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LEFSCHETZ FOR LOCAL PICARD GROUPS

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ABSTRACT. – We prove a strengthening of the Grothendieck-Lefschetz hyperplane theorem for local Picard groups conjectured by Kollár. Our approach, which relies on acyclicity results for absolute integral closures, also leads to a restriction theorem for higher rank bundles on projective varieties in positive characteristic.

RÉSUMÉ. – Nous prouvons un renforcement du théorème de l'hyperplan de Grothendieck-Lefschetz pour les groupes locaux de Picard conjecturés par Kollár. Notre approche, qui s'appuie sur des résultats en fermetures absolues, conduit également à un théorème de restriction pour les faisceaux de rang supérieur sur les variétés projectives en caractéristique positive.

A classical theorem of Lefschetz asserts that non-trivial line bundles on a smooth projective variety of dimension $\geq$ 3 remain non-trivial upon restriction to an ample divisor, and plays a fundamental role in understanding the topology of algebraic varieties. In [6], Grothendieck recast this result in more general terms using the machinery of formal geometry and deformation theory, and also stated a local version. With a view towards moduli of higher dimensional varieties, especially the deformation theory of log canonical singularities, Kollár recently conjectured [15] that Grothendieck's local formulation remains true under weaker hypotheses than those imposed in [6]. Our goal in this paper is to prove Kollár's conjecture for rings containing a field.

Statement of results

Let $(A, \mathfrak{m})$ be an excellent normal local ring containing a field. Fix some $0 \neq f \in \mathfrak{m}$. Let $V = \mathrm{Spec}(A) - \{\mathfrak{m}\}$, and $V_0 = \mathrm{Spec}(A/f) - \{\mathfrak{m}\}$. The following result is the key theorem in this paper; it solves [15, Problem 1.3] completely, and [15, Problem 1.2] in characteristic 0:

THEOREM 0.1. – *Assume* $\dim(A) \geq 4$. *The restriction map* $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ *is:*

1. *injective if* $\mathrm{depth}_{\mathfrak{m}}(A/f) \geq 2$ *and* A *has characteristic* 0*;*
2. *injective up to* p^∞*-torsion if* A *has characteristic* $p > 0$.

This result is sharp: surjectivity fails in general, while injectivity fails in general if $\dim(A) \leq 3$, in characteristic 0 if $\mathrm{depth}_{\mathfrak{m}}(A/f) < 2$, and in characteristic p if one includes p-torsion. Theorem 0.1 leads to a fibral criterion for a Weil divisor to be Cartier in a family, see Theorem 1.30. A stronger analogue of Theorem 0.1, including the mixed characteristic case, is due to Grothendieck [6, Expose XI] under the stronger condition $\mathrm{depth}_{\mathfrak{m}}(A/f) \geq 3$; complex analytic variants of Grothendieck's theorem are proven in [7], while topological analogues are discussed in [9]. Without this depth constraint, a previously known case of Theorem 0.1 was when A has log canonical singularities in characteristic 0, and $\{\mathfrak{m}\} \subset \mathrm{Spec}(A)$ is not an lc center (see [15, Theorem 19]).

Our approach to Theorem 0.1 relies on formal geometry over absolute integral closures [2, 11], and applies to higher rank bundles as well as projective varieties. This technique then leads to a short proof of the following result:

THEOREM 0.2. – *Let X be a normal projective variety of dimension $d \geq 3$ over an algebraically closed field of characteristic $p > 0$. If a vector bundle E on X is trivial over an ample divisor, then $(\mathrm{Frob}_X^e)^* E \simeq \mathcal{O}_X^{\oplus r}$ for $e \gg 0$.*

The numerical version of Theorem 0.2 for line bundles is due to Kleiman [13, Corollary 2, page 305]. The non-numerical version of the rank 1 case, with stronger assumptions on the singularities, is studied in [8]. This result may also be deduced from the boundedness [16] of semistable sheaves. We do not know the correct characteristic 0 analogue of this result.

An outline of the proof

Both theorems are similar in spirit, so we only discuss Theorem 0.1 here. We first prove the characteristic p result, and then deduce the characteristic 0 one by reduction modulo p and an approximation argument; the reduction necessitates the (unavoidable) depth assumption in characteristic 0. The characteristic p proof follows Grothendieck's strategy of decoupling the problem into two pieces: one in formal f-adic geometry, and the other an algebraization question. Our main new idea is to replace (thanks entirely to the Hochster-Huneke vanishing theorem [11]) our ring A with a very large extension $\overline{A}$ with better depth properties; Grothendieck's deformation-theoretic approach then immediately solves the formal geometry problem over $\overline{A}$. Next, we algebraize the solution over $\overline{A}$ by algebraically approximating formal sections of line bundles; the key here is to identify the cohomology of the formal completion of a scheme as the *derived* completion of the cohomology of the original scheme, i.e., a weak analogue of the formal functions theorem devoid of the usual finiteness constraints. Finally, we descend from $\overline{A}$ to A; this step is trivial in our context, but witnesses the torsion in the kernel.

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1. Local Picard groups

The goal of this section is to prove Theorem 0.1. In §1.1, we study formal geometry along a divisor on a (punctured) local scheme abstractly, and establish certain criteria for restriction map on Picard groups to be injective. These are applied in §1.2 to prove the characteristic p part of Theorem 0.1. Using the principle of "reduction modulo p" and a standard approximation argument (sketched in §1.4), we prove the characteristic 0 part of Theorem 0.1 in §1.3. The afore-mentioned fibral criterion is recorded in §1.5. Finally, in §1.6, we give examples illustrating the necessity of the assumptions in Theorem 0.1.

1.1. Formal geometry over a punctured local scheme

We establish some notation that will be used in this section.

Notation 1.1. – Let $(A, \mathfrak{m})$ be a local ring, and fix a regular element $f \in \mathfrak{m}$. Let $X = \mathrm{Spec}(A)$, $V = \mathrm{Spec}(A) - \{\mathfrak{m}\}$. For an X-scheme Y, write Y_n for the reduction of Y modulo f^{n+1}, and $\widehat{Y}$ for the formal completion[1] of Y along Y_0. Let $\mathrm{Vect}(Y)$ be the category of vector bundles (i.e., finite rank locally free sheaves) on Y, and write $\mathrm{Pic}(Y)$ and $\underline{\mathrm{Pic}}(Y)$ for the set and groupoid of line bundles respectively. Set $\underline{\mathrm{Pic}}(\widehat{Y}) := \lim \underline{\mathrm{Pic}}(Y_n)$ (where the limit is in the sense of groupoids), and $\mathrm{Pic}(\widehat{Y}) := \pi_0(\underline{\mathrm{Pic}}(\widehat{Y}))$. For any A-module M with associated quasi-coherent sheaf $\widetilde{M}$ on $\mathrm{Spec}(A)$, we *define* $H^i_{\mathfrak{m}}(M)$ as cohomology supported along $\{\mathfrak{m}\} \subset X$ of $\widetilde{M}$, i.e., as the ith cohomology of the complex $\mathrm{R}\Gamma_{\mathfrak{m}}(M)$ defined as the homotopy-kernel of the map $\mathrm{R}\Gamma(\mathrm{Spec}(A), \widetilde{M}) \to \mathrm{R}\Gamma(V, \widetilde{M})$.

We will use formal schemes associated to certain non-Noetherian X-schemes later in this paper. Rather than developing the general theory of such schemes, we simply define the concept that will be most relevant: cohomology.

Definition 1.2. – Fix an X-scheme Y. For $F \in D(\mathcal{O}_Y)$, set $\widehat{F} := \mathrm{R}\lim(F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$; we view $\widehat{F}$ as an $\mathcal{O}_{\widehat{Y}} := \lim_n \mathcal{O}_{Y_n}$-complex on $|\widehat{Y}| := Y_0$, so $\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) := \mathrm{R}\Gamma(Y_0, \widehat{F}) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(Y_0, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$.

