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Equivalence problem for minimal rational curves
with isotrivial varieties of minimal rational tangents

EQUIVALENCE PROBLEM FOR MINIMAL RATIONAL CURVES WITH ISOTRIVIAL VARIETIES OF MINIMAL RATIONAL TANGENTS

BY JUN-MUK HWANG

ABSTRACT. – We formulate the equivalence problem, in the sense of É. Cartan, for families of minimal rational curves on uniruled projective manifolds. An important invariant of this equivalence problem is the variety of minimal rational tangents. We study the case when varieties of minimal rational tangents at general points form an isotrivial family. The main question in this case is for which projective variety Z, a family of minimal rational curves with Z-isotrivial varieties of minimal rational tangents is locally equivalent to the flat model. We show that this is the case when Z satisfies certain projective-geometric conditions, which hold for a non-singular hypersurface of degree ≥ 4.

RÉSUMÉ. – Nous énonçons le problème d'équivalence, au sens de É. Cartan, pour des familles de courbes rationnelles minimales sur des variétés projectives uniréglées. Un invariant important de ce problème d'équivalence est la variété des tangentes rationnelles minimales. Nous étudions le cas où les variétés de tangentes rationnelles minimales aux points génériques forment une famille isotriviale. La question principale dans ce cas est : pour quelle variété projective Z une famille de courbes rationnelles minimales, dont les variétés de tangentes rationnelles minimales sont Z-isotriviales, est-elle localement équivalente au modèle plat ? Nous montrons que c'est le cas lorsque Z vérifie certaines conditions de géométrie projective qui sont satisfaites pour une hypersurface non singulière de degré ≥ 4.

1. Introduction

We will work over the complex numbers. For a uniruled projective manifold X, an irreducible component $\mathcal{K}$ of the space of rational curves on X is a *family of minimal rational curves* on X if the subvariety $\mathcal{K}_x$ consisting of members of $\mathcal{K}$ through a general point $x \in X$ is projective and non-empty. Minimal rational curves play an important role in the geometry of uniruled projective manifolds (cf. [6], [4]). We are interested in the following 'equivalence problems' in the sense of É. Cartan (cf. [3]) for families of minimal rational curves.

QUESTION 1.1. – Let X and X' be two uniruled projective manifolds with families of minimal rational curves $\mathcal{K}$ on X and $\mathcal{K}'$ on X'. Given two points $x \in X$ and $x' \in X'$, can we find open neighborhoods $x \in U \subset X$ and $x' \in U' \subset X'$ with a biholomorphic map $\varphi : U \to U'$ such that for each member C of $\mathcal{K}$ (resp. C' of $\mathcal{K}'$) there exists a member C' of $\mathcal{K}'$ (resp. C of $\mathcal{K}$) satisfying

$$\varphi(C \cap U) = C' \cap U' \quad (\text{resp. } \varphi^{-1}(C' \cap U') = C \cap U)?$$

If such a biholomorphic map φ exists, we will say that $(X, \mathcal{K}, x)$ is *equivalent* to $(X', \mathcal{K}', x')$. One motivation for studying this problem is the following theorem.

THEOREM 1.2. – *Let X (resp. X') be a Fano manifold with second Betti number 1 and let $\mathcal{K}$ (resp. $\mathcal{K}'$) be a family of minimal rational curves on X (resp. X'). Assume that $\dim \mathcal{K} = \dim \mathcal{K}' \geq \dim X = \dim X'$. Suppose for some $x \in X$ and $x' \in X'$, that $(X, \mathcal{K}, x)$ is equivalent to $(X', \mathcal{K}', x')$. Then the equivalence map $\varphi : U \to U'$ extends to a biregular morphism from X to X' sending x to x'.*

Theorem 1.2 follows from the argument in [7] although it was not explicitly stated there. Theorem 1.2 and its variations are useful in proving two Fano manifolds of second Betti number 1 are biregular (cf. [6], [4]). Thus Question 1.1 has interesting applications in algebraic geometry.

A natural approach to Question 1.1 is to find local properties of the family $\mathcal{K}$ near x which are invariant under the equivalence, i.e., *local invariants* of the family. An important invariant is provided by the variety of minimal rational tangents. Recall that given a general point $x \in X$, the *variety of minimal rational tangents* at x is the subvariety $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ defined as the union of the tangent directions of members of $\mathcal{K}$ through x. A great advantage of variety of minimal rational tangents $\mathcal{C}_x$ is that it is equipped with a projective embedding $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ and consequently all projective geometric invariants of the projective variety $\mathcal{C}_x$ give rise to invariants of the equivalence problem.

Throughout the paper we will consider only those $(X, \mathcal{K})$ for which the following condition holds.

ASSUMPTION 1.3. – $\dim X \geq 3$ and $\mathcal{C}_x$ at general point $x \in X$ is an irreducible non-singular variety and is not a linear subvariety in $\mathbb{P}T_x(X)$. In particular, it has positive dimension.

What happens if $\mathcal{C}_x$ is reducible is a very important and difficult issue requiring ideas and methods different from those considered below. One justification of making the assumption that $\mathcal{C}_x$ is irreducible is that there is a large class of examples satisfying it. As a matter of fact, all known examples with $\dim \mathcal{C}_x > 0$ satisfy the irreducibility assumption. The non-singularity assumption is not really restrictive. It is believed to be always true. Finally, the non-linearity assumption is harmless. When $\mathcal{C}_x$ is linear and irreducible, we can foliate X by projective spaces (e.g. [1, Theorem 3.1]) and the equivalence problem becomes trivial.

The main question in the equivalence problem for minimal rational curves is to study to what extent the equivalence is decided by the information of varieties of minimal rational tangents. More precisely, the main question is the following.

