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PARTICLE APPROXIMATION OF VLASOV EQUATIONS WITH SINGULAR FORCES: PROPAGATION OF CHAOS

BY MAXIME HAURAY AND PIERRE-EMMANUEL JABIN

ABSTRACT. – We justify the mean field approximation and prove the propagation of chaos for a system of particles interacting with a singular interaction force of the type $1/|x|^\alpha$, with $\alpha < 1$ in dimension $d \geq 3$. We also provide results for forces with singularity up to $\alpha < d-1$ but with a large enough cut-off. This last result thus almost includes the case of Coulombian or gravitational interactions, but it also allows for a very small cut-off when the strength of the singularity α is larger but close to one.

RÉSUMÉ. – Nous montrons la validité de l'approximation par champ moyen et prouvons la propagation du chaos pour un système de particules en interaction par le biais d'une force avec singularité $1/|x|^\alpha$, avec $\alpha < 1$ en dimension $d \geq 3$. Nous traitons également le cas de forces avec troncature et des singularités pouvant aller jusqu'à $\alpha < d-1$. Ce dernier résultat permet presque d'atteindre les cas d'interaction coulombiennes ou gravitationnelles et requiert seulement de très petits paramètres de troncature lorsque la singularité est proche de $\alpha = 1$.

1. Introduction

The N particles system. – The starting point is the classical Newton dynamics for N point-particles. We denote by $X_i \in \mathbb{R}^d$ and $V_i \in \mathbb{R}^d$ the position and velocity of the ith particle. For convenience, we also use the notation $Z_i = (X_i, V_i)$ and $Z = (Z_1, \ldots, Z_n)$. Assuming that particles interact two by two with the interaction force $F(x)$, one finds the classical

$$\begin{cases} \dot{X}_i = V_i, \\ \dot{V}_i = E_N(X_i) = \dfrac{1}{N} \displaystyle\sum_{j \neq i} F(X_i - X_j). \end{cases} \tag{1.1}$$

The (N-dependent) initial conditions Z^0 are given. We use the so-called mean-field scaling which consists in keeping the total mass (or charge) of order 1 thus formally enabling us to pass to the limit: this explains the $1/N$ factor in front of the force terms.

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There are many examples of physical systems following (1.1). The best known example concerns Coulombian or gravitational force $F(x) = -\nabla\Phi(x)$, with $\Phi(x) = C/|x|^{d-2}$ for $d \geq 3$ with $C \in \mathbb{R}^*$, which serves as a guiding example and reference. This system then describes particles (ions or electrons) in a one component plasma for $C > 0$, or gravitational interactions for $C < 0$. In the last case the system under study may be a galaxy, a smaller cluster of stars or much larger clusters of galaxies (and thus particles can be "stars" or even "galaxies").

For the sake of simplicity, we consider here only a basic form for the interaction. However the same techniques would apply to more complex models, for instance with several species (electrons and ions in a plasma), 3-particle (or more) interactions, models where the force also depends on the velocity as in swarming models like Cucker-Smale [9]… Indeed a striking feature of our analysis is that it is valid for a force kernel F not necessarily derived from a potential: In fact it never requires any Hamiltonian structure.

The potential and force used in this article. – Our first result applies to interaction forces that are smooth outside of the origin and "weakly" singular near zero, in the sense that they satisfy

$$(S^\alpha) \qquad \exists\, C > 0, \quad \forall\, x \in \mathbb{R}^d \backslash \{0\}, \quad |F(x)| \leq \frac{C}{|x|^\alpha}, \quad |\nabla F(x)| \leq \frac{C}{|x|^{\alpha+1}}, \tag{1.2}$$

for some $\alpha < 1$.

We refer to this condition as the "weakly" singular case because under this, the potential (when it exists) is continuous and bounded near the origin. It is reasonable to expect that the analysis is simpler in that case than with a singular potential.

The second type of potentials or forces that we are dealing with are more singular, satisfying the (S^α)-condition with $\alpha < d-1$, but with an additional cut-off η near the origin that will depend on N

$$(S^\alpha_m) \qquad \begin{array}{l} \text{i)}\ \ F \text{ satisfies a } (S^\alpha)\text{-condition for some } \alpha < d-1, \\ \text{ii)}\ \forall\, |x| \geq N^{-m},\ F_N(x) = F(x), \\ \text{iii)}\ \forall\, |x| \leq N^{-m},\ |F_N(x)| \leq N^{m\alpha}. \end{array} \tag{1.3}$$

We will refer to that case as the "strongly" singular case. Remark that the interaction kernel F in fact depends on the number of particles. This might seem strange from the physical point of view but it is in fact very common in numerical simulations in order to regularize the interactions.

As we shall see in more details later, we can choose the cut-off parameter smaller than typical inter particle distance (in position) if α is not too large (precisely smaller than $d/2$). In that case one would hope that the cut-off is actually rarely "used".

As the interaction force is singular, we first precise what we mean by solutions to (1.1) in the following definition

Definition 1. – *A (global) solution to* (1.1) *with initial condition*

$$Z^0 = (X_1^0,\ V_1^0, \ldots, X_N^0,\ V_N^0) \in \mathbb{R}^{2dN}$$

(at time 0) is a continuous trajectory $Z(t) = \big(X_1(t),\ V_1(t), \dots, X_N(t),\ V_N(t)\big)$ *such that*

$$\forall t \in \mathbb{R}^+,\ \forall i \le N, \qquad \begin{cases} X_i(t) = X_i^0 + \displaystyle\int_0^t V_i(s)\, ds \\ V_i(t) = V_i^0 + \displaystyle\frac{1}{N} \sum_{j \neq i} \int_0^t F(X_i(s) - X_j(s))\, ds. \end{cases} \tag{1.4}$$

Local (in time) solutions are defined similarly.

We always assume that such solutions to (1.1) exist, at least for almost all initial configurations of the particles and over any time interval $[0,\ T]$ under consideration. Of course as we use singular interaction forces, this is not completely obvious, but it holds under the assumption (1.2). This point is discussed at the end of the article in Subsection 6.1, and we now focus on the problem raised by the limit $N \to +\infty$.

Remark also that the uniqueness of such solutions is not important for our study. Only the uniqueness of the solution to the limit equation is crucial for the mean-field limit and the propagation of chaos.

The Jeans-Vlasov equation. – At first glance, the system (1.1) might seem quite reasonable. However many problems arise when one tries to use it for practical applications. In our case, the main issue is the number of particles, i.e., the dimension of the system. For example a plasma or a galaxy usually contains a very large number of "particles", typically from 10^9 to 10^{25}, which can make solving (1.1) numerically prohibitively costly.