The following two examples help explain the meaning of this definition:

Example 1.3. – If F is a quasicoherent $\mathcal{O}_X$-module associated to an A-module M, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F}) \simeq \mathrm{R}\lim(M \otimes^L_A A/(f^n))$. In particular, if M is A-flat, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F})$ is the f-adic completion of M in the usual sense. Note that if M is not A-flat, then $\mathrm{R}\Gamma(\widehat{X}, \widehat{F})$ could have cohomology in negative degrees.

Example 1.4. – Fix a quasicoherent flat $\mathcal{O}_V$-module F, assumed to be obtained from an A-module M via localization. Then $\mathrm{R}\Gamma(\widehat{V}, \widehat{F})$ is computed as follows. Fix an ideal $(g_1, \ldots, g_r) \subset A$ with $V(g_1, \ldots, g_r) = \{\mathfrak{m}\}$ set-theoretically (assumed to exist). Let $C(M; g_1, \ldots, g_r) := \bigotimes_{i=1}^r \left(M \xrightarrow{1} M_{g_i}\right)$ be the displayed Cech complex, and let $K(M)$ be the cone of the natural map $C(M; g_1, \ldots, g_r) \to M$. Then the (termwise) f-adic completion of K computes $\mathrm{R}\Gamma(\widehat{V}, \widehat{F})$. To see this, observe first that $K(M)/f^n K(M)$ computes

[1] The formal scheme $\widehat{Y}$ is used as a purely linguistic device to talk about compatible systems of sheaves on each Y_n, and not in a deeper manner.

$\mathrm{R}\Gamma(V_n, F \otimes^L_{\mathcal{O}_V} \mathcal{O}_{V_n})$. It follows that the term-wise f-adic completion of K computes $\mathrm{R}\lim \mathrm{R}\Gamma(V_n, F \otimes_{\mathcal{O}_V} \mathcal{O}_{V_n}) \simeq \mathrm{R}\Gamma(\widehat{V}, \widehat{F})$.

The derived completion functor $K \mapsto \mathrm{R}\lim(K \otimes^L_A A/f^n)$ already appears implicitly in the above definition. To access its values, recall the following definition:

Definition 1.5. – Given an A-module M, we define the *f-adic Tate module* as $T_f(M) := \lim M[f^n]$ with transition maps given by powers of f; note that $T_f(M) = 0$ if $f^N \cdot M = 0$ for some $N > 0$.

The Tate module leads to the second of the following two descriptions of the cohomology of a formal completion:

Lemma 1.6. – *Let Y be an X-scheme such that $\mathcal{O}_Y$ has bounded f^∞-torsion. For $F \in D(\mathcal{O}_Y)$, there are exact sequences*

$$1 \to \mathrm{R}^1 \lim H^{i-1}(Y_n, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}) \to H^i(\widehat{Y}, \widehat{F}) \to \lim H^i(Y, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}) \to 1,$$

and

$$1 \to \lim H^i(Y, F)/f^n \to H^i(\widehat{Y}, \widehat{F}) \to T_f(H^{i+1}(Y, F)) \to 1.$$

Proof. – We first give a proof when $\mathcal{O}_Y$ has no f-torsion (which will be the only relevant case in the sequel). The first sequence is then obtained from the formula

$$\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(Y, F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$$

and Milnor's exact sequence for $\mathrm{R}\lim$ (see [18]). Applying the projection formula (since A/f^n is A-perfect) to the above gives

$$\mathrm{R}\Gamma(\widehat{Y}, \widehat{F}) \simeq \mathrm{R}\lim \big(\mathrm{R}\Gamma(Y, F) \otimes^L_A A/f^n\big).$$

The second sequence is now obtained by applying the derived f-adic completion functor $\mathrm{R}\lim(- \otimes^L_A A/f^n)$ to the canonical filtration on $\mathrm{R}\Gamma(Y, F)$, which proves the claim. In general, the boundedness of f-torsion in $\mathcal{O}_Y$ shows that the map $\{\mathcal{O}_Y \xrightarrow{f^n} \mathcal{O}_Y\} \to \{\mathcal{O}_{Y_n}\}$ of projective systems is a (strict) pro-isomorphism, and hence $\{F \xrightarrow{f^n} F\} \to \{F \otimes^L_{\mathcal{O}_Y} \mathcal{O}_{Y_n}\}$ is also a pro-isomorphism. Now the previous argument applies. □

The following conditions on the data (A, f) will be assumed throughout this subsection; we do *not* assume A is Noetherian as this will not be true in applications.

Assumption 1.7. – Assume that the data from Notation 1.1 satisfies the following:

- X is integral, i.e., A is a domain;
- $j : V \hookrightarrow X$ is a quasi-compact open immersion, i.e., $\mathfrak{m}$ is the radical of a finitely generated ideal;
- $H^0(V, \mathcal{O}_V)$ is a finite A-module;
- $f^N \cdot H^1(V, \mathcal{O}_V) = 0$ for $N \gg 0$.

EXAMPLE 1.8. – Any S_2 Noetherian local domain $(A, \mathfrak{m})$ of dimension ≥ 3 admitting a dualizing complex satisfies Assumption 1.7: the A-module $H^2_{\mathfrak{m}}(A) \simeq H^1(V, \mathcal{O}_V)$ has finite length (see [6, Corollary VIII.2.3]), while $H^0(V, \mathcal{O}_V) \simeq A$ as A is S_2. The absolute integral closure of a *complete* Noetherian local domain of dimension ≥ 3 in characteristic p also satisfies these conditions (see Theorem 1.21), and is a key example for the sequel.

We now study formal geometry over $\widehat{V}$. The following elementary bound on the f^∞-torsion of certain cohomology groups will help relate sheaf theory on $\widehat{V}$ to that on V.

LEMMA 1.9. – *For $E \in \mathrm{Vect}(V)$, one has $f^k \cdot H^1(V, E) = 0$ for $k \gg 0$.*

Proof. – Fix an N with $f^N \cdot H^1(V, \mathcal{O}_V) = 0$, and set $\mathfrak{m}' := \mathrm{Ann}_A(f^N \cdot H^1(V, E)) \subset \mathfrak{m}$. For each $\mathfrak{p} \in V \subset \mathrm{Spec}(A)$, there is a $g \in \mathfrak{m} - \mathfrak{p}$ and an isomorphism $E|_{D(g)} \simeq (\mathcal{O}_V^{\oplus r})|_{D(g)}$. Clearing denominators gives an exact sequence

$$1 \to \mathcal{O}_V^{\oplus r} \to E \to Q \to 1$$

with $g^n \cdot Q = 0$ for some $n > 0$ (by quasi-compactness). Then $g^n \in \mathfrak{m}'$, so $\mathfrak{m}' \not\subset \mathfrak{p}$. Varying over all $\mathfrak{p} \in V$ shows that $A/\mathfrak{m}'$ is a local ring with a unique prime ideal $\mathfrak{m}/\mathfrak{m}'$, so $f^m \in \mathfrak{m}'$ for $m \gg 0$, and hence $f^{N+m} \cdot H^1(V, E) = 0$. □

We can now algebraically approximate formal sections of vector bundles on V:

LEMMA 1.10. – *For $E \in \mathrm{Vect}(V)$, one has $\widehat{H^0(V, E)} \simeq H^0(\widehat{V}, \widehat{E})$.*

Proof. – Lemma 1.9 shows that $\{H^1(V, E)[f^n]\}$ is essentially 0, so $T_f(H^1(V, E)) = 0$. It remains to observe that $\widehat{H^0(V, E)} \simeq \pi_0(\widehat{H^0(V, E)})$ since f is a non-zero divisor on $H^0(V, E)$. □