QUESTION 1.4. – Let X and X' be two projective manifolds with families of minimal rational curves $\mathcal{K}$ on X and $\mathcal{K}'$ on X' satisfying Assumption 1.3. Let $x \in X$ and $x' \in X'$ be general points in the sense of Assumption 1.3. Suppose there exist open neighborhoods $U \subset X, U' \subset X'$ and a commuting diagram

$$
\begin{array}{ccc}
\mathbb{P}T(U) & \xrightarrow{\psi} & \mathbb{P}T(U') \\
\downarrow & & \downarrow \\
U & \xrightarrow{\varphi} & U'
\end{array}
$$

where the vertical maps are natural projections and the horizontal maps are biholomorphisms satisfying

$$\psi_x(\mathcal{C}_x) = \mathcal{C}_{\varphi(x)} \text{ for each } x \in U.$$

Is $(X, \mathcal{K}, x)$ equivalent to $(X', \mathcal{K}', x')$?

We will see below that the answer is not affirmative in general. A general result toward Question 1.4 is provided by the following result, which is just a restatement of Theorem 3.1.4 of [6].

THEOREM 1.5. – *Let X and X' be two projective manifolds with families of minimal rational curves $\mathcal{K}$ on X and $\mathcal{K}'$ on X' satisfying Assumption 1.3. Let $x \in X$ and $x' \in X'$ be general points in the sense of Assumption 1.3. Suppose there exist open neighborhoods $U \subset X$, $U' \subset X'$ and a commuting diagram*

$$
\begin{array}{ccc}
\mathbb{P}T(U) & \xrightarrow{\psi} & \mathbb{P}T(U') \\
\downarrow & & \downarrow \\
U & \xrightarrow{\varphi} & U'
\end{array}
$$

where the vertical maps are natural projections and the horizontal maps are biholomorphisms satisfying

$$\psi_x(\mathcal{C}_x) = \mathcal{C}_{\varphi(x)} \text{ for each } x \in U$$

and $\psi = d\varphi$, the derivative of φ. Then $(X, \mathcal{K}, x)$ is equivalent to $(X', \mathcal{K}', x')$.

In comparison to Question 1.4, the crucial additional assumption in Theorem 1.5 is that the holomorphic map ψ comes from the derivative of φ. In this sense, the condition for Theorem 1.5 is *differential-geometric*. A central question is under what *algebraic-geometric* conditions on the varieties of minimal rational tangents, we can get this differential geometric condition. In this paper, we will concentrate on the following special case.

DEFINITION 1.6. – Let $Z \subset \mathbb{P}^{n-1}$ be a fixed irreducible non-singular non-linear projective variety. For an n-dimension projective manifold X and a family of minimal rational curves $\mathcal{K}$, we say that it has Z-isotrivial varieties of minimal rational tangents, if for a general point $x \in X$, $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is isomorphic to $Z \subset \mathbb{P}^{n-1}$ as projective varieties.

Note that for any Z, there exists $(X, \mathcal{K})$ with Z-isotrivial varieties of minimal rational tangents:

EXAMPLE 1.7. – Let $Z \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be a non-singular irreducible projective variety contained in a hyperplane. Let $\psi : X_Z \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ with center Z. Let $\mathscr{K}_Z$ be the family of curves which are proper transforms of lines in $\mathbb{P}^n$ intersecting Z. Then $\mathscr{K}_Z$ is a family of minimal rational curves on X_Z with Z-isotrivial varieties of minimal rational tangents. In fact, X_Z is quasi-homogeneous with an open orbit containing $\psi^{-1}(\mathbb{P}^n \setminus \mathbb{P}^{n-1})$.

Now we can formulate the following special case of Question 1.4.

QUESTION 1.8. – Let $Z \subset \mathbb{P}^{n-1}$ be an irreducible non-singular non-linear variety. Let X be an n-dimensional projective manifold and let $\mathscr{K}$ be a family of minimal rational curves on X with Z-isotrivial varieties of minimal rational tangents. Is $(X, \mathscr{K}, x)$ for a general $x \in X$ equivalent to that of Example 1.7?

The answer is not always affirmative:

EXAMPLE 1.9. – Let W be a 2ℓ-dimensional complex vector space with a symplectic form. Fix an integer k, $1 < k < \ell$ and let S be the variety of all k-dimensional isotropic subspaces of W. S is a uniruled homogeneous projective manifold. There is a unique family $\mathscr{K}$ of minimal rational curves, just the set of all lines on S under the Plücker embedding. The varieties of minimal rational tangents are Z-isotrivial where Z is the projectivization of the vector bundle $\mathcal{O}(-1)^{2\ell-2k} \oplus \mathcal{O}(-2)$ on $\mathbb{P}^{k-1}$ embedded by the dual tautological bundle of the projective bundle (cf. Proposition 3.2.1 of [9]). Let us denote it by $Z \subset \mathbb{P}V$. There is a distinguished hypersurface $R \subset Z$ corresponding to $\mathbb{P}\mathcal{O}(-1)^{2\ell-2k}$. Let D be the linear span of R in V. This D defines a distribution on S which is not integrable (cf. Section 4 of [9]). However, the corresponding distribution on X_Z of Example 1.7 is integrable. Thus $(S, \mathscr{K}, x)$ cannot be equivalent to $(X_Z, \mathscr{K}_Z, y)$ at general points x, y.

