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Semi-positivity in positive characteristics

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SEMI-POSITIVITY IN POSITIVE CHARACTERISTICS

BY ZSOLT PATAKFALVI

ABSTRACT. – Let $f : (X, \Delta) \to Y$ be a flat, projective family of sharply F-pure, log-canonically polarized pairs over an algebraically closed field of characteristic $p > 0$ such that $p \nmid \operatorname{ind}(K_{X/Y} + \Delta)$. We show that $K_{X/Y} + \Delta$ is nef and that $f_*(\mathscr{O}_X(m(K_{X/Y} + \Delta)))$ is a nef vector bundle for $m \gg 0$ and divisible enough. Some of the results also extend to non log-canonically polarized pairs. The main motivation of the above results is projectivity of proper subspaces of the moduli space of stable pairs in positive characteristics. Other applications are Kodaira vanishing free, algebraic proofs of corresponding positivity results in characteristic zero, and special cases of subadditivity of Kodaira-dimension in positive characteristics.

RÉSUMÉ. – Soit $f : (X, \Delta) \to Y$ une famille projective plate de paires nettement F-pures et log-canoniquement polarisées sur un corps algébriquement clos de caractéristique $p > 0$ tel que $p \nmid \operatorname{ind}(K_{X/Y} + \Delta)$. Nous montrons que $K_{X/Y} + \Delta$ est nef et que $f_*(\mathscr{O}_X(m(K_{X/Y} + \Delta)))$ est un fibré vectoriel nef pour $m \gg 0$ et qu'il est assez divisible. Certains des résultats s'étendent également aux couples non log-canoniquement polarisés. La principale motivation de ces résultats est la projectivité de sous-espaces propres de l'espace des modules des paires stables en caractéristiques positives. D'autres applications incluent des nouvelles preuves algébriques des résultats de positivité en caractéristique nulle, et un cas particulier de sous-additivité de la dimension de Kodaira de caractéristique positive.

1. Introduction

Results stating positivity of the (log-)relative canonical bundle and of the pushforwards of its powers played an important role in the development of modern algebraic geometry (e.g., [3] $\sim$ Corollary 1.9 , [11, 9, 19, 45, 21] $\sim$ Theorem 1.7, where $\sim$ denotes our statements of similar flavor). Applications are numerous: projectivity and quasi-projectivity of moduli spaces (e.g., [22, 46] $\sim$ Corollary 4.1), subadditivity of Kodaira-dimension (e.g., [45, 21] $\sim$ Corollary 4.6), Shafarevich type results about hyperbolicity of moduli spaces (e.g., [34, 1, 43]), Kodaira dimension of moduli spaces (e.g., [31, 4]), etc. Most of the proofs of the above mentioned general positivity results are either analytic or depend on Kodaira vanishing.

Either way, they work only in characteristic zero. The word "general" and "most" has to be stressed here: there are positivity results available for families of curves (e.g., [43, 22]), abelian varieties [5] and K3 surfaces [27] in positive characteristics. The aim of this article is to present positivity results available for arbitrary fiber dimensions in positive characteristics, bypassing the earlier used analytic or Kodaira vanishing type techniques. The strongest statements are in the case of (log)-canonically polarized fibers, but there are results for fibers with nef log-canonical bundles as well. As in characteristic zero, one also has to put some restrictions on singularities. Here we assume the fibers to be sharply F-pure, which corresponds to characteristic zero notion of log-canonical singularities via reduction mod p (see [42] for a survey on F-singularities, and Definition 2.4 for the definition of sharply F-pure singularities).

Some differences between our results and the characteristic zero statements mentioned above have to be stressed. First, we only claim the semi-positivity of $f_*\omega_{X/Y}^m$ for m big and divisible enough. This is a notable difference, since the characteristic zero results usually start with proving the $m = 1$ case and then deduce the rest from that. However, in positive characteristics there are known counterexamples for the semi-positivity of $f_*\omega_{X/Y}$ [30, 3.2]. So, any positivity result can hold only for $m > 1$, and its proof has to bypass the $m = 1$ case. Second, the characteristic zero results are birational in the sense that for example it is enough to assume that ω_F is big for a general fiber of F. In our results nefness of ω_F is essential, and for the semi-positivity of pushforwards we even need ω_F to be ample. Hence, our results give exactly what one needs for projectivity of moduli spaces (as in [22]), but yield subadditivity of Kodaira dimension only together with the log-Minimal Model Program in positive characteristics.

1.1. Results: normal, boundary free versions over a curve base

Here we state our results in a special, but less technical form. We assume that the spaces involved are normal and we do not add boundary divisors to our varieties. The base is also assumed to be a smooth projective curve. For the general form of the results, see Section 1.2.

We work over an algebraically closed field k of characteristic $p > 0$.

THEOREM 1.1. – *Let $f : X \to Y$ be a surjective, projective morphism from a normal variety to a smooth projective curve with normal generic fiber, such that rK_X is Cartier for some integer $r > 0$. Further assume that*

(a) *either $p \nmid r$ and the general fiber is sharply F-pure,*
(b) *or $p|r$ and the general fiber is strongly F-regular.*

Then:

(1) *If $K_{X/Y}$ is f-nef and K_{X_y} is semi-ample for generic $y \in Y$, then $K_{X/Y}$ is nef.*
(2) *If $K_{X/Y}$ is f-ample, then $f_*\mathcal{O}_X(mrK_{X/Y})$ is a nef vector bundle for $m \gg 0$.*
(3) *(A subadditivity of Kodaira dimension type corollary:) If $K_{X/Y}$ is f-semi-ample, K_{X_y} is big for generic $y \in Y$ and $g(Y) \geq 2$, then K_X is big as well.*

REMARK 1.2. – To explain the scope of the above results, let us mention a few facts about F-singularities. First, the usual singularities of the minimal model can be defined in arbitrary characteristics (e.g., [24]). Then, every S_2, G_1, sharply F-pure singularity is semi-log-canonical (i.e., the pair of its normalization and its conductor is log-canonical) and every strongly F-regular singularity is Kawamata log-terminal. Furthermore, if the (log-)canonical divisor is $\mathbb{Q}$-Cartier, then the difference between the two is "small" in both cases in a measurable sense via reductions mod p [13, 15, 44, 29, 33, 32].

For example, in dimension one sharply F-pure includes smooth and nodal singularities, and strongly F-regular includes smooth singularities. In particular, Theorem 1.1 applies to stable curves, recovering results of [43].

In dimension two, strongly F-regular singularities (without boundaries) are equivalent to Kawamata log-terminal singularities for $p > 5$ [14]. In particular Theorem 1.1 applies to stable degenerations with Kawamata log-terminal general fibers when $p > 5$, regardless of the index. Furthermore, in the sharply F-pure case, much worse singularities can be allowed in the general fibers. For example the general fiber can have nodes or a big portion of log-canonical singularities with index not divisible by p. See [14] and [28], for the actual list.

In higher dimensions one experiences similar behavior, but fewer explicitly worked out examples are known. Intuitively, the non-sharply F-pure but log canonical singularities can be thought of as being supersingular in a very strong sense. This phenomenon can be made more precise in particular cases. For example cones over abelian varieties are sharply F-pure exactly if the underlying abelian variety is ordinary.

Point (2) of Theorem 1.1 is the F-singularity version of the characteristic zero statement used to show projectivity of the moduli space of stable varieties [22, 6]. Therefore, it implies projectivity of coarse moduli spaces of certain sharply F-pure moduli functors. For the precise statement we refer the reader to Section 1.2.

Furthermore, Theorem 1.1 combined with lifting arguments gives a new algebraic proof of the following characteristic zero semi-positivity statement.

COROLLARY 1.3. – *Let $f : X \to Y$ be surjective, projective morphism from a Kawamata log terminal variety to a smooth projective curve over an algebraically closed field of characteristic zero. Let r be the index of K_X.*

(1) *If $K_{X/Y}$ is f-semi-ample, then $K_{X/Y}$ is nef.*
(2) *If $K_{X/Y}$ is f-ample, then $f_* \mathcal{O}_X(mrK_{X/Y})$ is a nef vector bundle for $m \gg 0$.*

1.2. Results: full generality

In algebraic geometry, one is frequently forced to work with pairs or even with non-normal pairs for various reasons: induction on dimension, compactification, working with non-proper varieties, etc. Hence, in the present article we put our results in a more general framework than that of Section 1.1. The actual framework that we work in is motivated by the main application, the projectivity of coarse moduli spaces, and is as follows.

NOTATION 1.4. – Let $f : X \to Y$ be a flat, relatively S_2 and G_1, equidimensional, projective morphism to a projective scheme over k and Δ a $\mathbb{Q}$-Weil divisor on X, such that

(1) Supp Δ contains neither codimension 0 points nor singular codimension 1 points of the fibers,
(2) there is a $p \nmid r > 0$, such that $r\Delta$ is a $\mathbb{Z}$-divisor, Cartier in relative codimension 1 and $\omega_{X/Y}^{[r]}(r\Delta)$ is a line bundle (note that $\omega_{X/Y}^{[r]}(r\Delta)$ is defined as $\iota_*(\omega_{U/Y}^r(r\Delta|_U))$ where $\iota : U \to X$ is the intersection of the relative Gorenstein locus and the locus where $r\Delta$ is Cartier) and
(3) for all but finitely many $y \in Y$, (X_y, Δ_y) is sharply F-pure (see Definition 2.4).

Notation 1.5. – Sometimes instead of the assumptions of Notation 1.4, we drop the assumption $p \nmid r$, but instead of sharply F-purity we assume strong F-regularity of (X_y, Δ_y) for all but finitely many $y \in Y$ (see [41, Definition 2.10] for the definition of strong F-regularity).

The main results of the paper are as follows.

Theorem 1.6. – *In the situation of Notation 1.4 or Notation 1.5, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-nef and for all but finitely many $y \in Y$, $K_{X_y} + \Delta_y$ is semi-ample, then $\omega_{X/Y}^{[r]}(r\Delta)$ is nef.*

Theorem 1.7. – *In the situation of Notation 1.4, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-ample, then $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all integers $m \gg 0$.*

Theorem 1.8. – *In the situation of Notation 1.5, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-ample and Y is a smooth curve, then $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all integers $m \gg 0$.*

Contrary to Theorem 1.6, in Theorem 1.7 we assumed that all but finitely many fibers are sharply F-pure. In fact, when $K_{X/Y} + \Delta$ is $\mathbb{Q}$-Cartier, the locus over which the (geometric) fibers are not sharply F-pure is closed [38, Theorem B]. Hence the seemingly weaker hypothesis of Theorem 1.6 is in fact only a specialization of Notation 1.4. Further, one cannot have assumptions only on the singularities of the generic fiber, if the goal is to prove nefness of $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$. Indeed, it is easy to construct examples of families over a curve with very singular fibers (i.e., projective cones over high genus curves) for which the above sheaf is not nef. On the other hand, if only the general fiber is required to be sharply F-pure, one can still try to prove weak-positivity of $f_*\left(\omega_{X/Y}^{[mr]}(mr\Delta)\right)$. This issue is addressed in other articles (e.g., [36, 38]).

Corollary 1.9. – *In the situation of Notation 1.4, if $\Delta = 0$, $K_{X/Y}$ is f-ample and for every $y \in Y$, X_y is sharply F-pure, $\operatorname{Aut}(X_y)$ is finite and there are only finitely many other $y' \in Y$ such that $X_y \cong X_{y'}$, then $\det\left(f_*\omega_{X/Y}^{[m]}\right)$ is an ample line bundle for all $m \gg 0$ and divisible enough.*

The author has evidence that taking determinant can be removed from the above corollary. I.e., it can be shown that $f_*\omega_{X/Y}^{[m]}$ is ample as a vector bundle. This issue will be also addressed in upcoming articles.

In addition to the above statements, the semi-ample assumption in Theorem 1.6 can be dropped on the expense that the index r has to be 1, as stated in the following theorem.

THEOREM 1.10. – *In the situation of Notation 1.4, if $r = 1$ and $\omega_{X/Y}(\Delta)$ is f-nef, then $\omega_{X/Y}(\Delta)$ is nef.*

For the proofs of the above statements, see Sections 3.3, 3.4 and 3.5. In the case that MMP and the moduli space of stable pairs work in positive characteristics as they do in characteristic zero, one would hope that the divisibility condition of Notation 1.4 could be removed and the sharply F-pure condition could be relaxed to semi-log canonical eventually. Unfortunately, the author has no evidence pro or against this (see Section 5).

The following are the main applications. The first one states the existence of a projective coarse moduli space for certain functors of stable varieties. Note that stable varieties are the higher dimensional analogues of stable curves. According to Corollary 4.1, the last step of the general scheme of proving existence of projective coarse moduli spaces initiated in [22] works if the singularities are at most sharply F-pure (see Section 4.1 for details).

COROLLARY 4.1. – *Let $\mathscr{F}$ be a subfunctor of*

$$Y \mapsto \left\{ \begin{array}{c|l} X & f : X \to Y \text{ is a flat, relatively } S_2 \text{ and } G_1\text{, equidimensional, projective} \\ {\scriptstyle f}\big\downarrow & \text{morphism with sharply } F\text{-pure fibers, such that there is a } p \nmid r > 0, \\ & \text{for which } \omega_{X/Y}^{[r]} \text{ is an } f\text{-ample line bundle, and } \mathrm{Aut}(X_y) \text{ is finite for} \\ Y & \text{all } y \in Y \end{array} \right\} \Big/ \cong \text{ over } Y.$$

If $\mathscr{F}$ admits

(1) *a coarse moduli space $\pi : \mathscr{F} \to V$, which is a proper algebraic space and*
(2) *a morphism $\rho : Z \to \mathscr{F}$ from a scheme, such that $\pi \circ \rho$ is finite,*

then V is a projective scheme.

The second application claims that the above positivity results hold in characteristic zero, assuming Conjecture 4.3 stating that semi-log canonical equals dense sharply F-pure type. Furthermore, in the Kawamata log-terminal case they hold unconditionally. It should be noted that recently Fujino gave an unconditional proof of Corollary 4.5 [6] using the Hodge-theoretic results of his joint paper with Fujisawa [8].

COROLLARY 4.5. – *Let (X, Δ) be a pair over an algebraically closed field of characteristic zero with $\mathbb{Q}$-Cartier $K_X + \Delta$ and $f : X \to Y$ a flat, projective morphism to a smooth projective curve. Further suppose that there is a $y_0 \in Y$, such that Δ avoids all codimension 0 and the singular codimension 1 points of X_{y_0}, and either*

(1) *(X_{y_0}, Δ_{y_0}) is Kawamata log terminal, or*
(2) *(X_{y_0}, Δ_{y_0}) is semi-log-canonical and for every model over a $\mathbb{Z}$-algebra A of finite type, it satisfies the statement of Conjecture 4.3.*

Assume also that $K_{X/Y} + \Delta$ is f-ample (resp. f-semi-ample). Then for $m \gg 0$ and divisible enough, $f_* \mathscr{O}_X(m(K_{X/Y} + \Delta))$ is a nef vector bundle (resp. $K_{X/Y} + \Delta$ is nef).

REMARK 1.11. – Most likely the methods of the article yield a stronger result in characteristic zero, if one works in characteristic zero throughout the proof and uses the Kodaira-vanishing for semi-log canonical schemes [7, Theorem 1.7] instead of Proposition 2.17. This would yield a result stating that for a flat family of slc pairs $f : (X, \Delta) \to Y$ with Δ avoiding the codimension zero and singular codimension one points of the fibers and $\omega_{X/Y}^{[r]}(r\Delta)$ an f-ample line bundle, $\omega_{X/Y}^{[r]}(r\Delta)$ and $f_* \omega_{X/Y}^{[mr]}(mr\Delta)$ are nef for $m \gg 0$. Notably this would yield a proof of the f-ample case of [6, Theorem 1.8] without using the variations of mixed Hodge structures techniques of [8]. We leave the details of such approach to later articles.