One can also prove the following Lefschetz-type result for π_1:

COROLLARY 1.11. – *The natural map $\pi_{1,\text{ét}}(V_0) \to \pi_{1,\text{ét}}(V)$ is surjective if A is Noetherian and f-adically complete.*

Proof. – We want $\pi_0(W) \simeq \pi_0(W_0)$ for any finite étale cover $W \to V$. If $\mathcal{A}$ is a finite flat quasi-coherent $\mathcal{O}_V$-algebra, then $H^0(V, \mathcal{A}) \simeq \widehat{H^0(V, \mathcal{A})} \simeq H^0(\widehat{V}, \widehat{\mathcal{A}}) \simeq \lim H^0(V_n, \mathcal{A}_n)$ by the Noetherian assumption and Lemma 1.10. Hence, if $\mathcal{O}_V \to \mathcal{A}$ is also étale, then $H^0(V, \mathcal{A}) \to H^0(V_n, \mathcal{A}_n) \to H^0(V_0, \mathcal{A}_0)$ induce bijections on idempotents. □

Next, we show that pullback along $\widehat{V} \to V$ is faithful on line bundles.

LEMMA 1.12. – *The natural map $\mathrm{Pic}(V) \to \mathrm{Pic}(\widehat{V})$ is injective.*

Proof. – Fix an $L \in \ker(\mathrm{Pic}(V) \to \mathrm{Pic}(V_0))$. Lemma 1.10 gives an injective map $s : L \to \mathcal{O}_V$ with $s|_{V_0}$ an isomorphism. Hence, if $Q = \mathrm{coker}(s)$, then multiplication by f is an isomorphism on Q, so $H^0(V, Q)$ is uniquely f-divisible. Lemma 1.9 shows $f^N \cdot H^1(V, L) = 0$ for $N \gg 0$, so $H^0(V, \mathcal{O}_V) \to H^0(V, Q)$ is surjective, and hence $H^0(V, Q)$ is a finitely generated f-divisible A-module. By Nakayama, $H^0(V, Q) = 0$, so $Q = 0$ as $\mathcal{O}_V$ is ample. □

REMARK 1.13. – The same argument shows $\mathrm{Vect}(V) \to \mathrm{Vect}(\widehat{V})$ is injective on isomorphism classes. If V_0 is S_2, then one can show that each $\widehat{E} \in \mathrm{Vect}(\widehat{V})$ algebraizes to some torsion free $E \in \mathrm{Coh}(V)$ (see [6, Theorem IX.2.2]); examples such as [15, Example12] show that E need not be a vector bundle, even in the rank 1 case.

The next observation is a manifestation of the formula $\widehat{V} = \mathrm{colim}_n V_n$ and some bookkeeping of automorphisms:

LEMMA 1.14. – *The natural map* $\mathrm{Pic}(\widehat{V}) \to \lim \mathrm{Pic}(V_n)$ *is bijective.*

Proof. – Since $\underline{\mathrm{Pic}}(\widehat{V}) \simeq \lim \underline{\mathrm{Pic}}(V_n)$ as groupoids, it suffices to show $\{\pi_1(\mathrm{Pic}(V_n))\} := \{H^0(V_n, \mathcal{O}^*_{V_n})\}$ satisfies the Mittag-Leffler (ML) condition. The assumption on V shows that $\{H^1(V, \mathcal{O}_V)[f^n]\}$ is essentially 0, and hence $\{H^0(V_n, \mathcal{O}_{V_n})\}$ satisfies ML. Since $|V_0| = |V_n|$, we have

$$\{H^0(V_n, \mathcal{O}^*_{V_n})\} = \{H^0(V_n, \mathcal{O}_{V_n}) \times_{H^0(V_0, \mathcal{O}_{V_0})} H^0(V_0, \mathcal{O}^*_{V_0})\}$$

as projective systems. The claim now follows from Lemma 1.15. □

LEMMA 1.15. – *If* $\{X_n\}$ *is a projective system of sets that satisfies ML, and* $Y_0 \to X_0$ *is some map, then the base change system* $\{Y_n\} := \{Y_0 \times_{X_0} X_n\}$ *also satisfies ML.*

Proof. – Let $Z_{n,k} \subset X_k$ be the image of $X_n \to X_k$ for any $k \leq n$. The assumption says: for fixed k, one has $Z_{n,k} = Z_{n+1,k}$ for $n \gg 0$. Since $\mathrm{im}(X_n \times_{X_0} Y_0 \to X_k \times_{X_0} Y_0) = Z_{n,k} \times_{X_0} Y_0$, the claim follows. □

We quickly recall the standard deformation-theoretic approach to studying line bundles on $\widehat{V}$:

LEMMA 1.16. – *The map* $\mathrm{Pic}(V_{n+1}) \to \mathrm{Pic}(V_n)$ *is injective if* $H^1(V_0, \mathcal{O}_{V_0}) = 0$*, and surjective if* $H^2(V_0, \mathcal{O}_{V_0}) = 0$.

Proof. – Standard using the exact sequence $1 \to \mathcal{O}_{V_0} \xrightarrow{a} \mathcal{O}^*_{V_{n+1}} \to \mathcal{O}^*_{V_n} \to 1$ where $a(g) = 1 + g \cdot f^n$. □

We end by summarizing the relevant consequences of the preceding discussion:

COROLLARY 1.17. – *For A satisfying Assumption 1.7, we have:*

1. *The map* $\mathrm{Pic}(V) \to \mathrm{Pic}(\widehat{V})$ *is injective.*
2. *The map* $\mathrm{Pic}(\widehat{V}) \to \lim \mathrm{Pic}(V_n)$ *is bijective.*
3. *The map* $\mathrm{Pic}(\widehat{V}) \to \mathrm{Pic}(V_0)$ *is injective if* $H^1(V_0, \mathcal{O}_{V_0}) = 0$.

Proof. – We simply combine Lemmas 1.12, 1.14, and 1.16. □

1.2. Characteristic p

We follow Notation 1.1. Our goal is to prove the following:

THEOREM 1.18. – *Fix an excellent normal local* $\mathbf{F}_p$*-algebra* $(A, \mathfrak{m})$ *of dimension* ≥ 4*, and some* $0 \neq f \in \mathfrak{m}$*. Then the kernel of* $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ *is* p^∞*-torsion.*

The rest of §1.2 is dedicated to proving Theorem 1.18, so we fix an $(A, \mathfrak{m}, f)$ as in Theorem 1.18 at the outset. The first reduction is to the complete case:

LEMMA 1.19. – *If* $\pi : \mathrm{Spec}(R) \to \mathrm{Spec}(A)$ *is the* $\mathfrak{m}$*-adic completion of* A*, then* $\mathrm{Pic}(V) \to \mathrm{Pic}(\pi^{-1}(V))$ *is injective.*

In the proof below, we write Mod_A^f for the category of finitely generated A-modules.

Proof. – A line bundle $L \in \mathrm{Pic}(V)$ extends to a unique finite A-module M with $\mathrm{depth}_{\mathfrak{m}}(M) \geq 2$, and similarly for line bundles on $\mathrm{Pic}(\pi^{-1}(V))$. Since $\pi^* : \mathrm{Mod}_A^f \to \mathrm{Mod}_R^f$ preserves depth, it suffices to prove: if $M \in \mathrm{Mod}_A^f$ with $M \otimes_A R \simeq R$, then $M \simeq A$. For this, we simply observe that an isomorphism $R \simeq M \otimes_A R$ can be approximated modulo $\mathfrak{m}$ by a map $A \to M$ which is injective (since A is a domain) and surjective by Nakayama, so $M \simeq A$. □

By Lemma 1.19 and the preservation of normality under completion of excellence rings, to prove Theorem 1.18, we can (and do) assume A is an $\mathfrak{m}$-adically complete Noetherian local normal ring. To proceed further, we define:

NOTATION 1.20. – Let $\overline{A}$ denote a fixed absolute integral closure of A. For any A-scheme Y, we write $\overline{Y} := Y_{\overline{A}}$.