Thus the correct formulation of Question 1.8 is to ask for which Z the answer to Question 1.8 is affirmative. Up to now the only result in this line is the following result of Ngaiming Mok in [10]:

THEOREM 1.10. – *Let S be an n-dimensional irreducible Hermitian symmetric space of compact type with a base point $o \in S$. If the projective variety $Z \subset \mathbb{P}^{n-1}$ is isomorphic to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$ for the family of minimal rational curves on S, then Question 1.8 has an affirmative answer.*

For example when S is the n-dimensional quadric hypersurface, $Z \subset \mathbb{P}^{n-1}$ is just an $(n-2)$-dimensional non-singular quadric hypersurface. Then $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ in Question 1.8 defines a conformal structure at general points of X. In this case, Theorem 1.10 says that this conformal structure is flat. In general, for each S, we can interpret the condition of Question 1.8 as a certain G-structure at general points of X and Theorem 1.10 says that this G-structure is flat.

It is worth recalling Mok's strategy for the proof of Theorem 1.10. The main point is to show that this G-structure which is defined at general points of X can be extended to a G-structure in a *neighborhood* of a general minimal rational curve. Once this extension is obtained, one can deduce the flatness by applying [5] which shows the vanishing of the

curvature tensor from *global* information of the tangent bundle of X on the minimal rational curve.

The projective variety $Z \subset \mathbb{P}^{n-1}$ treated in Theorem 1.10 is a homogeneous variety with reductive automorphism group. Our main result concerns the opposite case when the automorphisms of $Z \subset \mathbb{P}^{n-1}$ is 0-dimensional:

THEOREM 1.11. – *Assume that $Z \subset \mathbb{P}V$ satisfies the following conditions.*

(1) Z is non-singular, non-degenerate and linearly normal.

(2) The variety of tangent lines to Z, defined as a subvariety of $\mathbf{Gr}(2, V) \subset \mathbb{P}(\wedge^2 V)$, is non-degenerate in $\mathbb{P}(\wedge^2 V)$.

(3) $H^0(Z, T(Z) \otimes \mathcal{O}(1)) = H^0(Z, \mathrm{ad}(T(Z)) \otimes \mathcal{O}(1)) = 0$ where $\mathrm{ad}(T(Z))$ denotes the bundle of traceless endomorphisms of the tangent bundle of Z.

Then for a uniruled manifold X and a family $\mathcal{K}$ of minimal rational curves with Z-isotrivial varieties of minimal rational tangents, $(X, \mathcal{K}, x)$ at a general point x is equivalent to that of Example 1.7.

Note that the non-degeneracy and $H^0(Z, T(Z) \otimes \mathcal{O}(1)) = 0$ imply that the projective automorphism group of $(Z \subset \mathbb{P}V)$ is 0-dimensional. This means that we have a G-structure at general points of X with the group G isomorphic to the scalar multiplication group $\mathbb{C}^*$. The essential difference from Theorem 1.10 is the following: the G-structure *cannot* be extended to a neighborhood of a minimal rational curve. One way to see this is by directly checking it in the case of Example 1.7. There is a more conceptual way to see this as follows. Suppose it is possible to extend the G-structure to a neighborhood U of a general minimal rational curve. For simplicity, let us assume that U is simply connected and $Z \subset \mathbb{P}T(U)$ has no non-trivial automorphism. Then in $\mathbb{P}T(U)$ we have a submanifold $\mathcal{C} \subset \mathbb{P}T(U)$ with each fiber $\mathcal{C}_x \subset \mathbb{P}T_x(U)$ isomorphic to $Z \subset \mathbb{P}^{n-1}$. Since the automorphism group of $Z \subset \mathbb{P}^{n-1}$ is trivial, we get a unique trivialization of the projective bundle $\mathbb{P}T(U)$. But on a general minimal rational curve, $T(U)$ splits into $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^{n-1-p}$ for some $p > 0$, a contradiction. Thus in the case where the automorphism group of Z is 0-dimensional, we cannot use Mok's approach. In the setting of Theorem 1.11, the flatness of the G-structure, or the vanishing of the corresponding 'curvature tensors', must be proved only at general points. In other words, it must come from information on the geometry of minimal rational curves in a neighborhood of a general point. The crucial point of the proof of Theorem 1.11 lies in the use of such local information to prove flatness.

The variety of minimal rational tangents $Z \subset \mathbb{P}V$ in Example 1.9, satisfies the conditions (1) and (2) in Theorem 1.11. Thus some additional conditions like (3) are necessary. However we expect that the condition (3) can be weakened, hopefully, to $H^0(Z, T(Z)) = H^0(Z, \mathrm{ad}(T(Z))) = 0$.

Examples of $Z \subset \mathbb{P}V$ satisfying the conditions (1)-(3) of Theorem 1.11 are provided by non-singular hypersurfaces of degree ≥ 4 in $\mathbb{P}V$, $\dim V \geq 3$. In fact, (1) is standard for hypersurfaces and (2) follows from Proposition 2.6 of [4]. (3) can be checked from Theorem 4 (iii) of [2]. It is likely that the three conditions for Z also hold for a large class of complete intersections $Z \subset \mathbb{P}V$.

2. Coframes on a manifold and induced coframes on the tangent bundle

This section and the next section are concerned with the basic coframe formulation of Cartan's approach to equivalence problems. Its content must be well-known to differential geometers and essentially covered by [3]. However the special case we need had not been explicitly worked out. Also we expect some of the readers have background in algebraic geometry. So we will give a self-contained presentation with more coordinate-free notation. All functions, differential forms and tensors considered here are assumed to be holomorphic.