As usual in this kind of situation, one would like to replace the discrete system (1.1) by a "continuous" model. In our case this model is posed in the space $\mathbb{R}^{2d}$, i.e., it involves the distribution function $f(t, x, v)$ in time, position and velocity. The evolution of that function $f(t, x, v)$ is given by the Jeans-Vlasov equation (or collisionless Boltzmann equation)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E(x) \cdot \nabla_v f = 0\,, \\ E(x) = \displaystyle\int_{\mathbb{R}^d} \rho(t, y)\, F(x - y)\, dy, \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^d} f(t, x, v)\, dv, \end{cases} \tag{1.5}$$

where here ρ is the spatial density and the initial density f^0 is given.

Our purpose in this article is to understand when and in which sense, Equation (1.5) can be seen as a limit of system (1.1). This question is of importance for theoretical reasons, to justify the validity of the Vlasov equation for example. It also plays a role for numerical simulation in plasma physics [7, 30] and astrophysics [1], where a large class of methods (among which the "Particles in Cells" method) introduce a large number of "virtual" particles (roughly around 10^6 or 10^8, to compare with the real order mentioned above) in order to obtain a many particle system solvable numerically. The problem in that case is to explain why it is possible to correctly approximate the system by using much fewer particles. This would of course be ensured by the convergence of (1.1) to (1.5).

We make use of uniqueness results for the solution to Equation (1.5). The regularity theory for this equation is now well understood, even when the interaction F is singular, including the Coulombian case. The existence of weak solutions goes back to [3, 20]. Existence and uniqueness of global classical solutions in dimension up to 3 is proved in [51], [55] (see also [36]) and at the same time in [42]. Of course those results require some assumptions on the initial data f^0: for instance compact support and boundedness in [51]. We will state the precise result of existence and uniqueness we need in Proposition 2 in Section 3.2.

Formal derivation of Eq. (1.5) *from* (1.1). – One of the simplest way to understand formally how to derive Eq. (1.5) is to introduce the empirical measure

$$\mu_N^z(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t),V_i(t)}.$$

Remark that in the ODEs (1.1) there is no self-interaction: the force is summed over all the particles except the i-th. So if we want to define F everywhere, i.e., even in 0, the natural choice is $F(0)=0$. That convention $F(0)=0$ is natural for odd interaction kernels F which are consistent with the law of action-reaction, and we will use that convention in the rest of the article. Note anyway that because of the $1/N$ renormalization of the force term, the choice of $F(0)$ should only have a small impact on the dynamics.

Having defined F everywhere, measure valued solutions to the Vlasov Equation (1.5) are now well defined in the sense of distribution. In addition it is straightforward to check that $Z(t) = \big(X_i(t), V_i(t)\big)_{1\le i\le N}$ is a solution to (1.1), if and only if μ_N^z solves (1.5) in the sense of distribution.

At the level of measure valued solutions, the Newton system (1.1) is equivalent to the Vlasov Eq. (1.5). This remarkable fact suggests as a strategy to study the stability of measure valued solutions to (1.5) and to see the limit $N\to\infty$ as a stability result with respect to the convergence of the initial data. Unfortunately, except in special cases such as where F is Lipschitz [11, 49, 20], a well posedness result for measure valued solutions is not accessible.

Notice that when Eq. (1.5) is solved for smooth initial data (at least in L^p with $p>1$), its properties become quite different. Crucially for our purpose, it enjoys a reduction in complexity as it can efficiently be calculated numerically; in this sense calculating the solution for $N=\infty$ is easier than for a finite N. There are many other key differences, such as long time behavior, which are however beyond the scope of this introduction to describe.

The question of convergence and the mean-field limit. – The previous formal argument suggests a first way of rigorously deriving the Vlasov Equation (1.5). Take a sequence of initial conditions Z_N^0 (to be given for every number N or a sub-sequence of such numbers) and assume that the corresponding empirical measures at time 0 converge (in the usual weak-$*$ topology for measures)

$$\mu_N^z(0) \longrightarrow f^0(x,v).$$

One would then try to prove that the empirical measures at later times $\mu_N^z(t)$ weakly converge to a solution $f(t,x,v)$ to (1.5) with initial data f^0. In other words, is the following diagram commutative?

$$\begin{tikzcd} \mu_N^z(0) \arrow[r, rightsquigarrow, "\text{cvg}"] \arrow[d, "N\text{part}"'] & f(0) \arrow[d, "VP"] \\ \mu_N^z(t) \arrow[r, rightsquigarrow, "\text{cvg ?}"] & f(t). \end{tikzcd}$$

We refer to the *mean-field limit* for the question as to whether $\mu_N^z(t)$ converges to $f(t)$ for a given sequence of initial conditions Z_N^0 (or equivalently $\mu_N^0 = \mu_N^z(0)$). This is a purely deterministic problem. We give in Theorems 1 and 3 a quantitative version of the convergence of $\mu_N(t)$ towards $f(t)$, provided some assumptions on f^0 and on the initial configurations μ_N^0 are satisfied.

Propagation of molecular chaos. – In many physical settings, it is relevant to introduce randomness on the initial position and velocities Z of the particles. When the particles are indistinguishable (as in our model), a natural hypothesis is that the law should be invariant by permutation of the particles. This property is usually referred to as exchangeability (in probability), and precisely defined as

DEFINITION 2. – *A random vector-field with N component $Z = (Z_1, \dots, Z_N)$ is said to be exchangeable if for any permutation σ, Z and $Z_\sigma = (Z_{\sigma(1)}, \dots, Z_{\sigma(N)})$ have the same law.*

A particularly interesting situation arises when each couple $Z_i = (X_i, V_i)$ is chosen randomly and independently with law f^0, or when the independence is satisfied in an asymptotic sense, as N goes to infinity. The precise notion of "asymptotic independence" is usually given by the notion of *chaotic sequence*, restated in the following definition.

DEFINITION 3. – *Let E be a measurable metric space ($E = \mathbb{R}^{2d}$ in our applications), and f a probability measure on E. A sequence $(f^N)_{N\in\mathbb{N}}$ of exchangeable probability on E^N is said to be f-chaotic, if one of the following equivalent properties holds:*

i) *for all $k \in \mathbb{N}$, the k-marginals of f^N, defined as*

$$f_k^N(t, z_1, \dots, z_k) = \int_{R^{2d(N-k)}} f^N(t, z_1, \dots, z_N)\, dz_{k+1} \dots dz_N,$$

converge weakly towards $f^{\otimes k}$ as N goes to infinity: $f_k^N \rightharpoonup f^{\otimes k}$,

ii) *the second marginal f_2^N converges weakly towards $f^{\otimes 2}$: $f_2^N \rightharpoonup f^{\otimes 2}$,*

iii) *under the laws f^N, the (random) empirical measures μ_N^z converge in law towards f (with the weak topology of measure).*

The equivalence between i) and iii) holds because the marginals can be recovered from the expectations of moments of the empirical measure

$$f_k^N = \mathbb{E}(\mu_N^z(t, z_1) \dots \mu_N^z(t, z_k)) + O\left(\frac{k^2}{N}\right),$$

a result sometimes called Grunbaum lemma. We also mention that the fact that iii) holds when $f := f^{\otimes N}$ is known as the empirical law of large number, or Glivenko-Cantelli theorem; see Proposition 6 for precise statement.