Our strategy for proving Theorem 1.18 is to first prove that $\mathrm{Pic}(\overline{V}) \to \mathrm{Pic}(\overline{V}_0)$ is injective, and then descend to a finite level conclusion by norms. The situation over $\overline{V}$ is analyzed via the formal geometry of §1.1. The reason we work at the infinite level first is that formal geometry is easier over $\overline{V}$ than over V, thanks entirely to the following vanishing result:

THEOREM 1.21. – $\overline{A}$ *is Cohen-Macaulay, i.e.,* $H^i_{\mathfrak{m}}(\overline{A}) = 0$ *for* $i < \dim(A)$.

REMARK 1.22. – Strictly speaking, the local cohomology groups used in Theorem 1.21 are defined as the derived functors of sections supported at $\{\mathfrak{m}\} \subset \mathrm{Spec}(A)$ applied to $\overline{A}$. These do not *a priori* agree with those arising from the definition adopted in Notation 1.1. However, both approaches to local cohomology commute with filtered colimits. Hence, for both definitions, we have $H^i_{\mathfrak{m}}(\overline{A}) = \mathrm{colim}\, H^i_{\mathfrak{m}}(B)$ where the colimit ranges over finite extensions $A \to B$ contained in $\overline{A}$. By reduction to the Noetherian case, the two definitions of $H^i_{\mathfrak{m}}(\overline{A})$ coincide.

Theorem 1.21 is due to Hochster-Huneke [11], and can be found in [12, Corollary 2.3] in the form above. It implies $H^i(\overline{V}, \mathcal{O}_{\overline{V}}) = 0$ for $0 < i < \dim(A) - 1$, so $H^i(\overline{V}_0, \mathcal{O}_{\overline{V}_0}) = 0$ for $0 < i < \dim(A) - 2$. We use this to prove an infinite level version of Theorem 1.18:

PROPOSITION 1.23. – *The map* $\mathrm{Pic}(\overline{V}) \to \mathrm{Pic}(\overline{V}_0)$ *is injective if* $\dim(A) \geq 4$.

Proof. – This follows from Corollary 1.17 as $\overline{A}$ satisfies the relevant conditions by Theorem 1.21 since $\dim(A) \geq 4$. □

We can now descend down to prove the main theorem:

Proof of Theorem 1.18. – Fix an $L \in \ker(\mathrm{Pic}(V) \to \mathrm{Pic}(V_0))$. Proposition 1.23 shows $L \in \ker(\mathrm{Pic}(V) \to \mathrm{Pic}(\overline{V}))$. By expressing $\overline{A}$ as a filtered colimit of finite extensions, it follows that $L \in \ker(\mathrm{Pic}(V) \to \mathrm{Pic}(W))$ for a finite surjective map $W \to V$. As V is normal, using norms (see [3, §XVII.6.3]), we conclude that L is torsion. It now suffices to rule out the presence of prime-to-p torsion in $\ker(\mathrm{Pic}(V) \to \mathrm{Pic}(V_0))$. Corollary 1.17 shows that this kernel is contained in the kernel of $\lim \mathrm{Pic}(V_n) \to \mathrm{Pic}(V_0)$. The kernel of $\mathrm{Pic}(V_{n+1}) \to \mathrm{Pic}(V_n)$ is an $\mathbf{F}_p$-vector space for each n, so $\lim \mathrm{Pic}(V_n) \to \mathrm{Pic}(V_0)$ has no prime-to-p torsion in the kernel. □

Remark 1.24. – In the setting of Theorem 1.18, the proof above also shows: if $E \in \mathrm{Vect}(V)$ is trivial over V_0, i.e., satisfies $E|_{V_0} \simeq \mathcal{O}_{V_0}^{\oplus n}$, then E is trivialized by a finite extension of V.

Remark 1.25. – Using the strategy outlined in Remark 1.27 and the p-adic exponential, one can show the following *mixed characteristic* version of Theorem 1.18: if $(A, \mathfrak{m})$ is an excellent normal local flat $\mathbf{Z}_p$-algebra of dimension ≥ 4 which is S_3, and $f \in \mathfrak{m}$ satisfies $p \in \sqrt{(f)}$, then $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ is injective up to p^∞-torsion. As we do not know how to proceed further, we do not elaborate this argument here.

1.3. Characteristic 0

We follow Notation 1.1. Our goal is to prove the following:

Theorem 1.26. – *Fix an excellent normal local* $\mathbf{Q}$*-algebra* $(A, \mathfrak{m})$ *of dimension* ≥ 4*, and some* $0 \neq f \in \mathfrak{m}$*. Assume* $\mathrm{depth}_{\mathfrak{m}}(A/f) \geq 2$*. Then* $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$ *is injective.*

Proof. – By Lemma 1.29 below, we may assume that A is an essentially finitely presented $\mathbf{Q}$-algebra. The depth assumption implies that $\mathrm{depth}_{\mathfrak{m}}(A) \geq 3$ as f acts nilpotently $H^2_{\mathfrak{m}}(A)$ with kernel $H^1_{\mathfrak{m}}(A/f) = 0$. Now fix a line bundle L in the kernel of $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$. By spreading out (see [10, §2]), we can find:

1. A mixed characteristic dvr $(\mathcal{O}, (\pi))$ with perfect residue field of characteristic $p > 0$.
2. A normal Noetherian $\mathcal{O}$-flat local ring $\widetilde{A}$ satisfying:
    (a) There is a map $\widetilde{A}[1/\pi] \to A$.
    (b) $B := \widetilde{A}/\pi$ is normal of dimension $\dim(A)$ and has depth ≥ 3 at its closed point.
3. A section $\widetilde{A} \to \mathcal{O}$ of the structure map $\mathcal{O} \to \widetilde{A}$ defined by an ideal $\widetilde{\mathfrak{m}} \subset \widetilde{A}$ that, after inverting π, gives the image of the closed point under $\mathrm{Spec}(A) \to \mathrm{Spec}(\widetilde{A}[1/\pi])$.
4. An element $\widetilde{t} \in \widetilde{A}$ such that $\widetilde{A}/\widetilde{t}$ is $\mathcal{O}$-flat and maps to t along $\widetilde{A} \to \widetilde{A}[1/\pi] \to A$.
5. A line bundle $\widetilde{L}$ on $\widetilde{V}$ which induces L over V and lies in the kernel of $\mathrm{Pic}(\widetilde{V}) \to \mathrm{Pic}(\widetilde{V}_0)$; here $\widetilde{V} = \mathrm{Spec}(\widetilde{A}) - \{\widetilde{\mathfrak{m}}\}$, and the subscript 0 denoting passage to the $\widetilde{t} = 0$ fibre.