Let us start by recalling the notion of differential forms with values in a vector space. Section V.6 of [11] is a good reference for this notion. Given a vector space V, a V-valued differential k-form on M is just a holomorphic section ω of the vector bundle $\Omega_M^k \otimes V_M$ where V_M denotes the trivial bundle on M with each fiber V. The exterior derivative $d\omega$ is defined as a V-valued $(k+1)$-form and satisfies the usual properties of the exterior derivative including $d(d\omega) = 0$. For a V_1-valued k_1-form ω_1 and a V_2-valued k_2-form ω_2, their exterior product $\omega_1 \wedge \omega_2$ is defined as a $(V_1 \otimes V_2)$-valued $(k_1 + k_2)$-form. Let W be another vector space and ρ be a $\mathrm{Hom}(V, W)$-valued function on M. Given any V-valued k-form ω, $\rho_\sharp \omega$ is the W-valued k-form defined by the composition of ω with ρ. For another vector space W' and a $\mathrm{Hom}(W, W')$-valued function η on M, we have

$$(\eta \circ \rho)_\sharp \omega = \eta_\sharp(\rho_\sharp \omega).$$

Similarly, we have the notion of V-valued vector field D, as a holomorphic section of $T(M) \otimes V_M$. For a $\mathrm{Hom}(V, W)$-valued function ρ on M, $\rho_\sharp D$ is a W-valued vector field. For a V-valued vector field D and a W-valued 1-form ω, $D \rfloor \omega$ denotes the $(V \otimes W)$-valued function obtained by the natural pairing.

DEFINITION 2.1. – Let M be a complex manifold of dimension n. Fix a vector space V of dimension n. A V-valued 1-form ω on M is called a *coframe* if for each $x \in M$, the homomorphism $\omega_x : T_x(X) \to V$ is an isomorphism.

The following is immediate from a point-wise consideration.

LEMMA 2.2. – *Let ω be a coframe on a manifold M of dimension ≥ 3.*

1. *Let ξ and ξ' be two V^*-valued functions with $\xi_\sharp \omega = \xi'_\sharp \omega$. Then $\xi = \xi'$.*
2. *Let ξ and ξ' be two W-valued k-forms, $k = 1, 2$, for some finite-dimensional vector space W. Suppose that $\xi \wedge \omega = \xi' \wedge \omega$. Then $\xi = \xi'$.*
3. *Let ξ and ξ' be two W-valued functions, for some finite-dimensional vector space W. Suppose that $\xi_\sharp(\omega \wedge \omega) = \xi'_\sharp(\omega \wedge \omega)$. Then $\xi = \xi'$.*

DEFINITION 2.3. – Given a coframe ω, the *dual frame* of ω is a V^*-valued vector field D_ω on M, whose value at $x \in M$ is given by the inverse of the isomorphism $\omega_x : T_x(X) \to V$. In other words, the $(V^* \otimes V)$-valued function $D_\omega \rfloor \omega$ has constant value $\mathrm{Id}_V \in V^* \otimes V$.

The following can be checked easily.

LEMMA 2.4. – *For a coframe ω and its dual frame D_ω, the following hold.*

1. *For any function f, we have*
$$df = (D_\omega f)_\sharp \omega$$
   where $D_\omega f$ is the V^*-valued function obtained by differentiating f by D_ω.
2. *For any vector field v on M, the V-valued function $v \rfloor \omega$ satisfies*
$$(v \rfloor \omega)_\sharp D_\omega = v.$$

DEFINITION 2.5. – A coframe ω is *closed* if $d\omega = 0$. Given a coframe ω, there exists a $\mathrm{Hom}(V \otimes V, V)$-valued function σ^ω on M, called the *structure function* of ω such that
$$d\omega = \sigma_\sharp^\omega (\omega \wedge \omega).$$
In fact, σ^ω takes values in $\mathrm{Hom}(\wedge^2 V, V) \subset \mathrm{Hom}(V \otimes V, V)$ from the anti-symmetry in $d\omega$. By the canonical isomorphism $\mathrm{Hom}(V \otimes V, V) = \mathrm{Hom}(V^*, V^* \otimes V^*)$, we can view the structure function as a $\mathrm{Hom}(V^*, V^* \otimes V^*)$-valued function, in which case we will denote it by δ^ω. Then
$$[D_\omega, D_\omega] = \delta_\sharp^\omega D_\omega.$$

A coframe ω on a manifold M induces a coframe Ω on the manifold $T(M)$ in the following way.

DEFINITION 2.6. – View a coframe ω on M as a V-valued function μ on $T(M)$. Let $\pi : T(M) \to M$ be the projection and let $\theta := \pi^*\omega$ be the V-valued 1-form on $T(M)$ obtained by pulling back ω. Then the pair
$$\Omega := (\theta, \lambda := d\mu)$$
is a $(V \oplus V)$-valued 1-form on $T(M)$, which is in fact a coframe on $T(M)$. This is the *induced coframe* on $T(M)$.

PROPOSITION 2.7. – *To avoid confusion, we will rename $V \oplus V$ as $V_1 \oplus V_2$. Then the structure function of the coframe Ω, which is a $\mathrm{Hom}(\wedge^2(V_1 \oplus V_2), V_1 \oplus V_2)$-valued function on $T(M)$ takes values in $\mathrm{Hom}(\wedge^2 V_1, V_1)$. In fact,*
$$\sigma^\Omega = \pi^* \sigma^\omega.$$

Proof. – This is immediate from
$$d\Omega = (d\theta, d\lambda) = (\pi^* d\omega, 0).$$

$\square$

The following is straightforward from the definitions and Proposition 2.7.

PROPOSITION 2.8. – *The dual frame of the coframe Ω is given by*
$$D_\Omega = (D_\theta, D_\lambda)$$
where D_θ and D_λ are V^*-valued vector fields satisfying

(2.1) $\qquad D_\theta \rfloor \theta = D_\lambda \rfloor \lambda = D_\lambda \mu = \mathrm{Id}_V$ *and* $D_\theta \mu = D_\theta \rfloor \lambda = D_\lambda \rfloor \theta = 0.$

They have the following properties.

