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Robert L. JERRARD & Didier SMETS

On Schrödinger maps from T^1 to S^2

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

ON SCHRÖDINGER MAPS FROM T^1 TO S^2

BY ROBERT L. JERRARD AND DIDIER SMETS

ABSTRACT. – We prove an estimate for the difference of two solutions of the Schrödinger map equation for maps from T^1 to S^2 . This estimate yields some continuity properties of the flow map for the topology of $L^2(T^1, S^2)$, provided one takes its quotient by the continuous group action of T^1 given by translations. We also prove that without taking this quotient, for any $t > 0$ the flow map at time t is discontinuous as a map from $\mathcal{C}^\infty(T^1, S^2)$, equipped with the weak topology of $H^{1/2}$, to the space of distributions $(\mathcal{C}^\infty(T^1, \mathbb{R}^3))^*$. The argument relies in an essential way on the link between the Schrödinger map equation and the binormal curvature flow for curves in the euclidean space, and on a new estimate for the latter.

RÉSUMÉ. – Nous obtenons une estimation de la différence entre deux solutions de l'équation des Schrödinger maps de T^1 dans S^2 . Cette estimation fournit une propriété de continuité du flot associé pour la topologie de $L^2(T^1, S^2)$, quotientée par l'action continue du groupe T^1 via les translations. Nous démontrons également que sans cette prise de quotient, quel que soit $t > 0$ l'application flot au temps t est discontinue de $\mathcal{C}^\infty(T^1, S^2)$, équipé de la topologie faible de $H^{1/2}$, vers l'espace des distributions périodiques $(\mathcal{C}^\infty(T^1, \mathbb{R}^3))^*$. L'argument repose de manière essentielle sur le lien étroit entre l'équation des Schrödinger maps et celle du flot par courbure binormale pour une courbe dans l'espace, et sur une nouvelle estimation concernant ce dernier.

1. Introduction

We are interested in the Schrödinger map equation,

$$(1) \quad \partial_t u = \partial_s (u \times \partial_s u),$$

where $u : I \times T^1 \rightarrow S^2$, $0 \in I \subset \mathbb{R}$ is some open interval, $T^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ is the flat one dimensional torus, S^2 is the standard unit sphere in \mathbb{R}^3 , and \times is the vector product in \mathbb{R}^3 . The solutions we consider are all at least in $L^\infty(I, H^{1/2}(T^1, S^2))$, so that Equation (1) has a well-defined distributional meaning in $I \times T^1$. Such solutions are necessarily continuous with values into $H_{\text{weak}}^{1/2}(T^1, S^2)$, so that Cauchy problems are well-defined too.

Our main result provides an upper bound for the growth rate of the difference between two solutions, one of which being required to be sufficiently smooth.

THEOREM 1. – *Let $u \in \mathcal{C}(I, H^3(T^1, S^2))$ be a solution of the Schrödinger map Equation (1) on $I = (-T, T)$ for some $T > 0$. Given any other solution $v \in L^\infty(I, H^{1/2}(T^1, S^2))$ of (1), there exists a continuous function $\sigma : I \rightarrow T^1$ such that for every $t \in I$,*

$$\|v(t, \cdot) - u(t, \cdot + \sigma(t))\|_{L^2(T^1, \mathbb{R}^3)} \leq C \|v(0, \cdot) - u(0, \cdot)\|_{L^2(T^1, \mathbb{R}^3)},$$

where the constant $C \equiv C(\|\partial_{sss}u(0, \cdot)\|_{L^2}, T)$, in particular C does not depend on v .⁽¹⁾

For initial data in $H^k(T^1, S^2)$, with $k \geq 2$, Equation (1) possesses unique global solutions in $\mathcal{C}(\mathbb{R}, H^k(T^1, S^2))$. Those solutions preserve the energy

$$E(u(t, \cdot)) := \int_{T^1} |\partial_s u|^2(t, s) ds.$$

For initial data in the energy space $H^1(T^1, S^2)$, global solutions in $L^\infty(\mathbb{R}, H^1(T^1, S^2))$ are also known to exist. The latter are continuous with values in $H^1_{\text{weak}}(T^1, S^2)$, but are not known to be unique nor to preserve the energy.

REMARK 1. – *Let $D(\mathcal{F}) \subseteq H^{1/2}(T^1, S^2)$ be the domain of a hypothetical flow map \mathcal{F} for (1) with values into $\mathcal{C}(I, H^{1/2}_{\text{weak}}(T^1, S^2))$, and let \mathcal{G} be one of the (possibly unique) existing flow maps from $H^1(T^1, S^2)$ with values into $\mathcal{C}(I, H^1_{\text{weak}}(T^1, S^2))$. For $s \geq 0$, let H^s/S^1 and H^s_{weak}/S^1 denote the topological spaces obtained as the quotients of $H^s(T^1, \mathbb{R}^3)$ and $H^s_{\text{weak}}(T^1, \mathbb{R}^3)$ respectively by the continuous action group of S^1 through translations⁽²⁾. Let finally $\mathcal{F}_{/S^1}$ and $\mathcal{G}_{/S^1}$ denote the left composition of \mathcal{F} and \mathcal{G} respectively by the above quotient map⁽³⁾. As a consequence of Theorem 1, we see that:*

At any point in $H^3(T^1, S^2)$, $\mathcal{F}_{/S^1}$ is continuous as a map

$$\mathcal{F}_{/S^1} : \left(D(\mathcal{F}), \text{top}(L^2) \right) \longrightarrow \left(\mathcal{C}(H^{1/2}_{\text{weak}}), \text{top}(\mathcal{C}(L^2/T^1)) \right),$$

and $\mathcal{G}_{/S^1}$ is continuous as a map

$$\mathcal{G}_{/S^1} : \left(H^1, \text{top}(H^1_{\text{weak}}) \right) \longrightarrow \left(\mathcal{C}(H^1_{\text{weak}}), \text{top}(\mathcal{C}(H^1_{\text{weak}}/T^1)) \right).$$

We supplement Theorem 1 and Remark 1 by the following ill-posedness type result, which shows in particular that the function σ in the statement of Theorem 1 could not be removed.

THEOREM 2. – *For any given $t \neq 0$, the flow map for (1) at time t is not continuous as a map from $\mathcal{C}^\infty(T^1, S^2)$, equipped with the weak topology of $H^{1/2}$, to the space of distributions $(\mathcal{C}^\infty(T^1, \mathbb{R}^3))^*$. Indeed, for any $\sigma_0 \in \mathbb{R}$ there exist a sequence of smooth initial data $(u_{m, \sigma_0}(0, \cdot))_{m \in \mathbb{N}} \in \mathcal{C}^\infty(T^1, S^2)$ such that*

$$u_{m, \sigma_0}(0, \cdot) \rightharpoonup u^*(0, \cdot) \quad \text{in } H^{\frac{1}{2}}_{\text{weak}}(T^1, \mathbb{R}^3),$$

⁽¹⁾ See Section 7 for a more explicit version of C .

⁽²⁾ These are of course no longer vector spaces.

⁽³⁾ We mean pointwise in I . In other words $\mathcal{F}_{/S^1}$ and $\mathcal{G}_{/S^1}$ describe the profiles of the solutions, independently of their locations.

where $u^*(0, s) := (\cos(s), \sin(s), 0)$ is a stationary solution of (1), and for any $t \in \mathbb{R}$

$$u_{m, \sigma_0}(t, \cdot) \rightharpoonup u^*(t, \cdot + \sigma_0 t) \quad \text{in } H_{\text{weak}}^{\frac{1}{2}}(T^1, \mathbb{R}^3).$$

1.1. Three companions : Binormal flow, Schrödinger map, Cubic NLS

The binormal curvature flow equation for $\gamma : I \times \mathbb{R} \rightarrow \mathbb{R}^3$, where $0 \in I \subset \mathbb{R}$ is some open interval, is given by

$$(2) \quad \partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma,$$

where s is moreover required to be an arc-length parameter for the curves $\gamma(t, \cdot)$, for every $t \in I$. At least for smooth solutions, the arc-length parametrization condition $|\partial_s \gamma(t, s)|^2 = 1$ is compatible with Equation (2), since

$$\partial_t (|\partial_s \gamma|^2) = 2\partial_s \gamma \cdot \partial_{st} \gamma = 2\partial_s \gamma \cdot (\partial_s \gamma \times \partial_{sss} \gamma) = 0$$

whenever (2) is satisfied. Denoting by $(\tau, n, b) \equiv (\tau(t, s), n(t, s), b(t, s))$ the Frenet-Serret frame along the curve $\gamma(t, \cdot)$ at the point $\gamma(t, s)$, so that $\partial_s \gamma = \tau$ and $\partial_{ss} \gamma = \kappa n$ where κ is the curvature, Equation (2) may be rephrased as

$$\partial_t \gamma = \kappa b,$$

from which its name arises. In the sequel, we will only consider solutions of (2) which belong to $L^\infty(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$, so that (2) has a well-defined distributional meaning in $I \times \mathbb{R}$.

If $\gamma \in L^\infty(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$ is a solution to the binormal curvature flow Equation (2), then the map $u := \partial_s \gamma \in L^\infty(I, H_{loc}^{1/2}(\mathbb{R}, S^2))$, parametrizing the evolution in time of the unit tangent vectors to the curves $\gamma(t, \cdot)$, satisfies

$$\partial_t u = \partial_t \partial_s \gamma = \partial_s \partial_t \gamma = \partial_s (u \times \partial_s u)$$

in the sense of distributions on $I \times \mathbb{R}$. In other words, u is a solution of the Schrödinger map Equation (1) for maps from \mathbb{R} to S^2 , and the binormal curvature flow equation is therefore a primitive equation of the Schrödinger map equation.

Conversely, let $u \in L^\infty(I, H_{loc}^{1/2}(\mathbb{R}, S^2))$ be a solution to the Schrödinger map Equation (1) and define the function $\Gamma_u \in L^\infty(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$ by

$$(3) \quad \Gamma_u(t, s) := \int_0^s u(t, z) dz.$$

In the sense of distributions on $I \times \mathbb{R}$, we have

$$(4) \quad \partial_s \left(\partial_t \Gamma_u - \partial_s \Gamma_u \times \partial_{ss} \Gamma_u \right) = 0.$$

By construction, the primitive curves $\Gamma_u(t, \cdot)$ all have their base point $\Gamma_u(t, 0)$ fixed at the origin. If they were smooth, Equation (4) would directly imply the existence of a function c_u depending on time only and such that

$$\gamma_u(t, s) := \Gamma_u(t, s) + c(t)$$

is a solution to the binormal curvature flow Equation (2). The function c_u indeed represents the evolution in time of the actual base point of the curves. In Section 2 we will turn this argument into a statement for quasiperiodic solutions in $H^{3/2}$.

Our proofs of Theorem 1 and Theorem 2 spend most of their time in the binormal curvature flow world rather than in the Schrödinger map one.

Besides the bidirectional relation between the binormal curvature flow equation and the Schrödinger map equation presented just above, Hasimoto [9] exhibited in 1972 an intimate relation between the binormal curvature flow Equation (2) and the cubic focusing nonlinear Schrödinger equation. Let $\gamma : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ be a smooth and biregular solution of the binormal curvature flow Equation (2), and denote by κ and T respectively the curvature and torsion functions of γ . Then, the function Ψ defined on $I \times \mathbb{R}$ by the Hasimoto transform

$$\psi(t, s) := \kappa(t, s) \exp \left(i \int_0^s T(t, z) dz \right)$$

is a solution to

$$\partial_t \psi + \partial_{ss} \psi + \frac{1}{2} (|\psi|^2 - A(t)) \psi = 0$$

where

$$A(t) := \left(2 \frac{\partial_{ss} \kappa - \kappa T^2}{\kappa} + \kappa^2 \right) (t, 0).$$

If γ is 2π -periodic in s , that is if $\gamma : I \times T^1 \rightarrow \mathbb{R}^3$, then Ψ is only quasiperiodic unless $\int_0^{2\pi} T(t, z) dz \in 2\pi\mathbb{Z}$. Nevertheless, it is possible to recover a 2π -periodic function Ψ by means of a Galilean transform. One can also get rid of the $A(t)$ factor by means of a phase shift. More precisely, the function

$$\Psi(t, s) := \psi(t, s - \frac{b}{2}t) \exp \left(i \left(bs - b^2t - \int_0^t \frac{1}{2} A(z) dz \right) \right),$$

where

$$b := 1 - \frac{1}{2\pi} \int_0^{2\pi} T(0, z) dz = 1 - \frac{1}{2\pi} \int_0^{2\pi} T(t, z) dz$$

is well-defined on $I \times T^1$ and is a solution to the cubic focusing nonlinear Schrödinger equation

$$(5) \quad \partial_t \Psi + \partial_{ss} \Psi + \frac{1}{2} |\Psi|^2 \Psi = 0$$

on $I \times T^1$.

Equation (5) is integrable and known to be solvable by the inverse scattering method since the works of Zakharov and Shabat [21] in 1971 for the vanishing case and Ma and Ablowitz [14] in 1981 for the periodic case. Therefore the binormal curvature flow Equation (2) and the Schrödinger map Equation (1) are also integrable, in the weak sense that they can be mapped to an integrable equation. Notice however that the inverse of the Hasimoto transform, whenever it is well defined⁽⁴⁾, involves reconstructing a curve from its curvature and torsion functions, an operation which is both highly nonlinear and nonlocal. At least for that reason, translating an estimate available for the cubic focusing NLS to the binormal curvature flow equation or to the Schrödinger map equation, or even vice versa, does not look like a straightforward task.

⁽⁴⁾ Vanishing of Ψ yields underdetermination.

Our proofs of Theorem 1 and Theorem 2 make no use of the Hasimoto transform, nor, at least directly, of the induced integrability. It would be of interest to better understand the implications, if any, of Theorem 1 for the cubic focusing NLS.

1.2. Comments on the Cauchy theory

Schrödinger map equations arise in the literature not only for maps from T^1 to S^2 , but more generally for maps from a Riemannian manifold to a Kähler manifold of arbitrary dimensions. It is not our purpose here to cover the growing literature about the Schrödinger map equations in specific or general cases. Instead, we will briefly focus on those works that, when they appeared, brought an improvement at the level of the Cauchy theory when particularized (or adapted) to the case of maps from T^1 to S^2 only.

As far as the Cauchy theory is concerned, it seems that (1) was first studied by Zhou and Guo [22] in 1984 for maps u from an interval to S^2 , and by Sulem, Sulem and Bardos [20] in 1986 for maps u from \mathbb{R}^N to S^2 , for arbitrary $N \geq 1$. Their proofs, which do not make use of any dispersive estimate, and therefore their results, can be translated almost word for word⁽⁵⁾ in the case of maps from T^1 to S^2 . Both papers establish that (1) possesses weak solutions in $L^\infty(H^1)$. The method in [22] is based on a parabolic regularization and the Leray-Schauder fixed point theorem, whereas the method in [20] is based on discrete approximation schemes and a weak compactness argument. In particular none of them provides uniqueness in H^1 . The global character of their solutions, as stated in [20], follows from the global character of the approximating ones, since an energy is a Lyapunov functional for the latter.

In a second step, differentiating Equation (1) with respect to time and then performing energy estimates, it is proved in [20] that the growth rate of any H^k norm of a solution is controlled a priori by its $W^{2,\infty}$ norm. By Sobolev embedding and compactness, this yields local existence in $L^\infty(H^k)$ for any $k > N/2 + 2$, that is $k = 3$ in dimension 1. Working out some of the details in [20], one realizes (see e.g. Appendix) that in dimension 1 the H^2 norm of the solution suffices to control all the other higher norms, which yield local existence in $L^\infty(H^2)$ as well. Global existence in $L^\infty(H^k)$ for any $k \geq 2$ then follows, as remarked in [20] or in Zhou, Guo and Tan [23] in 1991, by the a priori control on the H^2 norm given by the further conservation law for

$$I(u(t, \cdot)) := \int_{T^1} |\partial_{ss}u + |\partial_s u|^2 u|^2(t, s) + |\partial_s u|^2(t, s) - \frac{3}{4} |\partial_s u|^4(t, s) ds,$$

which holds at least for solutions in $L^\infty(H^3)$. Pushing arguments of [20]⁽⁶⁾ a bit further, one also obtains global well-posedness in H^k , for $k \geq 3$, as well as Lipschitz continuity of the flow map in H^3 , for the H^1 or the H^2 norm, with a constant depending only on the H^3 norm (see also Appendix).

⁽⁵⁾ Since \mathbb{R}^N is non compact and functions take values in S^2 , the spaces involved in [20] are homogeneous Sobolev spaces.

⁽⁶⁾ Both [20] and [23] have let aside the study of the difference between two solutions, asserting that the uniqueness followed from regularity. Whereas this is generally true, it is sometimes desirable, and we actually needed that for Theorem 1, to know which second norm controls what first norm of the difference.

In 2001, Ding and Wang [7] used the parabolic approximation approach and gave complete proofs of existence of global solutions in $L^\infty(H^k)$, for any $k \geq 2$, and of uniqueness in $L^\infty(H^k)$, for any $k \geq 3$.

Uniqueness in $L^\infty(H^2)$ was proved by Chang, Shatah and Uhlenbeck [5] in 2000, some part being clarified by Nahmod, Shatah, Vega and Zeng [18] in 2006. The strategy used in those works consists in writing the components of the derivatives of a solution u in a special orthonormal frame in the tangent plane u^\perp , a frame adapted to the solution itself, in which the equation turns out to be of NLS type. In essence, this is very much alike the Hasimoto transform, mentioned in the previous subsection, but it has the advantage of additional flexibility. Borrowing known uniqueness results at the NLS level, uniqueness for the Schrödinger map equation follows, provided one can pull it back (which requires some care since the frame depends on the solution).

A different approach to uniqueness in H^2 , based on more classical energy methods, was successfully used by McGahagan [16, 15] in 2004. The key ingredient in her analysis, arguments earlier used by Shatah in related contexts (see e.g. [19]), is to measure the discrepancy between the derivatives of two different solutions not by embedding the target manifold in a big euclidean space and using euclidean distances, but by parallel transporting one derivative into the tangent plane of the other one and measuring there. Particularized to the case of T^1 and S^2 , [15] implies global well-posedness in $L^\infty(H^2)$, and Lipschitz continuity of the flow map in H^2 , for the H^1 norm, with a constant depending only on the H^2 norm.

At the time of writing, there does not seem to be any local well-posedness result available below the H^2 regularity threshold. On the other hand, as expressed in particular in the introduction of [18], in terms of scalings Equation (1) is sub-critical in $H^s(T^1, S^2)$ for any $s > 1/2$. Therefore, even though one would be tempted to believe that dimension $N = 1$ is the easiest case as far as the Cauchy theory is concerned, there remain substantial gaps between the criticality at the level of $H^{1/2}$, the existence at the level of H^1 , and the well-posedness at the level of H^2 .

Theorem 1 corresponds to a weak-strong type estimate, since one of the solutions is required to be smooth. Being such, there is no hope that it could be used alone to improve the Cauchy theory for (1). On the other hand, the estimate which it provides holds in a norm with very few regularity (L^2 is 1/2 derivative below the critical space).

Theorem 2 focuses on ill-posedness⁽⁷⁾ for low regularity norms. Such considerations have attracted attention in related contexts of dispersive equations, in particular since Bourgain [3] remarked that in cases it was possible to rule out regularity of the flow map even in spaces with more regularity than the critical space. A special attention was devoted to the cubic nonlinear Schrödinger equation in one spatial dimension, in particular in the works of Kenig-Ponce-Vega 2001 [12], Burq-Gérard-Tzvetkov 2002 [4] and Christ-Coliander-Tao 2003 [6]. In that case, the critical space in terms of scaling is $H^{-1/2}$, global well-posedness was proved in 1993 by Bourgain [2] in the space L^2 , and failure of continuity of the flow map arises in any H^s with $s < 0$. Counterexamples to continuity were essentially constructed by superposition

⁽⁷⁾ In higher dimensions Schrödinger maps are not always globally defined, and blow-up analysis is a current area of active investigation, but this is a different issue.

of a slow and a rapid modes, each of them being either explicit or well understood through the Galilean invariance of the equation. Our strategy for Theorem 2 is somewhat parallel though in a rather different context, and it does not really involve new ideas. In 2007, Molinet [17] gave further insight into the non continuity issue in the focusing cubic NLS case, and was even able to recover some form of continuity in L^2_{weak} , provided one is allowed to quotient by the action group of S^1 , through phase multiplication there. In some sense, Theorem 1 bears some common features with Molinet’s result⁽⁸⁾.

1.3. Strategy of the proofs and plan of the paper

The general strategy of the proof of Theorem 1 is as follows; we first voluntarily somewhat simplify the truth in order not to obscure the main ideas, and then indicate which additional issues have to be overcome in a second step.