Write $U = \mathrm{Spec}(B) - \{\widetilde{\mathfrak{m}} \cdot B\}$ for the punctured spectrum of B, and use the subscript 0 to indicate passage to the $\tilde{t} = 0$ fibre. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(\widetilde{V}) & \xrightarrow{a} & \mathrm{Pic}(\widetilde{V}_0) \\ \downarrow b & & \downarrow c \\ \mathrm{Pic}(U) & \xrightarrow{d} & \mathrm{Pic}(U_0) \end{array}$$

where the vertical maps are induced by reduction modulo π, while the horizontal maps are induced by reduction modulo $\widetilde{t}$. Theorem 1.18 tells us that the kernel of d is p^∞ torsion. Corollary 1.17 shows b is injective, so $\widetilde{L}$ (and hence L) is killed by a power of p. Repeating the above construction by spreading out over a mixed characteristic dvr whose residue characteristic is $\ell \neq p$, it follows that L is also killed by a power of ℓ, and is hence trivial. □

Remark 1.27. – We do not know a proof of Theorem 1.26 that avoids reduction modulo p except when A is S_3, where one can argue directly as follows. By Lemma 1.12, it suffices to prove $\mathrm{Pic}(\widehat{V}) \to \mathrm{Pic}(V_0)$ is injective. The kernel of this map is $H^1(\widehat{V}, 1+\widehat{I})$, where $I = (f) \subset \mathcal{O}_V$ is the ideal defining V_0. In characteristic 0, the exponential gives an isomorphism $\widehat{I} \simeq 1 + \widehat{I}$ of sheaves on $\widehat{V}$, so it suffices to prove $H^1(\widehat{V}, \widehat{I}) = 0$. Using $f : \mathcal{O}_V \simeq I$ and $H^1(V, \mathcal{O}_V) = 0$ (since $\mathrm{depth}_{\mathfrak{m}}(A) \geq 3$), it suffices to show $T_f(H^2(V, \mathcal{O}_V)) = 0$. The A-module $H^2(V, \mathcal{O}_V)$ has finite length as A is S_3, so $T_f(H^2(V, \mathcal{O}_V)) = 0$. If $\mathrm{depth}_{\mathfrak{m}}(A) \geq 3$ but A is not S_3, then the last step fails; in fact, there are examples [15, Example 12] of such A where $\mathrm{Pic}(\widehat{V}) \to \mathrm{Pic}(V_0)$ is not injective, rendering this approach toothless in general.

1.4. An approximation argument

We now explain the approximation argument used to reduce Theorem 1.26 to the case of essentially finitely presented algebras over $\mathbf{Q}$. First, we show how modules over the completion of an excellent ring can be approximated by modules over a smooth cover while preserving homological properties.

Lemma 1.28. – *Fix an excellent Henselian local ring $(P, \mathfrak{n})$ with $\mathfrak{n}$-adic completion $\widehat{P}$. Let I be the category of diagrams $P \to S \to \widehat{P}$ with $P \to S$ essentially smooth and S local. Then one has*

1. *I is filtered, and $\widehat{P} \simeq \mathrm{colim}_I\, S$.*
2. *$\mathrm{colim}_I\, \mathrm{Mod}^f S \simeq \mathrm{Mod}^f \widehat{P}$ via the natural functor.*
3. *If $M \in \mathrm{Mod}^f_{\widehat{P}}$ has $\mathrm{pd}_{\widehat{P}}(M) < \infty$, then there exist $S \in I$ and $N \in \mathrm{Mod}^f_S$ such that $N \otimes^L_S \widehat{P} \simeq M$.*

Recall that a map $P \to S$ of rings is *essentially smooth* if S is a localization of a smooth P-algebra.

Proof. – (1) is Popescu's theorem [20], while (2) is automatic from (1) as all rings in sight are Noetherian. Now pick $M \in \mathrm{Mod}^f_{\widehat{P}}$ as in (3) with a finite free resolution $K \to M$ over $\widehat{P}$. Then there exist an $S \in I$ and a finite free S-complex L such that $L \otimes_S \widehat{P} = K$ as complexes. It suffices to thus check that $L \in D^{\geq 0}(S)$. Write $j : P \to S$ and $a : S \to \widehat{P}$ for the given maps. As P is Henselian, for each integer c, there exists a section $S \to P$

of j such that the composite $b : S \to P \to \widehat{P}$ agrees with a modulo $\mathfrak{n}^c$. Then [4, Lemma 3.1] shows that $L \otimes_{S,b} \widehat{P}$ is acyclic outside degree 0 (for sufficiently large c). The same is also true for $L \otimes_S P$ by faithful flatness. If $I = \ker(S \to P)$, then I is a regular ideal contained in the Jacobson radical of S (since S is local and essentially P-smooth). Let $\widehat{S}$ be the I-adic completion of S, so $S \to \widehat{S}$ is faithfully flat. By the formula $L \otimes_S \widehat{S} \simeq \mathrm{R}\lim(L \otimes_S S/I^n)$, it suffices to show that the right hand side lies in $D^{\geq 0}(S)$. The regularity of I shows that each I^n/I^{n+1} is a free S/I-module (as $S/I = P$ is local), so $L \otimes_S S/I^n \in D^{\geq 0}(S)$ by devissage as $L \otimes_S S/I \in D^{\geq 0}(S)$. □

The approximation argument used above permits us to make the promised reduction:

Lemma 1.29. – *To prove Theorem 1.26, it suffices to do so when A is essentially finitely presented over* $\mathbf{Q}$.

Proof. – We may assume that the conclusion of Theorem 1.26 is known for all essentially finitely presented normal local k-algebras A of depth ≥ 3 over a characteristic 0 field k (the passage from $k = \mathbf{Q}$ to general k is routine and left to the reader). By Lemma 1.19 and excellence of A, it suffices to show the conclusion holds for all triples $(A, \mathfrak{m}, f)$ where $(A, \mathfrak{m})$ is a complete Noetherian local normal ring with $\mathrm{depth}_{\mathfrak{m}}(A) \geq 3$ in characteristic 0, and $0 \neq f \in \mathfrak{m}$.

If $k = A/\mathfrak{m}$, then we choose a Cohen presentation $A = \widehat{P}/I$ where P is the Henselisation at 0 over $k[x_1, \ldots, x_n]$, and $\widehat{P}$ is the completion. Choose an element $f \in \widehat{P}$ lifting $f \in A$, and a finite A-module M with $\mathrm{depth}_{\mathfrak{m}}(M) \geq 2$ corresponding to an element in the kernel of $\mathrm{Pic}(V) \to \mathrm{Pic}(V_0)$, where $V = \mathrm{Spec}(A) - \{\mathfrak{m}\}$, and $V_0 = V \cap \mathrm{Spec}(A/f)$. Observe that $\mathrm{pd}_{\widehat{P}}(A) \leq n - 3$ and $\mathrm{pd}_{\widehat{P}}(M) \leq n - 2$ by Auslander-Buschbaum. We will show $A \simeq M$.

By Lemma 1.28, we can find:

1. A factorization $P \xrightarrow{j} S \xrightarrow{a} \widehat{P}$ with $(S, \mathfrak{n})$ a local essentially smooth P-algebra.
2. A quotient $S \to B$ such that $B \otimes^L_S \widehat{P} \simeq A$.
3. A finite B-module M' invertible on $V_B = \mathrm{Spec}(B) - V(x_1, \ldots, x_n)$ such that $M' \otimes^L_B A \simeq M$.
4. A lift of f to $\mathfrak{n} \subset S$ such that M' is the trivial line bundle on $V_B \cap \mathrm{Spec}(B/f)$.