(a) *Under the projection $d\pi : T(T(M)) \to T(M)$, $d\pi(D_\theta) = D_\omega$.*
(b) *Any vector field $\tilde{v}$ on an open subset $U \subset T(M)$ tangent to the fibers of π is of the form $\tilde{v} = h_\sharp D_\lambda$ for some V-valued function h on U.*

(c) $[D_\theta, D_\lambda] = [D_\lambda, D_\lambda] = 0, \ [D_\theta, D_\theta] = (\pi^*\delta^\omega)_\sharp D_\theta.$

DEFINITION 2.9. – For a coframe ω on a manifold M, the vector field on $T(M)$

$$\gamma := \mu_\sharp D_\theta$$

is called the *geodesic flow* of the coframe ω.

PROPOSITION 2.10. – *Given a coframe ω on M, its geodesic flow γ on $T(M)$ has the following properties.*

(a) $[D_\lambda, \gamma] = D_\theta.$
(b) *For a point $v \in T(M)$, let $\gamma_v \in T_v(T(M))$ be the value of γ at v. Then*

$$d\pi_v(\gamma_v) = v \text{ for any vector } v \in T(M).$$

Proof. – From Proposition 2.8 (c) and (2.1),

$$[D_\lambda, \gamma] = [D_\lambda, \mu_\sharp D_\theta] = (D_\lambda \mu)_\sharp D_\theta = D_\theta,$$

proving (a).

For (b), let $\pi(v) = x \in M$. From

$$(\mu_\sharp D_\theta)_v = \mu(v)_\sharp (D_\theta)_v = \omega(v)_\sharp (D_\theta)_v$$

and $d\pi_v(D_\theta)_v = (D_\omega)_x,$

$$d\pi_v(\gamma_v) = d\pi_v((\mu_\sharp D_\theta)_v) = \omega(v)_\sharp (D_\omega)_x.$$

The latter must be v by Lemma 2.4.   □

3. Conformally closed coframes

Let ω be a coframe on a manifold M. For any function h on M, $(D_\omega h)_\sharp \omega$ is a closed 1-form by Lemma 2.4 (a). The following converse is just a restatement of the Poincaré lemma.

LEMMA 3.1. – *Given a coframe ω and a V^*-valued function ξ on M, suppose that $\xi_\sharp \omega$ is a closed 1-form. Then for any point $x \in M$, there exist a neighborhood $x \in U \subset M$ and a function h on U with $\xi = D_\omega h$.*

DEFINITION 3.2. – A coframe ω is *conformally closed* if for any point $x \in M$, there exist a neighborhood $x \in U \subset M$ and a non-vanishing function f on U such that $f\omega$ is closed on U.

We will need the following elementary fact from linear algebra. One can check it either by an explicit computation or from the fact that $\text{Hom}(\wedge^2 V, V)$ decomposes into two irreducible factors as a $\text{GL}(V)$-module.

PROPOSITION 3.3. – *For a vector space V, let $\Xi_V \subset \mathrm{Hom}(\wedge^2 V, V)$ be the subspace defined by*

$$\Xi_V := \{\sigma : V \otimes V \to V, \sigma(u,v) = -\sigma(v,u) \in \mathbb{C}u + \mathbb{C}v \text{ for any } u,v \in V\}.$$

Define the contraction homomorphism $\iota : V^* \to \mathrm{Hom}(V \otimes V, V)$ such that for a V^*-valued function η,

$$\iota(\eta)_\sharp(\omega \wedge \omega) = (\eta_\sharp \omega) \wedge \omega.$$

Then ι is injective and its image is Ξ_V.

The following theorem characterizes conformally closed coframes in terms of their structure functions.

THEOREM 3.4. – *A coframe ω on a manifold of dimension ≥ 3 is conformally closed if and only if its structure function σ_ω takes values in $\Xi_V \subset \mathrm{Hom}(\wedge^2 V, V)$.*

Proof. – Suppose that $f\omega$ is closed, i.e.,

$$0 = d(f\omega) = df \wedge \omega + f d\omega.$$

Then

$$(\sigma_\omega)_\sharp(\omega \wedge \omega) = d\omega = -\frac{df}{f} \wedge \omega = -d(\log f) \wedge \omega = -((D_\omega(\log f))_\sharp \omega) \wedge \omega.$$

This is equal to $\iota(-D_\omega(\log f))_\sharp(\omega \wedge \omega)$. By Lemma 2.2, we conclude that

$$\sigma^\omega = \iota(-D_\omega(\log f)).$$

Thus σ^ω takes values in Ξ_V.

Conversely, assume that σ^ω takes values in Ξ_V, i.e. there exists some V^*-valued function ξ on M such that

$$d\omega = \iota(\xi)_\sharp(\omega \wedge \omega) = (\xi_\sharp \omega) \wedge \omega.$$

Taking d-derivative on both sides, we get

$$0 = dd\omega = d(\xi_\sharp \omega) \wedge \omega + (\xi_\sharp \omega) \wedge d\omega.$$

Note that

$$(\xi_\sharp \omega) \wedge d\omega = (\xi_\sharp \omega) \wedge \iota(\xi)_\sharp(\omega \wedge \omega) = (\xi_\sharp \omega) \wedge (\xi_\sharp \omega) \wedge \omega.$$

Since $\xi_\sharp \omega$ is a (scalar-valued) 1-form on M, the right hand side must be 0. It follows that $d(\xi_\sharp \omega) \wedge \omega = 0$. From Lemma 2.2, this implies that $\xi_\sharp \omega$ is closed. Then by Lemma 3.1, there exist a neighborhood U and a function h on M such that $\xi = D_\omega h$ and, via Lemma 2.4,

$$d\omega = (\xi_\sharp \omega) \wedge \omega = ((D_\omega h)_\sharp \omega) \wedge \omega = dh \wedge \omega.$$

Write $h = -\log f$. Then

$$d(f\omega) = df \wedge \omega + f d\omega = df \wedge \omega + f dh \wedge \omega = df \wedge \omega + f(-\tfrac{df}{f}) \wedge \omega = 0.$$

Thus ω is conformally closed. $\qquad\square$

REMARK 3.5. – Theorem 3.4 is false when the dimension of the manifold is 2. In fact, when $\dim V = 2$, $\Xi_V = \mathrm{Hom}(\wedge^2 V, V)$ and the condition in Theorem 3.4 is empty.