The third application is a special case of subadditivity of Kodaira-dimension. It states that in our special setup, suited for moduli theory, if both the base and the general fiber is of (log-)general type then so is the total space.

COROLLARY 4.6. – *In the situation of Notation 1.4 or Notation 1.5, if furthermore Y is an S_2, G_1, equidimensional projective variety with K_Y $\mathbb{Q}$-Cartier and big, $K_{X/Y} + \Delta$ is f-semi-ample and $K_F + \Delta|_F$ is big for the generic fiber F, then $K_X + \Delta$ is big.*

1.3. Idea of the proof

To prove the above mentioned semi-positivity results first we show two general statements, Propositions 3.6 and 3.7, about semi-positivity of a line bundle and its pushforward. We consider the following situation, neglecting Δ at this time. Given a fibration $f : X \to Y$ and a Cartier divisor N on X with certain positivity (e.g., $N - K_{X/Y}$ is nef and f-ample), we want to prove positivity of $f_* \mathscr{O}_X(N)$. One way to approach this problem is to try to find sections of $f_* \mathscr{N}$, where $\mathscr{N} := \mathscr{O}_X(N)$. For that, notice that for nice Y and generic $y \in Y$, there is an isomorphism $f_* \mathscr{N} \otimes k(y) \to H^0(X_y, \mathscr{N})$. So lifting every element of $f_* \mathscr{N} \otimes k(y)$ to $H^0(Y, f_* \mathscr{N})$ is equivalent to lifting every element of $H^0(X_y, \mathscr{N})$ to $H^0(X, \mathscr{N})$. Fortunately, there is a nice lifting result available for F-singularities by Karl Schwede, see Proposition 2.17. This leads us to proving a global generation result for some twist of $f_* \mathscr{N}$ in Proposition 3.3, which then implies nefness of the same twist of $f_* \mathscr{N}$. The next step is to get rid of this twist. For that we use the product trick of Notation 2.11, i.e., we apply our global generation result for n-times fiber products of X with itself over Y. The upshot is that we obtain nefness of $\bigotimes_{i=1}^n f_* \mathscr{N}$ twisted by a line bundle. However, the twist is independent of n, which yields nefness of $f_* \mathscr{N}$ itself. This is done in Proposition 3.6. Then one can consider the natural morphism $f^* f_* \mathscr{N} \to \mathscr{N}$. If this is surjective enough and $f_* \mathscr{N}$ is a nef vector bundle, $\mathscr{N}$ is nef as well. This is Proposition 3.7.

Having shown the general semi-positivity statements, deducing the semi-positivities of Theorems 1.6, 1.7 and 1.10 is still a bit of work. The most tricky is Theorem 1.6, because the index can be an arbitrary integer not divisible by p. In the index one case, the rough idea is as follows. We take a very positive Cartier divisor L, and we prove by induction on $q > 0$ that $qK_{X/Y} + L$ is nef, using the general nefness result mentioned in the last sentence of the previous paragraph. Then, if this holds for all $q > 0$, $K_{X/Y}$ has to be nef as well. Unfortunately, this argument breaks down when $r := \operatorname{ind}(K_{X/Y}) > 1$. In that case we have to argue by contradiction. We choose a Cartier divisor B, which is the pullback of an ample Cartier divisor from Y, and we consider the smallest $t > 0$, such that $K_{X/Y} + tB$ is nef. Then similarly to the index one case, we prove inductively that $q(rK_{X/Y} + (r-1)tB) + L$ is nef for all $q > 0$. Therefore, so is $rK_{X/Y} + (r-1)tB$, and then also $K_{X/Y} + \frac{r-1}{r}tB$. However, $\frac{r-1}{r}t < t$, which contradicts the choice of t, unless $K_{X/Y}$ was nef originally. Unfortunately, there is a point where one has to be a bit more careful with this argument: $(r-1)tB$ has to be Cartier. Hence, we cannot really use t, we have to use a rational number $\frac{a}{b}$ which is slightly bigger than t and for which $(r-1)\frac{a}{b}$ is integer. However, then the question is whether $\frac{r-1}{r}\frac{a}{b} < t$ is going to hold or not. This is solved at least for $t > 1$ by Lemma 3.8 using elementary number theoretic considerations. Then, by pulling back our family $X \to Y$ via an adequate finite map $Y' \to Y$, we can maneuver ourselves into a situation where $t > 1$.

1.4. Notation

We fix an algebraically closed base field k of positive characteristic p. Every scheme is taken over this base field, and is assumed to be separated and Noetherian. The generic point of a subvariety W of a scheme X is denoted by η_W. For any scheme X over k, X_{sing} denotes the (reduced) closed set of X, where X is not regular.

In the present article, many schemes are not normal. For every such X we consider only Weil divisors that have no components contained in X_{sing}. By abuse of notation, a *Weil divisor* will always mean such a special divisor. They form a free $\mathbb{Z}$-module under addition, which we denote by $\text{Weil}^*(X)$. A *Weil divisorial sheaf*, on an S_2, G_1 scheme X is a rank one reflexive subsheaf of the total space of fractions $\mathscr{K}(X)$. In the present article every Weil divisorial sheaf is invertible in codimension one, hence by the abuse of notation *Weil divisorial sheaves* will mean Weil divisorial sheaves that are invertible in codimension one. The usual reference for such sheaves is [17], where they are called almost Cartier divisors. It is important to note that every reflexive sheaf on an S_2, G_1 scheme which is invertible in codimension one can be given a Weil divisorial sheaf structure. That is, one can find an embedding of it into $\mathscr{K}(X)$. For every $E = \sum a_D D \in \text{Weil}^*(X)$ one can associate a Weil divisorial sheaf:

(1.11a) $$\mathscr{O}_X(-E) := \{f \in \mathscr{K}(X) | \forall D : \text{ord}_D f \geq a_D\},$$

where if $D \subseteq X_{\text{sing}}$, then $a_D = 0$ necessarily, and then by $\text{ord}_D f \geq a_D$ we mean that $f \in \mathscr{O}_{X,\eta_D}$. Linear equivalence of Weil divisorial sheaves is defined by multiplying with an invertible element of $\mathscr{K}(X)$ and addition by multiplying them together and taking the reflexive hull. That is, $\mathscr{L} \sim \mathscr{K}$ if and only if there is a $f \in \mathscr{K}(X)^\times$, such that $\mathscr{L} = f \cdot \mathscr{K}$ and $\mathscr{L} + \mathscr{K} = (\mathscr{L} \cdot \mathscr{K})^{**}$. Weil-divisorial sheaves modulo linear equivalence form a group under the above addition, which is denoted by $\text{Pic}^*(X)$.

If X is of finite type over k and reduced, then there are only finitely many divisors contained in X_{sing}. Hence, by [17, Proposition 2.11.b], every Weil divisorial sheaf is linearly equivalent to one of the form (1.11a). Therefore the construction of (1.11a) yields a surjective homomorphism $\text{Weil}^*(X) \to \text{Pic}^*(X)$. One can show that the kernel consists of the $\sum a_D D \in \text{Weil}^*(X)$, associated to an $f \in \mathscr{K}$ (that is, $a_D = \text{ord}_D f$ for divisors D not contained in X_{sing} and f generates $\mathscr{O}_{X,\eta_D}$ otherwise). Summarizing, $\text{Pic}^*(X)$ is isomorphic to $\text{Weil}^*(X)$ modulo linear equivalence of Weil divisors. In the current article we mostly use the latter representation of $\text{Weil}^*(X)$.

Similarly as above, a $\mathbb{Q}$-*divisor* means a formal sum of codimension one points not contained in X_{sing} with rational coefficients. A $\mathbb{Q}$-divisor D is *effective*, i.e., $D \geq 0$, if all its coefficients are at least zero. For a $\mathbb{Q}$-divisor D, $\mathrm{ind}(D)$ is the smallest integer n, such that nD is an integer Cartier divisor. Given a flat projective morphism $f : X \to Y$ with X being S_2 and G_1 and Y Gorenstein, $\omega_X = \omega_{X/Y} \otimes f^*\omega_Y$ [35, Lemma 4.10]. Hence both ω_X and $\omega_{X/Y}$ are Weil divisorial sheaves [25, Corollary 5.69], [17, Theorem 1.9]. Any of their representing Weil divisors are denoted by K_X and $K_{X/Y}$.

Vector bundle means a locally free sheaf of finite rank. *Line bundle* means a locally free sheaf of rank one. When it does not cause any misunderstanding, pullback is denoted by lower index. E.g., if $\mathscr{F}$ is a sheaf on X, and $X \to Y$ and $Z \to Y$ are morphisms, then $\mathscr{F}_Z$ is the pullback of $\mathscr{F}$ to $X \times_Y Z$. This unfortunately is also a source of some confusion: $\mathscr{F}_y$ can mean both the stalk and the fiber of the sheaf $\mathscr{F}$ at the point y. We will try to use $\mathscr{F} \otimes k(y)$ for the fiber so that no confusion arises.

There are some important conventions of orders of operations, since expressions as $F^e_* \omega_X(\Delta) \otimes \mathscr{L}$ are used frequently. *Push-forward has higher priority than tensor product, but twisting with a divisor has higher priority than push-forward.* E.g., the above expression means $(F^e_*(\omega_X(\Delta))) \otimes \mathscr{L}$.

1.5. Organization

Section 3 is the core of the article. It contains the details of the argument outlined in Section 1.3. The necessary definitions, background material and technical statements can be found in Section 2. Many of these technical statements involve finding uniform bounds for certain behaviors experienced in the presence of very positive line bundles. Section 4 contains the proofs of the applications of the general positivity statements. Finally, we collected some of the many questions the article brings up in Section 5.

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2. Background

This section contains the necessary technical definitions and statements used in the arguments outlined in Section 1.3 and worked out in Section 3.

2.1. Definitions

Here we present the definitions used in the paper from the theory of F-singularities. As mentioned in the introduction, these are characteristic p counterparts of the singularities of the minimal model program. We give only the minimally needed definitions, we refer the reader to [42] for a general survey on the theory of F-singularities. As the singularities of the minimal model program, the F-singularities show up naturally in lifting statements for sections of line bundles. This is how they appear in the present article.

DEFINITION 2.1. – A *pair* (X, Δ) is an S_2, G_1, separated scheme essentially of finite type over k of pure dimension, with an effective $\mathbb{Q}$-divisor Δ. Note that, according to Section 1.4, a $\mathbb{Q}$-divisor is a formal sum with rational coefficients of codimension one points that are not contained in the singular locus of X. The index $\operatorname{ind}(X, \Delta)$ is defined as $\operatorname{ind}(K_X + \Delta)$, that is, the smallest positive integer r such that $r(K_X + \Delta)$ is an integer Cartier divisor.

NOTATION 2.2. – Let (X, Δ) be a pair, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $p \nmid \operatorname{ind}(X, \Delta)$. Set $g := \min\{e \in \mathbb{Z}^{>0} | (p^e - 1)(K_X + \Delta) \text{ is Cartier}\}$. (Note that there is an integer $e > 0$ for which $(p^e - 1)(K_X + \Delta)$ is Cartier, by the index assumption and Euler's theorem. Hence g exists and furthermore, these integers are exactly the multiples of g.) For any $e \geq 0$ such that $g \mid e$, define

$$\mathscr{L}_{e,\Delta} := \mathscr{O}_X((1 - p^e)(K_X + \Delta)).$$

Let $F^e : X \to X$ be the e-th iteration of the *absolute Frobenius* morphism, i.e., the map, which is the identity on points and is $r \mapsto r^{p^e}$ on the structure sheaf. Denote by $\phi^e : F^e_* \mathscr{L}_{e,\Delta} \to \mathscr{O}_X$ the unique extension from the Gorenstein locus of the composition of following maps:

- the embedding $F^e_* \mathscr{O}_X((1 - p^e)(K_X + \Delta)) \to F^e_* \mathscr{O}_X((1 - p^e)K_X)$ induced by $\mathscr{O}_X((1 - p^e)\Delta) \to \mathscr{O}_X$ (note that twisting with a divisor has higher priority than pushforward according to Section 1.4) and
- the twist $F^e_* \mathscr{O}_X((1 - p^e)K_X) \to \mathscr{O}_X$ of the Grothendieck trace by ω_X^{-1}.

PROPOSITION 2.3. – *In the situation of Notation 2.2,*

$$\phi^{e'} F^{e'}_* \mathscr{L}_{e',\Delta} \subseteq \phi^{e''} F^{e''}_* \mathscr{L}_{e'',\Delta}, \text{ for } e'' \geq e' \text{ such that } g | e'', e'. \tag{2.3a}$$

Proof. – Let $f := e' - e''$. Then,

$$\phi^{e'} F^{e'}_* \mathscr{L}_{e',\Delta} = \underbrace{\phi^{e''} F^{e''}_* \left(\phi^f \otimes \operatorname{id}_{\mathscr{L}_{e'',\Delta}} (F^f_* \mathscr{L}_{f,\Delta} \otimes \mathscr{L}_{e'',\Delta}) \right)}_{\substack{\phi^{e''} \circ F^{e''}_* (\phi^f \otimes \operatorname{id}_{\mathscr{L}_{e'',\Delta}}) = \phi^{e'} \text{ up to multiplication by a unit, because} \\ \text{of [40, Lemma 3.9] and the projection formula (using that } (F^e)^* \mathscr{L} \cong \mathscr{L}^{p^e})}} \subseteq \phi^{e''} F^{e''}_* \mathscr{L}_{e'',\Delta}. \qquad \square$$

DEFINITION 2.4. – In the situation of Notation 2.2, define the *non-F-Pure ideal* of (X, Δ) as

$$\sigma(X, \Delta) = \bigcap_{e \geq 0} \phi^{e \cdot g} F_*^{e \cdot g} \mathscr{L}_{e \cdot g, \Delta}.$$

According to [10, Lemma 13.1] (see also [41, Remark 2.9]) and Proposition 2.3 this intersection stabilizes, that is,

$$\phi^{e \cdot g} F_*^{e \cdot g} \mathscr{L}_{e \cdot g, \Delta} = \sigma(X, \Delta), \text{ for all } e \gg 0. \tag{2.4a}$$

Also by [41, Remark 2.9], if $e > 0$ is any integer such that $g|e$, then $\sigma(X, \Delta)$ is the unique largest ideal $\mathscr{I}$ such that

$$\phi^e F_*^e (\mathscr{L}_{e, \Delta} \cdot \mathscr{J}) = \mathscr{J}.$$

The pair (X, Δ) is *sharply F-pure* if $\sigma(X, \Delta) = \mathscr{O}_X$.

In the known counterexamples to Kodaira vanishing (e.g., [39]) one finds elements in adjoint linear systems that do not come from some high Frobenius. Hence, a technique to lift sections in positive characteristic is to consider only sections of adjoint bundles that come from arbitrary high Frobenius. One such collection of sections is the following subgroup of $H^0(X, \mathscr{L})$. For the main application, see Proposition 2.17.

DEFINITION 2.5. – In the situation of Notation 2.2, if $\mathscr{N}$ is a line bundle on X, then define

$$\begin{aligned} &S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}) \\ &:= \bigcap_{e \in \mathbb{Z}^{\geq 0}} \operatorname{im}(H^0(X, F_*^{e \cdot g}(\sigma(X, \Delta) \otimes \mathscr{L}_{e \cdot g, \Delta}) \otimes \mathscr{N}) \xrightarrow[H^0(\phi^{e \cdot g} \otimes \mathrm{id}_{\mathscr{N}})]{} H^0(X, \sigma(X, \Delta) \otimes \mathscr{N})). \end{aligned} \tag{2.5a}$$

REMARK 2.6. – It is very important to stress that $S^0(X, \sigma(X, \Delta) \otimes \mathscr{N})$ depends on Δ and $\mathscr{N}$, not only on $\sigma(X, \Delta) \otimes \mathscr{N}$ and not even on $\sigma(X, \Delta)$ and $\mathscr{N}$.