We remark that $\mathrm{pd}_S(B) \leq n - 3$ as $B \otimes^L_S \widehat{P} \simeq A$, and similarly $\mathrm{pd}_S(M') \leq n - 2$. As P is Henselian and S is P-smooth with a section over $\widehat{P}$, we may choose a large enough constant c (depending on M and A as $\widehat{P}$-modules) and a section $s_c : S \to P$ of j that coincides with a modulo $(x_1, \ldots, x_n)^c$. Set $A_c = B \otimes^L_S P$ and $M_c = M' \otimes^L_S P$. Then, by choice of c, both these complexes are in fact discrete, and hence A_c is a local quotient of P. Let $\mathfrak{m}_c \subset A_c$ be the maximal ideal; this is the image of $\mathfrak{n}$, and also generated by $\{x_1, \ldots, x_n\}$. We call this triple $(A_c, \mathfrak{m}_c, M_c)$ an approximation of $(A, \mathfrak{m}, M)$, and observe that better approximations can be found by replacing c with a larger integer. At the expense of performing this operation, we have:

1. $(A_c, \mathfrak{m}_c)$ coincides with $(A, \mathfrak{m})$ modulo $(x_1, \ldots, x_n)^c$, and $\dim(A_c) = \dim(A)$ as the Hilbert series of $(A_c, \mathfrak{m}_c)$ and $(A, \mathfrak{m})$ coincide (see [4, Theorem 3.2]).
2. $\mathrm{depth}_{\mathfrak{m}_c}(A_c) \geq 3$, and $\mathrm{depth}_{\mathfrak{m}_c}(M_c) \geq 2$ by Auslander-Buschbaum over P.
3. The singular locus of $\mathrm{Spec}(A_c)$ has codimension ≥ 2 by the Jacobian criterion.
4. M_c is invertible over $U := \mathrm{Spec}(A_c) - V(x_1, \ldots, x_n) = \mathrm{Spec}(A_c) - \{\mathfrak{m}_c\}$.

5. M_c restricts to the trivial line bundle over $U \cap \mathrm{Spec}(A_c/f)$.

By (2) and (3), such an A_c is in particular normal. As Theorem 1.26 is assumed to hold over A_c, we conclude that $M_c \simeq A_c$. Nakayama's lemma lifts this to a surjection $B \to M'$, which yields a surjection $A \to M$. As A is a domain and M is torsion free, we get $A \simeq M$, as desired. □

1.5. A fibral criterion

The results of the previous sections lead to a fibre-by-fibre criterion for a Weil divisor on the total space of a flat family to be Cartier:

THEOREM 1.30. – *Let S be the spectrum of a complete discrete valuation ring over a perfect field k. Let $f : X \to S$ be a separated flat finite type morphism with normal special fibre X_0. Let D be a Weil divisor on X satisfying:*

1. *D is Cartier on the fibres of f.*
2. *There exists a closed subset $Z \subset X_0$ of codimension ≥ 3 such that D is Cartier on $X - Z$.*

Then

1. *D is Cartier on X if k has characteristic 0.*
2. *$p^n D$ is Cartier on X for some $n > 0$ if k has characteristic $p > 0$.*

Proof. – We first give a proof in characteristic 0. As $Z \subset X_0$, we may replace X with its normal locus and assume that X is normal. Then, as X and X_0 are both normal and separated, we identify Cartier divisors with their corresponding Weil divisors on either scheme. If $Z = \varnothing$, we are done. If not, choose a generic point $z \in Z$, and let $A = \mathcal{O}_{X,z}$ with $f \in A$ defining X_0. Then $\mathrm{Spec}(A) \cap Z = \{z\}$, so $\dim(A) \geq 4$. If $V = \mathrm{Spec}(A) - \{z\}$ and $V_0 = V \cap X_0$, then $D|_V \in \ker(\mathrm{Pic}(V) \to \mathrm{Pic}(V_0))$. Theorem 1.26 then shows that $D|_V$ is trivial, so D is Cartier at $z \in X$. Noetherian induction then finishes the proof as the non-Cartier locus of D is a closed subset of X. In characteristic p, we simply use Theorem 1.18 in lieu of Theorem 1.26 in this argument. □

1.6. Examples

We now give examples illustrating the necessity of the depth assumption in Theorem 1.26 as well as the occurrence of p-torsion in Theorem 1.18. We begin with an example of the non-injectivity of the restriction map for coherent cohomology; this leads to the desired examples via the exponential.

EXAMPLE 1.31. – Fix a non-hyperelliptic smooth projective curve C of genus $g > 1$ over a field k. Let $L = \mathcal{O}_{\mathbf{P}^n}(1) \boxtimes K_C$ be the displayed line bundle on $\mathbf{P}^n \times C$ (for $n > 0$), and let $\mathbf{V}(L^{-1}) \to \mathbf{P}^n \times C$ be its total space. Let (X, x) be the affine cone over $\mathbf{P}^n \times C$ with respect to L, i.e., $X = \mathrm{Spec}(A)$ where $A := \Gamma(\mathbf{V}(L^{-1}), \mathcal{O}_{\mathbf{V}(L^{-1})}) = \oplus_{i \geq 0} H^0(\mathbf{P}^n \times C, L^i)$, x is the origin, and let $V = X - \{x\} \subset X$ be the punctured cone; note that L is very ample and A is normal. The affinization map $\mathbf{V}(L^{-1}) \to X$ is the contraction of the 0 section of $\mathbf{V}(L^{-1})$, so we can view V as the complement of the zero section in $\mathbf{V}(L^{-1})$. In particular, the Künneth formula shows

$$H^0(V, \mathcal{O}_V) = H^0(X, \mathcal{O}_X) \simeq \bigoplus_{i \geq 0} H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(i)) \otimes H^0(C, K_C^{\otimes i})$$

and

$$\begin{aligned} H^1(V, \mathcal{O}_V) &= \bigoplus_{i \in \mathbf{Z}} H^1(\mathbf{P}^n \times C, L^i) \\ &\simeq \Big(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \otimes H^1(C, \mathcal{O}_C)\Big) \oplus \Big(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \otimes H^1(C, K_C)\Big), \end{aligned}$$

with the evident $H^0(V, \mathcal{O}_V)$-module structure. Pick non-zero sections $s_1 \in H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ and $s_2 \in H^0(C, K_C)$, and set $f = s_1 \otimes s_2 \in A$. We will show that multiplication by f on $H^1(V, \mathcal{O}_V)$ has non-zero image. First, note that s_2 defines a map $\mathcal{O}_C \to K_C$ that induces a surjective non-zero map $H^1(C, \mathcal{O}_C) \to H^1(C, K_C)$. Since s_1 induces an injective map $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$, it follows $f = s_1 \otimes s_2$ induces a non-zero map

$$H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \otimes H^1(C, \mathcal{O}_C) \to H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \otimes H^1(C, K_C),$$

and hence a non-zero endomorphism of $H^1(V, \mathcal{O}_V)$ by the description above. In particular, if we set $V_0 = V \cap \mathrm{Spec}(A/f) \subset V$, then the map $H^1(V, \mathcal{O}_V) \to H^1(V_0, \mathcal{O}_{V_0})$ is not injective. The same calculation is valid after replacing X with its completion Y at x, and V and V_0 with their preimages U and U_0 respectively in Y (as $H^1(V, \mathcal{O}_V) \simeq H^1(U, \mathcal{O}_U)$, and similarly for V_0). Finally, since $H^1(V, \mathcal{O}_V)[f] \neq 0$, the inclusion $A/f \hookrightarrow H^0(V_0, \mathcal{O}_{V_0})$ is not surjective, so $\mathrm{depth}_x(A/f) = 1$; this reasoning also shows $\mathrm{depth}_x(A/g) = 1$ for any $0 \neq g \in A$ vanishing at x.

Remark 1.32. – The construction and conclusion of Example 1.31 works over any normal ring k, and specializes to the desired conclusion over the fibres as long as the sections s_i are chosen to be non-zero in every fibre.