4. Coframes adapted to an isotrivial cone structure

DEFINITION 4.1. – A *cone structure* on a complex manifold M is a submanifold $\mathscr{C} \subset \mathbb{P}T(M)$ such that the projection $\varpi : \mathscr{C} \to M$ is a smooth morphism with connected fibers. For each point $x \in M$, the fiber $\varpi^{-1}(x)$ will be denoted by $\mathscr{C}_x$.

DEFINITION 4.2. – Let V be an n-dimensional vector space and let $Z \subset \mathbb{P}V$ be a fixed projective subvariety. A cone structure $\mathscr{C} \subset \mathbb{P}T(M)$ on an n-dimensional manifold M is said to be Z-*isotrivial* if for each $x \in M$, the inclusion $(\mathscr{C}_x \subset \mathbb{P}T_x(M))$ is isomorphic to $(Z \subset \mathbb{P}V)$ up to projective transformations.

DEFINITION 4.3. – Given $Z \subset \mathbb{P}V$ and a Z-isotrivial cone structure on M, a coframe ω on M is said to be *adapted* to the cone structure if for each $x \in M$, the isomorphism $\omega_x : T_x(M) \to V$ sends $\mathscr{C}_x \subset \mathbb{P}T_x(M)$ to $Z \subset \mathbb{P}V$. Given any Z-isotrivial cone structure on a manifold M, an adapted coframe exists if we shrink M.

DEFINITION 4.4. – A Z-isotrivial cone structure $\mathscr{C} \subset T(M)$ is *locally flat* if for any point $x \in M$ there exist a neighborhood $x \in U \subset M$ and a biholomorphic map $\zeta : U \to V$ such that
$$\zeta_*(\mathscr{C}|_U) = \zeta(U) \times Z \subset V \times \mathbb{P}V$$
where $\zeta_* : \mathbb{P}T(M) \to \mathbb{P}T(V) = V \times \mathbb{P}V$ is the differential of ζ.

PROPOSITION 4.5. – *A Z-isotrivial cone structure is locally flat if and only if it has a conformally closed adapted frame.*

Proof. – The 'only if' part is obvious: the differential of the map ζ in Definition 4.4 defines a closed adapted coframe. For the 'if' part, since the question is local, we may assume that there is a closed adapted coframe. Then by Poincaré lemma, we can integrate it in a neighborhood to get a biholomorphic map $\zeta : U \to V$ satisfying the required condition. $\square$

We collect a few properties of adapted coframes.

PROPOSITION 4.6. – *Given a Z-isotrivial cone structure $\mathscr{C} \subset \mathbb{P}T(M)$, let $\hat{\mathscr{C}} \subset T(M)$ be the cone over $\mathscr{C}$. For an adapted coframe ω, the cone $\hat{\mathscr{C}} \subset T(M)$ is preserved by the geodesic flow γ of ω, i.e., at any point $u \in \hat{\mathscr{C}} \setminus 0$, $\gamma_u \in T_u(\hat{\mathscr{C}})$.*

Proof. – Let μ be the V-valued function on $T(M)$ defined by ω. Let $I_Z \subset \mathrm{Sym}^\bullet V^*$ be the ideal defining the projective variety Z. Since ω is adapted, $\mathscr{C} \subset T(M)$ is defined as the zero locus of the collection of functions
$$\{g \circ \mu : T(M) \xrightarrow{\mu} V \xrightarrow{g} \mathbb{C}, \ g \in I_Z\}.$$
Since $D_\theta \mu = 0$, we see that
$$\gamma(g \circ \mu) = \mu_\sharp D_\theta(g \circ \mu) = 0.$$
Thus γ is tangent to $\hat{\mathscr{C}}$. $\square$

PROPOSITION 4.7. – *Let ω be a coframe adapted to a Z-isotrivial cone structure $\mathscr{C} \subset \mathbb{P}T(M)$. Let u be a non-zero point of the affine cone $\hat{\mathscr{C}} \subset T(M)$ and $v \in T_u(\hat{\mathscr{C}}_x)$ with $x = \pi(u) \in M$. Then there exists a local V-valued function g in a neighborhood of u in $T(M)$ satisfying $D_\theta g = 0$ such that the vector field $g_\sharp D_\lambda$ is tangent to the fibers of $\hat{\mathscr{C}} \to M$ and $(g_\sharp D_\lambda)_u = v$.*

Proof. – Let $\bar{u} = \mu(u) \in \hat{Z} \subset V$ and

$$\mu_* : T_u(T(M)) \to T_{\bar{u}}(V)$$

be the differential of the function $\mu : T(M) \to V$. Choose a germ of vector field $\vec{v}$ at $\bar{u} \in V$ such that $\vec{v}$ is tangent to $\hat{Z}$ and $\vec{v}_{\bar{u}} = \mu_*(v)$. Under the canonical trivialization of $T(V) = V \times V$, the vector field $\vec{v}$ defines a V-valued function $\bar{g}$ in a neighborhood of $\bar{u}$. The V-valued function $g := \mu^* \bar{g}$ in a neighborhood of u satisfies $D_\theta g = 0$ from (2.1) in Proposition 2.8. The vector field $g_\sharp D_\lambda$ is tangent to the fibers of $\hat{\mathscr{C}} \to M$ and its value at u is v, from the choice of $\bar{g}$. $\qquad \square$