We refer to the lecture notes by Sznitman [58] for a rigorous proof of the equivalence of the three statements. For detailed explanations about quantification of the equivalence between convergence of the marginals f_k^N and the convergence in law of the empirical distributions μ_N^z, we refer to [35]. This quantified equivalence was for instance used in the recent and important work of Mischler and Mouhot about Kac's program in kinetic theory [48].

The introduction of these chaotic (or asymptotically independent) sequences of probability is justified in our mean-field setting by some previous works: several authors [12, 38, 39, 47, 40, 52] have already proved that equilibrium measures for these systems (1.1) do satisfy the chaoticity assumptions of Definition 3, the latter reference containing even quantitative chaos estimates for the Coulomb case.

The notion of propagation of molecular chaos was formalized by Kac's in [37] and goes back to Boltzmann and his "Stosszahlansatz". A standard reference is the famous course by Sznitman [58]. We also refer to the recent review of Golse [27], with the point of view of an analyst.

DEFINITION 4. – *Denoting by $f^N(t, z_1, \dots, z_N)$ the image by the dynamics* (1.1) *of a given initial law $f^N(0)$, Propagation of chaos holds when if starting with f^0-chaotic initial conditions, the sequence $f^N(t)$ is $f(t)$-chaotic for any time $t \geq 0$, where $f(t)$ is the unique solution (in a suitable sense) of the limit dynamics* (1.5).

Here, we will not prove this "full" propagation of chaos, starting from any f^0-chaotic sequence, but only a *partial* result valid only if the initial conditions have law $(f^0)^{\otimes N}$. That partial result is in some sense not a result of propagation (since we are of course not able to propagate the exact independence between particles), but we explain in Remark 5 what we are exactly able to propagate.

In the hard sphere problem, (partial) propagation of chaos towards the Boltzmann equation (in the Boltzmann-Grad scaling) was shown by Landford [41], with a non completely correct proof that was completed only recently by Gallagher, Saint-Raymond and Texier [23] (and extended to more general interactions). Unfortunately the deep techniques used in [23] do not seem to be applicable in our case.

The mean field limit results obtained in that article, see Theorems 1 and 3, imply (partial) quantified versions of the propagation of chaos, stated in Theorems 2 and 4, and Corollary 1 and 2.

Previous results in dimension one. – Let us shortly mention that in dimension one, the mean field limit and the propagation of chaos are better understood , even for the 1D Coulomb interaction. In fact, that case is in some sense simpler: the force $F(x) = \operatorname{sign}(x)$ is "only" discontinuous. The first mean field limit result in that case was obtained by Trocheris [59], and it was re-discovered by Cullen, Gangbo and Pisante as a particular case of semi-geostrophic equations [16]. We also refer to a simpler proof by the first author [33] using a weak-strong stability inequality for the 1D Vlasov-Poisson equation. All these mean-field results imply the propagation of chaos in a straightforward manner.

Previous results with cut-off or for smooth interactions. – The mean-field limit and the propagation of chaos are known to hold for smooth interaction forces ($F \in W^{1,\infty}_{\mathrm{loc}}$) since the end of the seventies and the works of Braun and Hepp [11], Dobrushin [20] and Neunzert and Wick [49]. Those articles introduce the main ideas and the formalism behind mean field limits; we also refer to the nice book by Spohn [57].

Their proofs however rely on Gronwall type estimates and are connected to the fact that Gronwall estimates are actually true for (1.1) uniformly in N if $F \in W^{1,\infty}$. This makes it impossible to generalize them to any case where F is singular, including Coulombian interactions and many other physically interesting models.

However, by keeping the same general approach, it is possible to deal with singular interactions with cut-off. For instance for Coulombian interactions, one could consider

$$F_N(x) = C\,\frac{x}{(|x|^2 + \varepsilon(N)^2)^{d/2}},$$

or other types of regularization at the scale $\varepsilon(N)$. The system (1.1) with such forces does not necessarily have much physical meaning but the corresponding studies are crucial to understand the convergence of numerical methods. For particles initially on a regular mesh, we refer to the works of Ganguly and Victory [25], Wollman [62] and Batt [6] (the latter gives a simpler proof, but valid only for larger cut-off than in the two first references). Unfortunately they had to impose that $\lim_{N\to\infty} \varepsilon(N) N^{1/d} = +\infty$, meaning that the cut-off for convergence results is usually larger than the one used in practical numerical simulations. Note that the scale $N^{-1/d}$ is the average distance between two neighboring particles in position.

These "numerically oriented" results do not imply the propagation of chaos, as the particles are on a mesh initially and hence (highly) correlated. Moreover, we emphasize that the two problems with initial particles on a mesh, or with initial particles independently and equally distributed seem to be very different. In the last case, Ganguly, Lee, and Victory [24] prove the convergence only for a much larger cut-off $\varepsilon(N) \approx (\ln N)^{-1}$.

Previous results for $2d$ Euler or other macroscopic equations. – A well known system, very similar at first sight with the question here, is the vortices system for the $2d$ incompressible Euler equation. One replaces (1.1) by

$$\dot{X}_i = \frac{1}{N} \sum_{j\neq i} \alpha_i\,\alpha_j\,\nabla^{\perp}\Phi(X_i - X_j), \tag{1.6}$$

where $\Phi(x) = (2\pi)^{-1} \ln|x|$ is still the Coulombian kernel (in 2 dimensions here) and $\alpha_i = \pm 1$. One expects this system to converge to the Euler equation in vorticity formulation

$$\partial_t\omega + \mathrm{div}\,(u\,\omega) = 0, \qquad \mathrm{div}\,u = 0, \qquad \mathrm{curl}\,u = \omega. \tag{1.7}$$

The same questions of convergence and propagation of chaos can be asked in this setting. Two results without regularization for the true kernel are already known. The work of Goodman, Hou and Lowengrub, [29, 28], has a numerical point of view but uses the true singular kernel in an interesting way. The work of Schochet [56] uses the weak formulation of Delort of the Euler equation and proves that empirical measures with bounded energy converge towards measures that are weak solutions to (1.7). Unfortunately, the possible lack of uniqueness of the vorticity Equation (1.7) in the class of measures does not allow to deduce the propagation of chaos.