Via the exponential, we obtain an example illustrating the depth condition in Theorem 1.26:

Example 1.33. – Consider Example 1.31 over $k = \mathbf{C}$. The exponential sequence shows $\mathrm{Pic}(V^{\mathrm{an}}) \to \mathrm{Pic}(V_0^{\mathrm{an}})$ is not injective as $H^1(V^{\mathrm{an}}, \mathbf{Z})$ is countable. One then also has non-injectivity of $\mathrm{Pic}(W) \to \mathrm{Pic}(W_0)$, where W is any link of $x \in X^{\mathrm{an}}$, i.e., $W = \overline{W} - \{x\}$ for a small contractible Stein analytic neighborhood $\overline{W}$ of x in X^{an}; this is because $H^1(V^{\mathrm{an}}, \mathbf{Z}) \simeq H^1(W, \mathbf{Z})$ (as both sides are homotopy equivalent to the circle bundle over $\mathbf{P}^n \times C$ defined by L^{-1}), and $H^1(V^{\mathrm{an}}, \mathcal{O}_{V^{\mathrm{an}}}) \simeq H^1(W, \mathcal{O}_W)$ (by excision and Cartan's Theorem B). By [19, Theorem 5], since any such $\overline{W}$ is normal of dimension ≥ 3, we may identify $\mathrm{Pic}(W)$ with isomorphism classes of analytic coherent S_2 sheaves on $\overline{W}$ free of rank 1 over W. Nakayama then shows non-injectivity of $\mathrm{Pic}(U) \to \mathrm{Pic}(U_0)$.

Remark 1.34. – The (punctured) local scheme of Example 1.33 is not essentially of finite type over k, but rather the (punctured) completion of such a scheme; an essentially finitely presented example can be obtained via Artin approximation. Note that *some* approximation is necessary to algebraically detect the analytic line bundles from Example 1.33 since $\mathrm{Pic}(V) = \mathrm{Pic}(C \times \mathbf{P}^2)/\mathbf{Z} \cdot L$ is smaller than $\mathrm{Pic}(V^{\mathrm{an}})$.

Reducing modulo p (suitably) shows that the map of Theorem 1.18 often has a non-trivial p-torsion kernel:

EXAMPLE 1.35. – Consider Example 1.31 over $k = \mathbf{Z}[1/N]$ for $n \geq 3$, and suitable choices of N, C, s_1, and s_2. Let B be the blowup of Y at x; this may be viewed as the base change to Y of the contraction $\mathbf{V}(L^{-1}) \to X$. Write $\widehat{B}$ for the formal completion of B along $i : \mathbf{P}^n \times C \hookrightarrow B$ (coming from the 0 section), and let $I \subset \mathcal{O}_B$ denote the ideal defining i, so $i^*(I) \simeq L$. Using formal GAGA for $B \to Y$, one can check that there is an exact sequence

$$1 \to H^1(\widehat{B}, 1+I) \to \mathrm{Pic}(B) \to \mathrm{Pic}(\mathbf{P}^n \times C) \to 1$$

with a canonical splitting provided by the composite projection $B \to \mathbf{V}(L^{-1}) \to \mathbf{P}^n \times C$. As $n \geq 3$, using Künneth, one computes

$$H^1(\widehat{B}, 1+I) \overset{\mathrm{can}}{\simeq} H^1(\widehat{B}, (1+I)/(1+I^2)) \overset{\log}{\simeq} H^1(\widehat{B}, I/I^2) \simeq H^1(\mathbf{P}^n \times C, L), \tag{1.1}$$

which, again thanks to Künneth, gives an exact sequence

$$1 \to \Big(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \otimes H^1(C, K_C)\Big) \to \mathrm{Pic}(B) \to \mathrm{Pic}(\mathbf{P}^n \times C) \to 1.$$

The restriction map $\mathrm{Pic}(B) \to \mathrm{Pic}(U)$ has kernel $\mathbf{Z} \cdot L \subset \mathrm{Pic}(\mathbf{P}^n \times C) \subset \mathrm{Pic}(B)$, where the last inclusion comes from the splitting. Thus, there is an injective map

$$\Big(H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1)) \otimes H^1(C, K_C)\Big) \hookrightarrow \mathrm{Pic}(U).$$

We leave it to the reader to check that this map coincides with the one coming from the exponential when specializing to $k = \mathbf{C}$. In particular, after replacing everything in sight with its base change along $k \to \mathbf{F}_p$ for suitable p, we see that $\mathrm{Pic}(U) \to \mathrm{Pic}(U_0)$ has a non-zero kernel; note that, as predicted by Theorem 1.18, this kernel is visibly p-torsion.

2. A vector bundle analogue

The main goal of this section is to explain how the techniques used to prove Theorem 1.18 can also be applied in a global setting. We will do so by explaining a quick proof of the following vector bundle analogue of Theorem 1.18:

THEOREM 2.1. – *Let X be a normal connected projective variety of dimension $d \geq 3$ over an algebraically closed field k of characteristic $p > 0$. If $E \in \mathrm{Vect}(X)$ is trivial over an ample divisor, then E is trivialized by a torsor for a finite connected k-group scheme. In particular, $(\mathrm{Frob}_X^e)^* E \simeq \mathcal{O}_X^{\oplus r}$ for $e \gg 0$.*

As it does not seem straightforward to deduce Theorem 2.1 from *the statement* of Theorem 1.18, we simply redo the relevant arguments in a slightly different setting. As explained below, Theorem 2.1 can be also easily deduced from the boundedness of semistable sheaves [16]. For the rest of this section, we adopt the following notation:

NOTATION 2.2. – Fix a normal connected projective variety X of dimension d over an algebraically closed field $k \supset \mathbf{F}_p$, and an ample divisor $H \subset X$. Let $\overline{X}$ be a fixed absolute integral closure of X. For any geometric object F over X, write $\overline{F}$ for its pullback to $\overline{X}$. For any X-scheme Y, we write Y_n for the n-th infinitesimal neighborhood of the inverse image of H, and $\widehat{Y}$ for the formal completion of Y along Y_0. For $K \in D(\mathcal{O}_Y)$, write $\widehat{K} \simeq \mathrm{R}\lim(K \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_n})$, viewed as an object on $\widehat{Y}$. We use $\underline{\mathrm{Vect}}(Y)$ to denote the *groupoid* of vector bundles on Y, i.e., the category whose objects are finite rank locally free $\mathcal{O}_Y$-modules, and morphisms are isomorphisms of $\mathcal{O}_Y$-modules. Finally, recall that a vector bundle on $\widehat{Y}$ is a compatible collection of vector bundles on each Y_n; hence, we set $\underline{\mathrm{Vect}}(\widehat{Y}) := \lim \underline{\mathrm{Vect}}(Y_n)$, where the limit is computed in sense of groupoids.

We first recall a classical observation, the Lemma of Enriques-Severi-Zariski:

LEMMA 2.3. – *Assume $d \geq 2$. For any $E \in \mathrm{Vect}(X)$, one has $H^i(X, E(-kH)) = 0$ for $i \leq 1$ and $k \gg 0$.*

Proof. – For any $E \in \mathrm{Vect}(X)$, Grothendieck-Serre duality gives $H^i(X, E(-kH))^\vee = H^{-i}(X, E^\vee \otimes \omega_X^\bullet(kH))$, where $\omega_X^\bullet$ is the dualizing complex of X normalized to have the dualizing sheaf in homological degree d. As X is normal, we have $\omega_X^\bullet \in D_{\mathrm{coh}}^{[-d,-2]}(\mathcal{O}_X)$. The claim now follows by Serre vanishing. ☐

Over $\overline{X}$, we have significantly better vanishing:

PROPOSITION 2.4. – *For $E \in \mathrm{Vect}(\overline{X})$, $i < d$ and $n \gg 0$, we have $H^i(\overline{X}, E(-n\overline{H})) = 0$.*

Proof. – If E is a finite direct sum of twists of $\mathcal{O}_{\overline{X}}$ by $\overline{H}$, then the claim follows from [11]. For the general case, fix a sufficiently large integer N. Then the standard construction of free resolutions (applied to the dual of E at some finite level) shows that one can find an exact triangle $E \to P \to Q$ in $D^{\geq 0}(\mathcal{O}_{\overline{X}})$ such that

1. $P = \left(P^0 \to P^1 \to \cdots \to P^N\right)$ with P^i a finite direct sum of twists of $\mathcal{O}_{\overline{X}}$ (in cohomological degree i).
2. Q lies in $D^{\geq N}(\mathcal{O}_{\overline{X}})$.