Given two solutions u and v of the Schrödinger map Equation (1), we construct the two primitive solutions γ_u and γ_v of the binormal curvature flow equation whose derivatives coincide with u and v respectively, as explained in Subsection 1.1 and proved in Section 2. Let us write $\gamma_u \equiv \gamma \equiv (\gamma_t)_{t \in I}$ and $\gamma_v \equiv \Gamma \equiv (\Gamma_t)_{t \in I}$; one should keep in mind that γ is the smooth solution and Γ is the rough one. If $u(0, \cdot)$ and $v(0, \cdot)$ are close in L^2 , then the curve Γ_0 is close to the curve γ_0 in many respects, but in particular for the discrepancy measure

$$F_{\gamma_0}(\Gamma_0) := \int_{\Gamma_0} 1 - f(d^2(x, \gamma_0)) \tau_{\Gamma_0}(x) \cdot \tau_{\gamma_0}(P(x)) d\mathcal{H}^1,$$

where d is the distance function, $f(d^2) \simeq 1 - d^2$ in a neighborhood of $d = 0$, P is the orthogonal projection on the curve γ_0 and τ_{Γ_0} and τ_{γ_0} denote the oriented unit tangents to γ_0 and Γ_0 respectively. It turns out, and this is the main ingredient in our proof, that since Γ and γ both solve the binormal curvature flow equation, we have the Gronwall type inequality

$$(6) \quad \frac{d}{dt} F_{\gamma_t}(\Gamma_t) \leq C F_{\gamma_t}(\Gamma_t)$$

almost everywhere on I , where the constant C is explicit in terms of the L^∞ norms of $\partial_{ss}\gamma_t$ and $\partial_{sss}\gamma_t$. In order to prove this inequality, we use in particular the fact that Γ satisfies some weak formulation of (2), namely

$$(7) \quad \frac{d}{dt} \int_{\Gamma_t} X \cdot \tau_{\Gamma_t} d\mathcal{H}^1 = \int_{\Gamma_t} \partial_t X \cdot \tau_{\Gamma_t} - D(\text{curl } X) : (\tau_{\Gamma_t} \otimes \tau_{\Gamma_t}) d\mathcal{H}^1,$$

for every smooth time-dependent vector field $X : I \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. In view of the definition of F , we apply (7) for the choice $X(x) := d^2(x, \gamma_t) \tau_{\gamma_t}(P(x))$. Once (6) is proved, we obtain from Gronwall’s inequality a control on $F_{\gamma_t}(\Gamma_t)$ in terms of $F_{\gamma_0}(\Gamma_0)$, and therefore of $\|u(0, t) - v(0, t)\|_2^2$. The last task we have to deal with is then to relate this discrepancy measure $F_{\gamma_t}(\Gamma_t)$ to the closeness in L^2 between $u(t, \cdot)$ and $v(t, \cdot)$. Notice the important fact that, by its definition, F does not care too much about the actual parametrizations of γ_t and Γ_t (only P is involved). In particular F is insensible to constant shifts in the space variables of

⁽⁸⁾ One could actually argue that the focusing cubic NLS on T^1 and Equation (1) are actually very much related, not to say equivalent, in particular in view of the Hasimoto transform (see Subsection 1.1 above) or the related gauge transforms of Chang, Shatah and Uhlenbeck [5]. Things are probably a little more complex though, since, as we already wrote, those transforms involve the solution itself and translating back and forth results in each framework requires at least some care.

one or both of the curves, and therefore it does not control the squared L^2 distance between $u(t, \cdot)$ and $v(t, \cdot)$. It turns out that F actually controls the squared L^2 distance between $v(t, \cdot)$ and a suitable translate $u(t, \cdot + \sigma(t))$ of $u(t, \cdot)$, and that one can keep track of $\sigma(t)$ in time; indeed, $\sigma(t)$ is essentially determined by $P(\Gamma_t(0)) = \gamma_t(\sigma(t))$. The conclusion of Theorem 1 then follows.

As mentioned, there are some issues coming into the program. First, we need the curves γ_u and γ_v to have the same pitch in order to perform the analysis at the binormal curvature flow level. If u and v do not have the same mean, we therefore first replace u by another smooth solution \tilde{u} which is sufficiently close to u and has the same mean as v . We apply our strategy to \tilde{u} and v and then estimate the difference between \tilde{u} and u by standard energy methods. Second, the orthogonal projection P involved in the definition of F may not be well-defined, in particular if the smooth curve γ has self-intersections or is close to be so. To overcome this difficulty, we need to keep track of the fact that at the initial time, the curves γ_0 and Γ_0 are not only close in terms of the Hausdorff distance or of F , but are also “parametrically” close. In doing so, we can guess at which place on γ_t we should orthogonally project each point of Γ_t , even when self-intersections are involved. The bottom line is that the actual definition of F gets somewhat more complex, as well as the weak formulation in (7), but the overall strategy remains the same.

The ideas behind the proof of Theorem 2 are part of the folklore on ill-posedness issues for dispersive equations in low regularity spaces. The functions u_{m,σ_0} mentioned in the statement of Theorem 2 are derived from explicit particular solutions $\gamma \equiv \gamma_{m,\bar{\sigma}_0}$ of the binormal curvature flow equation. Those special solutions satisfy the property that at any positive time t the solution curve γ_t is obtained from the initial curve γ_0 through a rigid motion. Such solutions have been studied and classified in a beautiful paper by Kida [13] in 1981. Among the many-parameters family of exact solutions that Kida constructed, we carefully single out a sequence of initial data that have the form of a helix of small pitch and small radius wrapped around a fixed circle of unit radius. The pitch and radius of those helices tend to zero along the sequence, with the radius being proportional to the power $3/2$ of the pitch. For each of these initial data, the corresponding solution γ_t at time t is obtained by the superposition of two rigid motions applied to γ_0 : a translational motion of the base circle (this corresponds to the “limit solution”, a circle being a traveling wave solution of Equation (2)) and a nontrivial sliding motion along the base circle induced by the curvature of the small helix. Recast into the framework of the Schrödinger map equation, this sliding motion gives us the continuity breaking parameter.

We now shortly describe the plan of the paper. In Section 2 we make precise the construction of a solution of (2) from a solution of (1), requiring as little regularity as possible. In Section 3, we introduce the adaptation of the functional F above and relate it to different types of discrepancy measures between two curves. In Section 4, we introduce and prove a weak formulation similar to (7) for parametrized solutions of the binormal curvature flow equation. In Section 5 we prove the basic estimate which is at the heart of (6). We gather all these informations in Section 6 to derive a discrepancy estimate for parametrized solutions of the binormal curvature flow equation. Finally, we prove Theorem 1 in Section 7 and Theorem 2

in Section 8. In the appendix we have gathered some presumably well-known continuity estimates for the flow map of (1) at the level of smooth solutions. We could not find them with sufficient precision about the dependence of various “constants” in the literature.

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2. Schrödinger maps and binormal curvature flow

In this section, we make precise the relation between the binormal curvature flow equation and the Schrödinger map equation which was outlined in Subsection 1.1 in the introduction. For later purposes, we wish to consider solutions of (1) defined on tori of length possibly different from the length 2π which we have arbitrarily chosen so far. Of course one can always use the scalings of the equation to turn back to such a situation, but in the section following this one we would like to explicitly keep track of some constants depending on the length. For $\ell > 0$, we therefore denote by T_ℓ^1 , or actually simply T^1 if no confusion occurs, the flat torus

$$T_\ell^1 := \mathbb{R}/\ell\mathbb{Z}.$$

LEMMA 1. – *Let $w \in L^\infty(I, H^{1/2}(T^1, S^2))$ be a solution of (1), and let Γ_w be defined by (3). There exists a unique continuous function $c_w : I \rightarrow \mathbb{R}^3$ satisfying $c_w(0) = 0$ and such that the function $\gamma_w \in L^\infty(I, H_{loc}^{3/2}(\mathbb{R}, \mathbb{R}^3))$ defined by*

$$(8) \quad \gamma_w(t, s) := \Gamma_w(t, s) + c_w(t),$$

is a solution of the binormal curvature flow Equation (2) on $I \times \mathbb{R}$.

Proof. – Since the function Γ_w is not a distribution on $I \times T^1$, we first consider instead its restriction $\tilde{\Gamma}_w$ to $I \times [0, \ell)$, which we view as a distribution on $I \times T^1$. We have $\tilde{\Gamma}_w \in \mathcal{C}(I, L^2(T^1, \mathbb{R}^3))$, but in general $\tilde{\Gamma}_w \notin L^\infty(I, H^{1/2}(T^1, \mathbb{R}^3))$, since for each $t \in I$ the function $\tilde{\Gamma}_w(t, \cdot)$ has a jump of height $-\int_{T^1} w(t, z) dz$ at the origin. We also define $S \in \mathcal{D}'(I \times T^1, \mathbb{R}^3)$ by

$$S := \partial_t \tilde{\Gamma}_w - w \times \partial_s w.$$

First, we have

$$(9) \quad \begin{aligned} \partial_s S &= \partial_t \partial_s \tilde{\Gamma}_w - \partial_s (w \times \partial_s w) \\ &= \partial_t w - \partial_s (w \times \partial_s w) - \partial_t \left(\left(\int_{T^1} w(t, z) dz \right) dt \otimes \delta_0 \right) \\ &= 0, \end{aligned}$$

since (1) holds and because (1) also implies that $\int_{T^1} w(t, z) dz$ does not depend on time.

Next, let $\varphi \in \mathcal{D}(I \times T^1, \mathbb{R}^3)$. We may write

$$\varphi(t, s) = \partial_s \chi(t, s) + m(t, s),$$

where

$$m(t, s) \equiv m(t) := \frac{1}{\ell} \int_{T^1} \varphi(t, s) ds \in \mathcal{D}(I \times T^1, \mathbb{R}^3)$$

and

$$\chi(t, s) := \int_0^s \varphi(t, z) - m(t) dz \in \mathcal{D}(I \times T^1, \mathbb{R}^3).$$

In view of (9), we obtain

$$(10) \quad \langle S, \varphi \rangle_{I \times T^1} = \langle S, \partial_s \chi + m \rangle_{I \times T^1} = -\langle \partial_s S, \chi \rangle_{I \times T^1} + \langle S, m \rangle_{I \times T^1} = \langle S, m \rangle_{I \times T^1},$$

where we denote by $\langle \cdot, \cdot \rangle_{I \times T^1}$ the duality between $\mathcal{D}'(I \times T^1, \mathbb{R}^3)$ and $\mathcal{D}(I \times T^1, \mathbb{R}^3)$. We expand

$$(11) \quad \begin{aligned} \langle S, m \rangle_{I \times T^1} &= -\langle \tilde{\Gamma}_w, \partial_t m \rangle_{I \times T^1} - \langle w \times \partial_s w, m \rangle_{I \times T^1} \\ &= -\int_I \int_0^\ell \left(\int_0^s w(t, z) dz \right) \cdot m'(t) ds dt - \int_I \int_0^\ell (w(t, s) \times \partial_s w(t, s)) \cdot m(t) ds dt \\ &= \int_I \int_0^\ell \left((s - \ell) w(t, s) + \int_0^t (w(\tau, s) \times \partial_s w(\tau, s)) d\tau \right) ds \cdot m'(t) dt \\ &= \int_I c_w(t) \cdot m'(t) dt, \end{aligned}$$

where

$$c_w(t) := \int_0^\ell \left((s - \ell)(w(t, s) - w(0, s)) + \int_0^t (w(\tau, s) \times \partial_s w(\tau, s)) d\tau \right) ds.$$

Notice that by construction $c_w(0) = 0$, and, since $w \in \mathcal{C}(I, H_{\text{weak}}^{1/2}(T^1, S^2))$, we have $c_w \in \mathcal{C}(I, \mathbb{R}^3)$. We will now prove that the function γ_w defined on $I \times \mathbb{R}$ by

$$\gamma_w(t, s) := \Gamma_w(t, s) + c_w(t)$$

is a solution to the binormal curvature flow Equation (2) on $I \times \mathbb{R}$. To that purpose, let $\psi \in \mathcal{D}(I \times \mathbb{R}, \mathbb{R}^3)$ be arbitrary. We first write

$$(12) \quad \langle \partial_t \gamma_w - \partial_s \gamma_w \times \partial_{ss} \gamma_w, \psi \rangle_{I \times \mathbb{R}} = \langle \partial_t \Gamma_w - w \times \partial_s w, \psi \rangle_{I \times \mathbb{R}} - \int_I c_w(t) \cdot n'(t) dt,$$

where

$$n(t) := \int_{\mathbb{R}} \psi(t, s) ds.$$

In order to compute the first term on the right-hand side of (12), we consider a partition of unity $1 = \sum_{j \in \mathbb{Z}} \lambda(\cdot + j\ell/2)$ on \mathbb{R} , where $\lambda \in \mathcal{D}(\mathbb{R}, \mathbb{R})$ has compact support in $(0, \ell)$, and we decompose ψ as $\psi = \sum_{j \in \mathbb{Z}} \psi_j$ where $\psi_j \in \mathcal{D}(I \times \mathbb{R}, \mathbb{R}^3)$ is defined by

$$\psi_j(t, s) := \psi(t, s) \lambda(s - j\ell/2).$$

For $j \in \mathbb{Z}$, we also define $\tilde{\psi}_j \in \mathcal{D}(I \times T^1, \mathbb{R}^3)$ by

$$\tilde{\psi}_j(t, s) := \sum_{k \in \mathbb{Z}} \psi_j(t, s + j\ell/2 + k\ell).$$

Since ψ_j has its support in $I \times [j\ell/2, (j/2 + 1)\ell]$, we have

$$(13) \quad \langle \partial_t \Gamma_w, \psi_j \rangle_{I \times \mathbb{R}} = - \int_I \int_{j\ell/2}^{(j/2+1)\ell} \tilde{\Gamma}_w(t, s) \cdot \partial_t \psi_j(t, s) \, ds dt.$$

Changing variables and using the equality $\Gamma_w(t, s + \ell) = \Gamma_w(t, s) + \int_{T^1} w$, valid for any $t \in I$ and $s \in \mathbb{R}$, we obtain, for each $t \in I$,

$$(14) \quad \int_{j\ell/2}^{(j/2+1)\ell} \tilde{\Gamma}_w(t, s) \cdot \partial_t \psi_j(t, s) \, ds = \int_0^\ell \tilde{\Gamma}_w(t, s) + H_j(s) \cdot \partial_t \tilde{\psi}_j(t, s) \, ds$$

where $H_j(s) \equiv (\int_{T^1} w)p$ for all $s \in [0, \ell]$ if $j = 2p \in 2\mathbb{Z}$, whereas $H_j(s) = (\int_{T^1} w)(p + 1)$ for $s \in [0, \ell/2]$ and $H_j(s) = (\int_{T^1} w)p$ for $s \in [\ell/2, \ell]$ if $j = 2p + 1 \in 2\mathbb{Z} + 1$. Since in any case H_j does not depend on time, integrating (14) in time and using (13) yields

$$(15) \quad \langle \partial_t \Gamma_w, \psi_j \rangle_{I \times \mathbb{R}} = \langle \partial_t \tilde{\Gamma}_w, \tilde{\psi}_j \rangle_{I \times T^1}.$$

By periodicity of w , we also have

$$(16) \quad \langle -w \times \partial_s w, \psi_j \rangle_{I \times \mathbb{R}} = \langle -w \times \partial_s w, \tilde{\psi}_j \rangle_{I \times T^1}.$$

Therefore, adding (16) to (15) and using (10) and (11) with the choice $\varphi = \tilde{\psi}_j$ we are led to

$$(17) \quad \langle \partial_t \Gamma_w - w \times \partial_s w, \psi_j \rangle_{I \times \mathbb{R}} = \langle S, \tilde{\psi}_j \rangle_{I \times T^1} = \int_I c_w(t) \cdot n'_j(t) \, dt,$$

where

$$n_j(t) := \int_{T^1} \tilde{\psi}_j(t, s) \, ds = \int_{\mathbb{R}} \psi_j(t, s) \, ds.$$

Summation in j inside (17) and substitution in (12) finally yield

$$\langle \partial_t \gamma_w - \partial_s \gamma_w \times \partial_{ss} \gamma_w, \psi \rangle_{I \times \mathbb{R}} = \int_I \left(\sum_j n'_j(t) \right) - n(t) \cdot c_w(t) \, dt = 0,$$

so that the claim is proved and hence the existence of c_w . Since c_w is required to be continuous with $c_w(0) = 0$, and since its distributional derivative is completely determined by (4) and (8), its uniqueness follows. \square

Unless $a := \int_{T^1} w = 0$, the “curves” γ_w constructed in the previous lemma are not closed, but merely invariant under translation by a . In the next section, we introduce various ways to measure the distance between such curves, and briefly study their relations.

3. Measuring the closeness between quasiperiodic curves

DEFINITION 1. – A curve in \mathbb{R}^3 parametrized by arc-length is a Lipschitz map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ such that

$$|\gamma'(s)| = 1 \quad \text{for a.e. } s \in \mathbb{R}.$$

A quasiperiodic curve in \mathbb{R}^3 of period $\ell > 0$ and pitch $a \in \mathbb{R}^3$ is a curve in \mathbb{R}^3 parametrized by arc-length and such that

$$\gamma(s + \ell) = \gamma(s) + a, \quad \forall s \in \mathbb{R}.$$

A closed curve in \mathbb{R}^3 of length $\ell > 0$ is a quasiperiodic curve in \mathbb{R}^3 of period ℓ and zero pitch.

REMARK 2. – i) When γ is a curve in \mathbb{R}^3 parametrized by arc-length, and when this does not lead to a confusion, we also denote by γ the subset $\{\gamma(s), s \in \mathbb{R}\}$ of \mathbb{R}^3 .

ii) If γ is a quasiperiodic curve of period ℓ and pitch a , then γ is also a quasiperiodic curve of period $n\ell$ and pitch na , for any integer $n \geq 1$. In the sequel, when we refer to “the” period and “the” pitch of a quasiperiodic curve γ , we implicitly assume that γ comes with a pre-assigned period ℓ , which might not be its minimal one.

DEFINITION 2. – Given two quasiperiodic curves γ and Γ in \mathbb{R}^3 , we define⁽⁹⁾

$$d_\varphi(\Gamma, \gamma) := \inf_{p \in \mathcal{P}(\ell, L)} \sup_{s \in \mathbb{R}} \left| \Gamma(s) - \gamma(p(s)) \right| = \inf_{p \in \mathcal{P}(\ell, L)} \sup_{s \in \mathbb{R}} \left| \gamma(s) - \Gamma(p(s)) \right|,$$

where ℓ and L are the periods of γ and Γ respectively, and for $0 < s_0, s_1$,

$$\mathcal{P}(s_0, s_1) = \{p \in \mathcal{C}(\mathbb{R}, \mathbb{R}) \text{ s.t. } p(s + s_0) = p(s) + s_1, \forall s \in \mathbb{R}\}.$$

One may verify that $d_\varphi(\Gamma, \gamma) < +\infty$ if and only if Γ and γ have the same pitch, and that the restriction of d_φ to quasiperiodic curves in \mathbb{R}^3 of fixed pitch is a distance on that space. Also,

$$d_{\mathcal{H}}(\Gamma, \gamma) \leq d_\varphi(\Gamma, \gamma),$$

where $d_{\mathcal{H}}$ is the Hausdorff distance.

In the sequel, $\gamma \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}^3)$ is a quasiperiodic curve of period $\ell > 0$ and pitch $a \in \mathbb{R}^3$, and Γ is a quasiperiodic curve of period $L > 0$ and with the same pitch as γ . We denote by r_γ the minimal radius of curvature of γ :

$$r_\gamma := \left(\max_{s \in [0, \ell]} |\gamma''(s)| \right)^{-1} \in (0, +\infty],$$

and by \mathcal{C}_γ the tubular neighborhood :

$$\mathcal{C}_\gamma := \{x \in \mathbb{R}^3, \text{ s.t. } d(x, \gamma) < r_\gamma/8\}.$$

LEMMA 2. – Let $s_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$ be such that $|x_0 - \gamma(s_0)| < r_\gamma/8$. There exists a unique $\xi_0 =: \xi(s_0, x_0) \in \mathbb{R}$ such that

1. $\xi_0 \in (s_0 - r_\gamma/4, s_0 + r_\gamma/4)$,
2. $(x_0 - \gamma(\xi_0)) \cdot \gamma'(\xi_0) = 0$.

Moreover, the function $\xi : \Xi_\gamma \rightarrow \mathbb{R}$, $(s, x) \mapsto \xi(s, x)$ defined on the open set

$$\Xi_\gamma := \{(s, x) \in \mathbb{R}^4 \text{ s.t. } |x - \gamma(s)| < r_\gamma/8\}$$

is of class \mathcal{C}^1 on Ξ_γ and for every $(s, x) \in \Xi_\gamma$ we have⁽¹⁰⁾

3. $(s + \ell, x + a) \in \Xi_\gamma$ and $\xi(s + \ell, x + a) = \xi(s, x) + \ell$,
4. $\partial_s \xi(s, x) = 0$,
5. $D_x \xi(s, x) = \left(1 + \frac{(x - \gamma(\xi(s, x))) \cdot \gamma''(\xi(s, x))}{1 - (x - \gamma(\xi(s, x))) \cdot \gamma''(\xi(s, x))} \right) \gamma'(\xi(s, x))$.

⁽⁹⁾ The P in d_φ stands for parametric.