The main difference between (1.1) and (1.6) is that System (1.1) is second order while (1.6) is first order. In particular given a test particle at the origin, one may consider the subset of the one particle phase space (or position-velocity space) corresponding to collisions; the one particle phase space is $\mathbb{R}^2$ for $2d$ Euler and $\mathbb{R}^{2d}$ in the Vlasov case. The subsets corresponding to collisions are very different in either cases: this subset is $\{0\} \subset \mathbb{R}^2$ for Euler versus $\mathbb{R}^d \subset \mathbb{R}^{2d}$ for Vlasov. And so proving convergence results leads to quite different difficulties in both situations as it requires to control collisions or "near" collisions.

The references mentioned above use the symmetry of the forces in the vortex case; a symmetry which cannot exist in our kinetic problem, independently of additional structural assumptions like $F = -\nabla\Phi$. The force is still symmetric with respect to the space variable, but there is now a velocity variable which breaks the argument used in the vortices case. For a more complete description of the vortices system, we refer to the references already quoted or to [32], which introduces in that case techniques similar to the one used here.

Our previous result for singular forces without cut-off. – To the best of our knowledge, the only mean field limit result available up to now for System (1.1) with singular forces is [34]. We proved the mean field limit (not the propagation of chaos) provided that:

- The interaction force F satisfies an (S^α)-condition with $\alpha < 1$.
- The particles are initially well distributed, meaning that the minimal inter-distance in $\mathbb{R}^{2d}$ is of the same order as the average distance between neighboring particles $N^{-1/2d}$.

The second assumption is all right for numerical purposes but does not allow to consider physically realistic initial conditions, as per the propagation of chaos property. This assumption is indeed not generic for empirical measures randomly chosen with law $(f^0)^{\otimes N}$, i.e., it is satisfied with probability going to 0 in the large N limit.

Organization of the paper. – In the next section, we precisely state our main theorems. In the third section, we introduce the notation, recall some results on the Vlasov-Poisson Equation (1.5) and give a short sketch of the proof. The fourth and longest section is devoted to the proof of the main field limit results, and we explain in the fifth section why those deterministic results imply the propagation of chaos. The sixth section contains two important discussions: one about the existence of solution to the system of ODE (1.1), and a second explaining why we cannot use the structure of the force term, when it is of potential form, attractive or repulsive. Finally, two useful propositions are proved in the appendix.

2. Main results

Before stating our main results, we recall the definition of the order one Monge-Kantorovitch-Wasserstein distance (MKW) denoted W_1: for two probability measures μ, ν on $\mathbb{R}^n$ with finite first moment, define

$$W_1(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|\pi(dx,dy),$$

where $\Pi(\mu,\nu)$ stands for the set of all probability measures on $\mathbb{R}^n\times\mathbb{R}^n$ with first marginal μ and second marginal ν (see also Definition 5). Here and everywhere, $|\cdot|$ stands for the usual Euclidean distance. Roughly speaking, the 1-MKW distance measures the minimal cost of the transport from a measure to the other one. We refer to Villani's book for more details [61].

2.1. The results without cut-off

Our main result in this article is deterministic: It proves that the mean field limit holds, provided that interaction forces still satisfy an (S^α)-condition (1.2) with $\alpha < 1$. The initial distributions of particles have to be uniformly compactly supported, and to satisfy a bound from above on a "discrete uniform norm" and again a bound from below on the minimal distance between particles (in position and speed) which is much less demanding than in [34].

THEOREM 1. – *Assume that* $d \geq 2$ *and that the interaction force* F *satisfies an* (S^α) *condition* (1.2)*, for some* $\alpha < 1$ *and let* $0 < \gamma < 1$.

Assume that $f^0 \in L^\infty(\mathbb{R}^{2d})$ *is non-negative, and has compact support and total mass one, and denote by* f *the unique non-negative, global, bounded, and compactly supported solution* f *of the Vlasov Equation* (1.5)*, see Proposition 2.*

Assume that the initial conditions Z^0 *are such that for each* N*, there exists a global solution* Z *to the* N *particle system* (1.1)*, and that the initial empirical distributions* μ_N^0 *of the particles satisfy*

i) *for a constant* C_∞ *independent of* N*,*

$$\sup_{z \in \mathbb{R}^{2d}} N^\gamma \mu_N^0\Big(B_{2d}\big(z, N^{-\frac{\gamma}{2d}}\big)\Big) \leq C_\infty, \quad \text{and} \quad \|f_0\|_\infty \leq C_\infty;$$

ii) *for some* $R_0 > 0$*,* $\forall N \in \mathbb{N}$*,* $\quad \operatorname{Supp} \mu_N^0 \subset B_{2d}(0, R_0)$*;*

iii) *for some* $r \in (0, r^*)$ *where* $r^* := \frac{d-1}{1+\alpha}$*,*

$$\inf_{i \neq j} |(X_i^0, V_i^0) - (X_j^0, V_j^0)| \geq N^{-\gamma(1+r)/2d}.$$

Then for any $T > 0$*, there exist two constants* $C_0(R_0, C_\infty, F, T)$ *and* $C_1(R_0, C_\infty, F, \gamma, r, T)$ *such that for* $N \geq C_1$*, the following estimate holds*

$$\forall t \in [0,\, T], \quad W_1(\mu_N(t), f(t)) \leq e^{C_0 t}\Big(W_1(\mu_N^0, f^0) + 2\, N^{-\frac{\gamma}{2d}}\Big). \tag{2.1}$$

REMARK 1. – *The conditions* (i)-(iii) *are fulfilled when the initial positions and velocities of the particles are chosen on a mesh. They are also fulfilled when one considers a finite number of particles inside cells of a mesh, as it is usually done in PIC method.*

To deduce from the previous theorem the propagation of chaos, it remains to show that we can apply its deterministic stability result to most of the random initial conditions. Precisely, we can show that when the initial positions and velocities are i.i.d. with law f^0, then the conditions (i)-(iii) of Theorem 1 are satisfied with a probability going to one in the limit. This leads to a quantitative version of propagation of chaos.