Then (2) shows that $H^i(\overline{X}, E(-n\overline{H})) \simeq H^i(\overline{X}, P(-n\overline{H}))$ for $i < d$ and any n. By (1), each $H^i(\overline{X}, P(-n\overline{H}))$ admits a *finite* filtration with graded pieces being subquotients of $H^{i-j}(\overline{X}, P^j(-n\overline{H}))$. Each of these subquotients vanishes for $i < d$ and $n \gg 0$. The desired conclusion follows as the filtration is finite. ☐

We can now algebraize some cohomology groups:

LEMMA 2.5. – *Assume $d \geq 2$. For any $E \in \mathrm{Vect}(\overline{X})$, we have $H^i(\overline{X}, E) \simeq H^i(\widehat{\overline{X}}, \widehat{E})$ for $i < d-1$. The analogous claim for $i = 0$ is also valid on X.*

Proof. – We first show the claim for $\overline{X}$. The projective system of exact sequences $1 \to E(-n\overline{H}) \to E \to E|_{\overline{X}_{n-1}} \to 1$ gives a triangle

$$\mathrm{R}\lim \mathrm{R}\Gamma(\overline{X}, E(-n\overline{H})) \to \mathrm{R}\Gamma(\overline{X}, E) \xrightarrow{a} \mathrm{R}\Gamma(\widehat{\overline{X}}, \widehat{E}).$$

The left hand side lies in $D^{[d,d+1]}(k)$ by Proposition 2.4, so $H^i(a)$ is an isomorphism for $i < d-1$. For $i = 0$, the same argument also applies for X using Lemma 2.3 instead of Proposition 2.4. ☐

Passage to formal completions of ample divisors faithfully reflects the geometry of bundles:

LEMMA 2.6. – *Assume $d \geq 2$. The functor $\underline{\mathrm{Vect}}(\overline{X}) \to \underline{\mathrm{Vect}}(\widehat{\overline{X}})$ is fully faithful, and similarly on X.*

Proof. – Lemma 2.5 shows that $\mathrm{Hom}(E,F) \simeq \mathrm{Hom}(\widehat{E},\widehat{F})$ for $E,F \in \mathrm{Vect}(\overline{X})$ (or $\mathrm{Vect}(X)$). It now suffices to check that if $f : E \to F$ induces an isomorphism $\widehat{f} : \widehat{E} \to \widehat{F}$, then f is itself an isomorphism. By taking determinants, we may assume E and F are line bundles. As the reduction $f_0 : E_0 \to F_0$ is an isomorphism, the support of $\mathrm{coker}(f)$ is a divisor that does not intersect $\overline{H}$, contradicting ampleness. □

We obtain a Lefschetz-type result for π_1:

COROLLARY 2.7. – *Assume $d \geq 2$. The map $\pi_1(X_0) \to \pi_1(X)$ is surjective.*

Proof. – We first observe that X_0 is connected by Lemma 2.3, so the notation is unambiguous. As $\pi_1(X_0) \simeq \pi_1(X_n) \simeq \pi_1(\widehat{X})$, it suffices to observe: for any finite étale $\mathcal{O}_X$-algebra $\mathcal{A}$, the natural map $H^0(X,\mathcal{A}) \to H^0(\widehat{X},\widehat{\mathcal{A}})$ is an isomorphism of algebras by Lemma 2.5, and hence identifies idempotents. □

Using the vanishing of cohomology on $\overline{X}$, deformations of the trivial bundle on $\overline{X}_0$ are easy to classify:

LEMMA 2.8. – *Assume $d \geq 3$. The fibre over the trivial bundle of $\underline{\mathrm{Vect}}(\widehat{\overline{X}}) \to \underline{\mathrm{Vect}}(\overline{X}_0)$ is contractible.*

Here a groupoid F is called "contractible" if it is equivalent, as a category, to the category with a single object and a single (identity) automorphism, i.e., if $\pi_0(F) = *$, and $\pi_1(F,f) = 0$ for any $f \in F$.

Proof. – Let $E = \mathcal{O}_{\overline{X}}^{\oplus r}$. It suffices to show that the fibre F_n over E_{n-1} of $\underline{\mathrm{Vect}}(\overline{X}_n) \to \underline{\mathrm{Vect}}(\overline{X}_{n-1})$ is contractible for $n \geq 1$. One has $\pi_0(F_n) = H^1(\overline{X}_0, \underline{\mathrm{End}}(E_0)(-n\overline{H})) \simeq H^1(\overline{X}_0, \mathcal{O}_{\overline{X}_0}(-n\overline{H}))^{\oplus r^2}$. This group vanishes by Proposition 2.4 and the exact sequence

$$1 \to \mathcal{O}_{\overline{X}}(-(n+1)\overline{H}) \to \mathcal{O}_{\overline{X}}(-n\overline{H}) \to \mathcal{O}_{\overline{X}_0}(-n\overline{H}) \to 1$$

as $d \geq 3$. A similar argument shows that $\pi_1(F_n) = \ker(H^0(\overline{X}_0, \underline{\mathrm{End}}(E_0)(-n\overline{H}))) = 0$, which proves the claim. □

We can now prove the promised result:

Proof of Theorem 2.1. – Fix an $E \in \mathrm{Vect}(X)$ with $E|_H \simeq \mathcal{O}_H^{\oplus r}$. Then Lemmas 2.6 and 2.8 show that $\overline{E}$ is the trivial bundle over $\overline{X}$. Hence, there is a finite cover of X trivialising E. By [1], there is a finite k-group scheme G such that E is trivialized by a G-torsor over X. Using Corollary 2.7 and the connected-étale sequence for G (see [21]), we may choose G to be connected, proving half the claim. The last part follows from the observation that any finite surjective purely inseparable map $Y \to X$ is dominated by a power of Frobenius on X. □

Langer's alternative proof (independent of Proposition 2.4) is:

Alternative proof of Theorem 2.1. – Since $E|_H \simeq \mathcal{O}_H^{\oplus r}$, the collection $\{(\mathrm{Frob}^e)^* E\}$ forms a bounded collection of semistable sheaves by [16] as $d \geq 3$. Hence, by Lemma 2.3 applied to the base of a quasi-compact parametrising family, there exists an integer $m \geq 0$ such that

$$H^i(X, (\mathrm{Frob}^e)^* E(-p^m H)) = 0$$

for all $e \geq 0$ and $i \leq 1$. Applying $(\mathrm{Frob}^m)^*$ to the sequence $1 \to E(-H) \to E \to E|_H \to 1$ then shows that $(\mathrm{Frob}^m)^* E \to (\mathrm{Frob}^m)^*(E|_H) \simeq \mathcal{O}_{p^m H}^{\oplus r}$ induces an isomorphism on global sections. An analogue of the argument in Lemma 2.6 then finishes the proof. □

We end by noting that the *proof* of Corollary 2.7, Fujita vanishing [5, Theorem 10], and standard representability results for Picard functors (see, for example, [14]) can be used to prove the following Lefschetz-type result for base-point free big divisors on normal varieties.

Theorem 2.9. – *Let X be a normal connected projective variety of dimension $d \geq 2$ over a field k, and fix a Cartier divisor $D \subset X$ such that $\mathcal{O}(D)$ is semiample and big. Then the restriction map* $\mathrm{Pic}^\tau(X) \to \mathrm{Pic}^\tau(D)$ *is:*

1. *injective if k has characteristic* 0*;*
2. *injective up to a finite and p^∞-torsion kernel if k has characteristic $p > 0$.*

In [17], one finds a stronger result with stronger assumptions: they completely describe the kernel and cokernel of $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ when X is a smooth projective variety in characteristic 0, and D is general in its linear system.

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