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THE STRONG ASYMPTOTIC FREENESS OF HAAR AND DETERMINISTIC MATRICES

BY BENOÎT COLLINS AND CAMILLE MALE

ABSTRACT. – In this paper, we are interested in sequences of q-tuple of $N \times N$ random matrices having a strong limiting distribution (i.e., given any non-commutative polynomial in the matrices and their conjugate transpose, its normalized trace and its norm converge). We start with such a sequence having this property, and we show that this property pertains if the q-tuple is enlarged with independent unitary Haar distributed random matrices. Besides, the limit of norms and traces in non-commutative polynomials in the enlarged family can be computed with reduced free product construction. This extends results of one author (C. M.) and of Haagerup and Thorbjørnsen. We also show that a p-tuple of independent orthogonal and symplectic Haar matrices have a strong limiting distribution, extending a recent result of Schultz. We mention a couple of applications in random matrix and operator space theory.

RÉSUMÉ. – Dans cet article, nous nous intéressons au q-tuple de matrices $N \times N$ qui ont une distribution limite forte (i.e., pour tout polynôme non commutatif en les matrices et leurs adjoints, sa trace normalisée et sa norme convergent). Nous partons d'une telle suite de matrices aléatoires et montrons que cette propriété persiste si on rajoute au q-tuple des matrices indépendantes unitaires distribuées suivant la mesure de Haar. Par ailleurs, la limite des normes et des traces en des polynômes non commutatifs en la suite élargie peut être calculée avec la construction du produit libre réduit. Ceci étend les résultats d'un des auteurs (C.M.) et de Haagerup et Thorbjørnsen. Nous montrons aussi qu'un p-tuple de matrices indépendantes orthogonales et symplectiques a une distribution limite forte, étendant par là-même un résultat de Schultz. Nous passons aussi en revue quelques applications de notre résultat aux matrices aléatoires et à la théorie des espaces d'opérateur.

1. Introduction and statement of the main results

Following random matrix notation, we call GUE the Gaussian Unitary Ensemble, i.e., any sequence $(X_N)_{N\geqslant 1}$ of random variables where X_N is an $N \times N$ selfadjoint random matrix whose distribution is proportional to the measure $\exp\big(-N/2\mathrm{Tr}(A^2)\big)\mathrm{d}A$, where $\mathrm{d}A$ denotes the Lebesgue measure on the set of $N \times N$ Hermitian matrices. We call a unitary Haar matrix

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of size N any random matrix distributed according to the Haar measure on the compact group of N by N unitary matrices.

We recall for readers' convenience the following definitions from free probability theory (see [4, 20]).

DEFINITION 1.1. – 1. *A* $\mathcal{C}^*$-probability space $(\mathcal{A}, .^*, \tau, \|\cdot\|)$ *consists of a unital* C^**-algebra* $(\mathcal{A}, .^*, \|\cdot\|)$ *endowed with a state* τ*, i.e., a linear map* $\tau : \mathcal{A} \to \mathbb{C}$ *satisfying* $\tau[\mathbf{1}_\mathcal{A}] = 1$ *and* $\tau[aa^*] \geqslant 0$ *for all* a *in* $\mathcal{A}$*. In this paper, we always assume that* τ *is a trace, i.e., that it satisfies* $\tau[ab] = \tau[ba]$ *for every* a, b *in* $\mathcal{A}$*. An element of* $\mathcal{A}$ *is called a (noncommutative) random variable. A trace is said to be* faithful *if* $\tau[aa^*] > 0$ *whenever* $a \neq 0$*. If* τ *is faithful, then for any* a *in* $\mathcal{A}$*,*

$$\|a\| = \lim_{k\to\infty} \Big(\tau\big[(a^*a)^k\big]\Big). \tag{1.1}$$

2. *Let* $\mathcal{A}_1, \dots, \mathcal{A}_k$ *be* $*$*-subalgebras of* $\mathcal{A}$ *having the same unit as* $\mathcal{A}$*. They are said to be* free *if for all* $a_i \in \mathcal{A}_{j_i}$ $(i = 1, \dots, k,\ j_i \in \{1, \dots, k\})$ *such that* $\tau[a_i] = 0$*, one has*

$$\tau[a_1 \cdots a_k] = 0$$

as soon as $j_1 \neq j_2, j_2 \neq j_3, \dots, j_{k-1} \neq j_k$*. Collections of random variables are said to be free if the unital subalgebras they generate are free.*

3. *Let* $\mathbf{a} = (a_1, \dots, a_k)$ *be a* k*-tuple of random variables. The joint distribution of the family* $\mathbf{a}$ *is the linear form* $P \mapsto \tau\big[P(\mathbf{a}, \mathbf{a}^*)\big]$ *on the set of polynomials in* $2k$ *noncommutative indeterminates. By* convergence in distribution*, for a sequence of families of variables* $(\mathbf{a}_N)_{N\geqslant 1} = (a_1^{(N)}, \dots, a_k^{(N)})_{N\geqslant 1}$ *in* $\mathcal{C}^*$*-algebras* $\big(\mathcal{A}_N, .^*, \tau_N, \|\cdot\|\big)$*, we mean the pointwise convergence of the map*

$$P \mapsto \tau_N\big[P(\mathbf{a}_N, \mathbf{a}_N^*)\big],$$

and by strong convergence in distribution*, we mean convergence in distribution, and pointwise convergence of the map*

$$P \mapsto \big\|P(\mathbf{a}_N, \mathbf{a}_N^*)\big\|.$$

4. *A family of noncommutative random variables* $\mathbf{x} = (x_1, \dots, x_p)$ *is called a* free semicircular system *when the noncommutative random variables are free, selfadjoint* $(x_i = x_i^*$*,* $i = 1, \dots, p)$*, and for all* k *in* $\mathbb{N}$ *and* $i = 1, \dots, p$*, one has*

$$\tau[x_i^k] = \int t^k d\sigma(t),$$

with $d\sigma(t) = \frac{1}{2\pi}\sqrt{4-t^2}\,\mathbf{1}_{|t|\leqslant 2}\,dt$ *the semicircle distribution.*

5. *A noncommutative random variable* u *is called a* Haar unitary *when it is unitary* $(uu^* = u^*u = \mathbf{1}_\mathcal{A})$ *and for all* n *in* $\mathbb{N}$*, one has*

$$\tau[u^n] = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In their seminal paper [14], Haagerup and Thorbjørnsen proved the following result.