⁽¹⁰⁾ One may believe at first sight that 3 and 4 are in contradiction if $a = 0$, but notice that the domain Ξ_γ does not contain the whole segment joining (s, x) and $(s + \ell, x)$.

COROLLARY 1. – Assume that $d_\varphi(\Gamma, \gamma) < r_\gamma/8$ and let $p \in \mathcal{P}(L, \ell)$ be such that

$$\sup_{s \in \mathbb{R}} |\Gamma(s) - \gamma(p(s))| < r_\gamma/8.$$

The function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$, $s \mapsto \sigma(s) := \xi(p(s), \Gamma(s))$ is the unique continuous function which satisfies, for any s in \mathbb{R} ,

1. $\sigma(s + L) = \sigma(s) + \ell$,
2. $|\Gamma(s) - \gamma(\sigma(s))| < r_\gamma/8$,
3. $(\Gamma(s) - \gamma(\sigma(s))) \cdot \gamma'(\sigma(s)) = 0$,
4. $|\sigma(s) - p(s)| < r_\gamma/4$.

Any continuous function σ that satisfies 1, 2 and 3 in Corollary 1 is called a reparametrization of γ for Γ . It also follows from Lemma 2 that

COROLLARY 2. – Assume that $d_\varphi(\Gamma, \gamma) < r_\gamma/8$. Any reparametrization σ of γ for Γ is a Lipschitz function on \mathbb{R} , and for almost every $s \in \mathbb{R}$ we have the formula

$$(18) \quad \sigma'(s) = \left(1 + \frac{(\Gamma(s) - \gamma(\sigma(s))) \cdot \gamma''(\sigma(s))}{1 - (\Gamma(s) - \gamma(\sigma(s))) \cdot \gamma''(\sigma(s))} \right) \Gamma'(s) \cdot \gamma'(\sigma(s)).$$

In particular,

$$(19) \quad \sigma'(s) \leq 1 + \frac{2}{r_\gamma} |\Gamma(s) - \gamma(\sigma(s))| \leq 1 + \frac{2}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty.$$

When each point x in \mathcal{C}_γ has a unique orthogonal projection $P(x)$ on γ (this can happen only if γ has no self-intersection), then there exists a unique reparametrization σ of γ for Γ (modulo a constant multiple of ℓ if $a = 0$), and it is determined by

$$\gamma(\sigma(s)) = P(\Gamma(s)).$$

We also have

LEMMA 3. – If σ is a reparametrization of γ for Γ , then

$$\sup_{s \in \mathbb{R}} |\sigma(s) - (s + \sigma(0))| \leq 2 \frac{L}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty + (L - \ell)_+,$$

where $(L - \ell)_+ := \max(L - \ell, 0)$.

We turn now to a different measure of the discrepancy between Γ and γ , which is better suited to our needs than d_φ .

DEFINITION 3. – For $r > 0$, we set

$$(20) \quad F_{\gamma,r}(\Gamma) := \inf_\sigma F_{\gamma,\sigma,r}(\Gamma) \equiv \inf_\sigma \int_0^L \mathcal{F}(\Gamma, \gamma, \sigma, r, s) ds,$$

where

$$\mathcal{F}(\Gamma, \gamma, \sigma, r, s) := 1 - f(|\Gamma(s) - \gamma(\sigma(s))|^2) \gamma'(\sigma(s)) \cdot \Gamma'(s),$$

the function $f \equiv f_r : [0, +\infty) \rightarrow [0, +\infty)$ is given by

$$f(d^2) := \begin{cases} 1 - \left(\frac{d}{r}\right)^2, & \text{for } 0 \leq d^2 \leq r^2, \\ 0, & \text{for } d^2 \geq r^2, \end{cases}$$

and the infimum in (20) is taken over all possible reparametrizations of γ for Γ .

It follows from Corollary 1 that $F_{\gamma,r}(\Gamma) < +\infty$ whenever $d_\varphi(\Gamma, \gamma) < r_\gamma/8$. On the other hand, since there does not exist any reparametrization of γ for Γ in that case, one has $F_{\gamma,r}(\Gamma) = +\infty$ whenever $d_\mathcal{H}(\Gamma, \gamma) \geq r_\gamma/8$. If $F_{\gamma,r}(\Gamma) < +\infty$ and each point x in \mathcal{C}_γ has a unique orthogonal projection $P(x)$ on γ , one can rewrite $F_{\gamma,r}(\Gamma)$ in the slightly more geometric form

$$F_{\gamma,r}(\Gamma) = \int_\Gamma 1 - f(d^2(x, \gamma)) \tau_\gamma(P(x)) \cdot \tau_\Gamma(x) d\mathcal{H}^1(x),$$

where τ_Γ and τ_γ denote the oriented tangent vectors to Γ and γ respectively. Notice finally that $0 \leq f \leq 1$ and that $f(d^2) = 1$ if and only if $d^2 = 0$. In particular, the integrand in (20) is non negative and moreover

$$\mathcal{F}(\Gamma, \gamma, \sigma, r, s) = 0 \iff (\Gamma(s) = \gamma(\sigma(s)) \text{ and } \Gamma'(s) = \gamma'(\sigma(s))).$$

The functional $F_{\gamma,r}$, which plays a central role in our analysis, can be related to more standard discrepancy measures, like the distances $d_\mathcal{H}$ or d_φ between the curves, or the L^2 distance between the parametrizations of their tangents. We devote the remaining part of this section to such relations, which we state one after the other before actually proving them.

LEMMA 4. – *If σ is a reparametrization of γ for Γ , then for $r \geq \|\Gamma - \gamma \circ \sigma\|_\infty$*

$$\|\Gamma - \gamma \circ \sigma\|_\infty^2 \leq \left(\sqrt{2}r + \frac{r^2}{L}\right) F_{\gamma,\sigma,r}(\Gamma).$$

In particular,

$$d_\varphi^2(\Gamma, \gamma) \leq \left(\sqrt{2}r + \frac{r^2}{L}\right) F_{\gamma,r}(\Gamma).$$

Combining Lemma 3 and Lemma 4 we will obtain

LEMMA 5. – *If σ is a reparametrization of γ for Γ , then for $r \geq \|\Gamma - \gamma \circ \sigma\|_\infty$*

$$\int_0^L |\Gamma'(s) - \gamma'(\frac{\ell}{L}s + \sigma(0))|^2 ds \leq \left(4 + 16 \frac{L^3}{r^4} \left(\sqrt{2}r + \frac{r^2}{L}\right)\right) F_{\gamma,\sigma,r}(\Gamma) + 16 \frac{L}{r^2} |L - \ell|^2.$$

In particular, if $L = \ell$,

$$\int_0^\ell |\Gamma'(s) - \gamma'(s + \sigma(0))|^2 ds \leq \left(4 + 16 \frac{\ell^3}{r^4} \left(\sqrt{2}r + \frac{r^2}{\ell}\right)\right) F_{\gamma,\sigma,r}(\Gamma).$$

Combining Corollary 1, the Cauchy-Schwarz inequality and Lemma 5 we will obtain

LEMMA 6. – *Assume that*

$$(21) \quad \Gamma(0) = \gamma(0) \quad \text{and} \quad \sqrt{L} \left(\int_0^L |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)|^2 ds \right)^{1/2} + |L - \ell| < r_\gamma/8.$$

Then $d_\varphi(\Gamma, \gamma) < r_\gamma/8$, there exists a unique reparametrization σ of γ for Γ that satisfies $\sigma(0) = 0$, and we have, for any $r > 0$,

$$F_{\gamma,\sigma,r}(\Gamma) \leq \left(1 + 2 \frac{L^2}{r^2} + 16 \frac{L^4}{r^4}\right) \int_0^L |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)|^2 ds + \left(\frac{2}{r^2} + \frac{8}{r^2} + 16 \frac{L^2}{r^4}\right) L |L - \ell|^2.$$

To complete this section, we now provide the proofs of Lemmas 2 to 6 and Corollaries 1 and 2.

Proof of Lemma 2. – If $r_\gamma = +\infty$, then γ is a straight line and the conclusion is straightforward. We assume next that $r_\gamma \in (0, +\infty)$.

Step 1: Existence. – If $(x_0 - \gamma(s_0)) \cdot \gamma'(s_0) = 0$, it suffices to choose $\xi_0 = s_0$. If not, assume that $(x_0 - \gamma(s_0)) \cdot \gamma'(s_0) > 0$, the other case being treated in a very similar manner. Define $h : \mathbb{R} \rightarrow [0, +\infty)$ by $h(z) := |\gamma(z) - x_0|$. For $z \in \mathbb{R}$ verifying $h(z) \neq 0$, we compute

$$h'(z) = \gamma'(z) \cdot \frac{\gamma(z) - x_0}{|\gamma(z) - x_0|}$$

and

$$(22) \quad h''(z) = \gamma''(z) \cdot \frac{\gamma(z) - x_0}{|\gamma(z) - x_0|} + \frac{1}{|\gamma(z) - x_0|} - \frac{((\gamma(z) - x_0) \cdot \gamma'(z))^2}{|\gamma(z) - x_0|^3}.$$

By assumption, we have $h(s_0) > 0$ and $h'(s_0) < 0$. If $h(s_1) = 0$ for some $s_1 \in [s_0, s_0 + r_\gamma/4]$, it suffices to choose $\xi_0 := s_1$. Assume next that $h(z) \neq 0$ for all $z \in [s_0, s_0 + r_\gamma/4]$. If $h'(z) < -1/2$ for all $z \in [s_0, s_0 + r_\gamma/4]$, then we would have $h(s_0 + r_\gamma/4) < h(s_0) - r_\gamma/8 < 0$, which is impossible. Let therefore $s_2 := \min\{z \geq s_0, h'(z) \geq -1/2\} < s_0 + r_\gamma/4$. We have $h(s_2) \leq h(s_0) - \frac{1}{2}(s_2 - s_0)$, and $h'(s_2) = -\frac{1}{2}$ if $s_2 \neq s_0$ or $-\frac{1}{2} \leq h'(s_2) < 0$ if $s_2 = s_0$. We claim that there exists $s_3 \in [s_2, s_0 + r_\gamma/4]$ such that $h'(s_3) = 0$, in which case it suffices to choose $\xi_0 := s_3$. Assume by contradiction that $h'(z) \neq 0$ for any $z \in [s_2, s_0 + r_\gamma/4]$, so that $h'(z) < 0$ for all $z \in [s_2, s_0 + r_\gamma/4]$, and in particular $h(z) \leq h(s_2) < r_\gamma/8$, for all $z \in [s_2, s_0 + r_\gamma/4]$. For $z \in [s_2, s_0 + r_\gamma/4]$ such that $h'(z) \geq -\frac{1}{2}$, since we also have by assumption $h'(z) < 0$, we infer from (22) and the definition of r_γ that

$$(23) \quad h''(z) \geq -\frac{1}{r_\gamma} + \frac{1}{h(z)} - \frac{1}{4h(z)} > \frac{1}{2h(z)} \geq \frac{1}{2h(s_2)} > 0.$$

Since $h'(s_2) \geq -1/2$, (23) first implies that in fact $h'(z) \geq -\frac{1}{2}$ for all $z \in [s_2, s_0 + r_\gamma/4]$, and then that

$$h'(s_0 + r_\gamma/4) > h'(s_2) + \frac{1}{2h(s_2)}(s_0 + r_\gamma/4 - s_2) \geq -\frac{1}{2} + \frac{s_0 + r_\gamma/4 - s_2}{2(h(s_0) - (s_2 - s_0)/2)} > 0,$$

since $h(s_0) < r_\gamma/8$. This is not compatible with the fact that $h'(z) < 0$ for all $z \in [s_2, s_0 + r_\gamma/4]$, and we therefore have reached our contradiction.

Step 2: Uniqueness. – Let $s \in (s_0 - r_\gamma/4, s_0 + r_\gamma/4)$ be such that

$$(24) \quad (x_0 - \gamma(s)) \cdot \gamma'(s) = 0.$$

On the one hand, if $h(s) \neq 0$, then $h'(s) = 0$, and therefore from (22)

$$h''(s) = \gamma''(s) \cdot \frac{\gamma(s) - x_0}{|\gamma(s) - x_0|} + \frac{1}{|\gamma(s) - x_0|} \geq -\frac{1}{r_\gamma} + \frac{2}{r_\gamma} > 0.$$

On the other hand, if $h(s) = 0$ then $\lim_{z \rightarrow s, z > s} h'(z) = 1$. In both cases, we infer that $h'(z) > 0$ for all $z > s$ in a sufficiently small neighborhood of s .

Let $s^+ := \sup\{u > s, h'(z) > 0 \text{ for } z \in (s, u)\} > s$. If $s^+ \neq +\infty$, then we must have $h'(s^+) = 0$ and $h''(s^+) \leq 0$. Therefore

$$0 \geq h''(s^+) = \gamma''(s^+) \cdot \frac{\gamma(s^+) - x_0}{|\gamma(s^+) - x_0|} + \frac{1}{|\gamma(s^+) - x_0|} \geq -\frac{1}{r_\gamma} + \frac{1}{|\gamma(s^+) - x_0|},$$

so that $|\gamma(s^+) - x_0| \geq r_\gamma$. Since $s \in (s_0 - r_\gamma/4, s_0 + r_\gamma/4)$, since γ is parametrized by arc-length, and since $|\gamma(s_0) - x_0| < r_\gamma/8$, we first infer that $|\gamma(s) - x_0| < 3r_\gamma/8$, and next that $s^+ \geq s + 5r_\gamma/8$. Hence $h(z) > h(s)$ for any $z \in (s, s + 5r_\gamma/8)$. A parallel argument leads to the conclusion that $h(z) > h(s)$ for any $z \in (s - 5r_\gamma/8, s)$. If $\tilde{s} \in (s_0 - r_\gamma/4, s_0 + r_\gamma/4)$ were such that $\tilde{s} \neq s$ and \tilde{s} is a solution to (24) with s replaced by \tilde{s} , then we would infer, since then $|\tilde{s} - s| < r_\gamma/2 < 5r_\gamma/8$, that $h(\tilde{s}) > h(s)$ and $h(s) > h(\tilde{s})$, which is not possible.

Step 3: Regularity and periodicity. – In Step 1 and Step 2, we have constructed a function ξ that satisfies and is uniquely defined by 1 and 2 for any $s \in \mathbb{R}$. Condition 3 also follows from the quasiperiodicity assumption on γ and the uniqueness of Step 2. Concerning regularity, let $(s_0, x_0) \in \Xi_\gamma$: The function

$$\psi : \Xi_\gamma \times \mathbb{R} \rightarrow \mathbb{R}, (s, x, z) \mapsto (x - \gamma(z)) \cdot \gamma'(z)$$

vanishes at $(s, x, z) = (s_0, x_0, \xi(s_0, x_0))$ and we have

$$\partial_z \psi(s_0, x_0, \xi(s_0, x_0)) = -|\gamma'(\xi(s_0, x_0))|^2 + (x_0 - \gamma(\xi(s_0, x_0))) \cdot \gamma''(\xi(s_0, x_0)) \leq -1 + \frac{r_\gamma}{8} \frac{1}{r_\gamma} < 0.$$

It follows from the implicit function theorem that there exists a continuously differentiable function $\tilde{\xi}$, defined in a neighborhood of (s_0, x_0) in Ξ_γ such that $\psi(s, x, \tilde{\xi}(s, x)) \equiv 0$. From Step 2, we infer that $\tilde{\xi} = \xi$. The implicit function theorem also yields, differentiating the relation

$$(x - \gamma(\xi(s, x))) \cdot \gamma'(\xi(s, x)) = 0$$

with respect to s and x respectively, the Formulas 4 and 5. \square

Proof of Corollaries 1 and 2. – Corollary 1 is straightforward. For the regularity stated in Corollary 2, notice that if σ is any reparametrization of γ for Γ then for all $s \in \mathbb{R}$ we have $\sigma(s) = \xi(\sigma(s), \Gamma(s))$. Since the function ξ is locally constant in s and continuously differentiable, and since Γ is Lipschitzian, it follows that σ is Lipschitzian. Formula (18) is then a direct consequence of 5 in Lemma 2 and the chain rule, and inequality (19) follows from (18), 2 in Corollary 1 and the definition of r_γ . \square

Proof of Lemma 3. – From (19), we deduce in particular that for $s \in [0, L]$,

$$(25) \quad \sigma(s) \leq \sigma(0) + s + \frac{2}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty s,$$

and

$$(26) \quad \sigma(L) \leq \sigma(s) + (L - s) + \frac{2}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty (L - s).$$

By assumption on σ , we have $\sigma(L) = \sigma(0) + \ell$, and therefore we deduce from (26) that

$$(27) \quad \sigma(s) \geq \sigma(0) + s + (\ell - L) - \frac{2}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty (L - s).$$

Combining (25) and (27) the conclusion follows. \square

Proof of Lemma 4. – For almost every $s \in \mathbb{R}$ such that $\Gamma'(s) \cdot \gamma'(\sigma(s)) \geq 0$, we have

$$\begin{aligned} \mathcal{F}(\Gamma, \gamma, \sigma, r, s) &\geq 1 - \Gamma'(s) \cdot \gamma'(\sigma(s)) \\ &= \frac{|\Gamma'(s)|^2}{2} + \frac{|\gamma'(\sigma(s))|^2}{2} - \Gamma'(s) \cdot \gamma'(\sigma(s)) \\ &= \frac{1}{2} |\Gamma'(s) - \gamma'(\sigma(s))|^2, \end{aligned}$$

while for almost every $s \in \mathbb{R}$ such that $\Gamma'(s) \cdot \gamma'(\sigma(s)) < 0$, we have

$$\mathcal{F}(\Gamma, \gamma, \sigma, r, s) \geq 1 \geq \frac{1}{2} |\Gamma'(s) - \gamma'(\sigma(s))|^2.$$

It follows in particular from the Pythagorean theorem that for almost every $s \in \mathbb{R}$,

$$(28) \quad \mathcal{F}(\Gamma, \gamma, \sigma, r, s) \geq \frac{1}{2} |\Gamma'(s) - \gamma'(\sigma(s))|^2 \geq \frac{1}{2} |Q_s(\Gamma'(s))|^2,$$

where Q_s denotes the orthogonal projection onto the plane $[\gamma'(\sigma(s))]^\perp$. For any $s \in \mathbb{R}$, we also have, since $r \geq \|\Gamma - \gamma \circ \sigma\|_\infty$,

$$(29) \quad \mathcal{F}(\Gamma, \gamma, \sigma, r, s) \geq 1 - f(|\Gamma(s) - \gamma(\sigma(s))|^2) \geq \frac{1}{r^2} |\Gamma(s) - \gamma(\sigma(s))|^2.$$

Combining (28) and (29), we are led, for almost every $s \in \mathbb{R}$, to

$$(30) \quad \mathcal{F}(\Gamma, \gamma, \sigma, r, s) \geq \frac{1}{\sqrt{2}r} |Q_s(\Gamma'(s))| |\Gamma(s) - \gamma(\sigma(s))| \geq \frac{1}{2\sqrt{2}r} |\partial_s |\Gamma(s) - \gamma(\sigma(s))|^2|.$$

Taking into account the L -periodicity of the map $s \mapsto \Gamma(s) - \gamma(\sigma(s))$, integration of (30) yields

$$(31) \quad \text{osc} \left(|\Gamma(\cdot) - \gamma \circ \sigma(\cdot)|^2 \right) \leq \sqrt{2}r F_{\gamma, \sigma, r}(\Gamma).$$

Integration of (29) and the Tchebichev inequality yield

$$(32) \quad \inf_{s \in \mathbb{R}} |\Gamma(s) - \gamma(\sigma(s))|^2 \leq \frac{r^2}{L} F_{\gamma, \sigma, r}(\Gamma),$$

and combining (31) and (32) we therefore obtain

$$\sup_{s \in \mathbb{R}} |\Gamma(s) - \gamma(\sigma(s))|^2 \leq \left(\sqrt{2}r + \frac{r^2}{L} \right) F_{\gamma, \sigma, r}(\Gamma),$$

which corresponds to the first claim.