THEOREM 2. – *Assume that* $d \geq 3$ *and that* F *satisfies an* (S^α)*-condition* (1.2) *with* $\alpha < 1$. *There exist a positive real number* $\gamma^* \in (0,1)$ *depending only on* (d, α) *and a function* $s^* : \gamma \in (\gamma^*, 1) \to s^*_\gamma \in (0, \infty)$ *such that:*

– for any non negative initial data $f^0 \in L^\infty(\mathbb{R}^{2d})$ *with compact support and total mass one, denoting by* f *the unique global, non-negative bounded, and compactly supported solution* f *to the Vlasov Equation* (1.5)*, see Proposition 2;*

– for each $N \in \mathbb{N}^*$*, denoting by* μ_N *the empirical measure corresponding to the solution to* (1.1) *with initial positions* $Z^0 = (X_i^0, V_i^0)_{i \leq N}$ *chosen randomly according to the probability* $(f^0)^{\otimes N}$*.*

Then, for all $T > 0$*, any*

$$\gamma^* < \gamma < 1 \quad \text{and} \quad 0 < s < s^*_\gamma,$$

there exist three positive constants $C_0(T, f, F)$*,* $C_1(\gamma, s, T, f, F)$ *and* $C_2(f^0, \gamma)$ *such that for* $N \geq C_1$

$$\mathbb{P}\left(\exists\, t \in [0,T],\ W_1(\mu_N^Z(t), f(t)) \geq 3\, e^{C_0 t}\, N^{-\frac{\gamma}{2d}}\right) \leq \frac{C_2}{N^s}. \tag{2.2}$$

The constants C_1 and C_2 blow up when γ or s approach their maximum value.

REMARK 2. – *We have explicit formulas for γ^* and s^*_γ namely*

$$\gamma^* := \frac{2+2\alpha}{d+\alpha} \quad \text{and} \quad s^*_\gamma := \frac{\gamma d - (2-\gamma)\alpha - 2}{2(1+\alpha)}. \tag{2.3}$$

Those conditions are not completely obvious, but it can be checked that if $\alpha < 1$ and $d \geq 3$, then $\gamma^* < 1$, so that admissible γ exists. And for an admissible γ, s^*_γ is also positive, so that admissible s also exists. The best choices for γ and s would be $\gamma = 1$ and $s = \frac{d-\alpha-2}{2(1+\alpha)}$ as those give the fastest convergence. Unfortunately the constant C_1 and C_2 would then be $+\infty$ hence the more complicated formulation.

REMARK 3. – *The expected distance between neighboring particles thrown at random independently and uniformly in a ball of radius 1 of $\mathbb{R}^{2d}$ is of order $N^{-1/2d}$. So roughly speaking, under the assumptions of Theorem 2, except for a small set of initial conditions, the deviation between the empirical measure and the limit is almost of order of the average inter-particle distance in position-velocity space.*

REMARK 4. – *The deterministic Theorem 1 is valid in dimension 2. Unfortunately, its assumptions are not generic in dimension 2 for initial conditions chosen randomly and independently. This is why we cannot prove the propagation of chaos for $d = 2$ in Theorem 2 even for small α. In fact, note for instance that if $d = 2$ then γ^* defined in* (2.3) *is larger than* 1 *so that it is never possible to find γ in $(\gamma^*, 1)$.*

REMARK 5. – *Theorem 2 is not exactly a result of "propagation", since the assumption on the independence of the initial particles cannot be strictly propagated in time for obvious reasons. However, we cannot provide a result of propagation of chaos in the usual sense considered in the literature (with a general chaotic sequence as initial distribution of particles). Indeed, we need assumptions (on the minimal distance between particles, on a discrete uniform norm,…) that are not implied by the chaoticity of Definition 3. However, we could still consider more general f^0-chaotic sequences provided additional assumptions are made. For instance, our techniques imply a result like*

Rough statement of propagation of f-chaotic sequences. – *Assume that the force F and the initial condition f_0 satisfy the assumptions of Theorem 2. Assume as well that the distribution f_0^N is f_0-chaotic, and that for some $C_\infty > 0, R > 0$ and $r > 1$, the points* i), ii), iii) *of Theorem 1 are satisfied with a probability going to* 1 *as N goes to infinity.*

Then for any $t > 0$, the same holds at time t: The distribution $f^N(t)$ of particles at time t is $f(t)$-chaotic and satisfies points i), ii), iii) *of Theorem 1 for some $C_\infty(t) > 0, R(t) > 0$ and $r(t) > 1$ with large probability.*

REMARK 6. – *The arguments in the proof of Theorem 2 prove that, at fixed N, there exists a global solution to* (1.4) *for a large set of initial conditions. In fact, in a very sketchy way, this theorem also propagates a control on the minimal inter-particles distance in position-velocity space. Used as is, it only says that asymptotically, the control is good with large probability. However for fixed N, if we let some constants increase as much as needed, it is possible to modify the argument and obtain a control for almost all initial configurations. Since the proof* *also implies that the only bad collisions are the collisions with vanishing relative velocities, we can obtain existence (and also uniqueness) for almost all initial data of the ODE* (1.1).

Theorem 2 states the propagation of chaos using the formulation iii) of Definition 3. Thanks to the equivalence of the three formulations, it may be restated in terms of convergence of marginals. In fact, using the notion of entropic chaotic sequence (introduced in [37], formalized in [13], see also [35, Definition 1.3]), we can even state a result of convergence of the marginals in L^1.

We first recall the definition of the entropy: for a probability measure with density f^N on $\mathbb{R}^{2dN}$,

$$H(f^N) := \int_{\mathbb{R}^{2dN}} \ln\big(f^N(z)\big)\, f^N(Z)\, dZ,$$

which is well defined in $\mathbb{R} \cup \{+\infty\}$ when f^N is compactly supported (see for instance [35, Lemma 3.1]). We emphasize that this definition also covers the case $N = 1$.

COROLLARY 1. – *Under the assumption of Theorem 2, the law $f^N(t)$ of the particle system $Z^N(t)$ at time t is entropically $f(t)$-chaotic:*

$$\forall\, t \geq 0,\ \forall\, k \in \mathbb{N},\ f_k^N(t) \xrightarrow[N\to+\infty]{} f(t)^{\otimes k}, \quad \textit{and} \quad \frac{1}{N} H\big(f^N(t)\big) \xrightarrow[N\to+\infty]{} H(f(t)). \tag{2.4}$$

This implies in particular a stronger version of the point i) in Definition 3. Precisely that

$$\forall\, k \in \mathbb{N},\ \big\| f_k^N(t) - f(t)^{\otimes k} \big\|_{L^1} \xrightarrow[N\to+\infty]{} 0.$$

The proof of Corollary 1 is performed after the proof of Theorem 2.

The improvements with respect to [34]. – The major improvement is the much weaker assumption in Theorem 1 on the initial distribution of positions and velocities, which enables us to prove the propagation of chaos.

The method of the proof is also quite different. It now relies on explicit bounds between the empirical measure and an appropriate solution to the limit Equation (1.5). This lets us easily use the properties of (1.5), and dramatically simplifies the proof in the long time case which was very intricate in [34] and does not require any special treatment here.