THEOREM 1.2 ([14] The strong asymptotic freeness of independent GUE matrices)
For any integer $N \geqslant 1$, *let* $X_1^{(N)}, \dots, X_p^{(N)}$ *be* $N \times N$ *independent GUE matrices and let* $(x_1, \dots, x_p)$ *be a free semicircular system in a* $\mathcal{C}^*$*-probability space with faithful state. Then, almost surely, for all polynomials* P *in* p *noncommutative indeterminates, one has*

$$\big\|P(X_1^{(N)}, \dots, X_p^{(N)})\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(x_1, \dots, x_p)\big\|,$$

where $\|\cdot\|$ *denotes the operator norm in the left hand side and the norm of the* $\mathcal{C}^*$*-algebra in the right hand side.*

This theorem is a very deep result in random matrix theory, and had an important impact. Firstly, it had significant applications to C^*-algebra theory [14, 21], and more recently to quantum information theory [5, 8]. Secondly, it was generalized in many directions. Schultz [24] has shown that Theorem 1.2 is true when the GUE matrices are replaced by matrices of the Gaussian Orthogonal Ensemble (GOE) or by matrices of the Gaussian Symplectic Ensemble (GSE). Capitaine and Donati-Martin [6] and, very recently, Anderson [3] have shown the analogue for certain Wigner matrices.

Another significant extension of Haagerup and Thorbjørnsen's result was obtained by one author (C. M.) in [18], where he showed that if in addition to independent GUE matrices, one also has an extra family of independent matrices with strong limiting distribution, the result still holds.

THEOREM 1.3 ([18] The strong asymptotic freeness of $\mathbf{X}_N, \mathbf{Y}_N$)
For any integer $N \geqslant 1$, *we consider*

- *a* p*-tuple* $\mathbf{X}_N$ *of* $N \times N$ *independent GUE matrices,*
- *a* q*-tuple* $\mathbf{Y}_N$ *of* $N \times N$ *matrices, possibly random but independent of* $\mathbf{X}_N$.

The above random matrices live in the $\mathcal{C}^*$*-probability space* $(\mathrm{M}_N(\mathbb{C}), .^*, \tau_N, \|\cdot\|)$, *where* τ_N *is the normalized trace on the set* $\mathrm{M}_N(\mathbb{C})$ *of* $N \times N$ *matrices. In a* $\mathcal{C}^*$*-probability space* $(\mathcal{A}, .^*, \tau, \|\cdot\|)$ *with faithful trace, we consider*

- *a free semicircular system* $\mathbf{x}$ *of* p *variables,*
- *a* q*-tuple* $\mathbf{y}$ *of noncommutative random variables, free from* $\mathbf{x}$.

If $\mathbf{y}$ *is the strong limit in distribution of* $\mathbf{Y}_N$, *then* $(\mathbf{x}, \mathbf{y})$ *is the strong limit in distribution of* $(\mathbf{X}_N, \mathbf{Y}_N)$.

In other words, if we assume that almost surely, for all polynomials P in $2q$ noncommutative indeterminates, one has

$$\tau_N\big[P(\mathbf{Y}_N, \mathbf{Y}_N^*)\big] \underset{N\to\infty}{\longrightarrow} \tau\big[P(\mathbf{y}, \mathbf{y}^*)\big], \tag{1.2}$$

$$\big\|P(\mathbf{Y}_N, \mathbf{Y}_N^*)\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(\mathbf{y}, \mathbf{y}^*)\big\|, \tag{1.3}$$

then, almost surely, for all polynomials P in $p+2q$ noncommutative indeterminates, one has

$$\tau_N\big[P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\big] \underset{N\to\infty}{\longrightarrow} \tau\big[P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\big], \tag{1.4}$$

$$\big\|P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)\big\| \underset{N\to\infty}{\longrightarrow} \big\|P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)\big\|. \tag{1.5}$$

The convergence in distribution, stated in (1.4), is the content of Voiculescu's asymptotic freeness theorem. We refer to [4, Theorem 5.4.10] for the original statement and for a proof. An alternative way to state (1.5) is the following interversion of limits: for any matrix $H_N = P(\mathbf{X}_N, \mathbf{Y}_N, \mathbf{Y}_N^*)$, where P is a fixed polynomial, if we denote $h = P(\mathbf{x}, \mathbf{y}, \mathbf{y}^*)$, then by the definition of the norm in terms of the state (1.1),

$$\lim_{N\to\infty} \lim_{k\to\infty} \Big(\tau_N \big[(H_N^* H_N)^k \big] \Big)^{\frac{1}{2k}} = \lim_{k\to\infty} \Big(\tau \big[(h^* h)^k \big] \Big)^{\frac{1}{2k}}.$$

It is natural to wonder whether, instead of GUE matrices, the same property holds for unitary Haar matrices. The main result of this paper is the following theorem.

THEOREM 1.4 (The strong asymptotic freeness of $U_1^{(N)}, \dots, U_p^{(N)}, \mathbf{Y}_N$)
For any integer $N \geqslant 1$, we consider

- *a p-tuple $\mathbf{U}_N$ of $N \times N$ independent unitary Haar matrices,*
- *a q-tuple $\mathbf{Y}_N$ of $N \times N$ matrices, possibly random but independent of $\mathbf{U}_N$.*

In a $\mathscr{C}^*$-probability space $(\mathscr{A}, .^*, \tau, \|\cdot\|)$ with faithful trace, we consider

- *a p-tuple $\mathbf{u}$ of free Haar unitaries,*
- *a q-tuple $\mathbf{y}$ of noncommutative random variables, free from $\mathbf{u}$.*

If $\mathbf{y}$ is the strong limit in distribution of $\mathbf{Y}_N$, then $(\mathbf{u}, \mathbf{y})$ is the strong limit in distribution of $(\mathbf{U}_N, \mathbf{Y}_N)$.

In order to solve this problem, it looks at first sight natural to attempt to mimic the proof of [14] and write a Master equation in the case of unitary matrices. However, even though such an identity can be obtained for unitary matrices, it is very difficult to manipulate it in the spirit of [14] in order to obtain the desired norm convergence. Part of the problem is that the unitary matrices are not selfadjoint, unlike the GUE matrices considered in [14], and in this context the linearization trick and the identities do not seem to fit well together. In order to bypass this problem, in this paper, we take a completely different route by building on Theorem 1.3 and using a series of folklore facts of classical probability and random matrix theory.

Our method applies to prove the strong convergence in distribution of Haar matrices on the orthogonal and the symplectic groups by building on the result of Schultz [24], which is the analogue of Theorem 1.2 for GOE or GSE matrices instead of GUE matrices. The analogue of Theorem 1.3 does not exist yet. If one shows that the estimates of matrix valued Stieltjes transforms in [18] can always be performed with the additional terms in the estimate of [24], then, following the lines of this paper, one gets Theorem 1.3 for Haar matrices on the orthogonal and the symplectic groups, instead of the unitary group only. Therefore, in the following Theorem, we stick to proving the strong convergence of independent unitary, orthogonal or symplectic Haar matrices, without "constant" matrices $\mathbf{Y}$:

THEOREM 1.5 (The strong asymptotic freeness of independent Haar matrices)
For any integer $N \geqslant 1$, let $U_1^{(N)}, \dots, U_p^{(N)}$ be a family of independent Haar matrices of one of the three classical groups. Let $u_1, \dots, u_p$ be free Haar unitaries in a $\mathscr{C}^*$-probability space with faithful state. Then, almost surely, for all polynomials P in $2p$ noncommutative indeterminates, one has

$$\big\| P(U_1^{(N)}, \dots, U_p^{(N)}, U_1^{(N)*}, \dots, U_p^{(N)*}) \big\| \underset{N\to\infty}{\longrightarrow} \big\| P(u_1, \dots, u_p, u_1^*, \dots, u_p^*) \big\|,$$

where $\|\cdot\|$ denotes the operator norm in the left hand side and the $\mathscr{C}^*$-algebra in the right hand side.