For the second claim, notice that if $F_{\gamma, r}(\Gamma) = +\infty$ there is nothing to prove, while if $F_{\gamma, r}(\Gamma) < +\infty$ it suffices to use the first claim and then to minimize over all the possible reparametrizations of γ for Γ . □

Proof of Lemma 5. – Integration of (28) for $s \in [0, L]$ yields

$$\int_0^L |\Gamma'(s) - \gamma'(\sigma(s))|^2 ds \leq 2F_{\gamma, \sigma, r}(\Gamma),$$

so that

$$(33) \quad \int_0^L |\Gamma'(s) - \gamma'(\sigma(0) + \frac{\ell}{L}s)|^2 ds \leq 4F_{\gamma, \sigma, r}(\Gamma) + 2 \int_0^L |\gamma'(\sigma(s)) - \gamma'(\sigma(0) + \frac{\ell}{L}s)|^2 ds.$$

By Lemma 3, we deduce

$$\sup_{s \in [0, L]} |\sigma(s) - (\sigma(0) + \frac{\ell}{L}s)| \leq \frac{2L}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty + 2|L - \ell|,$$

so that

$$(34) \quad \int_0^L |\gamma'(\sigma(s)) - \gamma'(\sigma(0) + \frac{\ell}{L}s)|^2 ds \leq L \left(\frac{2L}{r_\gamma} \|\Gamma - \gamma \circ \sigma\|_\infty + 2|L - \ell| \right)^2 \|\gamma''\|_\infty^2.$$

By Lemma 4 and replacing $\|\gamma''\|_\infty$ by $1/r_\gamma$, we deduce from (34)

$$(35) \quad \int_0^L |\gamma'(\sigma(s)) - \gamma'(\sigma(0) + \frac{\ell}{L}s)|^2 ds \leq \frac{8L}{r_\gamma^2} \left(\frac{L^2}{r_\gamma^2} \left(\sqrt{2}r + \frac{r^2}{L} \right) F_{\gamma, \sigma, r}(\Gamma) + |L - \ell|^2 \right).$$

Combining (33) and (35) the conclusion follows. \square

Proof of Lemma 6. – Set

$$G := \int_0^L |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)|^2 ds.$$

By Cauchy-Schwarz inequality and (21), we have, for any $\tau \in [0, L]$,

$$\begin{aligned} |\Gamma(\tau) - \gamma(\frac{\ell}{L}\tau)| &\leq \int_0^\tau |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)| ds + |1 - \frac{\ell}{L}| \int_0^\tau |\gamma'(\frac{\ell}{L}s)| ds \\ &\leq \sqrt{LG} + |L - \ell| < r_\gamma/8. \end{aligned}$$

Hence $d_\varphi(\Gamma, \gamma) \leq \sqrt{LG} + |L - \ell| < r_\gamma/8$. Moreover, the unique reparametrization σ of γ for Γ given by Corollary 1 for the choice $p(s) = Ls/\ell$ satisfies $\sigma(0) = 0$, and we have

$$(36) \quad \|\Gamma - \gamma \circ \sigma\|_\infty \leq \sqrt{LG} + |L - \ell|.$$

From Lemma 3 and (36), we obtain

$$(37) \quad \sup_{s \in [0, L]} |\sigma(s) - s| \leq \frac{2L}{r_\gamma} \left(\sqrt{LG} + |L - \ell| \right) + (L - \ell)_+.$$

We write, for almost every $s \in [0, L]$,

$$\begin{aligned} \mathcal{F}(\Gamma, \gamma, \sigma, r, s) &= 1 - \Gamma'(s) \cdot \gamma'(\sigma(s)) + (1 - f(|\Gamma(s) - \gamma(\sigma(s))|^2)) \Gamma'(s) \cdot \gamma'(\sigma(s)) \\ &\leq \frac{1}{2} |\Gamma'(s) - \gamma'(\sigma(s))|^2 + \frac{1}{r^2} |\Gamma(s) - \gamma(\sigma(s))|^2 \\ (38) \quad &\leq |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)|^2 + |\gamma'(\sigma(s)) - \gamma'(\frac{\ell}{L}s)|^2 + \frac{1}{r^2} \|\Gamma - \gamma \circ \sigma\|_\infty^2 \\ &=: |\Gamma'(s) - \gamma'(\frac{\ell}{L}s)|^2 + R. \end{aligned}$$

By (36) and (37), we infer that

$$\begin{aligned} R &\leq \left(\sup_{\tau \in [0, L]} |\sigma(\tau) - \frac{\ell}{L}\tau| \right)^2 \|\gamma''\|_\infty^2 + \frac{1}{r^2} (\sqrt{LG} + |L - \ell|)^2 \\ (39) \quad &\leq \frac{1}{r_\gamma^2} \left(\frac{2L}{r_\gamma} (\sqrt{LG} + |L - \ell|) + 2|L - \ell| \right)^2 + \frac{1}{r^2} (\sqrt{LG} + |L - \ell|)^2 \\ &\leq \left(\frac{2}{r^2} + 16 \frac{L^2}{r_\gamma^4} \right) LG + \left(\frac{2}{r^2} + \frac{8}{r_\gamma^2} + 16 \frac{L^2}{r_\gamma^4} \right) |L - \ell|^2. \end{aligned}$$

Integrating (38) for $s \in [0, L]$ and using (39), we are led to

$$F_{\gamma, \sigma, r}(\Gamma) \leq \left(1 + 2 \frac{L^2}{r^2} + 16 \frac{L^4}{r_\gamma^4} \right) G + \left(\frac{2}{r^2} + \frac{8}{r_\gamma^2} + 16 \frac{L^2}{r_\gamma^4} \right) L |L - \ell|^2,$$

and the proof is complete. □

4. Integral Formula for parametrized binormal curvature flows

The next proposition is the second main ingredient in our proof of Theorem 1. A very similar result⁽¹¹⁾ was obtained ten years ago by the first author in [10], where it was also used to define a weak notion of binormal curvature flow for objects that are only rectifiable currents of co-dimension 2 (notice that (43) only involves first order derivatives of γ). In [11] we provide an existence theory for a notion of binormal curvature flow related to the one introduced in [10], and also strongly based on Formula (43).

PROPOSITION 1. – *Let $I \subset \mathbb{R}$ be some open interval and let $\Gamma \in L^\infty(I, H_{loc}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}^3))$ be a solution of the binormal curvature flow equation on $I \times \mathbb{R}$. Assume moreover that Γ is L -quasiperiodic in the sense that*

$$(40) \quad \Gamma(t, s + L) = \Gamma(t, s) + a, \quad \forall t \in I, \forall s \in \mathbb{R},$$

for some period $L > 0$ and pitch $a \in \mathbb{R}^3$.

Then for any parametrized vector field $X \in \mathcal{C}^1(I \times \mathbb{R}, \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3))$ such that

$$(41) \quad X(t, s + L, x + a) = X(t, s, x) \quad \forall t \in I, s \in \mathbb{R}, \text{ and } x \in \mathbb{R}^3,$$

$$(42) \quad \partial_s X(t, s, \Gamma(t, s)) = \partial_{s_i} X(t, s, \Gamma(t, s)) = 0 \quad \forall i \in \{1, 2, 3\}, t \in I, \text{ and } s \in \mathbb{R},$$

we have

$$(43) \quad \frac{d}{dt} \int_0^L (X \circ \Gamma) \cdot \partial_s \Gamma \, ds = \int_0^L (\partial_t X \circ \Gamma) \cdot \partial_s \Gamma - (D \operatorname{curl} X \circ \Gamma) : (\partial_s \Gamma \otimes \partial_s \Gamma) \, ds$$

in the sense of distributions on I .

In (43), by abuse of notations we have used the expression $X \circ \Gamma$ to denote the function $(t, s) \mapsto X(t, s, \Gamma(t, s))$, and similarly for $\partial_s X \circ \Gamma$ and $D \operatorname{curl} X \circ \Gamma$, where D and curl apply to the x variable only.

Proof. – *Step 1.* – Assume that Γ is a smooth solution. We then write in coordinates, and with the same abuse of notations as the one mentioned here above,

$$\begin{aligned} & \frac{d}{dt} \int_0^L (X \circ \Gamma) \cdot \partial_s \Gamma \, ds - \int_0^L (\partial_t X \circ \Gamma) \cdot \partial_s \Gamma \, ds \\ &= \int_0^L \sum_{i,j=1}^3 (\partial_i X^j \circ \Gamma) \partial_t \Gamma^i \partial_s \Gamma^j \, ds + \int_0^L \sum_{j=1}^3 (X^j \circ \Gamma) \partial_{st} \Gamma^j \, ds \\ &= \int_0^L \sum_{i,j=1}^3 (\partial_i X^j \circ \Gamma) (\partial_t \Gamma^i \partial_s \Gamma^j - \partial_s \Gamma^i \partial_t \Gamma^j) \, ds - \int_0^L (\partial_s X \circ \Gamma) \cdot \partial_t \Gamma \, ds + \left[(X \circ \Gamma) \cdot \partial_t \Gamma \right]_0^L \\ &= \int_0^L \sum_{i,j=1}^3 (\partial_i X^j \circ \Gamma) (\partial_t \Gamma^i \partial_s \Gamma^j - \partial_s \Gamma^i \partial_t \Gamma^j) \, ds, \end{aligned}$$

⁽¹¹⁾ Without an s dependence for the vector field X but valid for binormal curvature flows of co-dimension 2 in arbitrary dimension $N \geq 3$.

where we have used (40), (41) and (42) for the last identity. By definition of the vector product,

$$(\partial_t \Gamma^i \partial_s \Gamma^j - \partial_t \Gamma^j \partial_s \Gamma^i) = \sum_{k=1}^3 \epsilon_{ijk} (\partial_t \Gamma \times \partial_s \Gamma)^k$$

where ϵ_{ijk} is the Levi-Civita symbol, therefore

$$(44) \quad \frac{d}{dt} \int_0^L (X \circ \Gamma) \cdot \partial_s \Gamma \, ds = \int_0^L (\partial_t X \circ \Gamma) \cdot \partial_s \Gamma \, ds + \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma) (\partial_t \Gamma \times \partial_s \Gamma)^k \, ds.$$

Next, we also write in coordinates

$$\int_0^L (D \operatorname{curl} X \circ \Gamma) : (\partial_s \Gamma \otimes \partial_s \Gamma) \, ds = \int_0^L \sum_{k,l=1}^3 (\partial_l (\operatorname{curl} X)^k) \circ \Gamma \partial_s \Gamma^l \partial_s \Gamma^k \, ds.$$

By definition of the rotational and the chain rule,

$$\begin{aligned} \int_0^L \sum_{k,l=1}^3 \partial_l (\operatorname{curl} X)^k \circ \Gamma \partial_s \Gamma^l \partial_s \Gamma^k \, ds &= \sum_{i,j,k,l=1}^3 \epsilon_{ijk} \int_0^L (\partial_{il} X^j \circ \Gamma) \partial_s \Gamma^l \partial_s \Gamma^k \, ds \\ &= \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_s (\partial_i X^j \circ \Gamma) - \partial_{si} X^j \circ \Gamma) \partial_s \Gamma^k \, ds \\ &= \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L \partial_s (\partial_i X^j \circ \Gamma) \partial_s \Gamma^k \, ds \end{aligned}$$

where we have used (42) for the last identity. Integration by parts and (40), (41) therefore yield

$$(45) \quad \int_0^L (D \operatorname{curl} X \circ \Gamma) : (\partial_s \Gamma \otimes \partial_s \Gamma) \, ds = - \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma) \partial_{ss} \Gamma^k \, ds.$$

Combining (44) and (45), and noticing that $\partial_t \Gamma \times \partial_s \Gamma = \partial_{ss} \Gamma$ whenever Γ solves the binormal curvature flow equation, the conclusion follows.

Step 2. – The general case. We consider a family of space-time radially-symmetric mollifiers $(\rho_\varepsilon)_{\varepsilon>0}$ and define the smooth functions $\Gamma_\varepsilon := \rho_\varepsilon * \Gamma$. Even though the Γ_ε are no longer solutions of the binormal curvature flow equation, we may reproduce the computations of Step 1 for Γ_ε up to before the conclusion, so as to obtain

$$(46) \quad \frac{d}{dt} \int_0^L (X \circ \Gamma_\varepsilon) \cdot \partial_s \Gamma_\varepsilon \, ds = \int_0^L (\partial_t X \circ \Gamma_\varepsilon) \cdot \partial_s \Gamma_\varepsilon \, ds - \int_0^L (D \operatorname{curl} X \circ \Gamma_\varepsilon) : (\partial_s \Gamma_\varepsilon \otimes \partial_s \Gamma_\varepsilon) \, ds + R_\varepsilon$$

where

$$R_\varepsilon := \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) (\partial_t \Gamma_\varepsilon \times \partial_s \Gamma_\varepsilon - \partial_{ss} \Gamma_\varepsilon)^k \, ds.$$

Using the (local in time) uniform convergence of Γ_ε to Γ , and the strong convergence in (e.g.) $\mathcal{C}_{\text{loc}}(I, L_{\text{loc}}^2(\mathbb{R}, \mathbb{R}^3))$ of $\partial_s \Gamma_\varepsilon$ to $\partial_s \Gamma$, we can pass to the limit in the left-hand side of (46) in the sense of distributions on I , as well as in the first and second terms on the right-hand side of (46) (pointwise in that case), and get the equivalent terms where Γ_ε is replaced by Γ . To conclude, we need therefore to prove that R_ε converges to zero in the sense of distributions

on I , as $\varepsilon \rightarrow 0$. First notice that since $\partial_{ss}\Gamma_\varepsilon$ converges in (e.g.) $\mathcal{C}_{\text{loc}}(I, H_{\text{loc}}^{-1}(\mathbb{R}, \mathbb{R}^3))$ to $\partial_{ss}\Gamma$ and since $\partial_i X^j \circ \Gamma_\varepsilon$ converges in (e.g.) $\mathcal{C}_{\text{loc}}(I, H_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^3))$ to $\partial_i X^j \circ \Gamma$ (indeed $\partial_i X^j$ is Lipschitzian), we have

$$(47) \quad \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) \partial_{ss} \Gamma_\varepsilon^k ds \xrightarrow{\varepsilon \rightarrow 0} \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) \partial_{ss} \Gamma^k ds,$$

in the sense of distributions on I . We will next show that in the sense of distributions on I ,

$$(48) \quad \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) (\partial_t \Gamma_\varepsilon \times \partial_s \Gamma_\varepsilon)^k ds \xrightarrow{\varepsilon \rightarrow 0} \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) (\partial_t \Gamma \times \partial_s \Gamma)^k ds,$$

after we first establish that the right hand side of (48) is indeed a distribution on I . Our argument is based on the fact that

$$\partial_s \Gamma \in L^\infty(I, H^{1/2}(T^1, \mathbb{R}^3)) \cap L^\infty(I \times T^1) = L^\infty(I, H^{1/2}(T^1, \mathbb{R}^3) \cap L^\infty(T^1, \mathbb{R}^3)) =: L^\infty(I, Y),$$

and on the algebra properties of this last space. Since $\partial_{ss}\Gamma \in L^\infty(I, H^{-1/2}(T^1, \mathbb{R}^3)) \subseteq L^\infty(Y^*)$, we firstly infer that

$$\partial_t \Gamma = \partial_s \Gamma \times \partial_{ss} \Gamma \in L^\infty(I, Y^*),$$

and then secondly that

$$\partial_t \Gamma \times \partial_s \Gamma \in L^\infty(I, Y^*),$$

the latter being therefore a well-defined distribution.

Let now $m \in \mathcal{D}(I, \mathbb{R})$ be arbitrary, and set

$$\chi_\varepsilon(t, s) := (\partial_i X^j \circ \Gamma_\varepsilon)(t, s) m(t) \in \mathcal{C}(I, H^{3/2}(T^1, \mathbb{R})).$$

For $p, q \in \{1, 2, 3\}$, we write

$$(49) \quad \partial_t \Gamma_\varepsilon^p \partial_s \Gamma_\varepsilon^q = \partial_t \Gamma^p \partial_s \Gamma^q + \partial_t (\Gamma_\varepsilon^p - \Gamma^p) \partial_s \Gamma_\varepsilon^q + \partial_t \Gamma^p \partial_s (\Gamma_\varepsilon - \Gamma)^q.$$

Denoting by $\langle \cdot, \cdot \rangle$ the $L^2(I, Y^*) - L^2(I, Y)$ duality, we have

$$(50) \quad \langle \partial_t (\Gamma_\varepsilon^p - \Gamma^p) \partial_s \Gamma_\varepsilon^q, \chi_\varepsilon \rangle = \langle \chi_\varepsilon \partial_t (\Gamma_\varepsilon^p - \Gamma^p), \partial_s \Gamma_\varepsilon^q \rangle = -\langle \rho_\varepsilon * (\chi_\varepsilon \partial_t (\Gamma_\varepsilon^p - \Gamma^p)), \partial_s \Gamma^q \rangle.$$

The family $(\rho_\varepsilon * (\chi_\varepsilon \partial_t (\Gamma_\varepsilon^p - \Gamma^p)))_\varepsilon$ being bounded in the dual Banach space $L^2(I, Y^*)$, the Banach-Alaoglu theorem yields a subsequence that converges weakly-* to some limit in $L^2(I, Y^*)$. Since it also converges to 0 in the sense of distributions on $I \times T^1$ (recall that χ_ε is strongly convergent in $\mathcal{C}(I, H^s(T^1, \mathbb{R}))$ for any $s < 3/2$), that limit must be zero. Hence, $m \in \mathcal{D}(I, \mathbb{R})$ being arbitrary, passing to the limit in the last term of (50) we deduce that

$$(51) \quad \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) \partial_t (\Gamma_\varepsilon^p - \Gamma^p) \partial_s \Gamma_\varepsilon^q ds \rightarrow 0$$

in the sense of distributions on I . Similarly we write

$$\langle \partial_t \Gamma^p \partial_s (\Gamma_\varepsilon^q - \Gamma^q), \chi_\varepsilon \rangle = \langle \partial_t \Gamma^p \chi_\varepsilon, \partial_s (\Gamma_\varepsilon^q - \Gamma^q) \rangle = -\langle (\text{Id} - \rho_\varepsilon *) (\partial_t \Gamma^p \chi_\varepsilon), \partial_s \Gamma^q \rangle,$$

and we deduce as above that

$$(52) \quad \int_0^L (\partial_i X^j \circ \Gamma_\varepsilon) \partial_t \Gamma^p \partial_s (\Gamma_\varepsilon^q - \Gamma^q) ds \rightarrow 0$$

in the sense of distributions on I . Combining (51) and (52) for various i, j, p and q in $\{1, 2, 3\}$ according to (49), we obtain (48). Combining (48) and (47) we are led to

$$R_\varepsilon \rightarrow \sum_{i,j,k=1}^3 \epsilon_{ijk} \int_0^L (\partial_i X^j \circ \Gamma) (\partial_t \Gamma \times \partial_s \Gamma - \partial_{ss} \Gamma)^k ds = 0,$$

and the proof is complete. \square

5. The heart of the matter : A pointwise estimate

In this section we present the key estimate that will allow us to control the growth of the discrepancy between a smooth solution γ and a less smooth solution Γ of the binormal curvature flow Equation (2). Nearly the same estimate plays an important role in a somewhat different setting in [11], where we study a class of non-parametrized weak solutions of the binormal curvature flow. For the sake of completeness, and because it is reasonably short, we present the proof both here and in [11], in full (and nearly identical) detail.

Let $(t_0, s_0) \in \mathbb{R}^2$ and $\gamma : \mathcal{O} \rightarrow \mathbb{R}^3$, where \mathcal{O} is some open neighborhood of (t_0, s_0) . Assume that $\partial_{ts}\gamma$ and $\partial_{sss}\gamma$ are continuous in \mathcal{O} , and that for any $(t, s) \in \mathcal{O}$,

$$(53) \quad |\partial_s \gamma(t, s)| = 1.$$

Let $x_0 \in \mathbb{R}^3$ be such that

$$(54) \quad |x_0 - \gamma(t_0, s_0)| |\partial_{ss}\gamma(t_0, s_0)| < 1/2$$

and

$$(55) \quad (x_0 - \gamma(t_0, s_0)) \cdot \partial_s \gamma(t_0, s_0) = 0.$$

The mapping $\Psi : \mathbb{R}^3 \times \mathcal{O} \rightarrow \mathbb{R}$

$$(x, t, s) \mapsto (x - \gamma(t, s)) \cdot \partial_s \gamma(t, s)$$

satisfies $\Psi(x_0, t_0, s_0) = 0$ and

$$(56) \quad \partial_s \Psi(x_0, t_0, s_0) = -|\partial_s \gamma(t_0, s_0)|^2 + (x_0 - \gamma(t_0, s_0)) \cdot \partial_{ss}\gamma(t_0, s_0) < -1/2,$$

where we have used (53) and (54) for the last inequality. From the implicit function theorem, we infer that there exist an open neighborhood \mathcal{U} of (x_0, t_0) in \mathbb{R}^4 , and a function $\zeta : \mathcal{U} \rightarrow \mathbb{R}$ such that

$$\Psi(x, t, \zeta(x, t)) = 0 \quad \forall (x, t) \in \mathcal{U}.$$

Moreover $\partial_t \zeta$ and $\partial_{x_i x_j} \zeta$ exist and are continuous in \mathcal{U} , for every $i, j \in \{1, 2, 3\}$.

We fix $r > 0$ and define the vector field $X \equiv X_{\gamma, r}$ on \mathcal{U} by

$$(57) \quad X(x, t) = f\left(|x - \gamma(t, \zeta(x, t))|^2\right) \partial_s \gamma(t, \zeta(x, t)),$$

where the function $f \equiv f_r : [0, +\infty) \rightarrow [0, +\infty)$ is given by

$$f(d^2) := \begin{cases} 1 - \left(\frac{d}{r}\right)^2, & \text{for } 0 \leq d^2 \leq r^2, \\ 0, & \text{for } d^2 \geq r^2. \end{cases}$$

The following pointwise estimate is the core of our analysis. It allows to control the terms which appear on the right-hand side of the identity in Proposition 1.