The following proposition gives an alternative description of $S^0(X, \sigma(X, \Delta) \otimes \mathscr{N})$, frequently used throughout the article. To prove it, use the consequence of (2.4a) that for all $e \gg 0$,

$$\left(\phi^{e' \cdot g} \otimes \mathrm{id}_{\mathscr{L}_{e \cdot g, \Delta}}\right) F_*^{e' \cdot g} \mathscr{L}_{(e'+e) \cdot g, \Delta} = (\phi^{e' \cdot g} F_*^{e' \cdot g} \mathscr{L}_{e' \cdot g, \Delta}) \otimes \mathscr{L}_{e \cdot g, \Delta} = \sigma(X, \Delta) \otimes \mathscr{L}_{e \cdot g, \Delta}.$$

PROPOSITION 2.7. – *In the situation of Notation 2.2, if $\mathscr{N}$ is a line bundle on X then*

$$S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}) = \bigcap_{e \in \mathbb{Z}^{\geq 0}} \operatorname{im}(H^0(X, F_*^{e \cdot g} \mathscr{L}_{e \cdot g, \Delta} \otimes \mathscr{N}) \xrightarrow[H^0(\phi^e \otimes \mathrm{id}_{\mathscr{N}})]{} H^0(X, \mathscr{N})). \tag{2.7a}$$

REMARK 2.8. – In the situation of Definition 2.5, if X is projective, then $H^0(X, \mathscr{N})$ is finite dimensional. Therefore,

$$\begin{aligned} S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}) &= \operatorname{im}\left(H^0(X, F_*^{e \cdot g}(\sigma(X, \Delta) \otimes \mathscr{L}_{e \cdot g, \Delta}) \otimes \mathscr{N}) \to H^0(X, \sigma(X, \Delta) \otimes \mathscr{N})\right) \\ &= \operatorname{im}\left(H^0(X, F_*^{e \cdot g} \mathscr{L}_{e \cdot g, \Delta} \otimes \mathscr{N}) \to H^0(X, \mathscr{N})\right) \end{aligned}$$

for all $e \gg 0$.

2.2. Cohomology and base change

Here, we list a couple of standard statements about cohomology and base change as a reference for the following sections. We also introduce the product construction in Notation 2.11, one of the main tricks of the article.

LEMMA 2.9. – *Let $f : X \to Y$ be a projective morphism over a Noetherian scheme, and $\mathscr{G}$ a coherent sheaf on X flat over Y, such that for all $i > 0$, $R^i f_* \mathscr{G} = 0$. Then, the natural morphisms*

$$f_* \mathscr{G} \otimes k(y) \to H^0(X_y, \mathscr{G}) \tag{2.9a}$$

are isomorphisms, and for all $y \in Y$ and $i > 0$,

$$H^i(X_y, \mathscr{G}) = 0.$$

Further, the above base-change holds more generally. That is, if $g : T \to Y$ is a morphism of Noetherian schemes, then the natural homomorphism $g^* f_* \mathscr{G} \to f_{T,*} \mathscr{G}_T$ is an isomorphism.

Proof. – Everything except the addendum follows from the statement of [16, Theorem III.12.11]. To see the addendum we have to understand slightly the proof of [16, Theorem III.12.11]. Let $\operatorname{Spec} A$ be an affine open set of Y. Using the notations of [16, Section III.12], by passing also to affine open sets of T we have to show that $T^0(A) \otimes_A M \to T^0(M)$ is an isomorphism for every A-module M. However, by [16, Proposition III.12.5] this is equivalent to proving that T^0 is right exact, which by [16, Proposition III.12.10] is satisfied if the natural map $T^0(A) \otimes_A k(y) \to T^0(k(y))$ is an isomorphism for every point $y \in \operatorname{Spec} A$. On the other hand the latter condition is exactly the base-change isomorphism (2.9a) that we have already showed. □

LEMMA 2.10 ([16, Theorem III.9.9 and Corollary III.12.9]). – *Let $f : X \to Y$ be a projective morphism over a Noetherian, integral scheme, and $\mathscr{G}$ a coherent sheaf on X flat over Y, such that for all $i > 0$, and $y \in Y$,*

$$H^i(X_y, \mathscr{G}) = 0.$$

Then $R^i f_* \mathscr{G} = 0$ for $i > 0$, $f_* \mathscr{G}$ is locally free, and the natural homomorphisms

$$f_* \mathscr{G} \otimes k(y) \to H^0(X_y, \mathscr{G})$$

are isomorphisms.

NOTATION 2.11. – For a morphism $f : X \to Y$ of schemes, define

$$X_Y^{(m)} := \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{m \text{ times}},$$

and let $f_Y^{(m)} : X_Y^{(m)} \to Y$ be the natural induced map. If $\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules, then

$$\mathscr{F}_Y^{(m)} := \bigotimes_{i=1}^m p_i^* \mathscr{F},$$

where p_i is the i-th projection $X_Y^{(m)} \to X$. Similarly, if Γ is a divisor on X and f is flat, then

$$\Gamma_Y^{(m)} := \sum_{i=1}^m p_i^* \Gamma,$$

In most cases, we omit Y from our notation. I.e., we use $X^{(m)}$, $\Gamma^{(m)}$, $f^{(m)}$ and $\mathscr{F}^{(m)}$ instead of $X_Y^{(m)}$, $\Gamma_Y^{(m)}$, $f_Y^{(m)}$ and $\mathscr{F}_Y^{(m)}$, respectively.

LEMMA 2.12. – *Let $f : X \to Y$ be a projective flat morphism over a Noetherian scheme and $\mathscr{G}$ a coherent sheaf on X flat over Y, such that $H^i(X_y, \mathscr{G}) = 0$ for all $i > 0$ and $y \in Y$. Then, using Notation 2.11, the natural morphisms*

$$f_*^{(n)}(\mathscr{G}^{(n)}) \to \bigotimes_{i=1}^{n} f_*\mathscr{G} \tag{2.12a}$$

and

$$f_*^{(n)}(\mathscr{G}^{(n)}) \otimes k(y) \to H^0(X_y^{(n)}, \mathscr{G}^{(n)}) \tag{2.12b}$$

are isomorphisms.

Proof. – First note, that since both $\mathscr{G}$ and X are flat over Y, $p_i^*\mathscr{G}$ is flat over Y as well. However then $\mathscr{G}^{(n)} = \bigotimes_{i=1}^n p_i^*\mathscr{G}$ is also flat over Y. By the assumptions and the Künneth formula, $H^i(X_y, \mathscr{G}^{(n)}) = 0$ for all $i > 0$ and $y \in Y$. Therefore, by applying Lemma 2.10 for both $\mathscr{G}^{(n)}$ and $\mathscr{G}$, one obtains that $R^i f_*^{(n)}\mathscr{G}^{(n)} = 0$ and $R^i f_*\mathscr{G} = 0$ for $i > 0$, $f_*\mathscr{G}$ is locally free, and that (2.12b) holds. The following isomorphisms show (2.12a).

$$f_*^{(n)}\mathscr{G}^{(n)} \cong \underbrace{Rf_*^{(n)}\mathscr{G}^{(n)}}_{R^i f_*^{(n)}\mathscr{G}^{(n)}=0 \text{ for } i>0} \cong \underbrace{\bigotimes_L{}_{i=1}^{n} Rf_*\mathscr{G}}_{\text{Künneth formula}} \cong \underbrace{\bigotimes_L{}_{i=1}^{n} f_*\mathscr{G}}_{R^i f_*\mathscr{G}=0 \text{ for } i>0} \cong \underbrace{\bigotimes_{i=1}^{n} f_*\mathscr{G}}_{f_*\mathscr{G} \text{ is locally free}}. \qquad \square$$

2.3. Global generation

Here we prove a Fujita type uniform global generation result for flat families in Proposition 2.15. The main tool is the relative version of Fujita vanishing, which is also crucial for many other statements of the article.

THEOREM 2.13 ([20, Theorem 1.5] Relative Fujita vanishing). – *Let $f : X \to Y$ be a projective morphism over a Noetherian scheme, and $\mathscr{N}$ an f-ample line bundle on X. Then for all coherent sheaves $\mathscr{F}$ on X there is an $m > 0$, such that for every $i > 0$ and f-nef line bundle $\mathscr{K}$,*

$$R^i f_*(\mathscr{F} \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0.$$

THEOREM 2.14 (e.g., [26, Theorem 1.8.5]). – *If on a projective scheme X, $\mathscr{N}$ is a globally generated ample line bundle and $\mathscr{F}$ a coherent sheaf such that $H^i(X, \mathscr{F} \otimes \mathscr{N}^{-i}) = 0$ for $i > 0$, then $\mathscr{F}$ is globally generated.*

PROPOSITION 2.15. – *Let $f : X \to Y$ be a projective morphism over a Noetherian scheme, and $\mathscr{N}$ an f-ample line bundle on X. Then for all coherent sheaves $\mathscr{F}$ on X flat over Y, there is an $m > 0$, such that for every $y \in Y$ and f-nef line bundle $\mathscr{K}$, $\mathscr{F} \otimes \mathscr{N}^m \otimes \mathscr{K}|_{X_y}$ is globally generated.*

Proof. – Let n be the biggest dimension of a fiber of f. Pick a globally generated ample line bundle $\mathscr{A}$ on X. Using Theorem 2.13, fix a $m > 0$, such that for every f-nef line bundle $\mathscr{K}$ on X,

$$R^i f_*(\mathscr{F} \otimes \mathscr{A}^{-n} \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0.$$

Then by Lemma 2.9 for all $y \in Y$ and f-nef $\mathscr{K}$,

$$H^i(X_y, \mathscr{F} \otimes \mathscr{A}^{-n} \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0.$$

In particular since $\mathscr{A}^{n-i}$ is nef for every $i \leq n$: for all $i \leq n$, $y \in Y$ and f-nef $\mathscr{K}$,

$$H^i(X_y, \mathscr{F} \otimes \mathscr{A}^{-i} \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0.$$

Therefore, by Theorem 2.14, $\mathscr{F} \otimes \mathscr{N}^m \otimes \mathscr{K}|_{X_y}$ is globally generated, which concludes our proof. □

2.4. Adjunction and surjectivity

In Proposition 2.17 another main ingredient of the article, the lifting statement, is stated for easier reference (see Section 1.3 for explanation, and Proposition 3.3 for the main application).

DEFINITION 2.16. – In the situation of Notation 2.2, a subvariety $Z \subseteq X$ is an F-pure center, if (X, Δ) is sharply F-pure at the generic point of Z and if for some (or equivalently all [40, Proposition 4.1]) $e > 0$,

$$\phi^{e\cdot g}(F_*^{e\cdot g}(\mathscr{I}_Z \cdot \mathscr{L}_{e\cdot g,\Delta})) \subseteq \mathscr{I}_Z. \tag{2.16a}$$

Furthermore, if Z is the union of F-pure centers, then (2.16a) still holds. In both situations for any $e > 0$, $\mathscr{L}_{e\cdot g,\Delta} \to \mathscr{O}_X$ descends then to

$$\phi^{e\cdot g}(F_*^{e\cdot g}(\mathscr{L}_{e\cdot g,\Delta}|_Z)) \subseteq \mathscr{O}_Z.$$

This defines a natural $\mathbb{Z}_{(p)}$-Weil divisorial sheaf, which then defines a $\mathbb{Q}$-Weil divisorial sheaf: the different Δ on Z, denoted by Δ_Z [41, Definition 5.1], [37, Definition 4.4]. The only situation where Δ_Z will be used in this article is if Z is an S_2, G_1 Cartier divisor and Δ is $\mathbb{Q}$-Cartier at the codimension one points of Z with index relatively prime to p. Then Δ_Z is the natural restriction $\Delta|_Z$ [37, Lemma 4.6]. Furthermore, if (Z, Δ_Z) is sharply F-pure, then Δ_Z is automatically a divisor in the sense of the current article (cf., paragraph before [37, Lemma 4.6]). That is, none of its components are contained in the singular locus of Z. In our situations this will always be the case.

PROPOSITION 2.17. – *In the situation of Notation 2.2, if X is projective, $Z \subseteq X$ is the union of F-pure centers of (X, Δ), and N a Cartier divisor, such that $N - K_X - \Delta$ is ample, then there is a commutative diagram as follows with surjective left vertical arrow.*

$$\begin{array}{ccc} S^0(X, \sigma(X,\Delta) \otimes \mathscr{O}_X(N)) & \hookrightarrow & H^0(X, \mathscr{O}_X(N)) \\ \downarrow & & \downarrow \\ S^0(Z, \sigma(Z,\Delta_Z) \otimes \mathscr{O}_X(N)) & \hookrightarrow & H^0(Z, \mathscr{O}_X(N)). \end{array}$$

Proof. – The statement is shown in [41, Proposition 5.3] for normal X. For S_2 and G_1 X, verbatim the same proof works. □

2.5. Fujita type version for $S_0 = H_0$

This section contains Fujita type results on the equality of H^0 with its subgroup S^0, introduced in Definition 2.5. As the statements of Section 3 need an absolute and a relative version as well, both are presented here.

NOTATION 2.18. – In the situation of Notation 2.2, define

$$\mathscr{B}_\Delta := \ker(F^g_*(\sigma(X,\Delta) \otimes \mathscr{L}_{g,\Delta}) \to \sigma(X,\Delta)).$$

Fix also an ample line bundle $\mathscr{N}$ on X and assume that X is projective over k.

REMARK 2.19. – Note that the definition $\mathscr{B}_\Delta$ makes sense if we replace Δ by any $\mathbb{Q}$-Weil divisor $0 \leq \Delta' \sim \Delta$, since then $\operatorname{ind}(K_X + \Delta') = \operatorname{ind}(K_X + \Delta)$. Here $\Delta \sim \Delta'$ means ordinary linear equivalence, not $\mathbb{Q}$-linear equivalence. That is, it means that $\Delta - \Delta'$ is the divisor of a $f \in \mathscr{K}(X)$.

LEMMA 2.20. – *In the situation of Notation 2.18, choose an integer $m > 0$ such that*

(1) *for every nef line bundle $\mathscr{K}$, $H^1(X, \mathscr{B}_\Delta \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0$ and*
(2) *$\mathscr{L}_{g,\Delta} \otimes \mathscr{N}^{m(p^g-1)}$ is nef.*

Then for every nef line bundle $\mathscr{K}$,

$$S^0(X, \sigma(X,\Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}) = H^0(X, \sigma(X,\Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}).$$

Proof. – Consider the exact sequence

$$0 \longrightarrow \mathscr{B}_\Delta \longrightarrow F^g_*(\mathscr{L}_{g,\Delta} \otimes \sigma(X,\Delta)) \longrightarrow \sigma(X,\Delta) \longrightarrow 0.$$

Applying the functor $F^{e\cdot g}_*(\_\_ \otimes \mathscr{L}_{e\cdot g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K}$ to it yields the following vertically drawn exact sequence.