Our paper is organized as follows. Section 2 consists of applications of the results stated above. Among other examples, we show that the limit of complicated random matrix models involving unitary random matrices have norms that converge towards (or are bounded by) values predicted by the theory of free probability. Sections 3 and 4 provide the proofs of Theorem 1.4 and Theorem 1.5 respectively. Section 5 is dedicated to the proof of Corollary 2.2, stated in the next section.

2. Applications

Our main result has the potential for many applications.

2.1. The spectrum of Hermitian random matrices

2.1.1. *Generalities on the strong convergence in distribution.* – We first recall for convenience some facts about the strong convergence in distribution, mainly an equivalent formulation. Given a self-adjoint variable h in a $\mathscr{C}^*$-probability space $(\mathcal{A}, .^*, \tau, \|\cdot\|)$, its spectral distribution μ_h is the unique probability measure that satisfies $\tau[h^k] = \int t^k \mathrm{d}\mu(t)$ for any $k \geqslant 1$. This measure has compact support included in $\big[-\|h\|, \|h\|\big]$. For any continuous map $f : \mathbb{R}$ to $\mathbb{C}$, the variable $f(h)$ is given by functional calculus, and coincides with the limit of $\big(P_n(h)\big)_{n\geqslant 1}$ in $\mathcal{A}$, where $(P_n)_{n\geqslant 1}$ is any Weierstrass's approximation of f by polynomials. Given a (non self-adjoint) variable x in $\mathcal{A}$, we set the self-adjoint variables $\Re x = \Big(\frac{x+x^*}{2}\Big)$ and $\Im x = \Big(\frac{x-x^*}{2i}\Big)$, so that $x = \Re x + i\Im x$.

PROPOSITION 2.1 (The strong convergence in distribution of self adjoint random variables)

Let $\mathbf{x}_N = (x_1^{(N)}, \ldots, x_p^{(N)})$ and $\mathbf{x} = (x_1, \ldots, x_p)$ be p-tuples of variables in $\mathscr{C}^*$-probability spaces, $(\mathcal{A}_N, .^*, \tau_N, \|\cdot\|)$ and $(\mathcal{A}, .^*, \tau, \|\cdot\|)$, with faithful states. Then, the following assertions are equivalent.

1. *$\mathbf{x}_N$ converges strongly in distribution to $\mathbf{x}$,*
2. *for any continuous map $f_i, g_i : \mathbb{R} \to \mathbb{C}$, $i = 1, \ldots, p$, the family of variables $\big(f_1(\Re x_1^{(N)}), g_1(\Im x_1^{(N)}), \ldots, f_p(\Re x_p^{(N)}), g_p(\Im x_p^{(N)})\big)$ converges strongly in distribution to $\big(f_1(\Re x_1), g_1(\Im x_1), \ldots, f_p(\Re x_p), g_p(\Im x_p)\big)$,*
3. *for any self-adjoint variable $h_N = P(\mathbf{x}_N)$, where P is a fixed polynomial, μ_{h_N} converges in weak-$*$ topology to μ_h where $h = P(\mathbf{x})$. Weak-$*$ topology means relatively to continuous functions on $\mathbb{C}$. Moreover, the support of μ_{h_N} converges in Hausdorff distance to the support of μ_h, that is: for any $\varepsilon > 0$, there exists N_0 such that for any $N \geqslant N_0$,*

$$\mathrm{Supp}\big(\mu_{h_N}\big) \subset \mathrm{Supp}\big(\mu_h\big) + (-\varepsilon, \varepsilon). \tag{2.1}$$

The symbol Supp *means the support of the measure.*

In particular, the strong convergence in distribution of a single self-adjoint variable is its convergence in distribution together with the Hausdorff convergence of its spectrum.

Proof. – Assuming (1), the Assertion (2) is obtained by Weierstrass's approximation of the functions f_i and g_i by polynomials in p complex variables on the centered ball of radius $\sup_{N\geqslant 0}\|x_N\|$. The converse is obvious.

Assuming (1), let us show (3). By Weierstrass's approximation, h_N converges strongly in distribution to h. The convergence in distribution of h_N to h implies the weak-$*$ convergence of μ_{h_N} to μ_h. For any $\varepsilon > 0$, let f_ε be a continuous map which takes the value 1 on the complementary of Supp $(\mu_h) + (-\varepsilon, \varepsilon)$ and 0 on Supp (μ_h). Then,

$$\|f_\varepsilon(\mathbf{x}_N)\| \underset{N\to\infty}{\longrightarrow} \|f_\varepsilon(\mathbf{x})\| = \lim_k \Big(\int f_\varepsilon(\mathbf{x})^k \mathrm{d}\mu_h\Big)^{\frac{1}{k}} = 0.$$

Hence, the support of μ_{h_N} is a subset of Supp $(\mu_h)+(-\varepsilon,\varepsilon)$ for N large enough, as otherwise one could have $\|f_\varepsilon(\mathbf{x}_N)\| = 1$ eventually.

Assuming (3), let us show (1). Let P be a polynomial in p variables and their conjugate. Denote $m_N = P(\mathbf{x}_N, \mathbf{x}_N^*)$ and $m = P(\mathbf{x}, \mathbf{x}^*)$. Then,

$$\tau_N[m_N] - \tau[m] = \tau_N[Q(\mathbf{x}_N, \mathbf{x}_N^*)] - \tau[Q(\mathbf{x}, \mathbf{x}^*)] + i(\tau_N[R(\mathbf{x}_N, \mathbf{x}_N^*)] - \tau[R(\mathbf{x}, \mathbf{x}^*)])$$

where $Q = \frac{1}{2}(P + P^*)$ and $R = \frac{1}{2i}(R - R^*)$ gives Hermitian variables. By the Assertion (3) and since the matrices are uniformly bounded in operator norm, we get the convergence in moments of the spectral distribution of $Q(\mathbf{x}_N, \mathbf{x}_N^*)$ and $R(\mathbf{x}_N, \mathbf{x}_N^*)$. Hence, we get the convergence in distribution of x_N to x. Then, the convergence holds in weak-$*$ topology since μ_h has bounded support. Furthermore,

$$\|m_N\|^2 = \|m_N^* m_N\| = \max \mathrm{Supp}\,\big(\mu_{m_N^* m_N}\big) \underset{N\to\infty}{\longrightarrow} \max \mathrm{Supp}\,\big(\mu_{m^* m}\big) = \|m^* m\| = \|m\|^2.$$