PROPOSITION 2. – Assume that

$$(58) \quad (\partial_t \gamma - \partial_s \gamma \times \partial_{ss} \gamma)(t_0, s_0) = \partial_s (\partial_t \gamma - \partial_s \gamma \times \partial_{ss} \gamma)(t_0, s_0) = 0.$$

Then for any $r > |x_0 - \gamma(t_0, s_0)|$ and $V \in S^2 \subset \mathbb{R}^3$ the estimate

$$|\partial_t X \cdot V - D(\text{curl} X) : (V \otimes V)| \leq K (1 - X \cdot V)$$

holds at the point (x_0, t_0) , where

$$K \equiv K(\gamma, r) := \frac{8}{r^2} + \frac{32}{r_0^2} + \left(2 + 32 \frac{r}{r_0}\right) \Sigma_0,$$

$$r_0 := 1/|\partial_{ss} \gamma(t_0, s_0)| \in (0, +\infty], \quad \text{and} \quad \Sigma_0 = |\partial_{sss} \gamma(t_0, s_0)| \in [0, +\infty).$$

(In case $r_0 = +\infty$, one can take $r = +\infty$ and then K should be understood as $K = 2\Sigma_0$.)

Proof. – We start by fixing some notations that will help keeping expressions of reasonable size in the sequel. For a function Y with values in \mathbb{R}^3 , and $i \in \{1, 2, 3\}$, we write Y^i to denote the i -th component of Y . We write d^2 to denote the function $(x, t) \mapsto |x - \gamma(t, \zeta(x, t))|^2$, $\gamma(\zeta)$ to denote the function $(x, t) \mapsto \gamma(t, \zeta(x, t))$, and similarly for $\partial_t \gamma(\zeta)$, $\partial_{ts} \gamma(\zeta)$, $\partial_s \gamma(\zeta)$, $\partial_{ss} \gamma(\zeta)$ and $\partial_{sss} \gamma(\zeta)$. When this does not lead to a possible confusion, we also denote by x the function $(x, t) \mapsto x$. Each of these functions are being defined on \mathcal{U} .

In view of (56), shrinking \mathcal{U} if necessary, we may assume that⁽¹²⁾

$$(59) \quad d^2(x, t) < r, \quad \forall (x, t) \in \mathcal{U},$$

and

$$(60) \quad \rho(x, t) := 1 - (x - \gamma(t, \zeta(x, t))) \cdot \partial_{ss} \gamma(t, \zeta(x, t)) > 1/2, \quad \forall (x, t) \in \mathcal{U}.$$

Step 1: First computation of $D(\text{curl} X) : (V \otimes V)$. – Differentiating (57) we obtain, pointwise on \mathcal{U} and for $i, j \in \{1, 2, 3\}$,

$$\partial_j X^i = \partial_j (f(d^2)) \partial_s \gamma(\zeta)^i + f(d^2) \partial_{ss} \gamma(\zeta)^i \partial_j \zeta$$

for the space derivatives, and

$$(61) \quad \partial_t X^i = \partial_t (f(d^2)) \partial_s \gamma(\zeta)^i + f(d^2) [\partial_{ss} \gamma(\zeta)^i \partial_t \zeta + \partial_{ts} \gamma(\zeta)^i]$$

for the time derivative. Also, for $i, j, \ell \in \{1, 2, 3\}$,

$$\begin{aligned} \partial_{\ell j} X^i &= \partial_{\ell j} (f(d^2)) \partial_s \gamma(\zeta)^i + \partial_{ss} \gamma(\zeta)^i [\partial_\ell (f(d^2)) \partial_j \zeta + \partial_j (f(d^2)) \partial_\ell \zeta] \\ &\quad + f(d^2) \partial_{sss} \gamma(\zeta)^i \partial_\ell \zeta \partial_j \zeta + f(d^2) \partial_{ss} \gamma(\zeta)^i \partial_{\ell j} \zeta. \end{aligned}$$

In particular, we may write

$$D(\text{curl} X) : (V \otimes V) =: A = A_1 + A_2 + A_3 + A_4,$$

⁽¹²⁾ In particular we will not need to deal with the lack of regularity of f at $d = r$.

where

$$\begin{aligned} A_1 &:= \epsilon_{ijk} \partial_{\ell i} (f(d^2)) \partial_s \gamma(\zeta)^j V^k V^\ell, \\ A_2 &:= \epsilon_{ijk} \partial_{ss} \gamma(\zeta)^j [\partial_\ell (f(d^2)) \partial_i \zeta + \partial_i (f(d^2)) \partial_\ell \zeta] V^k V^\ell, \\ A_3 &:= \epsilon_{ijk} f(d^2) \partial_{sss} \gamma(\zeta)^j \partial_\ell \zeta \partial_i \zeta V^k V^\ell, \\ A_4 &:= \epsilon_{ijk} f(d^2) \partial_{ss} \gamma(\zeta)^j \partial_{\ell i} \zeta V^k V^\ell, \end{aligned}$$

in which ϵ_{ijk} is the Levi-Civita symbol and we sum over repeated indices.

Step 2: Expressing derivatives of ζ in terms of γ . – Recall that by definition of ζ , we have

$$(62) \quad (x - \gamma(t, \zeta(t, x))) \cdot \partial_s \gamma(t, \zeta(t, x)) = 0$$

for every $(x, t) \in \mathcal{U}$. For $j \in \{1, 2, 3\}$, differentiating (62) with respect to x_j and using (53) we find

$$(63) \quad \partial_s \gamma(\zeta)^j - \partial_j \zeta + (x - \gamma(\zeta)) \cdot \partial_{ss} \gamma(\zeta) \partial_j \zeta = 0.$$

In view of (60), we may rewrite (63) as

$$(64) \quad \partial_j \zeta = \frac{1}{\rho} \partial_s \gamma(\zeta)^j.$$

For $\ell \in \{1, 2, 3\}$, differentiating (63) with respect to x_ℓ and using (53), we obtain

$$(65) \quad \partial_{\ell j} \zeta = \frac{1}{\rho} \left(\partial_{ss} \gamma(\zeta)^j \frac{\partial_s \gamma(\zeta)^\ell}{\rho} + \partial_{ss} \gamma(\zeta)^\ell \frac{\partial_s \gamma(\zeta)^j}{\rho} + (x - \gamma(\zeta)) \cdot \partial_{sss} \gamma(\zeta) \frac{\partial_s \gamma(\zeta)^j \partial_s \gamma(\zeta)^\ell}{\rho^2} \right).$$

Finally, differentiating (62) with respect to t we obtain

$$(66) \quad \partial_t \zeta = \frac{1}{\rho} (-\partial_t \gamma(\zeta) \cdot \partial_s \gamma(\zeta) + (x - \gamma(\zeta)) \cdot \partial_{ts} \gamma(\zeta)).$$

In particular, taking into account (58) it follows from (66) that, *at the point* (x_0, t_0) ,

$$\partial_t \zeta = \frac{1}{\rho} (x - \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) \times \partial_{sss} \gamma(\zeta)).$$

Step 3: Expressing derivatives of d^2 in terms of γ . – In view of the definition of d^2 , we have for $j \in \{1, 2, 3\}$,

$$(67) \quad \partial_j d^2 = 2(x - \gamma(\zeta))^j - 2(x - \gamma(\zeta)) \cdot \partial_s \gamma(\zeta) \partial_j \zeta = 2(x - \gamma(\zeta))^j,$$

where the last equality follows from (62). For $\ell \in \{1, 2, 3\}$, differentiating (67) with respect to x_ℓ and using (63), we obtain

$$(68) \quad \partial_{\ell j} d^2 = -2(\delta_{j\ell} - \partial_s \gamma(\zeta)^j \partial_\ell \zeta) = -2 \left(\delta_{j\ell} - \frac{\partial_s \gamma(\zeta)^j \partial_s \gamma(\zeta)^\ell}{\rho} \right),$$

where $\delta_{j\ell}$ is the Kronecker symbol. Also from the definition of d^2 , we have

$$(69) \quad \partial_t d^2 = -2(x - \gamma(\zeta)) \cdot (\partial_t \gamma(\zeta) + \partial_s \gamma(\zeta) \partial_t \zeta).$$

In particular, taking into account (62) and (58) it follows from (69) that, *at the point* (x_0, t_0) ,

$$(70) \quad \partial_t d^2 = -2(x - \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) \times \partial_{sss} \gamma(\zeta)).$$

Step 4: A reduced expression for $D(\text{curl}X) : (V \otimes V)$. – We substitute, in the terms A_1, A_2, A_3 and A_4 defined in Step 1, the expressions for the derivatives of d^2 and ζ which we obtained in Step 2 and Step 3. Some cancellations occur.

Examining A_1 , we first expand:

$$\begin{aligned} \partial_{\ell i}(f(d^2)) &= f''(d^2)\partial_{\ell}d^2\partial_i d^2 + f'(d^2)\partial_{\ell i}d^2 \\ &= \frac{2}{\rho}f'(d^2)\partial_s\gamma(\zeta)^{\ell}\partial_s\gamma(\zeta)^i - 2f'(d^2)\delta_{\ell i}, \end{aligned}$$

where we have used $f''(d^2) \equiv 0$ by (59) as well as (68) for the second equality. Next,

$$\epsilon_{ijk}\partial_s\gamma(\zeta)^{\ell}\partial_s\gamma(\zeta)^i\partial_s\gamma(\zeta)^jV^kV^{\ell} = (\partial_s\gamma(\zeta) \cdot (\partial_s\gamma(\zeta) \times V))(\partial_s\gamma(\zeta) \cdot V) = 0,$$

and

$$\epsilon_{ijk}\delta_{\ell i}\partial_s\gamma(\zeta)^jV^kV^{\ell} = \epsilon_{ijk}V^i\partial_s\gamma(\zeta)^jV^k = V \cdot (\partial_s\gamma(\zeta) \times V) = 0.$$

Hence,

$$A_1 = 0.$$

Concerning A_2 , (64) and (67) yield

$$\begin{aligned} A_2 &= \frac{2}{\rho}f'(d^2)\epsilon_{ijk}V^kV^{\ell}\partial_{ss}\gamma(\zeta)^j((x - \gamma(\zeta))^{\ell}\partial_s\gamma(\zeta)^i + (x - \gamma(\zeta))^i\partial_s\gamma(\zeta)^{\ell}) \\ &= \frac{2}{\rho}f'(d^2)\partial_{ss}\gamma(\zeta) \cdot (\partial_s\gamma(\zeta) \times V)(V \cdot (x - \gamma(\zeta))) + \\ &\quad \frac{2}{\rho}f'(d^2)(x - \gamma(\zeta)) \cdot (\partial_{ss}\gamma(\zeta) \times V)(V \cdot \partial_s\gamma(\zeta)) \\ &=: A_{2,1} + A_{2,2}. \end{aligned}$$

For A_3 , we invoke (64) to substitute $\partial_{\ell}\zeta$ and $\partial_i\zeta$ and obtain

$$A_3 = \frac{1}{\rho^2}f(d^2)(\partial_s\gamma(\zeta) \cdot V)\partial_s\gamma(\zeta) \cdot (\partial_{sss}\gamma(\zeta) \times V).$$

For A_4 finally, we invoke (65) to substitute $\partial_{\ell i}\zeta$ and obtain

$$\begin{aligned} A_4 &= \frac{1}{\rho^2}f(d^2)\partial_{ss}\gamma(\zeta) \cdot (\partial_{ss}\gamma(\zeta) \times V)(\partial_s\gamma(\zeta) \cdot V) \\ &\quad + \frac{1}{\rho^2}f(d^2)\partial_s\gamma(\zeta) \cdot (\partial_{ss}\gamma(\zeta) \times V)(\partial_{ss}\gamma(\zeta) \cdot V) \\ &\quad + \frac{1}{\rho^3}f(d^2)((x - \gamma(\zeta)) \cdot \partial_{sss}\gamma(\zeta))\partial_s\gamma(\zeta) \cdot (\partial_{ss}\gamma(\zeta) \times V)(\partial_s\gamma(\zeta) \cdot V) \\ &=: 0 + A_{4,1} + A_{4,2}. \end{aligned}$$

Step 5: Computation of $\partial_t X \cdot V$. – We expand (61) as

$$\partial_t X^i = f'(d^2)\partial_t d^2\partial_s\gamma(\zeta)^i + f(d^2)[\partial_{ss}\gamma(\zeta)^i\partial_t\zeta + \partial_{ts}\gamma(\zeta)].$$

Therefore, at the point (x_0, t_0) , we obtain from (58), (66) and (70)

$$\partial_t X \cdot V =: B = B_1 + B_2 + B_3,$$

where

$$\begin{aligned} B_1 &:= -2f'(d^2)((x - \gamma(\zeta)) \cdot (\partial_s\gamma(\zeta) \times \partial_{ss}\gamma(\zeta)))(\partial_s\gamma(\zeta) \cdot V), \\ B_2 &:= \frac{1}{\rho}f(d^2)((x - \gamma(\zeta)) \cdot (\partial_s\gamma(\zeta) \times \partial_{sss}\gamma(\zeta)))(\partial_{ss}\gamma(\zeta) \cdot V), \\ B_3 &:= f(d^2)(\partial_s\gamma(\zeta) \times \partial_{sss}\gamma(\zeta)) \cdot V. \end{aligned}$$

Step 6: Proof of Proposition 2 completed. – We write, at the point (x_0, t_0) ,

$$(71) \quad \begin{aligned} |B - A| &= |\partial_t X \cdot V - D(\operatorname{curl} X) : (V \otimes V)| \\ &\leq |A_1| + |A_{2,1}| + |A_{2,2} - B_1| + |A_3 - B_3| + |A_{4,1}| + |A_{4,2}| + |B_2|, \end{aligned}$$

and we will estimate each of the terms in the last line separately. We first observe the following elementary facts that hold at the point (x_0, t_0) (when they involve functions):

- a) $|V^\perp|, |V|, |\partial_s \gamma(\zeta)| \leq 1$ (indeed $V \in \mathcal{S}^2$ and (53) holds),
- b) $|f(d^2)| \leq 1, |f'(d^2)| = 1/r^2$ (follows from the definition of f),
- c) $\rho \geq 1/2, |1 - 1/\rho| \leq 2d/r_0, |1 - 1/\rho^2| \leq 6d/r_0$ (from (54) and the definition of ρ).

Also recall that by definition

$$r_0 = |\partial_{ss} \gamma(t_0, s_0)|^{-1}, \quad \text{and} \quad \Sigma_0 = |\partial_{sss} \gamma(t_0, s_0)|.$$

Taking into account (55), direct inspection yields

$$(72) \quad |A_1| = 0, \quad |A_{2,1}| \leq 4d(r^2 r_0)^{-1} |V^\perp|, \quad |A_{4,1}| \leq 4r_0^{-2} |V^\perp|^2,$$

as well as

$$|A_{4,2}| \leq 8dr_0^{-1} \Sigma_0 |V^\perp|, \quad |B_2| \leq 2dr_0^{-1} \Sigma_0 |V^\perp|.$$

Next, we write

$$\begin{aligned} |B_1 - A_{2,2}| &= \left| \frac{2}{\rho} f'(d^2) (\partial_s \gamma(\zeta) \cdot V) ((x - \gamma(\zeta)) \times \partial_{ss} \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) - \frac{V}{\rho}) \right| \\ &\leq 4d(r^2 r_0)^{-1} (|\partial_s \gamma(\zeta) - V| + 2dr_0^{-1}), \end{aligned}$$

and

$$(73) \quad \begin{aligned} |B_3 - A_3| &= |f(d^2) [(\partial_s \gamma(\zeta) \times \partial_{sss} \gamma(\zeta)) \cdot V] (1 - \frac{1}{\rho^2} V \cdot \partial_s \gamma(\zeta))| \\ &\leq \Sigma_0 |V^\perp| \left((1 - \partial_s \gamma(\zeta) \cdot V) + 6dr_0^{-1} \right) \\ &\leq \Sigma_0 \left((1 - \partial_s \gamma(\zeta) \cdot V) + 6dr_0^{-1} |V^\perp| \right). \end{aligned}$$

It remains to bound $d, |V^\perp|, |\partial_s \gamma(\zeta) - V|$ and $|1 - \partial_s \gamma(\zeta) \cdot V|$ in terms of $1 - X \cdot V$. To that purpose, first recall from the definition of f , from the fact that $|V| = |\partial_s \gamma(\zeta)| = 1$, and from the assumption $d \leq r$, that

$$(74) \quad 1 - X \cdot V \geq 1 - f(d^2) \geq d^2 r^{-2}.$$

Also, if $X \cdot V \geq 0$ then $1 - X \cdot V \geq 1 - \partial_s \gamma(\zeta) \cdot V$, and if $X \cdot V < 0$ then $1 - X \cdot V \geq 1 \geq (1 - \partial_s \gamma(\zeta) \cdot V)/2$. In any case, we have

$$(75) \quad 1 - X \cdot V \geq \frac{1}{2} (1 - \partial_s \gamma(\zeta) \cdot V).$$

Finally, by Hilbert's projection theorem

$$(76) \quad |V^\perp|^2 \leq |V - \partial_s \gamma(\zeta)|^2 = 2(1 - \partial_s \gamma(\zeta) \cdot V) \leq 4(1 - X \cdot V).$$

Inserting (74), (75), or (76) in (72)-(73), and writing $x \preceq y$ for $x \leq y(1 - X \cdot V)$, we obtain

$$|A_1| \preceq 0, \quad |A_{2,1}| \preceq \frac{8}{r_0 r}, \quad |A_{4,1}| \preceq \frac{16}{r_0^2}, \quad |A_{4,2}| \preceq 16 \frac{r}{r_0} \Sigma_0,$$

$$|B_2| \preceq 4 \frac{r}{r_0} \Sigma_0, \quad |B_1 - A_{2,2}| \preceq (1 + \frac{r}{r_0}) \frac{8}{r_0 r}, \quad |B_3 - A_3| \preceq (2 + 12 \frac{r}{r_0}) \Sigma_0,$$

and summation according to (71) yields the claim. \square

6. A Gronwall estimate for $F_{\gamma,\sigma,r}(\Gamma)$

Let $\gamma \in L^\infty(I, H^4_{loc}(\mathbb{R}, \mathbb{R}^3))$ and $\Gamma \in L^\infty(I, H^{\frac{3}{2}}_{loc}(\mathbb{R}, \mathbb{R}^3))$ be two solutions of the binormal curvature flow Equation (2) on $I \times \mathbb{R}$, where $I = (-T, T)$ for some $T > 0$. Assume moreover that γ and Γ are both quasiperiodic and that both have the same pitch, that is

$$(77) \quad \begin{aligned} \gamma(t, s + \ell) &= \gamma(t, s) + a, & \forall t \in J, \forall s \in \mathbb{R}, \\ \Gamma(t, s + L) &= \Gamma(t, s) + a, & \forall t \in J, \forall s \in \mathbb{R}, \end{aligned}$$

for some $\ell > 0, L > 0$, and $a \in \mathbb{R}^3$. It follows from (2) and our assumptions on γ and Γ that⁽¹³⁾ $\gamma \in \mathcal{C}(\bar{I}, \mathcal{C}^3(\mathbb{R}, \mathbb{R}^3))$ and $\Gamma \in \mathcal{C}(I, \mathcal{C}(\mathbb{R}, \mathbb{R}^3))$. For $t \in I$, we set

$$\gamma_t := \gamma(t, \cdot), \quad \Gamma_t := \Gamma(t, \cdot),$$

and define⁽¹⁴⁾

$$r_\gamma := \max_{t \in \bar{I}} \left(\max_{s \in \mathbb{R}} |\partial_{ss} \gamma(t, s)| \right)^{-1} \in (0, +\infty].$$

Assume that at time zero we have

$$d_\varphi(\Gamma_0, \gamma_0) < r_\gamma/8,$$

and let therefore σ_0 be a reparametrization of γ_0 for Γ_0 (the existence of which being ensured by Corollary 1).

THEOREM 3. – Fix $0 < r \leq r_\gamma/8$ and assume that

$$(78) \quad F(0) := F_{\gamma_0, \sigma_0, r}(\Gamma_0) < F_r := r \left(\sqrt{2} + \frac{r}{L} \right)^{-1}.$$

Define then

$$T_r := \frac{1}{K} \log \left(\frac{F_r}{F(0)} \right),$$

where

$$K := \frac{8}{r^2} + \frac{32}{r_\gamma^2} + \left(2 + 32 \frac{r}{r_\gamma} \right) \|\partial_{sss} \gamma\|_{L^\infty(I \times \mathbb{R})},$$

and set $J_r = [-T_r, T_r] \cap (-T, T)$. There exists a unique $\sigma \in \mathcal{C}(J_r \times \mathbb{R}, \mathbb{R})$ such that $\sigma(0, \cdot) = \sigma_0$ and $\sigma_t := \sigma(t, \cdot)$ is a reparametrization of γ_t for Γ_t , for every $t \in J_r$. The function F defined on J_r by $F(t) := F_{\gamma_t, \sigma_t, r}(\Gamma_t)$ is Lipschitz continuous on J_r and satisfies for almost every $t \in J_r$ the inequality

$$|F'(t)| \leq KF(t).$$

Therefore

$$F(t) \leq \exp(K|t|)F(0), \quad \forall t \in J_r,$$

and in particular

$$d_\varphi(\Gamma_t, \gamma_t) < r, \quad \forall t \in J_r.$$

⁽¹³⁾ Actually γ could be globally extended from I to \mathbb{R} as a solution, see Subsection 1.2, but the existence of Γ has to be assumed a priori.