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
F^{e\cdot g}_*(\mathscr{B}_\Delta \otimes \mathscr{L}_{e\cdot g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K} & & \\
\downarrow & & \\
F^{e\cdot g}_*\left(F^g_*(\mathscr{L}_{g,\Delta} \otimes \sigma(X,\Delta)) \otimes \mathscr{L}_{e\cdot g,\Delta}\right) \otimes \mathscr{N}^m \otimes \mathscr{K} & \cong & F^{(e+1)\cdot g}_*\left(\sigma(X,\Delta) \otimes \mathscr{L}_{(e+1)g,\Delta}\right) \otimes \mathscr{N}^m \otimes \mathscr{K} \\
\downarrow & \swarrow & \\
F^{e\cdot g}_*(\sigma(X,\Delta) \otimes \mathscr{L}_{e\cdot g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K} & & \\
\downarrow & & \\
0. & &
\end{array}$$

Hence,

$$H^0\left(X, F^{(e+1)\cdot g}_*(\sigma(X,\Delta) \otimes \mathscr{L}_{(e+1)g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K}\right) \to H^0\left(X, F^{e\cdot g}_*(\sigma(X,\Delta) \otimes \mathscr{L}_{e\cdot g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K}\right) \tag{2.20a}$$

is surjective if

$$H^1(X, F^{e\cdot g}_*(\mathscr{B}_\Delta \otimes \mathscr{L}_{e\cdot g,\Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K}) = 0. \tag{2.20b}$$

If we guarantee (2.20b) for all $e \geq 0$, then we obtain the surjectivity of (2.20a) for every $e \geq 0$, and consequently, that

$$S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}) = H^0(X, \sigma(X, \Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}).$$

However,

$$\begin{aligned} H^1 &(X, F_*^{e \cdot g}(\mathscr{B}_\Delta \otimes \mathscr{L}_{e \cdot g, \Delta}) \otimes \mathscr{N}^m \otimes \mathscr{K}) \\ &\cong H^1 \left(X, F_*^{e \cdot g}(\mathscr{B}_\Delta \otimes \mathscr{L}_{e \cdot g, \Delta} \otimes \mathscr{N}^{mp^{eg}} \otimes \mathscr{K}^{p^{eg}})\right) \\ &\cong H^1 \left(X, \mathscr{B}_\Delta \otimes \mathscr{L}_{e \cdot g, \Delta} \otimes \mathscr{N}^{mp^{eg}} \otimes \mathscr{K}^{p^{eg}}\right) \\ &\cong H^1 \left(X, (\mathscr{B}_\Delta \otimes \mathscr{N}^m) \otimes \left(\mathscr{L}_{g, \Delta} \otimes \mathscr{N}^{m(p^g - 1)}\right)^{\frac{p^{eg}-1}{p^g-1}} \otimes \mathscr{K}^{p^{eg}}\right), \end{aligned}$$

where the latest group is zero whenever the assumptions of the lemma hold. □

COROLLARY 2.21. – *In the situation of Notation 2.18, there is an $m > 0$ such that for every nef line bundle $\mathscr{K}$,*

(2.21a) $$S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}) = H^0(X, \sigma(X, \Delta) \otimes \mathscr{N}^m \otimes \mathscr{K}).$$

Proof. – Both conditions of Lemma 2.20 holds for $m \gg 0$. The first one by Fujita vanishing and the second by basic properties of ample line bundles. □

Recall that Notation 1.4 is the general framework in which our results were worded: a flat morphism $f : X \to Y$ and an effective Weil divisor Δ on X with adequate properties.

PROPOSITION 2.22. – *In the situation of Notation 1.4, if $\mathscr{N}$ is an f-ample line bundle on X and $\dim Y \geq 1$, then there is an $m > 0$ such that for every f-nef line bundle $\mathscr{K}$ and generic $y \in Y$ (i.e., contained in a dense open set),*

$$H^0(X_y, \mathscr{N}^m \otimes \mathscr{K}) = S^0(X_y, \sigma(X_y, \Delta_y) \otimes (\mathscr{N}^m \otimes \mathscr{K})_y).$$

Proof. – By discarding the finitely many points over which (X_y, Δ_y) is not sharply F-pure, we may assume that the pair (X_y, Δ_y) is sharply F-pure for every $y \in Y$. Further by pulling back to an open set of Y_{red} we may also assume that Y is regular. (For that, one also has to show that all assumptions of the corollary, after assuming that every (X_y, Δ_y) is sharply F-pure, hold for any pullback of the family. To see that the reflexive power $\omega_{X/Y}^{[r]}(r\Delta)$ of the relative log-canonical sheaf being a line bundle holds for the pulled back family, show that the corresponding sheaf of the pulled back family is isomorphic to the pullback of the sheaf of the original family and hence is a line bundle. To show the isomorphism, use that it holds in relative codimension one, and [18, Corollary 3.7]. The other conditions are immediate.) Let us fix then a $y = y'$. By Corollary 2.21, there is an $m > 0$, such that

$$H^0\left(X_{y'}, \mathscr{N}^m \otimes \mathscr{K}\right) = S^0\left(X_{y'}, \sigma(X_{y'}, \Delta_{y'}) \otimes (\mathscr{N}^m \otimes \mathscr{K})_{y'}\right),$$

for every f-nef line bundle $\mathscr{K}$ on X. However, then the above condition also holds for y' replaced by generic y, according to [38, Theorem C, (c)]. □

Proposition 2.23. – *In the situation of Notation 1.4, if $\mathscr{N}$ is an f-ample line bundle on X, then there is an $m > 0$ such that for every nef line bundle $\mathscr{K}$ and all $y \in Y$ for which (X_y, Δ_y) is sharply F-pure,*

$$H^0(X_y, \mathscr{N}^m \otimes \mathscr{K}) = S^0(X_y, \sigma(X_y, \Delta_y) \otimes \mathscr{N}_y^m \otimes \mathscr{K}_y).$$

Proof. – By discarding the points of y for which (X_y, Δ_y) is not sharply F-pure, we may assume that (X_y, Δ_y) is sharply F-pure for every $y \in Y$. By Proposition 2.22, there is an $m > 0$ and an open set $U_1 \subseteq Y$, such that the statement of the proposition holds for all $y \in U_1$, instead of all $y \in Y$. Using Proposition 2.22 again for the pullback of the family over $Y \setminus U_1$, one finds an open set $U_2 \subseteq Y \setminus U_1$ and possibly even bigger $m > 0$, such that the statement of the proposition holds also for all $y \in U_2$, and then for all $y \in U_1 \cup U_2$. Iterating this process, by the Noetherian property, one obtains finitely many $U_i \subseteq Y$ as above such that $\bigcup U_i = Y$. This finishes our proof. □

2.6. Auxiliary statements about the product construction

Here we present some statements about the construction of Notation 2.11.

Proposition 2.24. – *Using Notations 2.11, if $f : X \to Y$ is a morphism with Y Cohen-Macaulay and $\mathscr{F}$ a flat S_r coherent sheaf on X, then $\mathscr{F}^{(n)}$ is S_r as well.*

Proof. – Let $g : Z \to W$ be a morphism to a Cohen-Macaulay scheme, and $\mathscr{G}$ a flat coherent sheaf on Z. Then by [12, Proposition 6.3.1] and [12, Corollaire 6.1.2], $\mathscr{G}$ is S_r if and only if for each $Q \in W$, $\mathscr{G}|_{Z_Q}$ is $S_{r-\dim \mathscr{O}_{W,Q}}$ (if $r - \dim \mathscr{O}_{W,Q} < 0$ then we define $S_{r-\dim \mathscr{O}_{W,Q}}$ with the usual formula just allowing negative value inside the minimum as well). Getting back to the situation of our proposition, since $\mathscr{F}$ is S_r, $\mathscr{F}|_{X_Q}$ is $S_{r-\dim \mathscr{O}_{Y,Q}}$ for all $Q \in Y$. Then, by [37, Lemma 4.2], $\mathscr{F}^{(n)}|_{X_Q^{(n)}}$ is $S_{r-\dim \mathscr{O}_{Y,Q}}$ for all $Q \in Y$. Therefore, $\mathscr{F}^{(n)}$ is S_r on $X^{(n)}$. □

Proposition 2.25. – *Using Notations 2.11, if $f : X \to Y$ is a morphism from a G_1 scheme to a smooth curve, then $X^{(n)}$ is G_1 for any n.*

Proof. – Notice that for X to be G_1, X_y has to be G_1 for generic y, and G_0 for every $y \in Y$. However then for every $n \in \mathbb{Z}^+$, $X_y^{(n)}$ is G_1 for generic y, and G_0 for every $y \in Y$, which concludes our proof. □

Proposition 2.26. – *Using Notations 2.11, if $f : X \to Y$ is a projective, flat morphism from an S_2 and G_1 scheme to a smooth curve, then $\omega_{X/Y}^{(n)} \cong \omega_{X^{(n)}/Y}$.*

Proof. – First, notice that both $\omega_{X/Y}$ and $\omega_{X^{(n)}/Y}$ are reflexive. Furthermore, by [2, Lemma 2.11], so is $\omega_{X/Y}^{(n)}$. Therefore, it is enough to prove that $\omega_{X/Y}^{(n)} \cong \omega_{X^{(n)}/Y}$ in codimension one. However if U is the relative Gorenstein locus of f, then the isomorphism is clear over $U^{(n)}$. This concludes our proof since $\operatorname{codim}_{X^{(n)}} X^{(n)} \setminus U^{(n)} \geq 2$ by Proposition 2.25. □

LEMMA 2.27. – *In the situation of Notation 2.2, if $\mathscr{N}$ is a line bundle on X and X is reduced, then*

$$S^0\left(X^{(m)}, \sigma\left(X^{(m)}, \Delta^{(m)}\right) \otimes \mathscr{N}^{(m)}\right) \cong S^0(X, \sigma(X, \Delta) \otimes \mathscr{N})^{\bigotimes m}.$$

(Here $X^{(m)}$ and $\mathscr{N}^{(m)}$ are taken over Spec k.*)*

Proof. – A word of caution before starting the proof: for $S^0\left(X^{(m)}, \sigma(X^{(m)}, \Delta^{(m)}) \otimes \mathscr{N}^{(m)}\right)$ to be defined, $(X^{(m)}, \Delta^{(m)})$ has to be a pair. That is, $X^{(m)}$ has to be S_2 and G_1, and $\Delta^{(m)}$ has to be an element of $\mathrm{Weil}^*(X)$, i.e., none of the components of $\Delta^{(m)}$ can be contained in $\left(X^{(m)}\right)_{\mathrm{sing}}$. The conditions on $X^{(m)}$ follow from Propositions 2.24 and 2.25. For the condition on $\Delta^{(m)}$, notice that since X is reduced and we are working over an algebraically closed field, the components of X are generically smooth. It is immediate then that all generic points of $\Delta^{(m)}$ are contained in the smooth locus of $X^{(m)}$.

After the preliminary considerations note that for every $e > 0$ such that $g \mid e$,

$$(1 - p^e)(K_{X^{(m)}} + \Delta^{(m)}) = \sum_{i=1}^{m} p_i^*((1 - p^e)(K_X + \Delta))$$

over the Gorenstein locus and then by codimension argument it holds everywhere. Hence for every $e > 0$, such that $g \mid e$,

$$\mathscr{L}_{e,\Delta^{(m)}} \cong \bigotimes_{i=1}^{m} p_i^* \mathscr{L}_{e,\Delta}. \tag{2.27a}$$

Next we claim that, *for any collection of m coherent sheaves $\mathscr{F}_i$ on X*,

$$\bigotimes_{i=1}^{m} p_i^* F_*^e \mathscr{F}_i \cong F_*^e \left(\bigotimes_{i=1}^{m} p_i^* \mathscr{F}_i\right). \tag{2.27b}$$

Indeed, (2.27b) immediately follows from the natural isomorphism of $\bigotimes_k A_i$-modules

$$\begin{array}{ccc} \bigotimes_k F_*^e M_i & \cong & F_*^e\left(\bigotimes_k M_i\right), \\ \cup & & \cup \\ \bigotimes F_*^e(m_i) & \longmapsto & F_*^e(\bigotimes m_i) \end{array} \tag{2.27c}$$

where M_i are modules over a collection of m (commutative) algebras A_i over k, and F_*^e denotes the twist of the original module structure by taking p^e-th powers. The map of (2.27c) is an isomorphism of abelian groups almost by definition. One subtlety should be mentioned here: when applying the tensor product, the action of k on the M_i on the two sides is different. On the right side it acts regularly, while on the left side it acts after taking p^e-th powers. A priori, this could ruin the chances of getting even a bijection by the formula of (2.27c). However, k being perfect saves the day, since then $k^{p^e} = k$. Hence the relations on the tensors on the left are the same as on the right. To see that the map of (2.27c) is an isomorphism of $\bigotimes_k A_i$-modules, we have to check that the actions on the two sides match up. Indeed,

$$(\bigotimes a_i)(\bigotimes F_*^e(m_i)) = \bigotimes F_*^e\left(a_i^{p^e} m_i\right) \mapsto F_*^e\left(\bigotimes a_i^{p^e} m_i\right) = (\bigotimes a_i)(F_*^e(\bigotimes m_i)).$$

This concludes the proof of our claim. In particular,

$$F^e_* \mathscr{L}_{e,\Delta^{(m)}} \cong \underbrace{F^e_* \left(\bigotimes_{i=1}^m p_i^* \mathscr{L}_{e,\Delta} \right) \cong}_{(2.27a)} \underbrace{\bigotimes_{i=1}^m p_i^* F^e_* \mathscr{L}_{e,\Delta}}_{\text{by } (2.27b)}. \tag{2.27d}$$

Furthermore, the induced trace maps also respect the decomposition of (2.27d), since they respect it on the dense set of smooth points and then also globally since all sheaves involved are torsion free. Therefore, the following commutative diagram concludes the proof.

$$\begin{array}{ccc} H^0(X, F^e_* \mathscr{L}_{e,\Delta} \otimes \mathscr{N})^{\bigotimes m} & \longrightarrow & H^0(X, \mathscr{N})^{\bigotimes m} \\ \updownarrow \cong (2.27d) & & \updownarrow \cong \\ H^0\left(X^m, F^e_* \mathscr{L}_{e,\Delta^{(m)}} \otimes \mathscr{N}^{(m)}\right) & \longrightarrow & H^0\left(X^{(m)}, \mathscr{N}^{(m)}\right). \end{array}$$

□

3. Semi-positivity

In this section we present the main results of the article. For an outline of the arguments, see Section 1.3.

3.1. Generic global generation

We prove our most general results in the setting of Notation 1.4. However, since nefness is decided on curves, we are able to reduce this general setting to the special case when Y is a smooth curve (cf., Lemma 3.11, proof of Theorem 1.6, etc.). So, first we consider the special case of Notation 1.4 when Y is a smooth curve. Then we can replace the sheaf notation of Notation 1.4 with a divisorial one and further it is enough to have just one nice fiber. This is summarized below.

NOTATION 3.1. – We use the following notations in this section

(1) (X, Δ) is a pair, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $p \nmid \operatorname{ind}(K_X + \Delta)$,
(2) $f : X \to Y$ is a flat, projective morphism to a smooth projective curve,
(3) fix also a closed point $y_0 \in Y$ such that $X_0 := X_{y_0}$ is S_2, G_1 and reduced and Δ avoids all codimension 0 and the singular codimension 1 points of X_0,
(4) set $\Delta_0 := \Delta|_{X_0}$ and $r := \operatorname{ind}(K_X + \Delta)$.

REMARK 3.2. – In the situation of Notation 3.1, note the following:

(1) if a codimension one point ξ of X_0 is singular, then Δ avoids ξ hence it is Cartier there, otherwise if it is non-singular then K_X is Cartier at ξ and hence by the index assumption on $K_X + \Delta$, Δ is $\mathbb{Q}$-Cartier with index not divisible by p,
(2) therefore, at all codimension one points ξ of X_0, Δ can be sensibly restricted,
(3) $(X_0, \Delta|_{X_0})$ is a pair and
(4) by [37, Lemma 4.6], $\Delta|_{X_0}$ agrees with the F-different of Δ at X_0.