□

2.1.2. *The spectra of the sum and product of unitary invariant matrices.* – The following is a consequence of our main result:

COROLLARY 2.2. – *Let A_N, B_N be two $N \times N$ independent Hermitian random matrices. Assume that:*

1. *the law of one of the matrices is invariant under unitary conjugacy,*
2. *almost surely, the empirical eigenvalue distribution of A_N (respectively B_N) converges to a compactly supported probability measure μ (respectively ν),*
3. *almost surely, for any neighborhood of the support of μ (respectively ν), for N large enough, the eigenvalues of A_N (respectively B_N) belong to the respective neighborhood.*

Then, one has

- *almost surely, for N large enough, the eigenvalues of $A_N + B_N$ belong to a small neighborhood of the support of $\mu \boxplus \nu$, where $\boxplus$ denotes the free additive convolution (see* [20, Lecture 12]*).*
- *if moreover B_N is nonnegative, then the eigenvalues of $(B_N)^{1/2} A_N (B_N)^{1/2}$ belong to a small neighborhood of the support of $\mu \boxtimes \nu$, where $\boxtimes$ denotes the free multiplicative convolution (see* [20, Lecture 14]*).*

Corollary 2.2 is proved in Section 5. It can be applied in the following situation. Let A_N be an $N \times N$ Hermitian random matrix whose law is invariant under unitary conjugacy. Assume that, almost surely, the empirical eigenvalue distribution of A_N converges to a compactly supported probability measure μ and its eigenvalues belong to the support of μ for N large enough. Let Π_N be the matrix of the projection on first p_N coordinates, $\Pi_N = \mathrm{diag}\,(\mathbf{1}_{p_N}, \mathbf{0}_{N-p_N})$, where $p_N \sim tN$, $t \in (0,1)$. We consider the empirical eigenvalue distribution μ_N of the Hermitian random matrix

$$\Pi_n A_n \Pi_n.$$

Then, it follows from a theorem of Voiculescu [26] (see also [7]) that almost surely μ_N converges weakly to the probability measure $\mu^{(t)} = \mu \boxtimes [(1-t)\delta_0 + t\delta_1]$. This distribution is important in free probability theory because of its close relationship to the free additive convolution semigroup (see [20, Exercise 14.21]). Besides, the empirical eigenvalue distribution μ_N was proved to be a determinantal point process obtained as the push forward of a uniform measure in a Gelfand-Cetlin cone [9]. Very recently, it was proved by Metcalfe [19] that the eigenvalues satisfy universality property inside the bulk of the spectrum. Our result complements his, by showing that almost surely, for N large enough there is no eigenvalue outside of any neighborhood of the spectrum of $\mu^{(t)}$.

2.2. Questions from operator space theory

We present some examples of norms of large matrices we can compute by Theorem 1.4, as the norm of the limiting variables have been computed by other authors.

2.2.1. *The norm of the sum of unitary Haar matrices*. – The following question was raised by Gilles Pisier to one author (B.C.) ten years ago: let $U_1^{(N)}, \ldots, U_p^{(N)}$ be $N \times N$ independent unitary Haar random matrices, $p \geqslant 2$. Is it true that almost surely:

$$\Big\| \sum_{i=1}^{p} U_i^{(N)} \Big\| \underset{N\to\infty}{\longrightarrow} 2\sqrt{p-1}. \tag{2.2}$$

This question is very natural from the operator space theory point of view [21, Chapter 20], and was still open before this paper. Haagerup and Thorbjørnsen's theorem [14] have proved that the convergence (2.2) is true when $U_1^{(N)}, \ldots, U_p^{(N)}$ are certain sequence of independent large unitary matrices (non Haar distributed). Our main theorem implies that (2.2) is true almost surely when they are i.i.d. unitary Haar matrices. Indeed, $2\sqrt{p-1}$ is the norm of the sum of p free Haar unitaries by a result of Akemann and Ostrand [1]: they have proved that if u_i are the generators of the free group von Neumann algebra, then

$$\Big\| \sum_{i=1}^{p} a_i u_i \Big\| = \min_{t\geqslant 0} \Big\{ 2t + \sum_{i=1}^{p} \big(\sqrt{t^2 + |a_i|^2} - t \big) \Big\}. \tag{2.3}$$

And if $a_1 = \ldots = a_p = 1$ they prove that the minimizer of the right hand side is $2\sqrt{p-1}$.

By Theorem 1.5 and (2.3), we get that, for independent Haar matrices $U_1^{(N)}, \ldots, U_p^{(N)}$ on the orthogonal, unitary or symplectic group, almost surely one has

$$\Big\| \sum_{i=1}^{p} a_i U_i^{(N)} \Big\| \underset{N\to\infty}{\longrightarrow} \min_{t\geqslant 0} \Big\{ 2t + \sum_{i=1}^{p} \big(\sqrt{t^2 + |a_i|^2} - t \big) \Big\},$$

which is a generalization of (2.2).

2.2.2. *The sum of Haar matrices along with their conjugate*. – In the same vein, by a result of Kesten [16], the norm of the sum of p free Haar unitaries and of their conjugate equals $2\sqrt{2p-1}$. Hence, we get from our result that almost surely one has

$$\Big\| \sum_{i=1}^{p} \big(U_i^{(N)} + U_i^{(N)*}\big) \Big\| \underset{N\to\infty}{\longrightarrow} 2\sqrt{2p-1}.$$

Remark that this result is not true for random unitary matrices distributed according to the uniform measure on the set of permutation matrix. Indeed, in that case $2p$ is always an eigenvalue of the matrix since $\sum_i (U_i^{(N)} + U_i^{(N)*})$ is the adjacency matrix of a $2p$-regular graph. The convergence of the second largest eigenvalue to $2\sqrt{2p-1}$, known as Alon's conjecture [2], has been proved recently by Friedman [12].

2.2.3. *The sum of Haar matrices, matrix valued case*. – Lehner [17] has proved that for $u_1, \dots, u_p$ free Haar unitaries and $a_0, a_1, \dots, a_p$ Hermitian k by k matrices

$$\Big\| a_0 \otimes \mathbf{1} + \sum_{i=1}^{p} a_i \otimes u_i \Big\| = \inf_{b>0} \Big\| b^{\frac12} \Big(\big(\mathbf{1}_k + (b^{-\frac12} a_i b^{-\frac12})^2\big)^{\frac12} - \mathbf{1}_k \Big) b^{\frac12} \Big\|, \tag{2.4}$$

where the infimum is over all positive definite invertible k by k matrices b. Recall that from Theorem 1.5 we can deduce the following corollary (see [18, Proposition 7.3] for a proof).

