}^d b_{n_j l} h_l\right) t^{n_j}.$$

It is easy to see  $\nabla^{(0)}(f_j) = n_j f_j$ . Therefore, the rest is to show that  $\{f_j\}_{j=1}^d$  forms a basis of  $D_{e-\text{dif}}^+(V)$  over  $K_{\infty}^{(\text{pf})}[[t, t_1, \dots, t_e]]$ . To prove that  $\{f_j\}_{j=1}^d$  is a lifting of  $\{v_j\}_{j=1}^d$ , it suffices to show  $g_j - f_j \in (t, t_1, \dots, t_e) D_{e-\text{dif}}^+(V)$ . For each  $g_j$ , put  $s_k = (\sum_{l=1}^d b_{kl} h_l) t^k \in D_{e-\text{dif}}^+(V)$ (Lemma 4.7 above). Since we have  $\nabla^{(0)}(\overline{s_k}) = k \overline{s_k}$  (- denotes the reduction modulo  $(t, t_1, \dots, t_e)$ ) and this means that  $\overline{s_k}$  is an eigenvector of  $\nabla^{(0)}$ , it follows that the elements  $\{v_j, \overline{s_k} \neq 0\}_{k \neq n_j}$  are linearly independent over  $K_{\infty}^{(\text{pf})}$  in  $D_{\text{Bri}}(V)$ . Since we have  $v_j = \sum_{k=m}^M \overline{s_k}$  by definition, it follows that we obtain  $\overline{s_k} = 0$  for  $k \neq n_j$ . This means that we have  $s_k \in (t, t_1, \dots, t_e) D_{e-\text{dif}}^+(V)$   $(k \neq n_j)$  and  $g_j - f_j \in (t, t_1, \dots, t_e) D_{e-\text{dif}}^+(V)$ . Thus, this completes the proof.

REMARK 4.9. – In general, it is evident from the proof that, if L denotes a complete discrete valuation field of characteristic 0 with perfect residue field of characteristic p > 0and V is a de Rham representation of  $G_L = \text{Gal}(\overline{L}/L)$  where we choose an algebraic closure  $\overline{L}$  of L, we have a  $\nabla^{(0)}$ -equivariant isomorphism of  $L_{\infty}[[t]]$ -modules

$$D^+_{\mathrm{dif}}(V) \simeq_{\nabla^{(0)}} \bigoplus_{j=1}^{d=\mathrm{dim}_{\mathbb{Q}_p}V} L_{\infty}[[t]](n_j) \quad (n_j \in \mathbb{Z}).$$

THEOREM 4.10. – Let K be a complete discrete valuation field of characteristic 0 with residue field k of characteristic p > 0 such that  $[k : k^p] = p^e < +\infty$  and V be a p-adic representation of  $G_K$ . Let  $K^{pf}$  be the field extension of K defined as before. Then, V is a de Rham representation of  $G_K$  if and only if V is a de Rham representation of  $G_{K^{pf}}$ .

*Proof.* – We shall prove the main theorem in two parts.

(1) V:  $dR rep. of G_K \Rightarrow V: dR rep. of G_{K^{pf}}$ . – Since V is a de Rham representation of  $G_K$ , there exists a  $G_K$ -equivariant isomorphism of  $B_{dR,K}$ -modules

$$(4.4) B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K})^{d = \dim_{\mathbb{Q}_p} V}$$

Now, by tensoring  $B_{dR,K^{pf}} \otimes_{B_{dR,K}}$  (which is induced by the  $G_{K^{pf}}$ -equivariant surjection  $p: B_{dR,K} \twoheadrightarrow B_{dR,K^{pf}}: t_i \mapsto 0$ ) over (4.4), we obtain a  $G_{K^{pf}}$ -equivariant isomorphism of  $B_{dR,K^{pf}}$ -modules

$$B_{\mathrm{dR},K^{\mathrm{pf}}} \otimes_{\mathbb{Q}_p} V \simeq (B_{\mathrm{dR},K^{\mathrm{pf}}})^d.$$

This means that V is a de Rham representation of  $G_{K^{pf}}$ .