5. Characteristic connection

DEFINITION 5.1. – Given a cone structure $\mathscr{C} \subset \mathbb{P}T(M)$, denote by $\varpi : \mathscr{C} \to M$ the natural projection. Denote by $\mathcal{V} \subset T(\mathscr{C})$ the relative tangent bundle of the projection ϖ and by $\mathcal{T} \subset T(\mathscr{C})$ the tautological bundle whose fiber at $\alpha \in \mathscr{C}_x$ is $d\varpi_\alpha^{-1}(\hat{\alpha})$ where $\hat{\alpha} \subset T_x(M)$ is the 1-dimensional subspace corresponding to $\alpha \in \mathbb{P}T_x(M)$. A line subbundle $\mathcal{I} \subset T(\mathscr{C})$, with locally free quotient $T(\mathscr{C})/\mathcal{I}$, is called a *conic connection* if $\mathcal{I} \subset \mathcal{T}$ and $\mathcal{I} \cap \mathcal{V} = 0$, i.e., it splits the exact sequence

(5.1) $$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{V} \cong \mathcal{O}(-1) \longrightarrow 0$$

where $\mathcal{O}(1)$ denotes the relative hyperplane bundle on $\mathbb{P}T(M)$.

PROPOSITION 5.2. – *Let $\mathscr{C} \subset \mathbb{P}T(M)$ be a cone structure. Suppose that $H^0(\mathscr{C}_x, \mathcal{V} \otimes \mathcal{O}(1)) = 0$ for some $x \in M$. Then a conic connection is unique if it exists.*

Proof. – This follows from the fact that the set of splittings of (5.1) is $H^0(\mathscr{C}, \mathcal{V} \otimes \mathcal{O}(1))$. $\qquad \square$

Given a vector space V and a non-singular projective variety $Z \subset \mathbb{P}V$, the cone over Z will be denoted by $\hat{Z} \subset V$. For a point $\alpha \in Z$, the affine tangent space of Z at α is

$$\hat{T}_\alpha(Z) := T_u(\hat{Z}) \subset V \quad \text{a non-zero vector } u \in \hat{\alpha}.$$

This is independent of the choice of u.

Let $\mathscr{C} \subset \mathbb{P}T(M)$ be a cone structure. Denote by $\mathcal{P} \subset T(\mathscr{C})$ the subbundle whose fiber at $\alpha \in \mathscr{C}_x$ is $d\varpi_\alpha^{-1}(\hat{T}_\alpha(\mathscr{C}_x))$ where $\hat{T}_\alpha(\mathscr{C}_x) \subset T_x(M)$ is the affine tangent space of the projective subvariety $\mathscr{C}_x \subset \mathbb{P}T_x(M)$ at $\alpha \in \mathscr{C}_x$. The following is proved in Proposition 1 of [8].

PROPOSITION 5.3. – *Given a conic connection $\mathcal{I}$, regarding the subbundles of $T(\mathscr{C})$ as sheaves of vector fields on $\mathscr{C}$, we have $\mathcal{P} = [\mathcal{I}, \mathcal{V}]$.*

Proposition 5.4. – *Given a Z-isotrivial cone structure $\mathcal{C} \subset \mathbb{P}T(M)$ and an adapted coframe ω, the geodesic flow γ is tangent to the cone $\hat{\mathcal{C}} \subset T(M)$ by Proposition 4.6. Denote by $\Gamma \subset T(\mathcal{C})$ the line subbundle spanned by the image of γ. Then Γ is a conic connection on the cone structure $\mathcal{C} \subset \mathbb{P}T(M)$.*

Proof. – It suffices to show $\Gamma \subset \mathcal{I}$. This is immediate from Proposition 2.10 (b). $\square$

Let us introduce a distinguished class of conic connections.

Definition 5.5. – A conic connection $\mathcal{I} \subset T(\mathcal{C})$ is a *characteristic connection* if for any local section v of $\mathcal{P}$ and any local section w of $\mathcal{I}$, both regarded as local vector fields on the manifold $\mathcal{C}$, the Lie bracket $[v, w]$ is a local section of $\mathcal{P}$ again.

When $\mathcal{C} = \mathbb{P}T(M)$, any conic connection is a characteristic connection. On the other hand, when $\mathcal{C} \neq \mathbb{P}T(M)$, a characteristic connection is unique if it exists (Theorem 3.1.4 of [6]).

Proposition 5.6. – *Given a non-singular projective variety $Z \subset \mathbb{P}V$, define the subspace $\Xi_Z \subset \mathrm{Hom}(\wedge^2 V, V)$ by*

$$\Xi_Z := \{\sigma : V \otimes V \to V, \ \sigma(u,v) = -\sigma(v,u), \sigma(u,v) \in \hat{T}_\alpha(Z) \text{ if } \alpha \in Z, u \in \hat{\alpha} \text{ and } v \in \hat{T}_\alpha(Z)\}.$$

Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a Z-isotrivial cone structure and ω be an adapted coframe. Suppose the conic connection Γ on $\mathcal{C}$ induced by ω in the sense of Proposition 5.4 is a characteristic connection. Then the structure function σ^ω of ω takes values in Ξ_Z.