Corollary 2.3. – *Let $\mathbf{U}_N$ be a family of independent Haar matrices of one of the three classical groups. Let $\mathbf{u}$ be a family of free Haar unitaries. Let $k \geqslant 1$ be an integer. Then, for any polynomial P with coefficients in $\mathrm{M}_k(\mathbb{C})$, almost surely one has*

$$\|P(\mathbf{U}_N, \mathbf{U}_N^*)\| \underset{N\to\infty}{\longrightarrow} \|P(\mathbf{u}, \mathbf{u}^*)\|,$$

where $\|\cdot\|$ stands in the left hand side for the operator norm in $\mathrm{M}_{kN}(\mathbb{C})$ and in the right hand side for the $\mathcal{C}^*$-algebra norm in $\mathrm{M}_k(\mathcal{A})$.

We then deduce that the norm of block matrices of the form $a_0 \otimes \mathbf{1} + \sum_{i=1}^{p} a_i \otimes U_i^{(N)}$, where $a_0, \dots, a_1$ are Hermitian matrices, converges almost surely to the quantity (2.4) computed by Lehner.

2.2.4. *Application of Fell's absorption principle*. – For another application of Corollary 2.3, recall Fell's absorption principle [21, Proposition 8.1]: for any k by k unitary matrices $a_1, \dots, a_p$ and $u_1, \dots, u_p$ free Haar unitaries, one has

$$\Big\| \sum_{i=1}^{p} a_i \otimes u_i \Big\| = \Big\| \sum_{i=1}^{p} u_i \Big\| = 2\sqrt{p-1}.$$

By Corollary 2.3 we get for any $k \times k$ unitary matrices $a_1, \dots, a_p$, almost surely one has

$$\Big\| \sum_{i=1}^{p} a_i \otimes U_i^{(N)} \Big\| \underset{N\to\infty}{\longrightarrow} 2\sqrt{p-1},$$

which solves a question of Pisier in [21, Chapter 20].

2.3. Estimates on the norm of random matrices

2.3.1. *Haagerup's inequalities.* – Let $\mathbf{u} = (u_1, \ldots, u_p)$ be free Haar unitaries in a $\mathscr{C}^*$-probability space $(\mathscr{A}, .^*, \tau, \| \cdot \|)$ with faithful state. For any integer $d \geqslant 1$, we denote by W_d the set of reduced *-monomials in p indeterminates $\mathbf{x} = (x_1, \ldots, x_p)$ of length d:

$$W_d = \Big\{ P = x_{j_1}^{\varepsilon_1} \ldots x_{j_d}^{\varepsilon_d} \ \Big| \ j_1 \neq \cdots \neq j_d, \ \varepsilon_j \in \{1, *\} \ \forall j = 1, \ldots, d \Big\}.$$

In 1979, Haagerup [13] has shown that one has

$$\Big\| \sum_{n \geqslant 1} \alpha_n P_n(\mathbf{u}) \Big\| \leqslant (d+1) \|\alpha\|_2, \tag{2.5}$$

for any sequence $(P_n)_{n \geqslant 1}$ of elements in W_d and sequence $\alpha = (\alpha_n)_{n \geqslant 1}$ of complex numbers whose ℓ^2-norm is denoted by

$$\|\alpha\|_2 = \sqrt{\sum_{n \geqslant 1} |\alpha|^2}.$$

This result, known as Haagerup's inequality, has many applications (for example, estimates of return probabilities of random walks on free groups) and has been generalized in many ways. For instance, Buchholz has generalized (2.5) in an estimate of $\sum_{n \geqslant 1} a_n \otimes x_n$, where the a_n are now $k \times k$ matrices. Let $\mathbf{U}_N$ be a family of p independent $N \times N$ unitary Haar matrices. As a byproduct of our main result, we get that

$$\lim_{N \to \infty} \| \sum_{n \geqslant 1} \alpha_n P_n(\mathbf{U}_N) \| \leqslant (d+1) \|\alpha\|_2,$$

where for any $n \geqslant 1$, the polynomial P_n is in W_d.

2.3.2. *Kemp and Speicher's inequality.* – Kemp and Speicher [15] have generalized Haagerup's inequality for $\mathscr{R}$-diagonal elements in the so-called holomorphic case (with polynomials in the variables, but not their adjoint). Theorem 1.4 established, the consequence for random matrices sounds relevant since it allows to consider combinations of Haar and deterministic matrices, and then get a bound for its operator norm. The result of [15] we state below has been generalized by de la Salle [23] in the case where the noncommutative random variables have matrix coefficients. This situation could be interesting for practical applications, where block random matrices are sometimes considered (see [25] for applications of random matrices in telecommunication). Nevertheless, we only consider the scalar version for simplicity.

Recall that a noncommutative random variable a is called an $\mathscr{R}$-diagonal element if it can be written $a = uy$, for u a Haar unitary free from y (see [20]). Let $\mathbf{a} = (a_1, \ldots, a_p)$ be a family of free, identically distributed $\mathscr{R}$-diagonal elements in a $\mathscr{C}^*$-probability space $(\mathscr{A}, .^*, \tau, \| \cdot \|)$. We denote by W_d^+ the set of reduced monomials of length d in variables $\mathbf{x}$ (and not its conjugate), i.e.,

$$W_d^+ = \Big\{ x_{j_1} \ldots x_{j_d} \ \Big| \ j_1 \neq \cdots \neq j_d \Big\}.$$

Kemp and Speicher have shown the following, where the interesting fact is that the constant $(d+1)$ is replaced by a constant of order $\sqrt{d+1}$: for any sequence $(P_n)_{n \geqslant 1}$ of elements of W_d^+ and any sequence $\alpha = (\alpha_n)_{n \geqslant 1}$, one has

$$\Big\| \sum_{n \geqslant 1} \alpha_n P_n(\mathbf{a}) \Big\| \leqslant e\sqrt{d+1} \Big\| \sum_{n \geqslant 1} \alpha_n P_n(\mathbf{a}) \Big\|_2, \tag{2.6}$$

where $\|\cdot\|_2$ denotes the L^2-norm in $\mathcal{C}$, given by $\|x\|_2 = \tau[x^*x]^{1/2}$ for any a in $\mathcal{C}$. In particular, if $\mathbf{a} = \mathbf{u}$ is a family of free unitaries (i.e., $y = \mathbf{1}$) then we get $\| \sum_{n\geqslant 1} \alpha_n P_n(\mathbf{u})\Big\|_2 = \|\alpha\|_2$, so that (2.6) is already an improvement of (2.5) without the generalization on $\mathcal{R}$-diagonal elements.