(2) V:  $dR \operatorname{rep.} of G_{K^{pf}} \Rightarrow V$ :  $dR \operatorname{rep.} of G_K$ . – For simplicity, put  $R = K_{\infty}^{(pf)}[[t, t_1, \ldots, t_e]]$ . We shall construct the  $K_{\infty}^{(pf)}$ -linearly independent elements  $\{f_j^{(*)}\}_{j=1}^{d=\dim_{\mathbb{Q}_p} V}$  of  $R[1/t] \otimes_R D_{e-\operatorname{dif}}^+(V) \ (\subset B_{\mathrm{dR},K} \otimes_{\mathbb{Q}_p} V)$  such that  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $0 \le i \le e$  and  $1 \le j \le d$ .

(A) Construction of  $\{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$ . – From the presentation of Proposition 4.8 above, if we twist by some powers of t, we obtain a basis  $\{f_j\}_{j=1}^d$  of  $R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$  over R[1/t] such that  $\nabla^{(0)}(f_j) = 0$  for all  $1 \leq j \leq d$ . Thus, by applying Corollary 3.6 to the R[1/t]-module  $R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$  generated by  $\{f_j\}_{j=1}^d$ , we can deduce

(4.5) 
$$(\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (f_j) = t^{k_1 + \dots + k_e} \sum_{k=1}^d c_k f_k$$

where  $c_k$  is an element of R such that  $\nabla^{(0)}(c_k) = 0$ . Define an element  $f_j^{(*)}$  of  $R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$  by

$$f_j^{(*)} = \sum_{0 \le k_1, \dots, k_e} (-1)^{k_1 + \dots + k_e} \frac{t_1^{k_1} \cdots t_e^{k_e}}{k_1! \cdots k_e! t^{k_1 + \dots + k_e}} (\nabla^{(1)})^{k_1} \cdots (\nabla^{(e)})^{k_e} (f_j).$$

Note that this series converges in  $R[1/t] \otimes_R D^+_{e-\text{dif}}(V)$  for  $(t_1, \ldots, t_e)$ -adic topology by (4.5) and thus  $f_j^{(*)}$  actually defines an element of  $R[1/t] \otimes_R D^+_{e-\text{dif}}(V)$ . Then, it follows easily that we have  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $1 \leq i \leq e$  and  $1 \leq j \leq d$  by using the Leibniz rule. Furthermore, by using (4.5) and the fact  $\nabla^{(0)}(f_j) = 0$ , we can deduce that we have  $\nabla^{(0)}(f_j^{(*)}) = 0$  for all  $1 \leq j \leq d$ .

 $(B) \{f_j^{(*)}\}_{j=1}^d \in R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$  is linearly independent over  $K_{\infty}^{(\text{pf})}$ . – By the presentation of  $f_j^{(*)}$ , we have

$$f_j^{(*)} = f_j + g_j \qquad (g_j \in (t_1, \dots, t_e)(B_{\mathrm{dR}, K} \otimes_{\mathbb{Q}_p} V)).$$

Since  $\{f_j\}_{j=1}^d$  forms a basis of  $R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$  over R[1/t], it is, in particular, linearly independent over  $K_{\infty}^{(\text{pf})}$  ( $\subset R[1/t]$ ). Thus,  $\{\overline{f_j} = \overline{f_j}^{(*)}\}_{j=1}^d$  (– denotes the reduction modulo  $(t_1, \ldots, t_e)$ ) is linearly independent over  $K_{\infty}^{(\text{pf})}$  and we can see that  $\{f_j^{(*)}\}_{j=1}^d$  is linearly independent over  $K_{\infty}^{(\text{pf})}$  in  $R[1/t] \otimes_R D_{e-\text{dif}}^+(V)$ .

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(C) Conclusion. – Therefore, on the K-vector space generated by  $\{f_j^{(*)}\}_{j=1}^d$ ,  $\log(\gamma)$  and  $\{\log(\beta_i)\}_{i=1}^e$  act trivially ( $\Leftrightarrow \nabla^{(0)}(f_j^{(*)}) = 0$  and  $\nabla^{(i)}(f_j^{(*)}) = 0$  for all  $1 \le i \le e$  and  $1 \le j \le d$ ). Thus, this means that  $\Gamma_K$  acts on this K-vector space via finite quotient and there exists a finite field extension L/K in  $K_{\infty}^{(\text{pf})}$  such that  $\{f^{(*)}\}_{j=1}^d$  forms a basis of  $D_{dR,L}(V)$  over L. Since a potentially de Rham representation of  $G_K$  is a de Rham representation of  $G_K$ , this completes the proof.

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Kazuma MORITA Department of Mathematics Hokkaido University Sapporo 060-0810, Japan E-mail: morita@math.sci.hokudai.ac.jp