Proof. – By Proposition 5.3, if $\mathcal{I}$ is a characteristic connection, then given any local section $\tilde{v}$ of $\mathcal{V}$ and a local section $\tilde{w}$ of $\mathcal{I}$, $[[\tilde{v}, \tilde{w}], \tilde{w}]$ is a local section of $\mathcal{P}$. We can pull back this to the affine cone $\hat{\mathcal{C}} \subset T(M)$ by the projection $\hat{\mathcal{C}} \setminus 0 \to \mathcal{C}$. Let us denote the pullback distributions of $\mathcal{V}$ and $\mathcal{P}$ by $\hat{\mathcal{V}}$ and $\hat{\mathcal{P}}$. By Proposition 4.7, for any vector $v \in \hat{T}_\alpha(\hat{\mathcal{C}}_x)$, there exists a section $\tilde{v}$ of the relative tangent bundle $\hat{\mathcal{V}}$ of $\hat{\mathcal{C}} \to M$ such that

$$\tilde{v}_u = v, \ \tilde{v} = g_\sharp D_\lambda \text{ for some } V\text{-valued function } g \text{ satisfying } D_\theta g = 0.$$

From $\gamma(g) = 0$ and Proposition 2.10 (a),

$$[\tilde{v}, \gamma] = [g_\sharp D_\lambda, \gamma] = g_\sharp [D_\lambda, \gamma] - \gamma(g)_\sharp D_\lambda = g_\sharp D_\theta.$$

$$[[\tilde{v}, \gamma], \gamma] = [g_\sharp D_\theta, \gamma] = g_\sharp [D_\theta, \gamma] = g_\sharp [D_\theta, \mu_\sharp D_\theta] = g_\sharp(\mu_\sharp [D_\theta, D_\theta]) + g_\sharp(D_\theta \mu) D_\theta.$$

Since $D_\theta \mu = 0$, we see that

$$d\pi_u([[\tilde{v}, \gamma], \gamma]_u) = g(u)_\sharp (\mu(u)_\sharp [D_\theta, D_\theta]_u) = \sigma^\omega(v, u).$$

Since Γ is a characteristic connection, the latter must be in $\hat{T}_\alpha(Z)$ by Proposition 5.3. Thus $\sigma^\omega \in \Xi_Z$. $\square$

Proposition 5.7. – *Let $Z \subset \mathbb{P}V$ be a non-singular variety such that*

(1) *Z is linearly normal, i.e., $H^0(Z, \mathcal{O}(1)) = V^*$;*
(2) *the variety of tangent lines of Z is linearly non-degenerate in $\mathbb{P}(\wedge^2 V)$; and*
(3) *$H^0(Z, ad(T(Z)) \otimes \mathcal{O}(1)) = 0$.*

Then $\Xi_V = \Xi_Z$.

Proof. – From the definition of Ξ_Z in Proposition 5.6, there exists a natural homomorphism

$$j : \Xi_Z \to H^0(Z, T(Z) \otimes T^*(Z) \otimes \mathcal{O}(1)).$$

We claim that j is injective. If $\sigma \in \Xi_Z$ satisfies $j(\sigma) = 0$, then for any $u \in \hat{Z}$ and any $v \in T_u(\hat{Z})$, $\sigma(u, v) = 0$. But $\{u \wedge v, u \in \hat{Z}, v \in T_u(\hat{Z})\}$ generates $\wedge^2 V$ by the condition (2). Thus $\sigma = 0$ and j is injective.

By the conditions (1) and (3),

$$H^0(Z, T(Z) \otimes T^*(Z) \otimes \mathcal{O}(1)) = H^0(Z, ad(T(Z)) \otimes \mathcal{O}(1)) \oplus H^0(Z, \mathcal{O}(1)) \cong V^*.$$

Note that $\Xi_V \subset \Xi_Z$. Thus the injection

$$V^* \cong \Xi_V \subset \Xi_Z \xrightarrow{j} H^0(Z, T(Z) \otimes T^*(Z) \otimes \mathcal{O}(1)) \cong V^*$$

must be an isomorphism and $\Xi_V = \Xi_Z$. □

From Theorem 3.4, Proposition 5.6 and Proposition 5.7, we have the following differential geometric result.

THEOREM 5.8. – *Let $Z \subset \mathbb{P}V$ be a non-singular projective subvariety satisfying the conditions of Proposition 5.7. Let $\mathcal{C} \subset \mathbb{P}T(M)$ be a Z-isotrivial cone structure with an adapted frame ω. If the conic connection Γ induced by ω on $\mathcal{C}$ is a characteristic connection, then the structure function σ^ω is conformally closed. In particular, the cone structure is locally flat by Proposition 4.4.*

6. Proof of Theorem 1.11

The following is a restatement of Proposition 3.1.2 of [6] or Proposition 8 of [8].

PROPOSITION 6.1. – *Let X be a projective manifold with a family of minimal rational curves satisfying Assumption 1.3. There exists a connected open subset $M \subset X$ such that the varieties of minimal rational tangents define a cone structure $\mathcal{C} \subset \mathbb{P}T(M)$. Then $\mathcal{C}$ admits a characteristic connection $\mathcal{F}$. In fact, the leaves of $\mathcal{F}$ are given by tangent vectors to members of $\mathcal{K}$.*

We remark that Assumption 1.3 is crucial here. The assumption that $\mathcal{C}_x$ is non-singular implies that the tangent map in [8] is an embedding, which shows the existence of $\mathcal{F}$ as a line subbundle of $T(\mathcal{C})$ with locally free quotient.

Now we can finish the proof of Theorem 1.11 as follows. From Proposition 6.1, the cone structure has a characteristic connection $\mathcal{F} \subset T(\mathcal{C})$. Choose an adapted coframe ω. The induced connection $\Gamma \subset T(\mathcal{C})$ must agree with the characteristic connection $\mathcal{F}$ by Proposition 5.2. By Theorem 5.8, the cone structure is locally flat. It is easy to see that the corresponding cone structure in Example 1.7 is also locally flat. Thus Theorem 1.11 follows from Theorem 1.5.

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