Now let $\mathbf{U}_N = (U_1^{(N)}, \dots, U_p^{(N)}), \mathbf{V}_N = (V_1^{(N)}, \dots, V_p^{(N)})$ be families of $N \times N$ independent unitary Haar matrices and $\mathbf{Y}_N = (Y_1^{(N)}, \dots, Y_p^{(N)})$ be a family of $N \times N$ deterministic Hermitian matrices. Assume that for any $j = 1, \dots, p$, the empirical spectral distribution of $Y_j^{(N)}$ converges weakly to a measure μ (that does not depend on j) and that for N large enough, the eigenvalues of $Y_j^{(N)}$ belong to a small neighborhood of the support of μ. We set for any $j = 1, \dots, p$ the random matrix

$$A_j^{(N)} = U_j^{(N)} Y_j^{(N)} V_j^{(N)*}.$$

From Theorem 1.4 and [18, Corollary 2.1], we can deduce that almost surely the family $\mathbf{A}_N = (A_1^{(N)}, \dots, A_p^{(N)})$ converges strongly in law to a family $\mathbf{a}$ of free $\mathcal{R}$-diagonal elements, identically distributed. Hence, inequality (2.6) gives: for any polynomials P_n in W_d^+, $n \geqslant 1$,

$$\lim_{N\to\infty} \| \sum_{n\geqslant 1} \alpha_n P_n(A_1^{(N)}, \dots, A_p^{(N)}) \| \leqslant e\sqrt{d+1} \Big\| \sum_{n\geqslant 1} \alpha_n P_n(\mathbf{a}) \Big\|_2.$$

3. Proof of Theorem 1.4

We consider a unitary Haar matrix U_N, independent of a family of matrices $\mathbf{Y}_N$, having almost surely a strong limit in distribution. We show that almost surely $(U_N, \mathbf{Y}_N)$ has almost surely a strong limit in distribution. As it is known that $(U_N, \mathbf{Y}_N)$ converges in distribution [4, Theorem 5.4.10], the only thing we have to show is the convergence of norms. This will show Theorem 1.4 by recurrence on the number of Haar matrices. Moreover, the problem can be simplified in the following way (see [18, Section 3]):

- one can reason conditionally, and then assume that the matrices of $\mathbf{Y}_N$ are deterministic,
- one may assume that the matrices of $\mathbf{Y}_N$ are Hermitian by considering their Hermitian and anti-Hermitian parts,
- it is sufficient to prove that for any polynomial P, almost surely the norm of $\|P(\mathbf{U}_N, \mathbf{U}_N^*, \mathbf{Y}_N)\|$ converges, rather than "almost surely, for any polynomial".

The keystone of the proof is the use of a classical coupling of real random variables, namely the inverse transform sampling method, for Hermitian matrices (Lemma 3.1 below). It allows us to get the strong convergence of $(U_N, \mathbf{Y}_N)$ from the strong convergence of $(X_N, \mathbf{Y}_N)$, where X_N is a GUE matrix independent of $\mathbf{Y}_N$, for which we know the strong convergence by Theorem 1.3. For that purpose, we will first go through an intermediate problem. We use the coupling to prove in Lemma 3.2 the strong convergence of $(M_N, \mathbf{Y}_N)$, where M_N is the unitary invariant random matrix whose spectrum is $\{\frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N}\}$. From this, we deduce that the strong convergence holds for $(Z_N, \mathbf{Y}_N)$, where Z_N is any unitary invariant random matrix, independent of $\mathbf{Y}_N$, whose spectrum is $\{\gamma_N(\frac{1}{N}), \dots, \gamma_N(\frac{N-1}{N}), \gamma_N(\frac{N}{N})\}$ for $\gamma_N : [0,1] \to \mathbb{C}$ a random map converging uniformly to a continuous map. We finally use the coupling method again, to remark that a unitary Haar matrix could be written as above.

3.1. Preliminaries

Let a be a self-adjoint element in a $\mathcal{C}^*$-algebra, that is $a^* = a$. Denote by μ_a its spectral distribution, i.e., μ_a is the unique probability measure on $\mathbb{R}$ such that for any $k \geqslant 1$, $\int t^k \mathrm{d}\mu_a(t) = \tau[a^k]$. This measure has compact support included in $\big[-\|a\|, \|a\|\big]$. Denote by F_a its cumulative function, satisfying, for all t in $\mathbb{R}$:

$$F_a(t) := \mu\big(-]\infty, t]\big). \tag{3.1}$$

We set the generalized inverse of F_a: for any s in $]0,1]$

$$F_a^{-1}(s) = \inf\big\{t \in [-\pi, \pi] \,\big|\, F_a(t) \geqslant s\big\}. \tag{3.2}$$

By the inverse method for random variables [11, Chapter two], we get the following lemma.

LEMMA 3.1 (The coupling of self-adjoint variables and Hermitian matrices by cumulative functions)

1. *Let a, b be two self-adjoint noncommutative variables. Denote the self-adjoint variables $\tilde{b} = F_b^{-1} \circ F_a(a)$ given by functional calculus. If μ_a have no discrete part (i.e., $\mu_a\big(\{t\}\big) = 0$ for any t in $\mathbb{R}$), then $\tilde{b}$ has the same distribution as b.*
2. *Let A_N and B_N be two Hermitian matrices (living in the $\mathcal{C}^*$-probability space $(\mathrm{M}_N(\mathbb{C}), .^*, \tau_N, \|\cdot\|)$). Write the matrices $A_N = V_{A_N} \Delta_{A_N} V_{A_N}^*$ and $B_N = V_{B_N} \Delta_{B_N} V_{B_N}^*$, with V_{A_N}, V_{B_N} unitary matrices, such that the entries of Δ_{A_N} and Δ_{B_N} are non decreasing along the diagonal. Assume that diagonal entries of Δ_{A_N} are distinct. We set the matrix*

$$M_N := V_{A_N} \, diag\left(\frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N}\right) V_{A_N}^*. \tag{3.3}$$

Then $M_N = F_{A_N}(A_N)$ and $F_{B_N}^{-1}(M_N) = V_{A_N} \, \Delta_{B_N} \, V_{A_N}^*$.

3.2. Step 1: from the GUE to an intermediate model

LEMMA 3.2 (The strong asymptotic freeness of $M_N, \mathbf{Y}_N$). – *Define the random matrix*

$$M_N = V_N \, diag\left(\frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N}\right) V_N^*, \tag{3.4}$$

where V_N is a unitary Haar matrix, independent of $\mathbf{Y}_N$. Then, almost surely $(M_N, \mathbf{Y}_N)$ converges strongly in distribution to $(m, \mathbf{y})$, where $\mathbf{y}$ is the strong limit of $\mathbf{Y}_N$, free from a self adjoint variable m whose spectral distribution is the uniform measure on $[0,1]$.