⁽¹⁴⁾ We have already defined a quantity called r_γ for a curve with no time dependence. One is the natural extension of the other and there should not be any possible confusion.

Proof. – It follows from (78) and Lemma 4 that $\|\Gamma_0 - \gamma_0 \circ \sigma_0\|_\infty < r$. Since Γ and γ depend continuously on time, we have $\|\Gamma_t - \gamma_t \circ \sigma_0\|_\infty < r$ for any t in a sufficiently small neighborhood J of 0 in I . For $t \in J$, applying Corollary 1 to the curves γ_t and Γ_t with the choice $p = \sigma_0$ yields a reparametrization σ_t of γ_t for Γ_t that satisfies $\|\Gamma_t - \gamma_t \circ \sigma_t\|_\infty \leq \|\Gamma_t - \gamma_t \circ \sigma_0\|_\infty < r$ and which is the only reparametrization of γ_t for Γ_t that satisfy $\|\sigma_t - \sigma_0\|_\infty < r_\gamma/8$. Applying the implicit function theorem (exactly as we did in the proof of Lemma 2 but with an additional time dependence) to the function

$$(t, s, x, z) \mapsto (x - \gamma(t, z)) \cdot \gamma'(t, z),$$

we infer that $\sigma_t = \sigma(t, \cdot)$ for some continuous function σ on $J \times \mathbb{R}$ which is also uniquely determined locally and hence globally on $J \times \mathbb{R}$. In particular, $T_c > 0$, where

$$T_c := \sup\{0 < \tau < T \text{ s.t. } [\exists \sigma \in \mathcal{C}((-\tau, \tau) \times \mathbb{R}, \mathbb{R}) \text{ s.t. } \sigma(0, \cdot) = \sigma_0 \text{ and } \forall t \in (-\tau, \tau) \\ \sigma_t := \sigma(t, \cdot) \text{ is a reparametrization of } \gamma_t \text{ for } \Gamma_t \text{ satisfying } \|\Gamma_t - \gamma_t \circ \sigma_t\|_\infty < r]\}.$$

We claim that either $T_c > T_r$ or $T_c = T$. We argue by contradiction and assume that it is not the case. Fix an arbitrary $0 < T_1 < T_c$, set $R := \max_{t \in [-T_1, T_1]} \|\Gamma_t - \gamma_t \circ \sigma_t\|_\infty < r$, and fix $\chi_R \in \mathcal{C}^\infty(\mathbb{R}^+, [0, 1])$ such that $\chi_R \equiv 1$ on $[0, R]$ and $\chi_R \equiv 0$ outside $[r, +\infty)$. Define the parametrized vector field $X \in \mathcal{C}((-T_1, T_1) \times \mathbb{R}, \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}^3))$ by

$$(79) \quad X(t, s, x) = \chi_R(x - \gamma(\sigma(t, s))) f_{r_\gamma}(|x - \gamma(\sigma(t, s))|^2) \partial_s \gamma(t, \sigma(t, s)).$$

Note that X satisfies (41) and (42), by Corollary 1, Conclusion 1 and Lemma 2, Conclusion 4. Applying Proposition 1 to Γ and X on $(-T_1, T_1)$ and using Proposition 2 to estimate the right-hand side of (43)⁽¹⁵⁾, we obtain that F is Lipschitzian on $[-T_1, T_1]$ and

$$(80) \quad \left| \frac{d}{dt} F(t) \right| = \left| \frac{d}{dt} \int_0^L (X \circ \Gamma) \cdot \partial_s \Gamma ds \right| \leq KF(t),$$

where K is given in the statement and where we have also used the fact that by definition of R , $\chi_R(\Gamma(t, s) - \gamma(t, \sigma(t, s))) \equiv 1$. By Gronwall inequality and (80), we obtain $F(T_1) \leq \exp(KT_1)F(0) \leq \exp(KT_c)F(0) < F_c$, independently of T_1 . Lemma 4 applied to σ_{T_1} , γ_{T_1} and Γ_{T_1} with the r of the the statement then implies that

$$\begin{aligned} \|\Gamma_{T_1} - \gamma_{T_1} \circ \sigma_{T_1}\|_\infty^2 &\leq (\sqrt{2}r + \frac{r^2}{L})F(T_1) \\ &\leq (\sqrt{2}r + \frac{r^2}{L}) \exp(KT_c)F(0) \\ &< (\sqrt{2}r + \frac{r^2}{L}) \exp(KT_r)F(0) \leq r^2, \end{aligned}$$

independently of T_1 . Applying Corollary 1 as above for T_1 sufficiently close to T_c and with the choice $p_0 := \sigma_{T_1}$, and taking into account the fact that Γ and γ are uniformly continuous on $[-T_c, T_c] \times \mathbb{R}$ we would then be able to extend σ past T_c with the required properties, which is contradictory to the definition of T_c . \square

⁽¹⁵⁾ Indeed, the X defined in (79) is locally independent of s (this is again Lemma 2 conclusion 4), and, removing locally the variable s , coincides there with the vector field X defined in (57).

7. Proof of Theorem 1

We distinguish three cases. Case 1 is the base scenario where one can directly rely on the binormal flow estimate of Section 6. In Case 2 and Case 3, we will first modify the solution u into solutions \tilde{u} or \check{u} for which Case 1 may be applied, and then estimate the difference between u and \tilde{u} or \check{u} separately.

Case 1: Assume that $\int_{T^1} u = \int_{T^1} v$. – We first apply Lemma 1 to u and v successively. This yields solutions $\gamma_u \in \mathcal{C}(I, H_{\text{loc}}^4(\mathbb{R}, \mathbb{R}))$ and $\gamma_v \in \mathcal{C}(I, H_{\text{loc}}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}))$ of the binormal curvature flow equation, which for convenience we will call γ and Γ respectively. Since u and v are 2π -periodic with respect to the space variable, and since $\int_{T^1} u = \int_{T^1} v$, it follows that γ and Γ satisfy conditions (77) with $\ell = L := 2\pi$ and $a := \int_{T^1} u = \int_{T^1} v$. As in Section 6, for $t \in I$ we set

$$\gamma_t := \gamma(t, \cdot), \quad \Gamma_t := \Gamma(t, \cdot),$$

and define

$$r_u := \max_{t \in I} \left(\sup_{s \in \mathbb{R}} |\partial_{ss} \gamma(t, s)| \right)^{-1} = \max_{t \in I} \left(\sup_{s \in T^1} |\partial_s u(t, s)| \right)^{-1} \in (0, +\infty].$$

By construction, $\gamma_0(0) = \Gamma_0(0) = 0$. Applying Lemma 6 with $\Gamma := \Gamma_0$, $\gamma := \gamma_0$ and $L = \ell := 2\pi$, we infer that if

$$(81) \quad \int_{T^1} |v(0, s) - u(0, s)|^2 ds < \frac{r_u^2}{128\pi}$$

then $d_\varphi(\gamma_0, \Gamma_0) < r_u/8$, and there exists a reparametrization σ_0 of γ_0 for Γ_0 , such that, setting $r := r_u/8$, we have

$$F(0) := F_{\gamma_0, \sigma_0, r}(\Gamma_0) \leq \left(1 + \frac{8^3 \pi^2}{r_u^2} + \frac{2^8 \pi^4}{r_u^4}\right) \int_{T^1} |v(0, s) - u(0, s)|^2 ds.$$

Assume first that

$$(82) \quad \int_{T^1} |v(0, s) - u(0, s)|^2 ds < \frac{r_u}{8} \left(1 + \frac{8^3 \pi^2}{r_u^2} + \frac{2^8 \pi^4}{r_u^4}\right)^{-1} \left(\sqrt{2} + \frac{r_u}{16\pi}\right)^{-1} \\ = \left(\frac{1}{2\pi} + \frac{8\sqrt{2}}{r_u} + \frac{256\pi}{r_u^2} + \frac{512\sqrt{2}\pi^2}{r_u^3} + \frac{2^7 \pi^3}{r_u^4} + \frac{2^{11}\sqrt{2}\pi^4}{r_u^5}\right)^{-1}.$$

Then (78) holds in addition to (81), and therefore Theorem 3 provides the existence of $\sigma \in \mathcal{C}(J_r \times \mathbb{R}, \mathbb{R})$ such that $\sigma_t := \sigma(t, \cdot)$ is a reparametrization of γ_t for Γ_t for all $t \in J_r$, and the estimate

$$(83) \quad F_{\gamma_t, \sigma_t, r} \leq \exp(K_u |t|) F(0) \leq \exp(K_u |t|) \left(1 + \frac{8^3 \pi^2}{r_u^2} + \frac{2^8 \pi^4}{r_u^4}\right) \int_{T^1} |v(0, s) - u(0, s)|^2 ds$$

for every $t \in J_r$, where

$$K_u := \frac{544}{r_u^2} + 6 \|\partial_{ss} u\|_{L^\infty([-t, t] \times T^1)}, \quad J_r := [-T_r, T_r] \cap (-T, T), \quad \text{and} \quad T_r := \frac{1}{K_u} \log \left(\frac{F_r}{F(0)} \right).$$

Using Lemma 5, we infer from (83) that for every $t \in J_r$,

$$\|v(t, \cdot) - u(t, \cdot + \sigma(t, 0))\|_2^2 \leq C_{u,1} \exp(K|t|) \|v(0, \cdot) - u(0, \cdot)\|_2^2,$$

where

$$C_{u,1} := \left(4 + \frac{(1+2^{11})\pi^2}{r_u^2} + \frac{16\sqrt{2}\pi^3}{r_u^3} + \frac{2^9 3\pi^4}{r_u^4} + \frac{2^{13}\sqrt{2}\pi^5}{r_u^5} + \frac{2^8 \pi^6}{r_u^6} + \frac{2^{12}\sqrt{2}\pi^7}{r_u^7}\right).$$

On the other hand, if $T_r < T$ then for every t in $(0, T) \setminus J_r$ we simply write

$$\begin{aligned}
 \|v(t, \cdot) - u(t, \cdot + \sigma(T_r, 0))\|_2^2 &\leq 8\pi = 8\pi \exp(-K_u|t|) \exp(K_u|t|) \\
 (84) \qquad \qquad \qquad &\leq 8\pi \exp(-KT_r) \exp(K_u|t|) = 8\pi \frac{F(0)}{F_r} \exp(K_u|t|) \\
 &\leq C_{u,2} \exp(K_u|t|) \|v(0, \cdot) - u(0, \cdot)\|_2^2,
 \end{aligned}$$

where

$$C_{u,2} := \left(4 + \frac{2^6\sqrt{2}\pi}{r_u} + \frac{2^{11}\pi^2}{r_u^2} + \frac{2^{15}\sqrt{2}\pi^3}{r_u^3} + \frac{2^{10}\pi^4}{r_u^4} + \frac{2^{14}\sqrt{2}\pi^5}{r_u^5}\right).$$

For t in $(-T, 0) \setminus J_r$, we obtain the equivalent of (84) replacing $\sigma(T_r, 0)$ by $\sigma(-T_r, 0)$.

It remains to consider the situation where (82) does not hold. There again, we simply write, for any $t \in (-T, T)$,

$$\begin{aligned}
 \|v(t, \cdot) - u(t, \cdot)\|_2^2 &\leq 8\pi = 8\pi \|v(0, \cdot) - u(0, \cdot)\|_2^{-2} \|v(0, \cdot) - u(0, \cdot)\|_2^2 \\
 &\leq C_{u,2} \|v(0, \cdot) - u(0, \cdot)\|_2^2 \\
 &\leq C_{u,2} \exp(K_u|t|) \|v(0, \cdot) - u(0, \cdot)\|_2^2.
 \end{aligned}$$

Inspecting the constants $C_{u,1}$, $C_{u,2}$ and K_u , we conclude that for any $t \in (-T, T)$

$$\|v(t, \cdot) - u(t, \cdot + \sigma(t))\|_2^2 \leq C_u \exp(K_u|t|) \|v(0, \cdot) - u(0, \cdot)\|_2^2,$$

where

$$C_u := \left(4 + \frac{2^6\sqrt{2}\pi}{r_u} + \frac{(1+2^{11})\pi^2}{r_u^2} + \frac{2^{15}\sqrt{2}\pi^3}{r_u^3} + \frac{2^9 3\pi^4}{r_u^4} + \frac{2^{14}\sqrt{2}\pi^5}{r_u^5} + \frac{2^8\pi^6}{r_u^6} + \frac{2^{12}\sqrt{2}\pi^7}{r_u^7}\right)$$

and where σ is the continuous function on $(-T, T)$ defined by $\sigma \equiv 0$ if (82) does not hold and $\sigma(t) := \sigma(t, 0)$ for $t \in J_r$, $\sigma(t) := \sigma(-T_r, 0)$ for $t \in (-T, -T_r)$, and $\sigma(t) := \sigma(T_r, 0)$ for $t \in (T_r, T)$ if (82) holds.

The constants C_u and K_u above are completely explicit, but at this stage they involve estimates on the solution u on the whole interval $[-T, T]$. In view of the Cauchy theory at the level of smooth solutions (see Section), we may then estimate both C_u and $\exp(K_u|t|)$ by a function $C(\|\partial_{sss}u(0, \cdot)\|_2, T)$. This completes the proof of Theorem 1 in Case 1.

Notice that there would not be any particular difficulty in turning $C(\|\partial_{sss}u(0, \cdot)\|_2, T)$ into an explicit expression as well, but these expressions quickly become even more complicated (in particular they involve double exponentials in time). For Case 2 and Case 3 in the sequel, we will rely on additional levels of approximation; for convenience we will therefore abandon explicit expressions there even sooner than for Case 1. Without loss of generality, we will also consider u as being globally defined in time.

Case 2: Assume $|\int_{T^1} u| \geq 1/2$. – In that case we will take advantage of the symmetries of (1) and reduce it to a case similar to Case 1. More precisely, we first infer from the Cauchy-Schwarz inequality that

$$(85) \qquad \left| \int_{T^1} u - \int_{T^1} v \right| \leq \sqrt{2\pi} \|v(0, \cdot) - u(0, \cdot)\|_2.$$

As in Case 1, if the inequality

$$(86) \qquad \|v(0, \cdot) - u(0, \cdot)\|_2 < \frac{1}{4\sqrt{2\pi}}$$

does not hold, then we simply write for any $t \in (-T, T)$,

$$\begin{aligned} \|v(t, \cdot) - u(t, \cdot)\|_2^2 &\leq 8\pi = 8\pi \|v(0, \cdot) - u(0, \cdot)\|_2^{-2} \|v(0, \cdot) - u(0, \cdot)\|_2^2 \\ &\leq 2^8 \pi^2 \|v(0, \cdot) - u(0, \cdot)\|_2^2. \end{aligned}$$

Assume next that (86) holds. Then $|\int_{T^1} u - \int_{T^1} v| \leq 1/4$ and there exist $R_\theta \in SO(3)$ and $\kappa \in (1/2, 3/2)$ such that the function

$$\tilde{u}_0(s) := R_\theta u(0, \frac{s}{\kappa})$$

defined on T_ℓ^1 with $\ell = 2\pi\kappa$ satisfies $\int_{T_\ell^1} \tilde{u} = \int_{T^1} v$ and moreover⁽¹⁶⁾

$$(87) \quad \|R_\theta - \text{Id}\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)} \leq 2\sqrt{2\pi} \|v(0, \cdot) - u(0, \cdot)\|_2, \quad |\kappa - 1| \leq 2\sqrt{2\pi} \|v(0, \cdot) - u(0, \cdot)\|_2.$$

The unique (global) solution \tilde{u} of (1) with initial datum \tilde{u}_0 is given by

$$(88) \quad \tilde{u}(t, s) = R_\theta u\left(\frac{t}{\kappa^2}, \frac{s}{\kappa}\right).$$

We apply Lemma 1 to \tilde{u} and v successively, which yields solutions $\gamma_{\tilde{u}} \in \mathcal{C}(I, H_{\text{loc}}^4(\mathbb{R}, \mathbb{R}))$ and $\gamma_v \in \mathcal{C}(I, H_{\text{loc}}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}))$ of the binormal curvature flow equation. We then follow the exact same lines as in Case 1, the only minor difference being that here $2\pi =: L \neq \ell$ and we merely have, in view of (87),

$$|L - \ell|^2 \leq 32\pi^3 \|v(0, \cdot) - u(0, \cdot)\|_2^2,$$

whose only effect is to affect the value of some numerical constants (when we apply Lemma 5 and Lemma 6 only). Doing so, we obtain, for every $t \in (-T, T)$,

$$(89) \quad \int_0^{2\pi} |v(t, s) - \tilde{u}(t, \kappa s + \sigma(t))|^2 ds \leq C_{\tilde{u}} \exp(K_{\tilde{u}}|t|) \int_0^{2\pi} |v(0, s) - \tilde{u}(0, \kappa s)|^2 ds$$

for some function $\sigma \in \mathcal{C}(I, T_\ell^1)$, where

$$C_{\tilde{u}} \leq C(\|\tilde{u}_0'''\|_2, T) \leq C(\|\partial_{sss} u(0, \cdot)\|_2, T).$$

In view of (88), (89) translates into

$$(90) \quad \int_0^{2\pi} |v(t, s) - R_\theta u\left(\frac{t}{\kappa^2}, s + \frac{1}{\kappa}\sigma(t)\right)|^2 ds \leq C(\|\partial_{sss} u(0, \cdot)\|_2, T) \int_0^{2\pi} |v(0, s) - R_\theta u(0, s)|^2 ds.$$

Now, by (87) we estimate

$$\int_0^{2\pi} |(R_\theta - \text{Id})u\left(\frac{t}{\kappa^2}, s + \frac{1}{\kappa}\sigma(t)\right)|^2 ds \leq 2\pi \|R_\theta - \text{Id}\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)}^2 \leq 16\pi^2 \|v(0, \cdot) - u(0, \cdot)\|_2^2,$$

and

$$\int_0^{2\pi} |(R_\theta - \text{Id})u(0, s)|^2 ds \leq 2\pi \|R_\theta - \text{Id}\|_{\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)}^2 \leq 16\pi^2 \|v(0, \cdot) - u(0, \cdot)\|_2^2.$$

⁽¹⁶⁾ These can be checked from easy planar geometry.

Also,

$$(91) \quad \int_0^{2\pi} \left| u\left(\frac{t}{\kappa^2}, s + \frac{1}{\kappa}\sigma(t)\right) - u\left(t, s + \frac{1}{\kappa}\sigma(t)\right) \right|^2 ds \leq t^2 \left(1 - \frac{1}{\kappa^2}\right)^2 \|\partial_t u\|_{L^\infty((-4T, 4T), L^2(T^1))}^2 \\ \leq Ct^2 |1 - \kappa|^2 \|\partial_{ss} u\|_{L^\infty((-4T, 4T), L^2(T^1))}^2 \\ \leq C(\|\partial_{sss} u(0, \cdot)\|_2, T) \|v(0, \cdot) - u(0, \cdot)\|_2^2.$$

Combining (90)-(91) we complete the proof of Theorem 1 in Case 2.

Case 3: Assume $|\int_{T^1} u| < 1/2$. – As in Cases 1 and 2, if the inequality

$$(92) \quad \|v(0, \cdot) - u(0, \cdot)\|_2 < \min\left(\frac{1}{60} \sqrt{\frac{\pi}{2}} \|\partial_s u(0, \cdot)\|_\infty^{-1}, \frac{\sqrt{2\pi}}{5}\right)$$

does not hold, then we simply write for any $t \in (-T, T)$,

$$\|v(t, \cdot) - u(t, \cdot)\|_2^2 \leq 8\pi = 8\pi \|v(0, \cdot) - u(0, \cdot)\|_2^{-2} \|v(0, \cdot) - u(0, \cdot)\|_2^2 \\ \leq \max(2^8 3^2 5^2 \|\partial_s u(0, \cdot)\|_\infty^2, 2^3 5^2) \|v(0, \cdot) - u(0, \cdot)\|_2^2.$$

Assume next that (92) holds.