Finally, our analysis is now quantitative: for large enough N, Theorem 1 gives a precise rate of convergence in Monge-Kantorovitch-Wasserstein distance W_1, with important applications from the point of view of the numerical analysis (giving rates of convergence for particles' methods for instance). For more details about the novelties and improvements with respect to [34], we refer to the sketch of the proof in Subsection 3.3.

Unfortunately, the condition on the interaction force F is still the same and does not allow to treat Coulombian interactions. There are some physical reasons for this condition, which are discussed at the end of the article in Subsection 6.2. We refer to [4] for some ideas in how to go beyond this threshold in the repulsive case.

2.2. The results with cut-off

The result presented here is in one sense slightly weaker than the previously known result [24], since we just miss the critical case $\alpha = d-1$. But in that work the cut-off used is very large: $\varepsilon(N) \approx (\ln N)^{-1}$. Instead we are able to use cut-off that are some power of N and much more realistic from a physical point of view. For instance, astrophysicists doing gravitational simulations ($\alpha = d-1$) with "tree codes" usually use small cut-off parameters, lower than $N^{-1/d}$ by some order. See [17] for a physically oriented discussion about the optimal length of this parameter.

THEOREM 3. – *Assume that $d \geq 2$ and that the interaction force F_N satisfies an (S^α_m) condition* (1.3)*, for some $1 \leq \alpha < d-1$, with a cut-off order satisfying*

$$m < m^* := \frac{1}{2d} \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right),$$

and choose any $\gamma \in \left(\frac{m}{m^*}, 1\right)$.

Assume that $f^0 \in L^\infty(\mathbb{R}^{2d})$ is non-negative and has compact support and total mass one, and denote by f the unique, non-negative, bounded, and compactly supported solution f of the Vlasov Equation (1.5) *on the maximal time interval $[0,\ T^*)$, see Proposition 2.*

Assume also that for any N, the initial empirical distribution of the particles μ^0_N satisfies:

i) *for a constant C_∞ independent of N,*

$$\sup_{z\in\mathbb{R}^{2d}} N^\gamma \mu^0_N\Big(B_{2d}\big(z, N^{-\frac{\gamma}{2d}}\big)\Big) \leq C_\infty, \quad \textit{and} \quad \|f_0\|_\infty \leq C_\infty;$$

ii) *for some $R_0 > 0$, $\forall N \in \mathbb{N}$, $\quad \operatorname{Supp} \mu^0_N \subset B_{2d}(0, R_0)$.*

Then for any time $T < T^*$, there exist $C_0(R_0, C_\infty, F, T)$ and $C_1(R_0, C_\infty, F, \gamma, r, T)$ such that for $N \geq C_1$ the following estimate holds

$$\forall\, t \in [0,\ T], \quad W_1(\mu_N(t), f(t)) \leq e^{C_0 t}\Big(W_1(\mu^0_N, f_0) + 3\, N^{-\frac{\gamma}{2d}}\Big). \tag{2.5}$$

REMARK 7. – *One would like to take m as large as possible if we want to be close to the dynamics without cut-off.*

REMARK 8. – *As mentioned earlier, that theorem is interesting when the cut-off parameter is small in a suitable sense. For instance, it is interesting to known when a cut-off of order smaller than $N^{-1/d}$ is allowed, since the latter order is the one of the average inter-particle distance in position space. This happens in fact when $m^* > d^{-1}$ and leads to the condition $\alpha < \frac{d}{2}$.*

REMARK 9. – *Theorem 3 is also interesting for numerical simulations with particle's methods* [1, 7, 30] *because one obvious way to fulfill the assumption i) is to put particles initially on a mesh (with a grid length of $N^{-1/2d}$ in $\mathbb{R}^{2d}$). In that case, the result is even valid with $\gamma = 1$.*

As in the case without cut-off, the fact that the mean-field limit holds under "generic" conditions implies the propagation of molecular chaos.

THEOREM 4. – *Assume that* $d \geq 3$ *and that* F_N *satisfies a* (S_m^α)*-condition for some* $1 \leq \alpha < d-1$ *with a cut-off order* m *such that*

$$m < m^* := \frac{1}{2d} \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right),$$

and choose any $\gamma \in \left(\frac{m}{m^*}, 1\right)$.

Choose any non-negative initial condition $f^0 \in L^\infty$ *with compact support and total mass one for the Vlasov Equation* (1.5)*, and denote by* f *the unique non-negative and strong solution of the Vlasov equation* (1.5) *with initial condition* f^0 *on the maximal time interval* $[0, T^*)$*, given by Proposition 2.*

For each $N \in \mathbb{N}^*$*, consider the system* (1.1) *for* F_N *with initial positions* $(X_i, V_i)_{i \leq N}$ *chosen randomly according to the probability* $(f^0)^{\otimes N}$.

Then for any time $T < T^*$*, there exist positive constants* $C_0(T, f, F)$*,* $C_1(\gamma, m, T, f, F)$*,* $C_2(f)$ *and* $C_3(f)$ *such that for* $N \geq C_1$

$$\mathbb{P}\left(\exists\, t \in [0,T],\ W_1(\mu_N(t), f(t)) \geq 4\, e^{C_0 t} N^{-\frac{\gamma}{2d}}\right) \leq C_2 N^\gamma\, e^{-C_3 N^\lambda},$$

where $\lambda = 1 - \max\left(\gamma, \frac{1}{d}\right)$.

REMARK 10. – *Our result is valid only locally in time (but on the largest interval of time possible) in the case where blow-up may occur in the Vlasov equation, as for instance in dimension larger than or equal to four with attractive interactions. But it is valid for any time in dimension three, since in that case the strong solutions of the Vlasov equations we are dealing with are global, see Proposition 2 in Section 3.2.*

As in the case without cutoff, the result may be restated in term of convergence of the k-marginals.

COROLLARY 2. – *Under the assumption of Theorem 4, the law* $f^N(t)$ *of the particle system* $Z^N(t)$ *at time t is entropically* $f(t)$*-chaotic:*

$$\forall\, t \geq 0,\ \forall\, k \in \mathbb{N},\ f_k^N(t) \xrightarrow[N\to+\infty]{} f(t)^{\otimes k}, \quad \text{and} \quad \frac{1}{N} H\big(f^N(t)\big) \xrightarrow[N\to+\infty]{} H(f(t)).$$

This implies in particular a stronger version of the point i) in Definition 3. Precisely that

$$\forall\, k \in \mathbb{N},\ \big\|f_k^N(t) - f(t)^{\otimes k}\big\| \xrightarrow[N\to+\infty]{} 0.$$

2.3. Open problems and possible extensions

In dimension $d = 3$, the minimal cut-off is given by $m^* = \frac{\gamma}{6}\min((\alpha-1)^{-1}, 5\alpha^{-1})$. As γ can be chosen very close to one, for α larger but close to one, the previous bound tells us that we can choose cut-off of order almost $N^{-5/6}$, i.e., much smaller than the likely minimal inter-particles distance in position space (of order $N^{-2/3}$, see the third section). With such a small cut-off, one could hope that it is almost never used when we calculate the interaction forces between particles. Only a negligible number of particles will become that close to one another before the time T. This suggests that there should be some way to extend the result of convergence without cut-off at least to some $\alpha > 1$.