Proof. – Let X_N be a GUE matrix independent from $\mathbf{Y}_N$, such that $X_N = V_N \Delta_N V_N^*$, where Δ_N is a diagonal matrix, independent of V_N, with non decreasing entries along the diagonal (we recall a proof of that decomposition in Proposition 6.1, Section 6). Let x be a semicircular variable free from the strong limit $\mathbf{y}$ of $\mathbf{Y}_N$. Let F_{X_N} and F_x be the cumulative functions of the spectral measures of X_N and x respectively. By the coupling of Lemma 3.1, we get that $m = F_x(x)$ has the expected distribution and, since the eigenvalues of a GUE matrix are almost surely distinct, we get that almost surely $M_N = F_{X_N}(X_N)$. Then, for any polynomial P, almost surely

$$\Big| \|P(m, \mathbf{y})\| - \|P(M_N, \mathbf{Y}_N)\| \Big| \leqslant \Big| \|P(F_x(x), \mathbf{y})\| - \|P(F_x(X_N), \mathbf{Y}_N)\| \Big| + \|P(F_x(X_N), \mathbf{Y}_N) - P(F_{X_N}(X_N), \mathbf{Y}_N)\|. \tag{3.5}$$

The first term in the right hand side of (3.5) tends to zero almost surely by the strong convergence in distribution of $(X_N, \mathbf{Y}_N)$ to $(x, \mathbf{y})$ (Theorem 1.3) and Proposition 2.1 since F_x is continuous. For the second term, recall first that the convergence in distribution of X_N to x implies the pointwise convergence F_{X_N} to F_x (at any point of continuity of F_x, and so on $\mathbb{R}$). By Dini's theorem [22, Problem 127 Chapter II], F_{X_N} converges actually uniformly to F_x. Hence, since the matrices $X_N, \mathbf{Y}_N$ are uniformly bounded in operator norm and by local uniform continuity of P, the second term converges also to zero. □

3.3. Step 2: from the reference model to other unitary invariant models

LEMMA 3.3 (The strong asymptotic freeness of $Z_N, \mathbf{Y}_N$). – *Consider an N by N random matrix Z_N of the form*

$$Z_N := \gamma_N(M_N) = V_N \, diag\left(\gamma_N\Big(\frac{1}{N}\Big), \ldots, \gamma_N\Big(\frac{N-1}{N}\Big), \gamma_N\Big(\frac{N}{N}\Big)\right) V_N^*, \tag{3.6}$$

where M_N is the random matrix of Lemma 3.2 and $\gamma_N : [0,1] \to \mathbb{C}$ is a random map, independent of $\mathbf{Y}_N$. Assume that almost surely γ_N converges uniformly to a continuous map $\gamma : [0,1] \to \mathbb{C}$, that is

$$\|\gamma - \gamma_N\|_\infty := \sup_{t\in[0,1]} |\gamma(t) - \gamma_N(t)| \underset{N\to\infty}{\longrightarrow} 0.$$

We set the self adjoint variable $z = \gamma(m)$ given by functional calculus, with m as in Lemma 3.2 (it is well defined since $\|m\| \leqslant 1$). Then, almost surely, $(Z_N, \mathbf{Y}_N)$ converges strongly to $(z, \mathbf{y})$.

Proof. – For any polynomial P, one has

$$\Big| \|P(z, z^*, \mathbf{y})\| - \|P(Z_N, Z_N^* \mathbf{Y}_N)\| \Big| \tag{3.7}$$

$$\leqslant \Big| \big\|P\big(\gamma(m), \bar\gamma(m), \mathbf{y}\big)\big\| - \big\|P\big(\gamma(M_N), \bar\gamma(M_N), \mathbf{Y}_N\big)\big\| \Big| \tag{3.8}$$

$$+ \big\|P\big(\gamma(M_N), \bar\gamma(M_N), \mathbf{Y}_N\big) - P\big(\gamma_N(M_N), \bar\gamma_N(M_N), \mathbf{Y}_N\big)\big\|, \tag{3.9}$$

where $\bar\gamma$ denotes the complex conjugacy of γ. The first term of the right hand side of (3.7) tends to zero by Lemma 3.2, Proposition 2.1, and the continuity of γ. By the uniform convergence of γ_N, the continuity of polynomials, and the fact that the matrices $M_N, \mathbf{Y}_N$ are uniformly bounded in operator norm, the second term vanishes at infinity. □

3.4. Step 3: application to Haar matrices

Now, let U_N be a unitary Haar matrix, independent of $\mathbf{Y}_N$. By the spectral theorem (see Proposition 6.1 in Section 6), we can write $U_N = V_N \Delta_N V_N^*$, where the entries of $\Delta_N = \mathrm{diag}\,(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_N})$ have non decreasing argument in $[0, 2\pi[$ along the diagonal. Define as above $M_N = V_N \,\mathrm{diag}\,(\frac{1}{N}, \dots, \frac{N-1}{N}, \frac{N}{N})\, V_N^*$, where V_N is the unitary matrix in the decomposition of U_N. Denote by F_{U_N} the cumulative function of $\mathrm{diag}\,(\theta_1^{(N)}, \dots, \theta_N^{(N)})$. We get by the coupling of Lemma 3.1 that

$$U_N = \exp\big(2\pi i F_{U_N}^{-1}(M_N)\big). \tag{3.10}$$

By Lemma 3.3, to get the strong convergence of $(U_N, \mathbf{Y}_N)$ it remains to prove that almost surely $\gamma_N : t \to \exp\big(2\pi i F_{U_N}^{-1}(t)\big)$ converges uniformly to $\gamma : t \to \exp\big(2\pi i t\big)$.

From the convergence of U_N to a Haar unitary u, we get that almost surely F_{U_N} converges to F_u. Let t in $[0, 1[$. Almost surely, for any $0 < \alpha < 1 - t$, there exists $N_0 \geqslant 1$ such that for any $N \geqslant N_0$, $F_{U_N}(t + \alpha) \geqslant t + \frac{\alpha}{2}$. The points $\big(F_{U_N}^{-1}(t), t\big)$ and $\big(t + \alpha, F_{U_N}(t + \alpha)\big)$ belong to the graph of F_{U_N}, with vertical segments on points of discontinuity. Hence, since F_{U_N} is non decreasing we get $F_{U_N}^{-1}(t) \leqslant t + \alpha$. Hence, for any $0 \leqslant t < 1$, we get $\limsup_{N\to\infty} F_{U_N}^{-1}(t) \leqslant t$. With a symmetric reasoning we get $\liminf_{N\to\infty} F_{U_N}^{-1}(t) \geqslant t$. Now, remark that $F_{U_N}^{-1}(0) = \theta_1 \geqslant 0$ and $F_{U_N}^{-1}(1) = \theta_N \leqslant 1$. Hence, $F_{U_N}^{-1}$ converges pointwise to the identity map on $[0, 1]$. By Dini's theorem, it converges uniformly. Hence γ_N converges uniformly to γ.

4. Proof of Theorem 1.5

The proof of 1.5 is obtained by changing the words unitary, Hermitian and GUE into orthonormal, symmetric and GOE, respectively symplectic, self dual and GSE, and by taking $\mathbf{Y}_N$ a family of independent orthogonal, respectively symplectic, matrices. Instead of Theorem 1.3 we use the main result of [24]. In the symplectic case, we also have to consider matrices of even size.

5. Proof of Corollary 2.2

First recall the following consequence of [18, Corollary 2.1].