Recall that a coherent sheaf $\mathscr{F}$ on a scheme X is *generically globally generated*, if there is a homomorphism $\mathscr{O}_X^{\bigoplus m} \to \mathscr{F}$ which is surjective over a dense open set.

Proposition 3.3. – *In the situation of Notation 3.1, choose a Cartier divisor N and set $\mathscr{N} := \mathscr{O}_X(N)$. Assume that*

(1) *$N - K_{X/Y} - \Delta$ is an f-ample $\mathbb{Q}$-divisor,*
(2) *$H^0(X_0, \mathscr{N}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0})$,*
(3) *$N - K_{X/Y} - \Delta$ is nef.*

Then $f_*\mathscr{N} \otimes \omega_Y(2y_0)$ is generically globally generated.

Proof. – Set $M := N + f^*K_Y + 2X_0$ and $\mathscr{M} := \mathscr{O}_X(M)$. Consider the commutative diagram below.

(3.3a)

$$\begin{array}{ccccc} f_*\mathscr{M} & \longrightarrow & (f_*\mathscr{M}) \otimes k(y_0) & \hookrightarrow & H^0(X_0, \mathscr{M}|_{X_0}) \\ \uparrow & & & & \| \\ S^0(X, \sigma(X, \Delta + X_0) \otimes \mathscr{M}) \otimes \mathscr{O}_Y & & \longrightarrow & & S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{M}|_{X_0}), \end{array}$$

where $H^0(X_0, \mathscr{M}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{M}|_{X_0})$, because

$$H^0(X_0, \mathscr{M}|_{X_0}) \cong H^0(X_0, \mathscr{N}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0}) \cong S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{M}|_{X_0}).$$

Note that

(3.3b)

$$M - K_X - \Delta - X_0 = N + f^*K_Y + 2X_0 - K_{X/Y} - f^*K_Y - \Delta - X_0 = N - K_{X/Y} - \Delta + X_0.$$

Note also that $N - K_{X/Y} - \Delta$ is nef by assumption (3) and it is relatively ample by (1). Furthermore, X_0 is the pullback of an ample divisor from Y. Hence, $N - K_{X/Y} - \Delta + X_0$ is ample and then by (3.3b) so is $M - K_X - \Delta - X_0$. Note now the following:

– $p \nmid \operatorname{ind}(K_X + \Delta) = \operatorname{ind}(K_X + \Delta + X_0)$, and
– since X_0 is smooth at all its general points (by reducedness) and Δ contains no components of X_0, X_0 is a union of F-pure centers of $(X, \Delta + X_0)$.

Hence, Proposition 2.17 implies that the diagonal arrow in (3.3a) is surjective. This finishes our proof. □

3.2. Semi-positivity general case

Lemma 3.4. – *If $\mathscr{F}$ is a vector bundle on a smooth curve Y and $\mathscr{L}$ is a line bundle such that for every $m > 0$, $(\bigotimes_{i=1}^m \mathscr{F}) \otimes \mathscr{L}$ is generically globally generated, then $\mathscr{F}$ is nef.*

Proof. – Take a finite cover $\tau : Z \to Y$ by a smooth curve and a quotient line bundle $\mathscr{E}$ of $\tau^*\mathscr{F}$. Since $(\bigotimes_{i=1}^m \mathscr{F}) \otimes \mathscr{L}$ is generically globally generated, so is $(\bigotimes_{i=1}^m \tau^*\mathscr{F}) \otimes \tau^*\mathscr{L}$ and hence $\mathscr{E}^m \otimes \tau^*\mathscr{L}$ as well. Therefore $m \deg(\mathscr{E}) + \deg(\tau^*\mathscr{L}) \geq 0$ for all $m > 0$. In particular then $\deg(\mathscr{E}) \geq 0$. Since this is true for arbitrary τ and $\mathscr{E}$, $\mathscr{F}$ is nef indeed. □

Lemma 3.5. – *In the situation of Notation 3.1 (using the product notations introduced in Notation 2.11) $f^{(n)} : (X^{(n)}, \Delta^{(n)}) \to Y$ also satisfies the assumptions of Notation 3.1.*

Proof. – We show every assumption of Notation 3.1 for $f^{(n)} : X^{(n)} \to Y$ one by one.

– $X^{(n)}$ is S_2 and G_1 by Propositions 2.24 and 2.25.

- Since all the components of X_0 are generically smooth, the same holds for a generic fiber and then for X itself. Therefore, $\Delta^{(n)} \in \mathrm{Weil}^*(X^{(n)})$, in particular, $(X^{(n)}, \Delta^{(n)})$ is a pair.
- $f : X^{(n)} \to Y$ is flat, projective.
- $X_0^{(n)}$ is reduced, S_2 (by [37, Lemma 4.2]) and G_1.
- $\Delta^{(n)}|_{X_{y_0}^{(n)}} = \sum_{i=1}^n p_i^* \Delta_0$ avoids the codimension 0 and the singular codimension one points of $X_{y_0}^{(n)}$, because the same holds for Δ_0 and every component of X_0 is generically smooth.
- By Proposition 2.26,

$$\begin{aligned}\mathrm{ind}(K_X + \Delta) &= \mathrm{ind}(K_{X/Y} + \Delta) = \mathrm{ind}((K_{X/Y} + \Delta)^{(n)}) \\ &= \mathrm{ind}(K_{X^{(n)}/Y} + \Delta^{(n)}) = \mathrm{ind}(K_{X^{(n)}} + \Delta^{(n)})\end{aligned}$$

and hence $K_{X^{(n)}} + \Delta^{(n)}$ is $\mathbb{Q}$-Cartier and $p \nmid \mathrm{ind}(K_{X^{(n)}} + \Delta^{(n)})$. □

Proposition 3.6. – *In the situation of Notation 3.1, choose a Cartier divisor N and set $\mathscr{N} := \mathscr{O}_X(N)$. Assume that*

- $R^i f_* \mathscr{N} = 0$ *for all* $i > 0$,
- $N - K_{X/Y} - \Delta$ *is an* f*-ample* $\mathbb{Q}$*-divisor,*
- $H^0(X_0, \mathscr{N}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0})$,
- $N - K_{X/Y} - \Delta$ *is nef.*

Then $f_* \mathscr{N}$ *is a nef vector bundle.*

Proof. – The proof uses the notations introduced in Notation 2.11. Let $n > 0$ be an integer. First, notice the following:

- by Lemma 3.5, the assumptions of Notation 3.1 are satisfied for $f^{(n)} : (X^{(n)}, \Delta^{(n)}) \to Y$,
- $N^{(n)} - K_{X^{(n)}/Y} - \Delta^{(n)} = (N - K_{X/Y} - \Delta)^{(n)}$ is $f^{(n)}$-ample,
- by Lemma 2.27 and the Künneth formula,

$$\begin{aligned} H^0\left(X_0^{(n)}, \mathscr{N}^{(n)}|_{X_0^{(n)}}\right) &= H^0\left(X_0, \mathscr{N}|_{X_0}\right)^{\bigotimes n} \\ &\cong S^0\left(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0}\right)^{\bigotimes n} \\ &\cong S^0\left(X_0^{(n)}, \sigma\left(X_0^{(n)}, \Delta^{(n)}\right) \otimes \mathscr{N}^{(n)}|_{X_0^{(n)}}\right),\end{aligned}$$

- $N^{(n)} - K_{X^{(n)}/Y} - \Delta^{(n)} = (N - K_{X/Y} - \Delta)^{(n)}$ is nef.

Hence Proposition 3.3 applies to $(X^{(n)}, \Delta^{(n)})$ and $N^{(n)}$, and consequently, $f_*^{(n)}(\mathscr{N}^{(n)}) \otimes \omega_Y(2y_0)$ generically globally generated for every $n > 0$.

By assumption (3.6) and Lemmas 2.9, 2.10 and 2.12, $f_*^{(n)}(\mathscr{N}^{(n)}) \cong \bigotimes_{i=1}^n f_* \mathscr{N}$ and $f_* \mathscr{N}$ is a vector bundle. Therefore, $f_* \mathscr{N}$ is a vector bundle, such that $(\bigotimes_{i=1}^n f_* \mathscr{N}) \otimes \omega_Y(2y_0)$ is generically globally generated for every $n > 0$. Hence, by Lemma 3.4, $f_* \mathscr{N}$ is a nef vector bundle. This concludes our proof. □

Proposition 3.7. – *In the situation of Proposition 3.6, if furthermore $\mathscr{N}$ is f-nef and $\mathscr{N}_y$ globally generated except possibly finitely many $y \in Y$, then $\mathscr{N}$ is nef.*

Proof. – Consider the following commutative diagram for every $y \in Y$

(3.7a)
$$\begin{array}{ccc} f^* f_* \mathscr{N} & \longrightarrow & \mathscr{N} \\ \downarrow & & \downarrow \\ H^0(X_y, \mathscr{N}) \otimes \mathscr{O}_{X_y} & \longrightarrow & \mathscr{N}_y. \end{array}$$

The left vertical arrow is surjective because of assumption (3.6) of Proposition 3.6 and Lemmas 2.9 and 2.10. The bottom horizontal arrow is surjective except finitely many $y \in Y$. Hence $f^* f_* \mathscr{N} \to \mathscr{N}$ is surjective except possibly at points lying over finitely many points of $y \in Y$. To show that $\mathscr{N}$ is nef, we have to show that $\deg \mathscr{N}|_C \geq 0$ for every smooth projective curve C mapping finitely to X. By assumption this follows if C is vertical. So, we may assume that C maps surjectively onto Y. However, then $(f^* f_* \mathscr{N})|_C \to \mathscr{N}|_C$ is generically surjective. Since $f_* \mathscr{N}$ is nef by Proposition 3.6, so is $f^* f_* \mathscr{N}$ and hence $\deg(\mathscr{N}|_C) \geq 0$ has to hold. □

3.3. Semi-positivity when the relative log-canonical divisor is relatively semi-ample

LEMMA 3.8. – *Let $r > 0$ be an integer. If $t > 1$ is a real number, then there is a rational number $\frac{a}{b}$, such that $r \mid b+1$ and*

(3.8a)
$$\frac{a}{b+1} < t < \frac{a}{b}.$$

Furthermore, both a and b can be chosen to be arbitrarily big.

Proof. – (3.8a) is equivalent to

$$\frac{b+1}{a} > \frac{1}{t} > \frac{b}{a}.$$

Since $b+1 = cr$ has to hold for some integer c, (3.8a) together with $r \mid b+1$ is equivalent to finding a rational number $\frac{c}{a}$, such that

$$\frac{c}{a} > \frac{1}{tr} > \frac{c}{a} - \frac{1}{ar},$$

which is equivalent to finding positive integers a and c, such that

$$c > \frac{1}{tr} a > c - \frac{1}{r},$$

which is equivalent to finding a positive integer a, such that

$$1 > \left\{ \frac{1}{tr} a \right\} > 1 - \frac{1}{r}.$$

However there is such an a, since $t > 1$ and hence $\frac{1}{tr} < \frac{1}{r}$. □

LEMMA 3.9. – *If $X \to Y$ is a flat, projective morphism to a smooth, projective curve, $\mathscr{N}$ is an f-nef line bundle on X such that $\mathscr{N}_y$ is semi-ample for generic $y \in Y$ and $\mathscr{A}$ an ample line bundle on Y, then $\mathscr{N} \otimes f^* \mathscr{A}^l$ is nef for $l \gg 0$.*

Proof. – By [16, Corollary 12.9] on an open set (depending on n) of Y, $f_*\mathscr{N}^n$ is locally free and

$$(f_*\mathscr{N}^n) \otimes k(y) \to H^0(X_y, \mathscr{N}^n)$$

is an isomorphism. Therefore, for $n \gg 0$ and divisible enough, $f^*f_*\mathscr{N}^n \to \mathscr{N}^n$ is surjective over an open set of Y (cf., (3.7a)). Choose $l > 0$ such that $(f_*\mathscr{N}^n) \otimes \mathscr{A}^{nl}$ is globally generated. Choose also any curve C on X. If C is vertical, $\deg \mathscr{N} \otimes f^*\mathscr{A}^l|_C \geq 0$ by the assumption that $\mathscr{N}$ is f-nef. Otherwise if C is horizontal, by the choice of $\mathscr{A}$, $(f^*f_*\mathscr{N}^n) \otimes f^*\mathscr{A}^{nl}$ is globally generated. Hence, by the homomorphism $(f^*f_*\mathscr{N}^n) \otimes f^*\mathscr{A}^{nl} \to \mathscr{N}^n \otimes f^*\mathscr{A}^{nl}$, which is surjective over an open set of Y, $\mathscr{N}^n \otimes f^*\mathscr{A}^{nl}|_C$ is generically globally generated. Therefore $\deg \mathscr{N} \otimes f^*\mathscr{A}^l|_C \geq 0$. □

THEOREM 3.10. – *In the situation of Notation 3.1, if (X_0, Δ_0) is sharply F-pure, $K_{X/Y} + \Delta$ is f-nef and $K_{X_y} + \Delta_y$ is semi-ample for generic $y \in Y$ (e.g., this is satisfied if $K_{X/Y} + \Delta$ is f-ample or f-semi-ample), then $K_{X/Y} + \Delta$ is nef.*

Proof. – *Assume that $K_{X/Y} + \Delta$ is not nef.* Choose a general member of a complete linear system of a very ample divisor on Y and let B be its pullback to X. By Lemma 3.9 for $s \gg 0$, $K_{X/Y} + \Delta + sB$ is nef. Let then

$$t := \min\{s \in \mathbb{R} | 0 \leq s \text{ and } K_{X/Y} + \Delta + sB \text{ is nef}\}.$$

If $t = 0$, there is nothing to prove. Hence we may assume $t > 0$.

First, we claim that in fact $t > 1$ can be assumed. The reason is that if $t \leq 1$, then by taking a degree d smooth cyclic cover(1) $Y' \to Y$ of Y for some $d \gg 0$ and $p \nmid d$, and pulling back everything there, we may replace B by a Cartier divisor B', such that $B'd = B$. Indeed, set $X' := X \times_Y Y'$ and let $\pi : X' \to X$ be the natural projection. Then, one has to verify that the pulled back family still satisfies the assumptions of Notation 3.1: for example X' is S_2 because according to [12, Corollaire 6.1.2 and Proposition 6.3.1], a flat fibration over a smooth curve is S_2 exactly if the generic fiber is S_2 and the special fibers are S_1. This is stable under pullback. One has to be also careful about pulling back Δ. It is not $\mathbb{Q}$-Cartier, but according to Definition 2.1, none of its components is contained in X_{sing} and hence Δ is Cartier outside of a codimension at least two open set. Then one can pull back by pulling back over this open set, and then extending it in the unique way. The only trap in this process is that a priori one of the components of $\pi^*\Delta$ can end up in the singular locus. However, this cannot happen, since outside of finitely many general fibers, π is étale, and Δ does not contain any components of the branched and therefore general fibers by assumption. The other conditions are immediate(2). Note that we also have to replace y_0 by any of its preimages $y_0' \in Y'$.