Unfortunately, we do not know how to make rigorous the previous argument on the close encounters. First it is highly difficult to translate for particles system that are highly correlated. To state it properly we need L^∞ bounds on the 2-particle marginal. But obtaining such a bound for singular interactions seems difficult. Moreover, it remains to control the influence of particles that have had a close encounter (their trajectories after an encounter are not well controlled) on the other particles.

Many particles systems with diffusion. – It would be very natural to try to adapt our techniques to the stochastic case of Langevin equations

(2.6)
$$\forall i \leq N, \quad \begin{cases} X_i(t) = X_i^0 + \displaystyle\int_0^t V_i(s)\,ds \\ V_i(t) = V_i^0 + \displaystyle\frac{1}{N}\sum_{j\neq i}\int_0^t F(X_i(s) - X_j(s))\,ds - \lambda\int_0^t V_i(s)\,ds + \nu B_i(t), \end{cases}$$

where the B_i are independent Brownian motions, and $\nu, \lambda > 0$. Solutions of that system should formally converge to solutions of the Jeans-Vlasov-Fokker-Planck equation

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + E(x)\cdot\nabla_v f = \dfrac{\nu^2}{2}\Delta_v f + \lambda\,\mathrm{div}(vf)\,, \\ E(x) = \displaystyle\int_{\mathbb{R}^d}\rho(t,y)\,F(x-y)\,dy. \end{cases} \tag{2.7}$$

It was shown by McKean in [46] that the propagation of chaos holds when $F \in W^{1,\infty}$. But to the best of our knowledge, there is no similar result when the interaction force is singular, even weakly. Our techniques, which rely on strong controls on the trajectories and on the minimal inter-particle distance are very sensitive to noise, and cannot be directly adapted to the stochastic case.

Remark that the situation is in some way "opposite" in the vortex case. The propagation of chaos for the stochastic vortex system (the system (1.6) with independent noises) was first proved by Osada in the eighties [50], and recently generalized by Fournier, Mischler and the first author [22]. But what happens for the Euler equation is still not well understood despite interesting efforts [56].

3. Notation, useful results and sketch of the proof

3.1. ∞-MKW distance, blob distributions and more notation

In the sequel (and before), we always use the Euclidean distance on $\mathbb{R}^d$ for positions or velocities, or on $\mathbb{R}^{2d}$ for couples of "position-velocity". It will simply be denoted by $|x|$, $|v|$, $|z|$. The notation $B_n(a,R)$ will always stand for the ball of center a and radius R in dimension $n = d$ or $2d$. The Lebesgue measure of a measurable set A will also be denoted by $|A|$.

Empirical distribution μ_N and minimal inter-particle distance d_N. – Given a configuration $Z = (X_i, V_i)_{i\leq N}$ of the particles in the phase space $\mathbb{R}^{2dN}$, the associated empirical distribution is the measure

$$\mu_N^Z = \frac{1}{N}\sum \delta_{X_i,V_i}.$$

An important remark is that if $(X_i(t,), V_i(t))_{i\leq N}$ is a solution of the system of ODE (1.1), then the measure $\mu_N^Z(t)$ is a solution of the Vlasov Equation (1.5) in a weak sense, provided that we define also the interaction force at zero with $F(0) = 0$. As said before, this condition is necessary since there is no self interaction in the Newton system (1.1).

For every configuration (or equivalently for every empirical measure), we define the minimal distance d_N^Z between particles in $\mathbb{R}^{2d}$

$$d_N^Z = d_N(\mu_N^Z) := \min_{i \neq j} |Z_i - Z_j| = \min_{i \neq j} \big(|X_i - X_j|^2 + |V_i - V_j|^2\big)^{\frac{1}{2}}. \tag{3.1}$$

This is not an averaged quantity, contrary to most of the relevant quantities in statistical physics and thermodynamics, but an extremal one. However it is crucial to control the possible concentrations of particles and we will need to bound that quantity from below.

In the following we often omit the Z superscript, in order to keep "simple" notation.

Infinite MKW distance. – We use many times the Monge-Kantorovitch-Wasserstein distances of order one and infinite. The order one distance, denoted by W_1, is classical and we refer to the very clear book of Villani for definition and properties [61]. The second one denoted W_∞ is not widely used, so we recall its definition. We start with the definition of transference plane

Definition 5. – *Given two probability measures μ and ν on $\mathbb{R}^n$ for any $n \geq 1$, a transference plane π from μ to ν is a probability measure on $X \times X$ such that*

$$\int_X \pi(x, dy) = \mu(x), \qquad \int_X \pi(dx, y) = \nu(y),$$

that is the first marginal of π is μ and the second marginal is ν.

With this we may define the W_∞ distance

Definition 6. – *For two probability measures μ and ν on $\mathbb{R}^n$, with $\Pi(\mu,\nu)$ the set of transference planes from μ to ν:*

$$W_\infty(\mu,\nu) = \inf \big\{\pi - \operatorname{esssup} |x-y| \;\big|\; \pi \in \Pi\big\}.$$

There is also another notion, called the transport map. A transport map is a measurable map $T : \operatorname{Supp} \mu \to \mathbb{R}^n$ such that $T_\# \mu = \nu$, where the pushforward of a measure m by a transform L is defined by

$$L_\# m(O) = m(L^{-1}(O)), \qquad \text{for any measurable set } O.$$

In fact, it can be checked that to any transport map, one may associate a transference plane through the measure $(\mathrm{Id}, T)_\# \, \mu \in \Pi$; this measure is actually the pushforward of μ via the map $x \to \big(x, T(x)\big)$. In one of the few works on the subject [15] Champion, and De Pascale and Juutineen prove that if μ is absolutely continuous with respect to the Lebesgue measure $\mathscr{L}$, then at least one optimal transference plane for the infinite MKW distance is given by an optimal transport map, i.e., there exists T such that $T_\# \mu = \nu$ (and thus $(\mathrm{Id}, T)_\# \, \mu \in \Pi$), and

$$W_\infty(\mu,\nu) = \mu - \operatorname{esssup}_x |Tx - x|.$$

Although that is not mandatory (we could actually work with optimal transference planes), we will use this result and work in the sequel with transport maps. That will greatly simplify the notation in the proof.