Lemma 5.1. – *Let $D_1^{(N)}$ and $D_2^{(N)}$ be two diagonal matrices having a strong limit in distribution separately. Then, there exists diagonal matrices $\tilde{D}_1^{(N)}$ and $\tilde{D}_2^{(N)}$, with the same eigenvalues as $D_1^{(N)}$ and $D_2^{(N)}$ respectively, such that $(\tilde{D}_1^{(N)}, \tilde{D}_2^{(N)})$ converges strongly in distribution.*

Let A_N and B_N be as in Corollary 2.2. Without loss of generality, we can assume that the laws of A_N and B_N are invariant under unitary conjugacy. Let Δ_{A_N} and Δ_{B_N} be diagonal matrices of eigenvalues of A_N and B_N respectively. By Proposition 2.1, the assumptions on A_N and B_N mean their strong convergence in distribution separately, and so the strong convergence of Δ_{A_N} and Δ_{B_N} separately. With the notations of Lemma 5.1, consider $\tilde{\Delta}_{A_N}$ and $\tilde{\Delta}_{B_N}$. Let (U_N, V_N) be independent unitary Haar matrices, independent of $(\tilde{\Delta}_{A_N}, \tilde{\Delta}_{B_N})$. Then (A_N, B_N) and $(U_N\tilde{\Delta}_{A_N}U_N^*, V_N\tilde{\Delta}_{B_N}V_N^*)$ are pairs of random matrices with the same probability law (see Proposition 6.1). By Theorem 1.4, we get the almost sure strong convergence of $(U_N, V_N, \tilde{\Delta}_{A_N}, \tilde{\Delta}_{B_N})$, and then of $(U_N\tilde{\Delta}_{A_N}U_N^*, V_N\tilde{\Delta}_{B_N}V_N^*)$. Hence, we obtain that (A_N, B_N) has a strong limit in distribution (a, b). The spectral distribution of a is μ, the one of b is ν, and a and b are free. The strong convergence implies the convergence of the spectrum of $A_N + B_N$ to the support of $\mu \boxplus \nu$ (which is the spectral distribution of $a + b$) by Proposition 2.1. We then get the first point of Corollary 2.2.

We get the second point of Corollary 2.2 with the same reasoning on $(\Delta_{A_N}, \Delta_{B_N}^{1/2})$. The application stated after Corollary 2.2 follows by taking $\Pi_N = B_N$, which satisfies the assumptions since $t \in (0,1)$, and remarking that $\Pi_N^{1/2} = \Pi_N$.

6. Appendix: The spectral theorem for unitary invariant random matrices

This result seems to be folklore in the literature of Random Matrix Theory, but we are not able to find an exact reference, so we include a proof for the convenience of the readers.

PROPOSITION 6.1 (The spectral theorem for unitary invariant random matrices)
Let M_N be an $N \times N$ Hermitian or unitary random matrix whose distribution is invariant under conjugacy by unitary matrices. Then, M_N can be written $M_N = V_N\Delta_N V_N^*$ almost surely, where

- *V_N is distributed according to the Haar measure on the unitary group,*
- *Δ_N is the diagonal matrix of the eigenvalues of M_N, arranged in increasing order if M_N is Hermitian, and in increasing order with respect to the set of arguments in $[-\pi, \pi[$ if M_N is unitary,*
- *V_N and Δ_N are independent.*

We actually use the proposition only for unitary Haar and GUE matrices, which are two cases where almost surely the eigenvalues are distinct. The fact that multiplicities of eigenvalues are almost surely one brings slight conceptual simplifications in the proof, but nevertheless does not change the result. Hence, we choose to state the proposition without any restriction on the multiplicity of the matrices.

Proof. – By reasoning conditionally, one can always assume that the multiplicities of the eigenvalues of M_N are almost surely constant. We denote by $(N_1, \dots, N_K)$ the sequence of multiplicities when the eigenvalues are considered in the natural order in $\mathbb{R}$ or in increasing order with respect to their argument in $[-\pi, \pi[$.

Since M_N is normal, it can be written $M_N = \tilde{V}_N\Delta_N\tilde{V}_N$, where $\tilde{V}_N$ is a random unitary matrix and Δ_N is as announced. The choice of $\tilde{V}_N$ can be made in a measurable way, see for instance [10, Section 5.3], with minor modifications.

Let $(u_1, \dots, u_K)$ be a family of independent random matrices, independent of $(\Delta_N, \tilde{V}_N)$ and such that for any $k = 1, \dots, K$, the matrix u_k is distributed according to the Haar measure on $\mathcal{U}(N_k)$, the group of $N_k \times N_k$ unitary matrices. We set

$$V_N = \tilde{V}_N \operatorname{diag}(u_1, \dots, u_K),$$

and claim that the law of V_N depends only on the law of M_N, not in the choice of the random matrix $\tilde{V}_N$. Indeed, let $M_N = \bar{V}_N\Delta_N\bar{V}_N$ be an other decomposition, where $\bar{V}_N$

is a unitary random matrix, independent of $(u_1, \ldots, u_K)$. The multiplicities of the eigenvalues being $N_1, \ldots, N_K$, there exists $(v_1, \ldots, v_K)$ in $\mathcal{U}(N_1) \times \cdots \times \mathcal{U}(N_K)$, independent of $(u_1, \ldots, u_K)$, such that $\bar{V}_N = \tilde{V}_N \operatorname{diag}(v_1, \ldots, v_K)$. Hence, we get $\bar{V}_N \operatorname{diag}(u_1, \ldots, u_K) = \tilde{V}_N \operatorname{diag}(v_1 u_1, \ldots, v_K u_K)$, which is equal in law to V_N. This proves the claim.

Let W_N be an $N \times N$ unitary matrix. Then $W_N M_N W_N^* = (W_N \tilde{V}_N)\Delta_N (W_N \tilde{V}_N)^*$. By the above, since M_N and $W_N M_N W_N^*$ are equal in law, then V_N and $W_N V_N$ are also equal in law. Hence V_N is Haar distributed in $\mathcal{U}(N)$.

It remains to show the independence between V_N and Δ_N. Let $f : \mathcal{U}(N) \to \mathbb{C}$ and $g : \mathrm{M}_N(\mathbb{C}) \to \mathbb{C}$ two bounded measurable functions such that g depends only on the eigenvalues of its entries. Then one has $\mathbb{E}\big[f(V_N)g(\Delta_N)\big] = \mathbb{E}\big[f(V_N)g(M_N)\big]$. Let W_N be Haar distributed in $\mathcal{U}(N)$, independent of (V_N, Δ_N). Then by the invariance under unitary conjugacy of the law of M_N, one has

$$\begin{aligned}
\mathbb{E}\big[f(V_N)g(\Delta_N)\big] &= \mathbb{E}\big[f(W_N V_N)g(W_N M_N W_N^*)\big] \\
&= \mathbb{E}\big[f(W_N V_N)g(\Delta_N)\big] \\
&= \mathbb{E}\Big[\mathbb{E}\big[f(W_N V_N)\big|V_N, \Delta_N\big]g(\Delta_N)\Big] \\
&= \mathbb{E}\big[f(W_N)\big]\mathbb{E}\big[g(\Delta_N)\big] = \mathbb{E}\big[f(V_N)\big]\mathbb{E}\big[g(\Delta_N)\big]. \qquad \square
\end{aligned}$$

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