After applying the above pullback and replacing B by B', t changes to $d \cdot t$, because:

$$K_{X'/Y'} + \pi^*\Delta + d \cdot tB' = \pi^*(K_{X/Y} + \Delta + tB)$$

(1) To be precise, if $B = f^*D$, then we choose a general very ample, effective divisor H on Y, such that there is a divisor G, for which $D + H \sim dG$. Then $Y' := \operatorname{Spec}_Y \left(\bigoplus_{i=0}^{d-1} \mathscr{O}_Y(-iG)\right)$, where the ring structure is given by the embedding $\mathscr{O}_Y(-dG) \hookrightarrow \mathscr{O}_Y$ using $D + H \sim dG$. Y' will be smooth because $D + H$ is smooth.
(2) Remember that since D and H are general, π is étale over X_0.

is nef, and if there was a $s < dt$, such that $K_{X'/Y'} + \pi^*\Delta + sB'$ was nef, then for every curve C on X', $K_{X'/Y'} + \pi^*\Delta + sB'|_C \geq 0$ would hold. However, then also

$$0 \leq K_{X'/Y'} + \pi^*\Delta + sB'|_C = \pi^* \left(K_{X/Y} + \Delta + \frac{s}{d}B\right)\Big|_C = K_{X/Y} + \Delta + \frac{s}{d}B\Big|_{\pi_*(C)}$$

would hold. Since every curve on X is (a fraction of) the pushforward of a curve from X', this would mean that $K_{X/Y} + \Delta + \frac{s}{d}B$ was nef, which would contradict the definition of t. Hence as claimed, t changes to $d \cdot t$. Therefore, since d can be chosen to be arbitrary big, we may indeed assume that $t > 1$. This finishes the proof of our claim.

So, from now we assume that $t > 1$. Then, by Lemma 3.8, there is a rational number $\frac{a}{b}$, such that $r \mid b + 1$ and $\frac{a}{b+1} < t < \frac{a}{b}$. Set $A := \frac{a}{b}B$. By Theorem 2.13, Corollary 2.21 and Proposition 2.15, there is an ample Cartier divisor Q on X such that for all $i > 0$ and f-nef Cartier divisor L,

$$R^i f_*(\mathscr{O}_X(Q + L)) = 0, \tag{3.10a}$$

$$H^0(X_0, \mathscr{O}_{X_0}(Q + L)) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{O}_{X_0}(Q + L)) \tag{3.10b}$$

and

$$\mathscr{O}_{X_y}(Q + L) \text{ is globally generated for all } y \in Y. \tag{3.10c}$$

We prove by induction that $q((b + 1)(K_{X/Y} + \Delta) + bA) + Q$ *is nef for every integer* $q \geq 0$. For $q = 0$ the statement is true by the choice of Q. Hence, we may assume that $(q - 1)((b + 1)(K_{X/Y} + \Delta) + bA) + Q$ is nef. Now, we verify that the conditions of Proposition 3.7 hold for $N := q((b + 1)(K_{X/Y} + \Delta) + bA) + Q$ and $\mathscr{N} := \mathscr{O}_X(N)$. Indeed:

– N is Cartier by the choice of b and A,

– $R^i f_* \mathscr{N} = 0$ for all $i > 0$ because of (3.10a) and that

$$N - Q = q((b + 1)(K_{X/Y} + \Delta) + bA) \tag{3.10d}$$

is an f-nef Cartier divisor,

– the $\mathbb{Q}$-divisor

$$N - K_{X/Y} - \Delta = \Big(b(K_{X/Y} + \Delta + A)\Big) + \Big((q - 1)((b + 1)(K_{X/Y} + \Delta) + bA) + Q\Big)$$

is not only f-ample, but also nef, because of the inductional hypothesis and that $A = \frac{a}{b}B \geq tB$,

– using (3.10b) and the f-nefness of (3.10d),

$$H^0(X_0, \mathscr{N}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0}),$$

– since all the summands of N are f-nef, so is N,

– $N|_{X_y}$ is globally generated, by (3.10c) and since $N - Q$ is f-nef according to (3.10d).

Hence Proposition 3.7 implies that N is nef. This finishes our inductional step, and hence the proof of the nefness of $q((b + 1)(K_{X/Y} + \Delta) + bA) + Q$ for every $q \geq 0$. However, then $(b + 1)(K_{X/Y} + \Delta) + bA$ has to be nef as well. Therefore, so is

$$\begin{aligned}\frac{1}{b+1}((b + 1)(K_{X/Y} + \Delta) + bA) &= (K_{X/Y} + \Delta) + \frac{b}{b+1}A \\ &= (K_{X/Y} + \Delta) + \frac{b}{b+1}\frac{a}{b}B = (K_{X/Y} + \Delta) + \frac{a}{b+1}B.\end{aligned}$$

However, $\frac{a}{b+1} < t$, which contradicts the definition of t. Therefore our assumption was false, $K_{X/Y} + \Delta$ is nef indeed. □

Lemma 3.11. – *In the situation of Notation 1.4, let $\tau : Y' \to Y$ be a finite morphism from a smooth curve and set $X' := X \times_Y Y'$ and $\pi : X' \to X$, $f' : X' \to Y'$ the induced morphisms. Let Δ' be the pullback of Δ, which is defined by the following procedure. Take the open set U of Notation 1.4 and notice that $r\Delta|_U$ is a Cartier divisor. Then, pull it back to $\pi^{-1}U$, extend it uniquely over X' and finally divide all its coefficients by r. The extension is unique, since* $\operatorname{codim}_{X'} X' \setminus \pi^{-1}U \geq 2$. *In the above situation we claim that*

$$\pi^* \omega_{X/Y}^{[r]}(r\Delta) \cong \omega_{X'/Y'}^{[r]}(r\Delta').$$

In particular, $p \nmid r = \operatorname{ind}(K_{X'/Y'} + \Delta')$.

Proof. – Notice that by construction $\pi^* \omega_{X/Y}^{[r]}(r\Delta)$ and $\omega_{X'/Y'}^{[r]}(r\Delta')$ agree over $\pi^{-1}U$, that is, in relative codimension one. Notice also that since $\omega_{X/Y}^{[r]}(r\Delta)$ is assumed to be a line bundle, so is $\pi^* \omega_{X/Y}^{[r]}(r\Delta)$, and therefore $\pi^* \omega_{X/Y}^{[r]}(r\Delta)$ is reflexive in the sense of [18]. On the other hand, since $\omega_{X'/Y'}^{[r]}(r\Delta')$ is defined as a pushforward of a line bundle from relative codimension one, it is reflexive by [18, Corollary 3.7]. Therefore by [18, Proposition 3.6], $\pi^* \omega_{X/Y}^{[r]}(r\Delta)$ and $\omega_{X'/Y'}^{[r]}(r\Delta')$ are isomorphic everywhere. In particular, then $\omega_{X'/Y'}^{[r]}(r\Delta')$ is a line bundle. □

Recall that Theorem 1.6 states that in the situation of Notation 1.4 or Notation 1.5, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-nef and for all but finitely many $y \in Y$, $K_{X_y} + \Delta_y$ is semi-ample, then $\omega_{X/Y}^{[r]}(r\Delta)$ is nef. Theorem 3.10 proves this statement when Y is a smooth curve, to which we reduce the general case below.

Proof of Theorem 1.6. – Choose any curve C on X. We are supposed to prove that $\deg \omega_{X/Y}^{[r]}(r\Delta)\big|_C \geq 0$. If C is vertical this is immediate since $\omega_{X/Y}^{[r]}(r\Delta)$ is assumed to be f-nef. Otherwise, let Y' be the normalization of C. It is enough to prove that $\deg \omega_{X/Y}^{[r]}(r\Delta)\big|_{Y'} \geq 0$. Let $\tau : Y' \to Y$ be the induced morphism. Then, we are in the situation of Lemma 3.11. Using the notation introduced there, it is enough to prove that $\pi^* \omega_{X/Y}^{[r]}(r\Delta)$ is nef. However, according to Lemma 3.11, this is the same as proving that $\omega_{X'/Y'}^{[r]}(r\Delta')$ is nef. But, $f' : (X', \Delta') \to Y'$ satisfy all assumptions of Theorem 3.10 (or Theorem 3.16 in the case of Notation 1.5). Therefore, $\omega_{X'/Y'}^{[r]}(r\Delta')$ is nef indeed. □

Theorem 3.12. – *In the situation of Notation 3.1, assume that (X_0, Δ_0) is sharply F-pure and $K_{X/Y} + \Delta$ is f-ample. Further choose an $M > 0$ such that for all $i > 0$ and $m \geq M$,*

$$R^i f_* (\mathscr{O}_X(mr(K_{X/Y} + \Delta))) = 0 \text{ and} \tag{3.12a}$$

$$H^0(X_0, \mathscr{O}_{X_0}(mr(K_{X/Y} + \Delta))) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{O}_{X_0}(mr(K_{X/Y} + \Delta))). \tag{3.12b}$$

Then $f_*(\mathscr{O}_X(mr(K_{X/Y} + \Delta)))$ is a nef vector bundle for every $m \geq M$.

Proof. – Let $N := mr(K_{X/Y} + \Delta)$ for some $m \geq M$, and $\mathscr{N} := \mathscr{O}_X(N)$. Then the assumptions of Proposition 3.6 for this N are satisfied:

- $R^i f_* \mathscr{N} = 0$ for all $i > 0$ by (3.12a),
- $N - K_{X/Y} - \Delta$ is an f-ample $\mathbb{Q}$-divisor,
- $H^0(X_0, \mathscr{N}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0})$ by (3.12b),
- $N - K_{X/Y} - \Delta$ is nef by the nefness of $K_{X/Y} + \Delta$ granted by Theorem 3.10.

Therefore, Proposition 3.6 applies indeed and hence $f_* \mathscr{N} = f_* \mathscr{O}_X(mr(K_{X/Y} + \Delta))$ is a nef vector bundle for all $m \geq M$. This finishes our proof. □

REMARK 3.13. – Note that by Theorem 2.13 and Corollary 2.21, there is an M as in the statement of Theorem 3.12.

Recall that Theorem 1.7 states that in the situation of Notation 1.4, if $\omega_{X/Y}^{[r]}(r\Delta)$ is f-ample, then $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all integers $m \gg 0$. Theorem 3.12 proves this statement when Y is a smooth curve, to which we reduce the general case below.

Proof of Theorem 1.7. – First, using Theorem 2.13 and Proposition 2.23, choose an $M > 0$ such that for every $m \geq M$, $i > 0$ and all but finitely many $y \in Y$,

$$R^i f_*(\omega_{X/Y}^{[mr]}(mr\Delta)) = 0 \text{ and} \tag{3.13a}$$

$$H^0(X_y, \mathscr{O}_{X_y}(mr(K_{X_y} + \Delta_y))) = S^0(X_y, \sigma(X_y, \Delta_y) \otimes \mathscr{O}_{X_y}(mr(K_{X_y} + \Delta_y))). \tag{3.13b}$$

Choose now any finite map $\tau : Y' \to Y$ from a smooth, projective curve. We are supposed to prove that $\tau^* f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ is nef for all $m \geq M$. By Lemma 2.9, $f_*(\omega_{X/Y}^{[mr]}(mr\Delta))$ commutes with base change for every $m \geq M$. I.e., using the notations of Lemma 3.11, for every $m \geq M$,

$$\tau^* f_*(\omega_{X/Y}^{[mr]}(mr\Delta)) \cong f'_*(\pi^* \omega_{X/Y}^{[mr]}(mr\Delta)).$$

Therefore, it is enough to prove that the latter is nef. However, again by Lemma 3.11, this is equivalent to proving that $f'_* \omega_{X'/Y'}^{[mr]}(mr\Delta')$ is nef for all $m \geq M$, which is exactly the statement of Theorem 3.12. □

Proof of Corollary 1.9. – By Theorem 1.7 there is a $i_0 > 0$ such that for all $i \geq i_0$, $f_* \omega_{X/Y}^{[ir]}$ is a nef vector bundle. By possibly increasing i_0 we may also assume that there is a $d_0 > 0$ such that for all $d \geq d_0$, for all $i \geq i_0$ and all $y \in Y$,

1. the formation of $f_* \omega_{X/Y}^{[ir]}$ commutes with base-change,
2. $i_0 r K_{X_y}$ is very ample,
3. $\xi_y^d : S^d H^0\left(X_y, \omega_{X_y}^{[i_0 r]}\right) \to H^0\left(X_y, \omega_{X_y}^{[d i_0 r]}\right)$ is surjective and
4. $\operatorname{Ker} \xi_y^d$, which can be identified with $H^0(X_y, \mathscr{I}_y(d))$ for the ideal $\mathscr{I}_y$ of the embedding $X_y \hookrightarrow \mathbb{P}^{h^0\left(X_y, \omega_{X_y}^{[i_0 r]}\right) - 1}$, globally generates $\mathscr{I}_y(d)$ for all $y \in Y$.

By (1) and (3) the following natural map is surjective:

$$S^d\left(f_* \omega_{X/Y}^{[i_0 r]}\right) \twoheadrightarrow f_* \omega_{X/Y}^{[d i_0 r]}. \tag{3.13c}$$

The fiber of the kernel of this map at y is $\operatorname{Ker} \xi_y^d$. Hence using assumption (4), the fiber of the "classifying map" of [22, 3.9 Ampleness lemma] associated to the surjection (3.13c) are the sets $\{y' \in Y | X_{y'} \cong X_y\}$. By the assumption of the corollary these sets and hence the fibers of the classifying map are finite. In particular then by [22, 3.9 Ampleness lemma], using the finiteness of the automorphism groups, det $\left(f_* \omega_{X/Y}^{[d i_0 r]}\right)$ is ample for all $d \geq d_0$. □

3.4. Semi-positivity when the relative log-canonical divisor is relatively nef

THEOREM 3.14. – *In the situation of Notation 3.1, if* $\operatorname{ind}(X, \Delta) = 1$, (X_0, Δ_0) *is sharply* F*-pure and* $K_{X/Y} + \Delta$ *is* f*-nef, then* $K_{X/Y} + \Delta$ *is nef.*

Proof. – Using Theorem 2.13, Corollary 2.21 and Proposition 2.15, there is an ample enough line bundle $\mathscr{Q}$ on X, such that for all $i > 0$ and f-nef line bundle $\mathscr{K}$,

(3.14a) $$R^i f_*(\mathscr{Q} \otimes \mathscr{K}) = 0,$$

(3.14b) $$H^0(X_0, \mathscr{Q} \otimes \mathscr{K}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes (\mathscr{Q} \otimes \mathscr{K})|_{X_0})$$

and

(3.14c) $$\mathscr{Q} \otimes \mathscr{K}|_{X_y} \text{ is globally generated for all } y \in Y.$$

Let Q be a divisor of $\mathscr{Q}$. We prove by induction that $q(K_{X/Y} + \Delta) + Q$ is nef for all $q \geq 0$. For $q = 0$ the statement is true by the choice of Q. Hence, we may assume that $(q-1)(K_{X/Y} + \Delta) + Q$ is nef. Now, we verify that the conditions of Proposition 3.7 hold for $N := q(K_{X/Y} + \Delta) + Q$ and $\mathscr{N} := \mathscr{O}_X(N)$. Indeed:

- N is Cartier by the index assumption,
- $R^i f_* \mathscr{N} = 0$ for all $i > 0$ because of (3.14a) and that $K_{X/Y} + \Delta$ is an f-nef Cartier divisor,
- furthermore, the $\mathbb{Q}$-divisor
$$N - K_{X/Y} - \Delta = (q-1)(K_{X/Y} + \Delta) + Q$$
is not only f-ample, but also nef by the inductional hypothesis,
- using the f-nefness of $K_{X/Y} + \Delta$ and (3.14b),
$$H^0(X_0, \mathscr{N}|_{X_0}) = S^0(X_0, \sigma(X_0, \Delta_0) \otimes \mathscr{N}|_{X_0}),$$
- since all the summands of N are f-nef, so is N,
- for every $y \in Y$, $N|_{X_y}$ is globally generated by (3.14c).