For $e \in S^2$ and $r \geq 0$, we denote by $M_{e,r}$ the Möbius transform on S^2 given by $M_{e,r} := S_e^{-1} \circ D_\alpha \circ S_e$, where $\alpha := 1 + r$, S_e is the stereographic projection from $S^2 \setminus \{e\}$ to the equatorial plane of S^2 (with e as north pole), and D_α is the dilation of magnitude α in that equatorial plane. More explicitly, in spherical coordinates (φ, θ) on S^2 such that $\varphi = 0$ corresponds the point e we have

$$M_{e,r}(\varphi, \theta) := \left(2 \arctan\left(\frac{1}{\alpha} \tan\left(\frac{\varphi}{2}\right)\right), \theta\right).$$

Claim. – There exist $e_0 \in S^2$ and $r_0 \geq 0$ such that $\check{u}_0 := M_{e_0, r_0} \circ u(0, \cdot)$ satisfies $\int_{T^1} \check{u}_0 = \int_{T^1} v$ and moreover we have the estimate

$$(93) \quad r_0 \leq \max(60\sqrt{2/\pi} \|\partial_s u(0, \cdot)\|_\infty, 5/\sqrt{2\pi}) \|v(0, \cdot) - u(0, \cdot)\|_2.$$

Proof of the claim. – Fix first $e \in S^2$ and $0 \leq r \leq 1$. Because of the assumption $|\int_{T^1} u| < 1/2$, the image of $u(0, \cdot)$ cannot be contained entirely in the spherical cap $\{0 \leq \varphi \leq \arccos(1/2) = \pi/3\}$, nor in its opposite spherical cap $\{2\pi/3 \leq \varphi \leq \pi\}$. Let therefore $s_e \in T^1$ be such that $\pi/3 \leq \varphi(u(0, s_e)) \leq 2\pi/3$. For every $s \in T^1$ such that $|s - s_e| \leq \pi/(12\|\partial_s u(0, \cdot)\|_\infty)$, we have $\pi/4 \leq \varphi(u(0, s)) \leq 3\pi/4$. For every point p on S^2 , we have $M_{e,r}(p) \cdot e \geq p \cdot e$, and for those points p such that $\pi/4 \leq \varphi(p) \leq 3\pi/4$ we even have $M_{e,r}(p) \cdot e \geq p \cdot e + cr$, where⁽¹⁷⁾

$$c := \frac{4 - \sqrt{17 - 12\sqrt{2}}}{7 - 2\sqrt{2}} - \frac{\sqrt{2}}{2} \geq \frac{1}{5}.$$

In particular, we have

$$e \cdot \int_{T^1} M_{e,r}(u(0, s)) ds \geq e \cdot \int_{T^1} u(0, s) ds + \min\left(\frac{\pi}{60} \|\partial_s u(0, \cdot)\|_\infty^{-1}, \frac{2\pi}{5}\right) r,$$

⁽¹⁷⁾ One can compute the value of c noticing that the worst case scenario corresponds to $r = 1$ and $\varphi(\bar{p}) = \pi/4$, so that $M_{e,1}(\bar{p}) \cdot e = \cos(\varphi(M_{e,1}(\bar{p}))) =: \cos(\bar{\varphi})$ is the solution of the equation $\sin(\bar{\varphi})/(1 - \cos(\bar{\varphi})) = 2 \sin(\pi/4)/(1 - \cos(\pi/4)) = 2(\sqrt{2} - 1)$ which is obtained by taking squares of both sides and easy algebra.

and therefore by (85)

$$(94) \quad e \cdot \int_{T^1} M_{e,r}(u(0, s)) - v(0, s) \, ds \geq -\sqrt{2\pi} \|v(0, \cdot) - u(0, \cdot)\|_2 + \min\left(\frac{\pi}{60} \|\partial_s u(0, \cdot)\|_\infty^{-1}, \frac{2\pi}{5}\right) r.$$

Set $\bar{r} := \max(60\sqrt{2/\pi} \|\partial_s u(0, \cdot)\|_\infty, 5/\sqrt{2\pi}) \|v(0, \cdot) - u(0, \cdot)\|_2$; in particular from (92) we see that $\bar{r} \leq 1$. The function

$$G : B[0, \bar{r}] \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad x \mapsto \begin{cases} \int_{T^1} M_{x/|x|, |x|}(u(0, s)) - v(0, s) \, ds & \text{if } x \neq 0, \\ \int_{T^1} u(0, s) - v(0, s) \, ds & \text{if } x = 0, \end{cases}$$

is continuous on $B[0, \bar{r}]$, and in view of (94), it satisfies

$$G(x) \cdot x \geq 0 \quad \forall x \in \partial B[0, \bar{r}].$$

It follows from the Poincaré-Bohl theorem [8] that G has at least one zero x_0 in $B[0, \bar{r}]$. Our claim is proved choosing $r_0 := |x_0|$ and $e_0 := x_0/|x_0|$ (If $|x_0| = 0$ we are in Case 1). \square

Let $\tilde{u} \in \mathcal{C}(\mathbb{R}, H^3(T^1, S^2))$ denote the solution of (1) with initial datum \tilde{u}_0 . We first apply Theorem 1 Case 1 to \tilde{u} and v . This yields the estimate

$$(95) \quad \int_0^{2\pi} |v(t, s) - \tilde{u}(t, s + \sigma(t))|^2 \, ds \leq C_{\tilde{u}} \exp(K_{\tilde{u}}|t|) \int_0^{2\pi} |v(0, s) - \tilde{u}(0, s)|^2 \, ds$$

for every $t \in I$ and for some function $\sigma \in \mathcal{C}(I, T^1)$, where, since $M_{e,r} \in \mathcal{C}^\infty(S^2, S^2)$,

$$C_{\tilde{u}} \exp(K_{\tilde{u}}|t|) \leq C(\|\partial_{sss} \tilde{u}_0\|_2, T) \leq C(\|\partial_{sss} u(0, \cdot)\|_2, T).$$

Next, using (93) we infer that

$$\|u(0, \cdot) - \tilde{u}_0\|_{H^1}^2 \leq C(\|\partial_s u(0, \cdot)\|_\infty) \|v(0, \cdot) - u(0, \cdot)\|_2^2,$$

and therefore from Corollary A.3 we obtain, for every $t \in I$,

$$(96) \quad \begin{aligned} \|u(t, \cdot + \sigma(t)) - \tilde{u}(t, \cdot + \sigma(t))\|_2^2 &= \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_2^2 \\ &\leq \|u(t, \cdot) - \tilde{u}(t, \cdot)\|_{H^1}^2 \\ &\leq C(\|\partial_{sss} u(0, \cdot)\|_2, T) \|v(0, \cdot) - u(0, \cdot)\|_2^2. \end{aligned}$$

Combining (95) and (96), the proof of Theorem 1 in Case 3 is completed. \square

8. Proof of Theorem 2

We begin with a simpler example than the one stated in Theorem 2. This example in itself does not imply the kind of non-continuity of the flow map which we point at, but hopefully it raises the intuition sufficiently so that the actual example will appear natural.

Consider, for ε and p positive, the helix $\gamma_0 \equiv \gamma_{0,\varepsilon,p}$ of radius ε and pitch $2\pi p$ oriented along the e_1 direction:

$$\gamma_0(s) = (ps, \varepsilon \cos(s), \varepsilon \sin(s)),$$

where $\mathfrak{s} := s/\sqrt{\varepsilon^2 + p^2}$ so that γ_0 is parametrized by arc-length. Explicit computations yield the identity

$$\left(\partial_s \gamma_0 \times \partial_{ss} \gamma_0\right)(s) = -\frac{p}{\varepsilon^2 + p^2} \partial_s \gamma_0(s) + \frac{1}{\sqrt{\varepsilon^2 + p^2}}(1, 0, 0).$$

If we denote by $\gamma \equiv \gamma_{\varepsilon,p}$ the solution of the binormal curvature flow Equation (2) with initial datum γ_0 , then it follows that

$$\gamma(t, s) = \gamma_0(\xi) + \frac{t}{\sqrt{\varepsilon^2 + p^2}}(1, 0, 0),$$

where $\xi := s - pt/(\varepsilon^2 + p^2)$. In particular⁽¹⁸⁾

$$(97) \quad \gamma(t, 0) = \left(-\frac{p^2}{(\varepsilon^2 + p^2)^{\frac{3}{2}}} + \frac{1}{(\varepsilon^2 + p^2)^{\frac{1}{2}}}\right)(t, 0, 0) = \frac{\varepsilon^2}{(\varepsilon^2 + p^2)^{\frac{3}{2}}}(t, 0, 0).$$

Given $\sigma_0 > 0$, if we consider now sequences $(\varepsilon_m)_{m \in \mathbb{N}}$ and $(p_m)_{m \in \mathbb{N}}$ such that

$$\varepsilon_m \rightarrow 0, \quad p_m \rightarrow 0, \quad \text{and} \quad \varepsilon_m^2/p_m^3 \rightarrow \sigma_0,$$

as $m \rightarrow +\infty$, then the sequence $(\gamma_{0,\varepsilon_m,p_m})_{m \in \mathbb{N}}$ is bounded in $H_{\text{loc}}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}^3)$ and weakly converges in $H_{\text{loc}}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}^3)$ to the function $\gamma_0^*(s) := (s, 0, 0)$, which is a stationary solution of Equation (2). On the other hand, from (97) we infer that for every $t \in \mathbb{R}$ the sequence $(\gamma_{\varepsilon_m,p_m}(t, \cdot))_{m \in \mathbb{N}}$ weakly converges in $H_{\text{loc}}^{\frac{3}{2}}(\mathbb{R}, \mathbb{R}^3)$ to the function $\gamma_0^*(s + \sigma_0 t)$. If we require moreover that $1/\sqrt{\varepsilon_m^2 + p_m^2} \in \mathbb{N}$ (e.g. by choosing $\varepsilon_m^2 := \sigma_0 p_m^3$ and p_m as the unique positive solution of $\sigma_0 p_m^3 + p_m^2 = m^{-2}$) then the functions $u_n := \partial_s \gamma_{\varepsilon_m,p_m}$ are solutions of the Schrödinger map Equation (1) such that $u_m(0, \cdot)$ weakly converge in $H^{\frac{1}{2}}(T^1, S^2)$ as $m \rightarrow +\infty$ to the stationary solution $u^*(0, s) := (1, 0, 0)$ and such that $u_m(t, \cdot)$ weakly converge in $H^{\frac{1}{2}}(T^1, S^2)$ as $m \rightarrow +\infty$ to $u^*(t, \cdot + \sigma_0 t)$.

Of course here, u^* is a constant function of space and time, and in particular $u^*(t, \cdot + \sigma_0 t) = u^*(t, \cdot)$. In order to exhibit a sequence having the properties stated in Theorem 2, we build up on the previous example modifying it in such a way that the limit solution u^* is no longer constant but satisfies $u^*(t, s) = (\cos(s), \sin(s), 0)$. Roughly speaking, it suffices to wrap the original helix γ_0 around a fixed unit circle in the (e_1, e_2) -plane instead of letting it extend along the e_1 -axis. One can probably do so just by a simple ansatz and then some possibly more delicate analysis. Instead, we will rely on known explicit traveling wave solutions of (2) that behave exactly as wrapped helices. We describe them now.

In 1981, Kida [13] studied the set of initial curves in \mathbb{R}^3 for which the solution map of the binormal curvature flow equation reduces to a family of rigid motions. Using the symmetries of (2) and its conservation laws, such motions are necessarily the superposition of a constant speed rotation around a fixed axis (which we may always assume to be the x_3 -axis after a fixed rotation) and a constant speed translation parallel to that same axis. Following [13], we

⁽¹⁸⁾ The two fractions inside the large parenthesis correspond to what will be called $-C$ and Ω in the actual example later.

denote by Ω and V the speeds of rotation around e_3 and of translation along e_3 respectively, and by C the speed of the slipping motion of the curve along itself⁽¹⁹⁾, so that

$$\gamma(t, s) = \gamma(0, s - Ct) \cdot \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) & 0 \\ -\sin(\Omega t) & \cos(\Omega t) & 0 \\ 0 & 0 & 1 \end{pmatrix} + V(0, 0, t)$$

and therefore γ satisfies the additional equation

$$(98) \quad \partial_t \gamma = -C \partial_s \gamma + \Omega e_3 \times \gamma + V e_3.$$

Combining (2) and (98) we write

$$(99) \quad \partial_s \gamma \times \partial_{ss} \gamma = -C \partial_s \gamma + \Omega e_3 \times \gamma + V e_3.$$

Taking the scalar product of (99) with $\partial_s \gamma$ yields

$$(100) \quad -C + \Omega r^2 \theta' + V z' = 0,$$

where we wrote $(\gamma^1(0, \cdot), \gamma^2(0, \cdot)) =: (r(\cdot) \cos(\theta(\cdot)), r(\cdot) \sin(\theta(\cdot)))$ and $\gamma^3(0, \cdot) =: z(\cdot)$.

Taking the vector product of (99) with $\partial_s \gamma$ instead yields

$$(101) \quad z'' = -\Omega r r' = -\frac{\Omega}{2} R',$$

where $R = r^2$. After integration, (101) leads to

$$(102) \quad z' = \frac{\Omega}{2} [A - R]$$

for some integration constant $A \in \mathbb{R}$. Assume that $\Omega \neq 0$. Combining (100) and (102) we obtain

$$(103) \quad \theta' = \frac{1}{2} V + (C - \frac{1}{2} AV \Omega) / (\Omega R),$$

and then combining (2) with (102) and (103) we finally obtain

$$(104) \quad (R')^2 + f(R) = 0,$$

where

$$(105) \quad f(R) = \Omega^2 R^3 + (V^2 - 2A\Omega^2)R^2 + (4V(C - \frac{1}{2}AV\Omega)/\Omega + \Omega^2 A^2 - 4)R + 4(C - \frac{1}{2}AV\Omega)^2 / \Omega^2.$$

We require the parameters A, C, V, Ω to be such that f has two positive roots and one negative root, and we then write

$$(106) \quad f(R) =: \Omega^2 (R - \alpha)(R - \beta)(R + \delta)$$

where $0 < \beta \leq \alpha$ and $0 < \delta$. The solution of Equation (104) is given in terms of the Jacobi elliptic function sn by

$$(107) \quad R(s) = \alpha + (\beta - \alpha) \text{sn}^2\left(\frac{1}{2}(\alpha + \delta)^{\frac{1}{2}} \Omega s | k\right)$$

⁽¹⁹⁾ Even though the speed given by the binormal curvature flow equation is perpendicular to the tangent vector, in a non-orthogonal frame like $(\partial_s \gamma, \gamma \times e_3, e_3)$ the component C of $\partial_t \gamma$ along $\partial_s \gamma$ may not be zero. For the rigid motions we consider, C is a constant function of space and time.

where $k := ((\alpha - \beta)/(\alpha + \delta))^{\frac{1}{2}} \in [0, 1)$. Inserting (107) in (102) and then integrating in s yields

$$(108) \quad z(s) = z(0) + \frac{\Omega}{2}(A + \delta)s - (\alpha + \delta)^{\frac{1}{2}}E\left(\frac{1}{2}(\alpha + \delta)^{\frac{1}{2}}\Omega s|k\right),$$

where $E(u|k) := \int_0^u \operatorname{dn}^2(u', k) du' = \int_0^u (1 - k^2 \operatorname{sn}^2(u', k)) du'$ is the incomplete elliptic integral of second kind. Inserting (107) in (103) and then integrating in s yields

$$\theta(s) = \theta(0) + \frac{1}{2}Vs + \frac{2C - AV\Omega}{\alpha(\alpha + \delta)^{\frac{1}{2}}\Omega^2} \Pi\left(\frac{1}{2}(\alpha + \delta)^{\frac{1}{2}}\Omega s \left| \frac{\alpha - \beta}{\alpha}, k\right.\right),$$

where $\Pi(u|\ell, k) := \int_0^u (1 - \ell \operatorname{sn}^2(u', k))^{-1} du'$ is the incomplete elliptic integral of third kind. Notice that if $\alpha = \beta$ then $k = 0$, and if moreover $\alpha = \beta = A = 1$ then the corresponding solution γ is a circle of radius one in the (x_1, x_2) -plane. For $0 < k \ll 1$, in particular if $\alpha + \delta \simeq 1$ and $0 < \alpha - \beta \ll 1$, the Jacobi elliptic function $\operatorname{sn}(\cdot, k)$ resembles the sin function. We wish therefore to find coefficients A, C, V, Ω such that f vanishes on the negative axis, such that $\alpha + \delta \simeq 1$, $0 < \alpha - \beta \ll 1$, $\Omega \gg 1$, and $(\alpha - \beta)^2 \simeq \Omega^{-3}$, so that the corresponding solutions γ behave like the wrapped helices which we discussed at the beginning of this section, with pitch $\simeq 2\pi\Omega^{-1}$ and radius $\simeq \alpha - \beta$. We need also that $\partial_s \gamma$ is well-defined on T^1 . We explain the details now.

The function R is periodic with period

$$(109) \quad T_R := \frac{4K(k)}{(\alpha + \delta)^{\frac{1}{2}}\Omega}.$$

The function z satisfies

$$z(s + T_R) = z(s) + \frac{\Omega}{2}(A + \delta)T_R - 2(\alpha + \delta)^{\frac{1}{2}}E(k),$$

where $E(k) := E(K(k)|k)$ is the complete elliptic integral of the second kind. In particular

$$(110) \quad z'(s + T_R) = z'(s) \quad \forall s \in \mathbb{R}.$$

The function θ satisfies

$$(111) \quad \theta(s + T_R) = \theta(s) + \frac{1}{2}VT_R + 2\frac{2C - AV\Omega}{\alpha(\alpha + \delta)^{\frac{1}{2}}\Omega^2} \Pi\left(\frac{\alpha - \beta}{\alpha}, k\right)$$

where $\Pi(\ell, k) := \Pi(K(k)|\ell, k)$ is the complete elliptic integral of the third kind.

We fix $m \in \mathbb{N} \setminus \{0\}$ and we require that

$$(112) \quad \theta(s + T_R) = \theta(s) + \frac{2\pi}{m},$$

so that both R and $\exp(i\theta)$ are mT_R -periodic. Since z satisfies (110), we infer that if (112) holds, then $\partial_s \gamma$ is well-defined on T_ℓ^1 , where

$$\ell := mT_R.$$

At the end, we will also require that $\ell = 2\pi$, but we found more convenient to let ℓ as a free parameter at this stage and to use scaling properties at the very last step. In view of (111), our requirement (112) translates into

$$(113) \quad \frac{2K(k)V}{(\alpha + \delta)^{\frac{1}{2}}\Omega} + 2\frac{2C - AV\Omega}{\alpha(\alpha + \delta)^{\frac{1}{2}}\Omega^2} \Pi\left(\frac{\alpha - \beta}{\alpha}, k\right) = \frac{2\pi}{m}.$$

Equating the coefficients of order zero in R in (105) and (106), we obtain

$$(114) \quad C - \frac{1}{2}AV\Omega = \frac{\Omega^2}{2}(\alpha\beta\delta)^{\frac{1}{2}}.$$

Inserting (114) in (113), we obtain

$$(115) \quad \frac{V}{\Omega} = \left[\frac{2\pi}{m} - 2 \frac{\beta^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(\alpha+\delta)^{\frac{1}{2}}} \Pi\left(\frac{\alpha-\beta}{\alpha}, k\right) \right] \frac{(\alpha+\delta)^{\frac{1}{2}}}{2K(k)},$$

which determines the ratio V/Ω in terms of α, β and δ only⁽²⁰⁾.

Equating the coefficients of order two in R in (105) and (106), we obtain

$$(116) \quad A = \frac{1}{2} \left(\alpha + \beta - \delta + \frac{V^2}{\Omega^2} \right),$$

and therefore using (115) we obtain

$$(117) \quad A = \frac{1}{2} \left(\alpha + \beta - \delta + \left[\frac{2\pi}{m} - 2 \frac{\beta^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(\alpha+\delta)^{\frac{1}{2}}} \Pi\left(\frac{\alpha-\beta}{\alpha}, k\right) \right]^2 \frac{\alpha+\delta}{4K^2(k)} \right)$$

so that A too is determined in terms of α, β and δ only.

Finally, equating the coefficients of order one in R in (105) and (106), we obtain

$$(118) \quad \Omega^2(\alpha\beta - \alpha\delta - \beta\delta) = 4V(C - \frac{1}{2}AV\Omega)/\Omega + \Omega^2A^2 - 4,$$

and therefore using (114), (115) and (117) we obtain the equation

$$(119) \quad g(\alpha, \beta, \delta)\Omega^2 = 4,$$

where

$$\begin{aligned} g(\alpha, \beta, \delta) := & \frac{1}{4} \left(\alpha + \beta - \delta + \left[\frac{2\pi}{m} - 2 \frac{\beta^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(\alpha+\delta)^{\frac{1}{2}}} \Pi\left(\frac{\alpha-\beta}{\alpha}, k\right) \right]^2 \frac{\alpha+\delta}{4K^2(k)} \right)^2 \\ & + 2\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\delta^{\frac{1}{2}} \left[\frac{2\pi}{m} - 2 \frac{\beta^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}(\alpha+\delta)^{\frac{1}{2}}} \Pi\left(\frac{\alpha-\beta}{\alpha}, k\right) \right] \frac{(\alpha+\delta)^{\frac{1}{2}}}{2K(k)} \\ & - \alpha\beta + \alpha\delta + \beta\delta. \end{aligned}$$

Hence, Ω is determined in terms of α, β, δ , provided $g \neq 0$.

We assume $m \geq 2$ (we are ultimately interested only in the case $m \rightarrow +\infty$) and we start from the solution⁽²¹⁾ of (119) given by

$$\alpha = \beta := 1, \quad \delta := \frac{1}{m^2 - 1}, \quad \Omega := \sqrt{m^2 - 1},$$

which corresponds to

$$A = 1, \quad V = 1, \quad C = \sqrt{m^2 - 1},$$

and for which $\ell = mT_R = 2\pi$ and $g(\alpha, \beta, \delta) = g(1, 1, \frac{1}{m^2-1}) = \frac{4}{m^2-1}$. By continuity, g does not vanish for (α, β, δ) in a neighborhood⁽²²⁾ of $(1, 1, \frac{1}{m^2-1})$, and as long as it remains so,

⁽²⁰⁾ Recall that k is a function of α, β, δ .