Hence Proposition 3.7 implies that N is nef. This finishes our inductional step, and hence the proof of the nefness of $q(K_{X/Y} + \Delta) + Q$ for every $q \geq 0$. However, then $K_{X/Y} + \Delta$ has to be nef as well. This concludes our proof. □

Proof of Theorem 1.10. – The proof is identical to that of Theorem 1.6 after setting $r := 1$ and using Theorem 3.14 instead of Theorem 3.10. □

3.5. The case of indices divisible by p

Given a pair (X, Δ) with $p|\operatorname{ind}(K_X + \Delta)$, one can perturb Δ carefully to obtain another pair (X, Δ') such that $p \nmid \operatorname{ind}(K_X + \Delta')$. This method can be used to move some of our results to the situation where the index of the log-canonical divisor is divisible by p. There is one price to be paid: since the perturbed pair has to still satisfy the adequate sharply F-pure assumptions, slightly stronger singularity assumptions have to be imposed on the original pair. The adequate class of singularities is strongly F-regular singularities, positive characteristic analogues of Kawamata log terminal singularities. These, contrary to sharply F-pure singularities are closed under small perturbations, and furthermore form a subset of sharply F-pure singularities. In particular, their perturbations are guaranteed to be sharply F-pure. For the definition of strongly F-regular singularities we refer to [41, Definition 2.10]. Here, we only use the property that given a Cartier divisor $A \geq 0$ and a strongly F-regular pair (X, Δ), $(X, \Delta + \varepsilon A)$ is strongly F-regular as well, for every $0 < \varepsilon \ll 1$. In fact, this property can be even used to define strongly F-regular singularities for quasi-projective X. That is, when X is quasi-projective, (X, Δ) is strongly F-regular if and only if for every effective Cartier divisor A there is an $\varepsilon > 0$ such that $(X, \Delta + \varepsilon A)$ is sharply F-pure.

LEMMA 3.15. – *Let (X, Δ) be a pair with a flat morphism to a Gorenstein scheme Y, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier and $p|\operatorname{ind}(K_X + \Delta)$. Choose also an effective $\mathbb{Z}$-divisor D which is linearly equivalent to $K_{X/Y} + A$ for some Cartier divisor A on X. Then $K_X + \Delta + \frac{1}{p^v - 1}(D + \Delta)$ is $\mathbb{Q}$-Cartier with index not divisible by p for every $v \gg 0$.*

Proof. – First, since Y is Gorenstein, it is enough to show that $K_{X/Y} + \Delta + \frac{1}{p^v-1}(D + \Delta)$ $\mathbb{Q}$-Cartier with index not divisible by p for every $v \gg 0$. For proving that, we may choose for $K_{X/Y}$ the representative $D - A$. Let r be an integer such that $r(K_{X/Y} + \Delta)$ is Cartier. Then (here the $(\_, \_)$ sign stands for greatest common divisor),

$$\begin{aligned} \frac{r}{(r, p^v)}(p^v - 1)\left(K_{X/Y} + \Delta + \frac{1}{p^v - 1}(D + \Delta)\right) & \\ \sim \frac{r}{(r, p^v)}(p^v - 1)\left(D - A + \Delta + \frac{1}{p^v - 1}(D + \Delta)\right) & \\ = r\frac{p^v}{(r, p^v)}(D + \Delta) - \frac{r}{(r, p^v)}(p^v - 1)A & \\ \sim r\frac{p^v}{(r, p^v)}(K_{X/Y} + \Delta) + \left(r\frac{p^v}{(r, p^v)} - \frac{r}{(r, p^v)}(p^v - 1)\right)A, & \end{aligned}$$

which is Cartier. Furthermore, for $v \gg 0$, $\frac{r}{(r,p^v)}(p^v - 1)$ is an integer not divisible by p. This concludes our proof. □

THEOREM 3.16. – *In the situation of Notation 3.1 by possibly allowing $p|\operatorname{ind}(K_X + \Delta)$, if (X_0, Δ_0) is strongly F-regular, $K_{X/Y} + \Delta$ is f-nef and $K_{X_y} + \Delta_y$ is semi-ample for generic $y \in Y$, then $K_{X/Y} + \Delta$ is nef.*

Proof. – Fix an ample integer Cartier divisor A on X such that there is an effective $\mathbb{Z}$-divisor D linearly equivalent to $A + K_{X/Y}$ and furthermore D avoids the codimension 0 and the singular codimension 1 points of X_0 as well as the singular codimension 1 points of X. Define for $v \gg 0$, $\Delta' := \Delta + \frac{1}{p^v-1}(D + \Delta)$. Then if we replace Δ by Δ', the assumptions of Notation 3.1 are still satisfied for every $v \gg 0$, even $p \nmid \operatorname{ind}(K_X + \Delta')$ by Lemma 3.15. Furthermore, since (X_0, Δ_0) was strongly F-regular and $\Delta'_0 := \Delta'|_{X_0}$ differs from Δ_0 in a small enough divisor, (X_0, Δ'_0) is sharply F-pure for every $v \gg 0$. Also, the f-nefness and the semi-ampleness assumptions hold for $K_{X/Y} + \Delta'$, since

$$K_{X/Y} + \Delta' \sim_{\mathbb{Q}} \frac{p^v}{p^v - 1}(K_{X/Y} + \Delta) + \frac{1}{p^v - 1}A. \tag{3.16a}$$

Hence, Theorem 3.10 implies that $K_{X/Y} + \Delta'$ is nef. However then using (3.16a), so is

$$\frac{p^v - 1}{p^v}\left(\frac{p^v}{p^v - 1}(K_{X/Y} + \Delta) + \frac{1}{p^v - 1}A\right) = (K_{X/Y} + \Delta) + \frac{1}{p^v}A.$$

Since this holds for all $v \gg 0$, $K_{X/Y} + \Delta$ is nef. □

The following lemma follows from Definition 2.5, using the natural embedding $\mathscr{L}_{e,\Delta'} \to \mathscr{L}_{e,\Delta}$ given for every integer $e \geq 0$ and effective divisors $\Delta \leq \Delta'$.

Lemma 3.17. – *Let (X, Δ) and (X, Δ') be two pairs with the same underlying spaces, such that both $K_X + \Delta$ and $K_X + \Delta'$ are $\mathbb{Q}$-Cartier with indices not divisble by p. Assume furthermore that $\Delta \leq \Delta'$. Then for any line bundle $\mathscr{N}$ on X,*

$$S^0(X, \sigma(X, \Delta) \otimes \mathscr{N}) \supseteq S^0(X, \sigma(X, \Delta') \otimes \mathscr{N}).$$

Theorem 3.18. – *In the situation of Notation 3.1 by possibly allowing $p \mid \operatorname{ind}(K_X + \Delta)$, assume also that (X_0, Δ_0) is strongly F-regular and $K_{X/Y} + \Delta$ is f-ample. Then $f_* \mathscr{O}_X(mr(K_{X/Y} + \Delta))$ is a nef vector bundle for every $m \gg 0$.*

Proof. – First, note that by Theorem 3.16, $K_{X/Y} + \Delta$ is nef. Let A be the pullback of any ample Cartier divisor B from Y. Then, by the f-ampleness of $K_{X/Y} + \Delta$, $r(K_{X/Y} + \Delta) + A$ is ample. Therefore, there is a $d > 0$, such that $dr(K_{X/Y} + \Delta) + dA + K_{X/Y}$ is linearly equivalent to an effective $\mathbb{Z}$-divisor D, which avoids the codimension 0 and the singular codimension 1 points of X_0 as well as the singular codimension 1 points of X. Let $n = p^v - 1$ for arbitrary $v \gg 0$ (which notation will be used throughout the proof) and define $\Delta_n := \Delta + \frac{1}{n}(D + \Delta)$. By Lemma 3.15, (X, Δ_n) satisfy the assumptions of Notation 3.1 for $v \gg 0$, even the divisibility condition on the index. Furthermore the same holds for $(X^{(n)}, \Delta_n^{(n)})$ by Lemma 3.5. Note that here the upper and the lower indices agree on purpose. Similarly, throughout the proof the n's in the upper and lower indices agree on purpose.

Fix now a $n' = p^{v'} - 1 \gg 0$. Then, since $\Delta_{n'} - \Delta$ is a small effective divisor, $(X_0, (\Delta_{n'})_0)$ is sharply F-pure. In particular, by Theorem 2.13 and Corollary 2.21 we may choose an $M > 0$ such that for all $i > 0$ and $m \geq M$,

$$R^i f_*(\mathscr{O}_X(mr(K_{X/Y} + \Delta))) = 0 \text{ and} \tag{3.18a}$$

$$H^0(X_0, \mathscr{O}_{X_0}(mr(K_{X/Y}+\Delta))) = S^0(X_0, \sigma(X_0, (\Delta_{n'})_0) \otimes \mathscr{O}_{X_0}(mr(K_{X/Y}+\Delta))). \tag{3.18b}$$

Define then for any $m \geq \max\{2, M\}$ and $v \geq v'$ (still keeping the notation $n = p^v - 1$), $N := mr\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right) + \frac{d}{n}A^{(n)}$. Then the assumptions of Proposition 3.6 hold for $\left(X^{(n)}, \Delta_n^{(n)}\right)$ and N, because if $\mathscr{N} := \mathscr{O}_{X^{(n)}}(N)$:

– N is Cartier, since $\frac{d}{n}A^{(n)} = \left(f^{(n)}\right)^*(dB)$.

– $R^i f_*^{(n)} \mathscr{N} = 0$ for all $i > 0$ by (3.18a), Lemmas 2.9, 2.10 and the Künneth formula. We also use here that $\mathscr{O}_{X^{(n)}}(mr(K_{X^{(n)}/Y} + \Delta^{(n)})) \cong \mathscr{O}_X(mr(K_{X/Y} + \Delta))^{(n)}$.
– $N - \left(K_{X^{(n)}/Y} + \Delta_n^{(n)}\right)$ is an $f^{(n)}$-ample and nef $\mathbb{Q}$-divisor, because of the following computation and the nefness of $K_{X/Y} + \Delta$ granted by Theorem 3.16. Note that we also use that $\frac{n+1+dr}{n} < 2$ because $v \geq v'$ and $v' \gg 0$.

$$\begin{aligned}
N - &\left(K_{X^{(n)}/Y} + \Delta_n^{(n)}\right) \\
&= mr\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right) + \frac{d}{n}A^{(n)} - \left(K_{X^{(n)}/Y} + \Delta^{(n)} + \frac{1}{n}\left(D^{(n)} + \Delta^{(n)}\right)\right) \\
&\sim_{\mathbb{Q}} mr\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right) + \frac{d}{n}A^{(n)} \\
&\qquad - \left(K_{X^{(n)}/Y} + \frac{n+1}{n}\Delta^{(n)} + \frac{1}{n}(dr(K_{X/Y} + \Delta) + dA + K_{X/Y})^{(n)}\right) \\
&= mr\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right) - \frac{n+1}{n}\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right) - \frac{dr}{n}\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right), \\
&= \left(mr - \frac{n+1+dr}{n}\right)\left(K_{X^{(n)}/Y} + \Delta^{(n)}\right),
\end{aligned}$$

– $H^0\left(X_0^{(n)}, \mathscr{N}|_{X_0^{(n)}}\right) = S^0\left(X_0^{(n)}, \sigma\left(X_0^{(n)}, (\Delta_n^{(n)})_0\right) \otimes \mathscr{N}|_{X_0^{(n)}}\right)$ by (3.18b), Lemma 3.17 and Lemma 2.27.

Therefore, applying Proposition 3.6 yields that the following vector bundle is nef for every $v \gg 0$.

$$\begin{aligned}
f_*^{(n)} \mathscr{N} &\cong f_*^{(n)} \mathscr{O}_{X^{(n)}}\left(mr(K_{X^{(n)}/Y} + \Delta^{(n)}) + \frac{d}{n}A^{(n)}\right) \\
&\cong f_*^{(n)} \mathscr{O}_{X^{(n)}}(mr(K_{X^{(n)}/Y} + \Delta^{(n)})) \otimes \mathscr{O}_Y(dB) \\
&\cong \underbrace{\left(\bigotimes_{i=1}^{n} f_* \mathscr{O}_X(mr(K_{X/Y} + \Delta))\right) \otimes \mathscr{O}_Y(dB).}_{\text{by Lemma 2.12}}
\end{aligned}$$

Since this holds for every $n = p^v - 1 \gg 0$, $f_* \mathscr{O}_X(mr(K_{X/Y} + \Delta))$ is nef. □

Proof of Theorem 1.8. – The statement is a special case of Theorem 3.18. □

4. Applications

4.1. Projectivity of proper moduli spaces

Recently there have been great advances in constructing moduli spaces of varieties (or pairs) of (log-)general type in characteristic zero, cf., [23]. The method used in these results has been worked out in [22] and is as follows. First, one defines a subfunctor of the functor of all families of (log-)canonically polarized varieties. Second, one proves nice properties of this functor: openness, separatedness, properness, boundedness and tame automorphisms. Then it follows that the chosen functor admits a coarse moduli space, which is a proper algebraic space. Third, by exhibiting a semi-positive line bundle that descends to the coarse moduli space, one proves that the coarse moduli space is a projective scheme. Hence, Theorem 1.7 implies that in positive characteristics if for a subfunctor as above of families of sharply F-pure varieties the first two steps are known, then the third step works as well. That is, the subfunctor admits a coarse moduli space, which is a projective variety.

COROLLARY 4.1. – *Let $\mathscr{F}$ be a subfunctor of*

$$Y \mapsto \left\{ \begin{array}{c|l} X & f : X \to Y \text{ is a flat, relatively } S_2 \text{ and } G_1\text{, equidimensional, projective} \\ f\big\downarrow & \text{morphism with sharply } F\text{-pure fibers, such that there is a } p \nmid r > 0, \\ & \text{for which } \omega_{X/Y}^{[r]} \text{ is an } f\text{-ample line bundle, and } \operatorname{Aut}(X_y) \text{ is finite for} \\ Y & \text{all } y \in Y \end{array} \right\} \Big/ \cong \text{ over } Y.$$

If $\mathscr{F}$ admits

(1) *a coarse moduli space $\pi : \mathscr{F} \to V$, which is a proper algebraic space and*
(2) *a morphism $\rho : Z \to \mathscr{F}$ from a scheme, such that $\pi \circ \rho$ is finite,*

then V is a projective scheme.

REMARK 4.2. – As mentioned in the introduction of this section, by [22, Theorem 2.2], the assumptions of Corollary 4.1 is satisfied if $\mathscr{F}$ belongs to an open class with tame automorphisms and is separated, bounded and complete.

Proof of Corollary 4.1. – To prove that V is a projective scheme, one has to exhibit an ample line bundle on it. Let this be in our situation the descent of $\det\left(g_*\omega_{W/Z}^{[mr]}\right)$ to V for some divisible enough m, where $g : W \to Z$ is the element of $\mathscr{F}(Z)$ corresponding to the morphism $\rho : Z \to \mathscr{F}$. (Here Z should really be thought of as the functor $\operatorname{Hom}(\_, Z)$. Then $Z \to \mathscr{F}$ is a functor $\operatorname{Hom}(\_, Z) \to \mathscr{F}$, so g is the image of $\mathrm{id}_Z \in \operatorname{Hom}(Z, Z) = \operatorname{Hom}(\_, Z)(Z)$ via the map $\rho(Z) : \operatorname{Hom}(\_, Z) \to \mathscr{F}(Z)$.) Note that for divisible enough m, $\det\left(g_*\omega_{W/Z}^{[mr]}\right)$ descends indeed to V by the finiteness of the automorphism groups of the fibers, cf., [22, 2.5]. To prove that it is ample, it is enough to show that its pullback via $\pi \circ \rho$ is ample. Therefore we are supposed to prove that $\det\left(g_*\omega_{W/Z}^{[mr]}\right)$ is ample for some m divisible enough. However, since the isomorphism equivalence classes of the closed fibers of g are exactly the fibers of $\pi \circ \rho$, this ampleness is shown in Corollary 1.9. □

4.2. Characteristic zero implications

Fix an algebraically closed field k' of characteristic zero throughout this section. The following is a major conjecture in the theory of F-singularities (cf., [32, 33], [29, Conjecture 1 and Corollary 4.5]).

CONJECTURE 4.3. – *Given a pair (X, Δ) with semi-log canonical singularities over k', consider a model (X_A, Δ_A) of it over a $\mathbb{Z}$-algebra $A \subseteq k'$ of finite type. Then there is a dense set of closed points $S \subseteq \operatorname{Spec} A$, such that $(X_s, \Delta_s) := ((X_A)_s, (\Delta_A)_s)$ is sharply F-pure for all $s \in S$.*

REMARK 4.4. – If in the above conjecture semi-log canonical is replaced by Kawmata log terminal and sharply F-pure by strongly F-regular, then the statement is known [44].

Hence, the results of the paper has the following consequences in characteristic zero. We emphasize this is a completely new algebraic method of obtaining such positivity results in characteristic zero.

COROLLARY 4.5. – *Let (X, Δ) be a pair over k' with $\mathbb{Q}$-Cartier $K_X + \Delta$ and $f : X \to Y$ a flat, projective morphism to a smooth projective curve. Further suppose that there is a $y_0 \in Y$, such that Δ avoids all codimension 0 and the singular codimension 1 points of X_{y_0}, and either*

(1) *(X_{y_0}, Δ_{y_0}) is Kawamata log terminal, or*
(2) *(X_{y_0}, Δ_{y_0}) is semi-log-canonical and for every model over a $\mathbb{Z}$-algebra A of finite type, it satisfies the statement of Conjecture 4.3.*

Assume also that $K_{X/Y} + \Delta$ is f-ample (resp. f-semi-ample). Then for $m \gg 0$ and divisible enough, $f_* \mathscr{O}_X(m(K_{X/Y} + \Delta))$ is a nef vector bundle (resp. $K_{X/Y} + \Delta$ is nef).

Proof. – Consider a model $f_A : (X_A, \Delta_A) \to Y_A$ of $f : (X, \Delta) \to Y$ over a $\mathbb{Z}$-algebra $A \subseteq k'$ of finite type. By normalizing and then further localizing A, we may assume that

(1) Spec A is Gorenstein,
(2) f_A is flat,
(3) Δ_A avoids the codimension 0 and the singular codimension 1 points of the fibers X_s,
(4) (X_s, Δ_s) is a pair, i.e., X_s is S_2 and G_1, for all $s \in \operatorname{Spec} A$,
(5) $K_{X_s} + \Delta_s$ is $\mathbb{Q}$-Cartier (note that if we have $r = \operatorname{ind}(K_{X_s} + \Delta_s)$, then $r(K_{X_A} + \Delta_A)|_{X_s} = r(K_{X_s} + \Delta_s)$ in codimension one and then everywhere, therefore $\operatorname{ind}(K_{X_s} + \Delta_s) \mid \operatorname{ind}(K_{X_A} + \Delta_A)$),
(6) $\operatorname{char}(s) \nmid \operatorname{ind}(K_{X_A} + \Delta_A)$ for every $s \in S$ (and then by the above considerations, $\operatorname{char}(s) \nmid \operatorname{ind}(K_{X_s} + \Delta_s)$),
(7) Y_s is smooth for every $s \in S$,
(8) Δ avoids all codimension 0 and the singular codimension 1 points of $X_{(y_0,s)}$ for all $s \in \operatorname{Spec} A$,
(9) $K_{X_A/Y_A} + \Delta_A$ is f_A-ample (resp. f_A-semi-ample) and
(10) in the f_A-ample case, we may also assume that m is chosen big and divisible enough such that $f_{A,*}\mathscr{O}_X(m(K_{X_A/Y_A} + \Delta_A))|_{Y_s} \cong (f_s)_* \mathscr{O}_X(m(K_{X_s/Y_s} + \Delta_s))$ and the same for s replaced by $\bar{s}$ (if s was given by a morphism $A \to k''$, then $\bar{s}$ denotes a morphism given by $A \to \bar{k''}$, where $\bar{k''}$ is any algebraic closure of k'').

Note also that by the assumptions there is a dense set $S \subseteq \operatorname{Spec} A$ of closed points for which $(X_{(y_0,s)}, \Delta_{(y_0,s)})$ is F-pure. In particular then for every $s \in S$, $f_s : (X_s, \Delta_s) \to Y_s$ satisfy the assumptions of Notation 3.1 and Theorem 3.12 (resp. Theorem 3.10) except that the base field is not algebraically closed. However, the above assumptions are stable under passing to the algebraic completion of the base-field. Therefore, $f_{\bar{s}} : (X_{\bar{s}}, \Delta_{\bar{s}}) \to Y_{\bar{s}}$ satisfies the assumptions of Notation 3.1 and Theorem 3.12 (resp. Theorem 3.10) for all $s \in S$ including the algebraic closedness of the base field. In particular, by Theorem 3.12 (resp. Theorem 3.10), $(f_{A,*}\mathscr{O}_{X_A}(m(K_{X_A/Y_A} + \Delta_A)))_{\bar{s}}$ is a nef vector bundle (resp. $(K_{X_A/Y_A} + \Delta_A)_{\bar{s}}$ is nef) for every $s \in S$. However, then so is $(f_{A,*}\mathscr{O}_{X_A}(m(K_{X_A/Y_A} + \Delta_A)))_s$ (resp. $(K_{X_A/Y_A} + \Delta_A)_s$). By [26, Proposition 1.4.13], nefness at a point implies nefness at all its generalizations. Hence $f_{A,*}\mathscr{O}_{X_A}(m(K_{X_A/Y_A} + \Delta_A))$ (resp. $K_{X_A/Y_A} + \Delta_A$) and then also $f_* \mathscr{O}_X(m(K_{X/Y} + \Delta))$ (resp. $K_{X/Y} + \Delta$) is nef. □

4.3. Subadditivity of Kodaira-dimension

Subadditivity of Kodaira dimension was one of the major applications in characteristic zero of the semipositivity of $f_*\omega^m_{X/Y}$ (e.g., [45, 21]). We present here a similar result in positive characteristic. However, we would like to draw the reader's attention that in positive characteristic, there is already some ambiguity to the notion of Kodaira dimension. See Question 5.1 for explanation. Hence we have to phrase the statement slightly differently, involving the bigness of (log-)canonical divisors instead of the (log-)Kodaira dimension.

COROLLARY 4.6. – *In the situation of Notation 1.4 or Notation 1.5, if furthermore Y is an S_2, G_1, equidimensional projective variety with K_Y $\mathbb{Q}$-Cartier and big, $K_{X/Y} + \Delta$ is f-semi-ample and $K_F + \Delta|_F$ is big for the generic fiber F, then $K_X + \Delta$ is big.*

Proof. – Since, K_Y is big, there is a $m > 0$, such that $mK_Y = A + E$ for integer very ample and effective divisors A and E. It is enough to prove that $f^*A + m(K_{X/Y} + \Delta)$ is big. By Theorem 1.6, $K_{X/Y} + \Delta$ is nef. So, since $f^*A + m(K_{X/Y} + \Delta)$ is nef, it is enough to show that $(f^*A + m(K_{X/Y} + \Delta))^{\dim X} > 0$. However then the following computation concludes our proof.

$$(f^*A + m(K_{X/Y} + \Delta))^{\dim X} \geq \underbrace{f^*A^{\dim Y} \cdot m(K_{X/Y} + \Delta)^{\dim F}}_{\text{both } f^*A \text{ and } K_{X/Y}+\Delta \text{ are nef}}$$
$$= A^{\dim Y} \cdot (m(K_F + \Delta|_F))^{\dim F} > \underbrace{0.}_{K_F+\Delta|_F \text{ is big}} \qquad \square$$

5. Questions

Here we list questions that are left open by the article and we feel are important. The first question is motivated by the absence of resolution of singularities in positive characteristics. Recall, that the Kodaira dimension of a variety X in characteristic zero is defined as the Kodaira dimension of $K_{X'}$ for a projective smooth birational model X' of X.

QUESTION 5.1. – Is there a birational invariant in positive characteristics, which specializes to Kodaira dimension in the particular case when there is a smooth birational model? In particular this question would be solved if we had resolution of singularities in positive characteristics.

The following few questions concern sharpness of Theorem 1.7.

QUESTION 5.2. – Can one drop the index not divisible by p assumption in Theorem 1.7?

QUESTION 5.3. – Is there a family $f : X \to Y$ of semi-log canonical but not sharply F-pure schemes over a projective smooth curve with K_X $\mathbb{Q}$-Cartier f-ample, such that $f_*\omega^{[m]}_{X/Y}$ is not nef for every $m \gg 0$?

QUESTION 5.4. – Can one give an effective bound on m for which Theorem 1.7 holds? Is it possibly true for $m \geq 2$?

BIBLIOGRAPHY

[1] S. J. Arakelov, Families of algebraic curves with fixed degeneracies, *Izv. Akad. Nauk SSSR Ser. Mat.* **35** (1971), 1269–1293.

[2] B. Bhatt, W. Ho, Zs. Patakfalvi, C. Schnell, Moduli of products of stable varieties, *Compositio Math.* **149** (2013), 2036–2070.

[3] H. Cartan, *Plongements projectifs*, Séminaire Henri Cartan; 10e année : 1957/1958. Fonctions automorphes, Secrétariat mathématique, 1958.

[4] D. Eisenbud, J. Harris, The Kodaira dimension of the moduli space of curves of genus ≥ 23, *Invent. Math.* **90** (1987), 359–387.

[5] G. Faltings, C.-L. Chai, *Degeneration of abelian varieties*, Ergebn. Math. Grenzg. **22**, Springer, 1990.

[6] O. Fujino, Semi-positivity theorems for moduli problems, preprint, 2012.

[7] O. Fujino, Fundamental theorems for semi log canonical pairs, *Algebr. Geom.* **1** (2014), 194–228.

[8] O. Fujino, T. Fujisawa, Variations of Mixed Hodge Structure and Semipositivity Theorems, *Publ. Res. Inst. Math. Sci.* **50** (2014), 589–661.

[9] T. Fujita, On Kähler fiber spaces over curves, *J. Math. Soc. Japan* **30** (1978), 779–794.

[10] O. Gabber, Notes on some t-structures, in *Geometric aspects of Dwork theory. Vol. I, II*, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, 711–734.

[11] P. A. Griffiths, Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping, *Publ. Math. I.H.É.S.* **38** (1970), 125–180.

[12] A. Grothendieck, Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II, *Publ. Math. I.H.É.S.* **24** (1965), 5–231.

[13] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps, *Amer. J. Math.* **120** (1998), 981–996.

[14] N. Hara, Classification of two-dimensional F-regular and F-pure singularities, *Adv. Math.* **133** (1998), 33–53.

[15] N. Hara, K.-I. Watanabe, F-regular and F-pure rings vs. log terminal and log canonical singularities, *J. Algebraic Geom.* **11** (2002), 363–392.

[16] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Math. **52**, Springer, 1977.

[17] R. Hartshorne, Generalized divisors on Gorenstein schemes, in *Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part III (Antwerp, 1992)*, **8**, 1994, 287–339.

[18] B. Hassett, S. J. Kovács, Reflexive pull-backs and base extension, *J. Algebraic Geom.* **13** (2004), 233–247.

[19] Y. Kawamata, Characterization of abelian varieties, *Compositio Math.* **43** (1981), 253–276.

[20] D. S. Keeler, Ample filters of invertible sheaves, *J. Algebra* **259** (2003), 243–283.

[21] J. Kollár, Subadditivity of the Kodaira dimension: fibers of general type, in *Algebraic geometry, Sendai, 1985*, Adv. Stud. Pure Math. **10**, North-Holland, 1987, 361–398.

[22] J. Kollár, Projectivity of complete moduli, *J. Differential Geom.* **32** (1990), 235–268.

[23] J. KOLLÁR, Moduli of varieties of general type, in *Handbook of moduli, vol. II*, Adv. Lect. Math. **25**, Int. Press, 2013, 131–157.

[24] J. KOLLÁR, *Singularities of the minimal model program*, Cambridge Tracts in Mathematics **200**, 2013.

[25] J. KOLLÁR, S. MORI, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics **134**, Cambridge Univ. Press, 1998.

[26] R. LAZARSFELD, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebn. Math. Grenzg. **48**, Springer, 2004.

[27] D. MAULIK, Supersingular K3 surfaces for large primes, *Duke Math. J.* **163** (2014), 2357–2425.

[28] V. B. MEHTA, V. SRINIVAS, Normal F-pure surface singularities, *J. Algebra* **143** (1991), 130–143.

[29] L. E. MILLER, K. SCHWEDE, Semi-log canonical vs F-pure singularities, *J. Algebra* **349** (2012), 150–164.

[30] L. MORET-BAILLY, Familles de courbes et de variétés abéliennes sur $\mathbb{P}^1$ (II. Exemples), *Astérisque* **86** (1981), 125–140.

[31] D. MUMFORD, Hirzebruch's proportionality theorem in the noncompact case, *Invent. Math.* **42** (1977), 239–272.

[32] M. MUSTAŢĂ, Ordinary varieties and the comparison between multiplier ideals and test ideals II, *Proc. Amer. Math. Soc.* **140** (2012), 805–810.

[33] M. MUSTAŢĂ, V. SRINIVAS, Ordinary varieties and the comparison between multiplier ideals and test ideals, *Nagoya Math. J.* **204** (2011), 125–157.

[34] A. N. PARŠIN, Algebraic curves over function fields, *Dokl. Akad. Nauk SSSR* **183** (1968), 524–526.

[35] ZS. PATAKFALVI, Fibered stable varieties, to appear in *Trans. of the A.M.S.*

[36] ZS. PATAKFALVI, On subadditivity of Kodaira dimension in positive characteristic, preprint arXiv:1308.5371.

[37] ZS. PATAKFALVI, K. SCHWEDE, Depth of F-singularities and base change of relative canonical sheaves, *Journal of the Institute of Mathematics of Jussieu* **13** (2014), 43–63.

[38] ZS. PATAKFALVI, K. SCHWEDE, W. ZHANG, F-singularities in families, preprint arXiv:1305.1646.

[39] M. RAYNAUD, Contre-exemple au "vanishing theorem" en caractéristique $p > 0$, in *C. P. Ramanujam—a tribute*, Tata Inst. Fund. Res. Studies in Math. **8**, Springer, 1978, 273–278.

[40] K. SCHWEDE, F-adjunction, *Algebra Number Theory* **3** (2009), 907–950.

[41] K. SCHWEDE, A canonical linear system associated to adjoint divisors in characteristic $p > 0$, preprint arXiv:1107.3833, to appear in *J. reine angew. Math.*

[42] K. SCHWEDE, K. TUCKER, A survey of test ideals, in *Progress in commutative algebra 2*, Walter de Gruyter, Berlin, 2012, 39–99.

[43] L. SZPIRO, Sur le théorème de rigidité de Parsin et Arakelov, in *Journées de Géométrie Algébrique de Rennes (Rennes, 1978), Vol. II*, Astérisque **64**, Soc. Math. France, 1979, 169–202.

[44] S. Takagi, An interpretation of multiplier ideals via tight closure, *J. Algebraic Geom.* **13** (2004), 393–415.

[45] E. Viehweg, Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, in *Algebraic varieties and analytic varieties (Tokyo, 1981)*, Adv. Stud. Pure Math. **1**, North-Holland, 1983, 329–353.

[46] E. Viehweg, *Quasi-projective moduli for polarized manifolds*, Ergebn. Math. Grenzg. **30**, Springer, 1995.

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