⁽²¹⁾ Notice that $\Pi(0, 0) = K(0) = \pi/2$.

⁽²²⁾ A neighborhood in the sector $\{\alpha \geq \beta > 0\} \cap \{\delta > 0\}$, that is the domain of g .

Ω and then V , A and C successively defined by (119), (115), (117) and (114) are coefficients such that both (105) and (106) are satisfied and moreover (112) holds. Since we wish to be able to reach solutions of the form $(\alpha - \beta)^2 \simeq m^{-3}$ for m large, we need slightly more than just a continuity argument. For that purpose, we restrict our attention to the one-parameter family

$$\alpha := \alpha_{\epsilon, m} \equiv 1, \quad \beta := \beta_{\epsilon, m} \equiv 1 - \epsilon, \quad \delta := \delta_{\epsilon, m} \equiv \frac{1}{m^2 - 1},$$

where $\epsilon > 0$. We compute after elementary algebra

(120)

$$\begin{aligned} g(\alpha_{\epsilon, m}, \beta_{\epsilon, m}, \delta_{\epsilon, m}) &= \frac{4}{m^2 - 1} + \frac{\epsilon^2}{4} - \frac{\epsilon}{m^2 - 1} \\ &+ \frac{1}{4(m^2 - 1)^2} \left[\left(\frac{\pi - \sqrt{1 - \epsilon} \Pi(\epsilon, \epsilon \sqrt{m^2/(m^2 - 1)})}{K(\epsilon \sqrt{m^2/(m^2 - 1)})} \right)^2 - 1 \right]^2 \\ &+ \frac{1}{2} (2 - \epsilon) \frac{1}{m^2 - 1} \left[\left(\frac{\pi - \sqrt{1 - \epsilon} \Pi(\epsilon, \epsilon \sqrt{m^2/(m^2 - 1)})}{K(\epsilon \sqrt{m^2/(m^2 - 1)})} \right)^2 - 1 \right] \\ &+ \frac{2}{m^2 - 1} \left[\frac{\pi - \sqrt{1 - \epsilon} \Pi(\epsilon, \epsilon \sqrt{m^2/(m^2 - 1)})}{K(\epsilon \sqrt{m^2/(m^2 - 1)})} \sqrt{1 - \epsilon} - 1 \right]. \end{aligned}$$

Each of the squared brackets in the previous expression vanishes for $\epsilon = 0$, and therefore we may estimate them as $O(\epsilon)$ as $\epsilon \rightarrow 0$, independently of $m \geq 2$ (indeed $m^2/(m^2 - 1)$ remains bounded as $m \rightarrow +\infty$ and Π and K are continuously differentiable functions up to $(0, 0)$ and 0 respectively). Hence,

$$\left| g(\alpha_{\epsilon, m}, \beta_{\epsilon, m}, \delta_{\epsilon, m}) - \frac{4}{m^2 - 1} \right| = O(\epsilon^2) + O\left(\frac{\epsilon}{m^2}\right), \quad \text{as } \epsilon \rightarrow 0 \text{ and } m \rightarrow +\infty.$$

In particular, if

$$(121) \quad \epsilon \equiv \epsilon(m) = o\left(\frac{1}{m}\right) \quad \text{as } m \rightarrow +\infty,$$

then setting $\alpha_m := \alpha_{\epsilon(m), m}$, $\beta_m := \beta_{\epsilon(m), m}$, $\delta_m := \delta_{\epsilon(m), m}$, we have

$$g_m := g(\alpha_m, \beta_m, \delta_m) = \frac{1}{m^2 - 1} (4 + o(1)) \quad \text{as } m \rightarrow +\infty,$$

which yields

$$\Omega_m = \sqrt{m^2 - 1} (1 + o(1)) \quad \text{as } m \rightarrow +\infty,$$

and therefore from (109)

$$\ell_m = mT_R = 2\pi + o(1) \quad \text{as } m \rightarrow +\infty.$$

We are now in position to complete the proof of Theorem 2. For $\tilde{\sigma}_0 > 0$ fixed and for $m \geq 2$, we set $\epsilon(m) := \tilde{\sigma}_0 m^{-\frac{3}{2}}$. Provided m is sufficiently large, $g_m \neq 0$ and the above construction yields an initial data γ_m for (2) such that $\partial_s \gamma_m$ is periodic with period $2\pi + o(1)$ and such that the corresponding solution of (2) is a rigid motion consisting of a parallel translation at speed C_m along e_3 and a rotation of angular speed Ω_m around e_3 . By an homothety centered at zero and of factor $1 + o(1)$, we transform γ_m to $\tilde{\gamma}_m$ such that $\partial_s \tilde{\gamma}_m$ becomes exactly 2π -periodic. In view of (98) and the fact that $\partial_s \tilde{\gamma}_m = e_3 \times \tilde{\gamma}_m + o(1)$ in our example, we are particularly

interested in the asymptotic of $\Omega_m - C_m$, which will yield σ_0 in the statement of Theorem 2. Using (114), we write

$$\Omega_m - C_m = \Omega_m - \frac{1}{2}A_m \frac{V_m}{\Omega_m} \Omega_m^2 - \frac{\Omega_m^2}{2}(\alpha_m \beta_m \delta_m)^{\frac{1}{2}}.$$

We expand using (115), (117), (120) and (119),

$$A_m = 1 - \frac{1}{2}\tilde{\sigma}_0 m^{-\frac{3}{2}} + O(m^{-\frac{5}{2}}), \quad \frac{V_m}{\Omega_m} = \frac{1}{m} + O(m^{-\frac{3}{2}}), \quad \Omega_m^2 = m^2 - \frac{\tilde{\sigma}_0^2}{16}m + O(1),$$

$$\Omega_m = m - \frac{\tilde{\sigma}_0^2}{32} + O(m^{-1}), \quad \text{and} \quad (\alpha_m \beta_m \delta_m)^{\frac{1}{2}} = 1/m + O(m^{-5/2}),$$

as $m \rightarrow +\infty$, so that finally

$$\Omega_m - C_m = \frac{\tilde{\sigma}_0^2}{4} + o(1) \quad \text{as } m \rightarrow +\infty.$$

Therefore, for $\sigma_0 := \tilde{\sigma}_0^2/4$ the statement of Theorem 2 follows defining $u_{m,\sigma_0} := \partial_s \tilde{\gamma}_{m,\tilde{\sigma}_0}$. For negative σ_0 , it suffices to use the $\Omega \mapsto -\Omega$, $C \mapsto -C$ symmetry, and for the case $\sigma_0 = 0$ it suffices to consider the trivial example $u_{m,0} := u^*$. \square

Appendix

On the flow map for initial data in H^3

In this appendix, we briefly recall some properties of the flow map of (1) for initial data in $H^3(T_\ell^1, S^2)$ for some $\ell > 0$. Most if not all of them are presumably well-known, and we certainly do not claim originality nor optimality here. Nevertheless, since for some of them we could not always find the arguments which we wanted in the literature, we included sketch of proofs at the end of the section for the sake of completeness. We state them first for solutions that are a priori infinitely smooth; then the stability estimate of Corollary A.3 allows to extend some of them to solutions with initial data in H^3 .

We denote by C , followed possibly by some arguments between round parentheses, a numerical constant whose actual value may change from line to line, but to which a (sometimes complicated) explicit value could be attributed a posteriori, depending only on its arguments.

LEMMA A.1. – For a smooth solution u of (1), we have the following conserved quantities⁽²³⁾

$$E(u) := \int_{T^1} |\partial_s u|^2 ds \quad \text{and} \quad I(u) := \int_{T^1} |\partial_t u|^2 + |\partial_{ss} u|^2 - \frac{3}{4} |\partial_s u|^4 ds.$$

COROLLARY A.1. – For a smooth solution u of (1) with initial datum u^0 and for every $t \in \mathbb{R}$,

$$\|\partial_{ss} u(t, \cdot)\|_2^2 \leq 4\|\partial_{ss} u^0\|_2^2 + 2\|\partial_s u^0\|_2^6.$$

⁽²³⁾ As mentioned in the introduction, there are actually an infinite number of independent conserved quantities. The additional ones, involving higher and higher order derivatives of the solution, seem however to lack some form of coerciveness in order to control these derivatives.

LEMMA A.2. – For a smooth solution u of (1) and for every $t \in \mathbb{R}$,

$$\frac{d}{dt} (\|\partial_{sss}u\|_2^2 + \|\partial_{ts}u\|_2^2) \leq C(\ell) \|\partial_{ss}u\|_2^2 (\|\partial_{sss}u\|_2^2 + \|\partial_{ts}u\|_2^2).$$

COROLLARY A.2. – For a smooth solution u of (1) with initial data u^0 , for every $T > 0$ and for every $t \in [-T, T]$

$$\begin{aligned} \|\partial_{sss}u(t, \cdot)\|_2^2 &\leq C(\ell) \left(\|\partial_{sss}u^0\|_2^4 + \|\partial_{sss}u^0\|_2^2 \right) \exp \left(C(\ell) \left[\|\partial_s u^0\|_2^6 + \|\partial_{ss}u^0\|_2^2 \right] |t| \right) \\ &\leq C(\ell, \|\partial_{sss}u^0\|_2, T). \end{aligned}$$

LEMMA A.3. – Let u, \tilde{u} be two smooth solutions of (1). Set $w := u - \tilde{u}$. Then for every $t \in \mathbb{R}$,

$$\begin{aligned} \frac{d}{dt} \|w\|_2^2 &\leq C(\ell, \|\partial_{sss}u\|_2, \|\partial_{sss}\tilde{u}\|_2) \left(\|w\|_2^2 + \|\partial_s w\|_2^2 \right), \\ \frac{d}{dt} \|\partial_s w\|_2^2 &\leq C(\ell, \|\partial_{sss}u\|_2, \|\partial_{sss}\tilde{u}\|_2) \left(\|w\|_2^2 + \|\partial_s w\|_2^2 \right), \\ \frac{d}{dt} \left(\|\partial_{ss}w\|_2^2 + \|\partial_t w\|_2^2 \right) &\leq C(\ell, \|\partial_{sss}u\|_2, \|\partial_{sss}\tilde{u}\|_2) \left(\|\partial_{ss}w\|_2^2 + \|\partial_t w\|_2^2 \right). \end{aligned}$$

COROLLARY A.3. – Let u, \tilde{u} be two smooth solutions of (1) with initial data u^0 and \tilde{u}^0 respectively. Then for every $T > 0$ and for every $t \in [-T, T]$

$$\begin{aligned} \|u - \tilde{u}\|_{H^1}^2 &\leq C(\ell, \|\partial_{sss}u^0\|_2, \|\partial_{sss}\tilde{u}^0\|_2, T) \|u^0 - \tilde{u}^0\|_{H^1}^2, \\ \|u - \tilde{u}\|_{H^2}^2 &\leq C(\ell, \|\partial_{sss}u^0\|_2, \|\partial_{sss}\tilde{u}^0\|_2, T) \|u^0 - \tilde{u}^0\|_{H^2}^2. \end{aligned}$$

Corollary A.3 implies that the restriction of the flow map to smooth initial data is Lipschitzian both for the H^1 and the H^2 norm, with a Lipschitz constant depending on the H^3 norm⁽²⁴⁾. By approximation, the same remains true for the restriction of the flow map to initial data in H^3 . In turn, this implies that E and I are conserved quantities for initial data in H^3 , and that Corollary A.1 and A.2 hold for u and \tilde{u} in H^3 .

Proof of Lemma A.1. – This is direct differentiation. □

Proof of Corollary A.1. – Having in mind the conservation law for I , we first write

$$\int_{T^1} |\partial_s u|^4 ds \leq \int_{T^1} |\partial_s u^1|^4 + |\partial_s u^3|^4 + |\partial_s u^3|^4 ds,$$

and since $\partial_s u^j$ has zero mean for each $j \in \{1, 2, 3\}$, the Gagliardo-Nirenberg inequality yields (see Agueh [1] to compute the optimal⁽²⁵⁾ constant)

$$\int_{T^1} |\partial_s u^j|^4 \leq \frac{1}{\sqrt{3}} \left(\int_{T^1} |\partial_s u^j|^2 \right)^{\frac{3}{2}} \left(\int_{T^1} |\partial_{ss} u^j|^2 \right)^{\frac{1}{2}},$$

⁽²⁴⁾ It follows from McGahagan [15] that the Lipschitz constant of the flow map for the H^1 norm depends only on a bound on the H^2 norm (instead of the H^3 norm). In particular this allows to claim uniqueness and Lipschitz continuity of the flow map in H^2 for the H^1 norm. The proof in [15] is more involved than the direct proof of Corollary A.3 below, and the latter is sufficient for our needs here.

⁽²⁵⁾ And, in particular, “some”!

so that after summation

$$\int_{T^1} |\partial_s u|^4 ds \leq \sqrt{3} \left(\int_{T^1} |\partial_s u|^2 \right)^{\frac{3}{2}} \left(\int_{T^1} |\partial_{ss} u|^2 \right)^{\frac{1}{2}}.$$

Hence, using the conservation law for E ,

$$\begin{aligned} \|\partial_{ss} u(t, \cdot)\|_2^2 &\leq I(u) + \frac{3\sqrt{3}}{4} \|\partial_s u(t, \cdot)\|_2^3 \|\partial_{ss} u(t, \cdot)\|_2 \\ &\leq I(u) + \frac{3^3}{2^5} \|\partial_s u(t, \cdot)\|_2^6 + \frac{1}{2} \|\partial_{ss} u(t, \cdot)\|_2^2 \\ &= I(u) + \frac{3^3}{2^5} \|\partial_s u(0, \cdot)\|_2^6 + \frac{1}{2} \|\partial_{ss} u(t, \cdot)\|_2^2, \end{aligned}$$

and we may absorb the last term on the right-hand side by the left-hand side. Since conversely

$$I(u) = I(u(0, \cdot)) \leq \int_{T^1} |u(0, \cdot) \times \partial_{ss} u(0, \cdot)|^2 + |\partial_{ss} u(0, \cdot)|^2 ds \leq 2 \|\partial_{ss} u(0, \cdot)\|_2^2,$$

the conclusion follows. □

Proof of Lemma A.2. – Differentiating (1) with respect to t and using the identity $|u| \equiv 1$ yields the equation

$$(A.1) \quad \partial_{tt} u + \partial_{ssss} u = -2\partial_{ss} (|\partial_s u|^2 u) + \partial_s (|\partial_s u|^2 \partial_s u).$$

Differentiating (A.1) with respect to s , taking the L^2 -scalar product of the resulting equation by $\partial_{ts} u$ and integrating by parts leads to

$$(A.2) \quad \frac{d}{dt} \int_{T^1} \frac{|\partial_{ts} u|^2}{2} + \frac{|\partial_{ssss} u|^2}{2} ds \leq -2 \int_{T^1} \partial_{ss} (|\partial_s u|^2 u) \partial_{ts} u ds + \int_{T^1} \partial_{ss} (|\partial_s u|^2 \partial_s u) \partial_{ts} u ds.$$

We estimate the right-hand side of (A.2) by applying the chain-rule to the ∂_{ss} and ∂_{ss} operators involved. Among the resulting terms, the only one that cannot be straightforwardly bounded in absolute value by $C(\ell) \|\partial_{ss} u\|_2^2 (\|\partial_{ssss} u\|_2^2 + \|\partial_{ts} u\|_2^2)$ using Hölder’s inequality and Sobolev’s embedding is

$$D_1 := -2 \int_{T^1} \partial_{ss} (|\partial_s u|^2) u \partial_{ts} u ds.$$

We integrate by parts

$$D_1 = 2 \int_{T^1} \partial_{ss} (|\partial_s u|^2) \partial_s (u \partial_{ts} u) ds$$

and, in view of the equality $0 = \partial_t |u|^2 = u \partial_t u$, we develop

$$\partial_s (u \partial_{ts} u) = \partial_s u \partial_{ts} u + u \partial_{tss} u = \partial_s u \partial_{ts} u + u \partial_{tss} u - \partial_{ss} (u \partial_t u) = -\partial_s u \partial_{ts} u - \partial_{ss} u \partial_t u$$

so that

$$D_1 = -2 \int_{T^1} \partial_{ss} (|\partial_s u|^2) (\partial_s u \partial_{ts} u + \partial_{ss} u \partial_t u) ds$$

and here again Hölder’s inequality and Sobolev’s embedding yield the claim. □

Proof of Corollary A.2. – This follows from Lemma A.2, Corollary A.1 and the inequality

$$\|\partial_{ts}u(0, \cdot)\|_2^2 = \|\partial_s u^0 \times \partial_{ss} u^0 + u^0 \times \partial_{sss} u^0\|_2^2 \leq C(\ell) \|\partial_{ss} u^0\|_2^4 + 2\|\partial_{sss} u^0\|_2^2,$$

where we used Sobolev's embedding. \square

Proof of Lemma A.3. – We subtract (1) for u and \check{u} , and then also (A.1) for u and \check{u} . This yields

$$(A.3) \quad \partial_t w = w \times \partial_{ss} z + z \times \partial_{ss} w,$$

and

$$(A.4) \quad \partial_{tt} w + \partial_{ssss} w = -2\partial_{ss}(Qw) - 4\partial_{ss}((\partial_s z \cdot \partial_s w)z) + \partial_s(Q\partial_s w) + 2\partial_s((\partial_s z \cdot \partial_s w)\partial_s z),$$

where $z := (u + \check{u})/2$ and $Q := \frac{|\partial_s u|^2 + |\partial_s \check{u}|^2}{2}$. Multiplying (A.3) by w and integrating by parts on T^1 we obtain

$$\frac{d}{dt} \int_{T^1} \frac{|w|^2}{2} = \int_{T^1} (w \times z) \cdot \partial_{ss} w = - \int_{T^1} (w \times \partial_s z) \cdot \partial_s w$$

so that Hölder's inequality and Sobolev's embedding yield the desired claim at the L^2 level. Next, differentiating (A.3) with respect to s , multiplying the resulting equation by $\partial_s w$ and integrating by parts on T^1 we obtain, after some cancellation,

$$\frac{d}{dt} \int_{T^1} \frac{|\partial_s w|^2}{2} = \int_{T^1} (w \times \partial_{sss} z) \times \partial_s w,$$

and the desired claim at the H^1 level follows. Finally, multiplying (A.4) by $\partial_t w$ and integrating by parts on T^1 yields

$$\frac{d}{dt} \int_{T^1} \left(\frac{|\partial_{ss} w|^2}{2} + \frac{|\partial_t w|^2}{2} \right) = -2E_1 - 4E_2 + E_3 + 2E_4,$$

where

$$E_1 := \int_{T^1} \partial_{ss}(Qw) \partial_t w, \quad E_2 := \int_{T^1} \partial_{ss}((\partial_s z \cdot \partial_s w)z) \partial_t w, \\ E_3 := \int_{T^1} \partial_s(Q\partial_s w) \partial_t w, \quad E_4 := \int_{T^1} \partial_s((\partial_s z \cdot \partial_s w)\partial_s z) \partial_t w.$$

The terms E_1 , E_3 and E_4 can straightforwardly be treated by Hölder's inequality and Sobolev's embedding. We expand ∂_{ss} in E_2 through the chain-rule as

$$E_2 = \int_{T^1} (\partial_{sss} w \partial_s z)(z \partial_t w) + E_{2,1} + E_{2,2}$$

and once more $E_{2,1}$ and $E_{2,2}$ are treated by Hölder's inequality and Sobolev's embedding. Finally, since $0 = |u|^2 - |\check{u}|^2$, we have $z \partial_t w = -w \partial_t z$ and therefore

$$\int_{T^1} (\partial_{sss} w \partial_s z)(z \partial_t w) = - \int_{T^1} (\partial_{sss} w \partial_s z)(w \partial_t z) = \int_{T^1} \partial_{ss} w \partial_s (\partial_s z (w \partial_t z)),$$

which is now in a suitable form to be estimated as above. This yields the desired claim at the H^2 level. \square

Proof of Corollary A.3. – It follows directly from Lemma A.3, Corollary A.2 and the fact that

$$\begin{aligned} \|\partial_t u(0, \cdot) - \partial_t \check{u}(0, \cdot)\|_2^2 &= \frac{1}{4} \|(u^0 - \check{u}^0) \times \partial_{ss}(u^0 + \check{u}^0) + (u^0 + \check{u}^0) \times \partial_{ss}(u^0 - \check{u}^0)\|_2^2 \\ &\leq C(\ell, \|\partial_{sss} u^0\|_2, \|\partial_{sss} \check{u}^0\|_2) \|u^0 - \check{u}^0\|_{H^2}^2. \quad \square \end{aligned}$$

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Robert L. JERRARD
Department of Mathematics
University of Toronto
Toronto, Ontario M5S 2E4, Canada
E-mail: rjerrard@math.utoronto.ca

Didier SMETS
Laboratoire Jacques-Louis Lions
Université Pierre & Marie Curie
4 place Jussieu
BC 187
75252 Paris Cedex 05, France
E-mail: smets@ann.jussieu.fr