Optimal transport is useful to compare the discrete sum appearing in the force induced by the N particles to the integral of the mean-field force appearing in the Vlasov equation. For instance, if f is a continuous distribution and μ_N an empirical distribution we may rewrite the interaction force of μ_N using a transport map $T = (T_x, T_v)$ of f onto μ_N

$$\frac{1}{N}\sum_{i\neq j} F(X_i^0 - X_j^0) = \int F(X_i^0 - T_x(y,w))f(y,w)\,dydw.$$

Note that in the equality above, the function F is singular at $x = 0$, and that we impose $F(0) = 0$. The interest of the infinite MKW distance is that the singularity is still localized "in a ball" after the transport : The term under the integral in the right-hand side has no singularity out of a ball of radius $W_\infty(f, \mu_N)$ in x. Other MKV distances of order $p < +\infty$ destroy that simple localization after the transport, which is why it seems more difficult to use them.

The use of that distance yields rates of convergence, improving in that way our previous work on the subject [34]. To our knowledge however this distance was only seldom used in the literature: It was used by McCann [45] for a problem of stability of rotating binary stars, and by the first authors and collaborators [32, 14] for mean-field limits for the Euler equation and the aggregation equation.

The scale ε. – We also introduce a scale

$$\varepsilon(N) = N^{-\gamma/2d}, \tag{3.2}$$

for some $\gamma \in (0,1)$ to be fixed later but close enough to 1. Remark that this scale is larger than the average distance between a particle and its closest neighbor, which is of order $N^{-1/2d}$. We will often define quantities directly in term of ε rather than N. For instance, the cut-off order m used in the (S^α_m)-condition may be rewritten in term of ε, with $\bar m := \frac{2d}{\gamma} m \in \left(1, \min(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha})\right)$.

$$(S^\alpha_m) \qquad \begin{array}{l} \text{i)} \ \ F \text{ satisfies an } (S^\alpha)\text{-condition},\\ \text{ii)} \ \ \forall\,|x| \ge \varepsilon^{\bar m},\ F_N(x) = F(x),\\ \text{iii)} \ \ \forall\,|x| \le \varepsilon^{\bar m},\ |F_N(x)| \le \varepsilon^{-\bar m\alpha}. \end{array} \tag{3.3}$$

The solution f_N of Vlasov equation with blob initial condition. – We define a smoothing of μ_N at the scale $\varepsilon(N)$. For this, we choose a bounded kernel $\phi : \mathbb{R}^{2d} \to \mathbb{R}$ radial with compact support in $B_{2d}(0,1)$ and total mass one, and denote $\phi_\varepsilon(\cdot) = \varepsilon^{-2d}\phi(\cdot/\varepsilon)$. The precise choice of ϕ is not very relevant here, and the simplest one is maybe $\phi = \frac{1}{|B_{2d}(0,1)|}\mathbf{1}_{B_{2d}(0,1)}$. We use this to smooth μ_N and define

$$f_N^0 = \mu_N^0 * \phi_{\varepsilon(N)}, \tag{3.4}$$

and denote by $f_N(t,x,v)$ the solution to the Vlasov Eq. (1.5) for the initial condition f_N^0.

With f_N, the assumption of point i) in Theorems 1 and 3 may be rewritten

$$\|f_N^0\|_\infty \le C_\infty,$$

independently of N. Since solutions of the Vlasov equation are transported along the characteristic of the associated vector-field, even in the case where that vector-field is only in $W^{1,1}$ (see [2] for details), the uniform L^∞ bound also holds for any time. That L^∞ bound together with an estimate on the growth of the support lets us use standard stability estimates to control the W_1 distance of f_N to another solution of the Vlasov equation, see Loeper's result [43] recalled in Proposition 3.

A key point in the rest of the article is that f_N^0 and μ_N^0 are very close in W_∞ distance as per

PROPOSITION 1. – *For any $\phi : \mathbb{R}^{2d} \to \mathbb{R}$ radial with compact support in $B_{2d}(0,1)$ and total mass one we have for any $\mu_N^0 = \frac{1}{N}\sum_{i=1}^N \delta_{(X_i^0, V_i^0)}$*

$$W_\infty(f_N^0, \mu_N^0) = c_\phi \varepsilon(N)$$

where c_ϕ is the smallest c for which $\operatorname{Supp}\phi \subset \overline{B_{2d}(0,c)}$.

Proof. – Unfortunately even in such a straightforward case, it is not possible to give a simple explicit formula for the optimal transport map. But there is a rather simple optimal transference plane. Define

$$\pi(x,v,y,w) = \frac{1}{N}\sum_i \phi_\varepsilon(x-y, v-w)\,\delta_{(X_i^0,\,V_i^0)}(y,w).$$

Note that

$$\int_{\mathbb{R}^{2d}} \pi(x,v,dy,dw) = [\mu_N^0 * \phi_\varepsilon](x,v) = f_N^0(x,v),$$

and since ϕ_ε has mass 1

$$\int_{\mathbb{R}^{2d}} \pi(dx,dv,y,w) = \frac{1}{N}\sum_i \delta_{(X_i^0,\,V_i^0)}(y,w) = \mu_N^0(y,w).$$

Therefore π is a transference plane between f_N^0 and μ_N^0. Now take any (x,v,y,w) in the support of π. By definition there exists i such that $y = X_i^0$, $w = V_i^0$ and (x,v) is in the support of $\phi_\varepsilon(. - X_i^0, v - V_i^0)$. Hence by the assumption on the support of ϕ

$$|x-y|^2 + |v-w|^2 \le c_\phi^2 [\varepsilon(N)]^2,$$

which gives the upper bound.

We turn to the lower bound. Remark that the fact that ϕ is assumed to be radial and the definition of c_ϕ imply that $\phi > 0$ on $B_{2d}(0, c_\phi)$. Choose $X_i^0,\ V_i^0$ any extremal point of the cloud $(X_j^0, V_j^0)_{j\le N}$. Denote $u_i \in S^{2d-1}$ a vector separating the cloud at $X_i^0,\ V_i^0$, i.e.,

$$u_i \cdot (X_j^0 - X_i^0,\ V_j^0 - V_i^0) < 0, \qquad \forall j \ne i.$$

Now define $(x,v) = (X_i^0,\ V_i^0) + \lambda\,\varepsilon(N)\,u_i$. Since ϕ_ε is radial and $\phi_\varepsilon > 0$ on $B(0, c_\phi\,\varepsilon)$ then $f_N^0(x,v) > 0$ when $\lambda < c_\phi$. Denote by T the optimal transference map. $T(x,v)$ has to be one of the $(X_j^0,\ V_j^0)$. Hence by the definition of u_i, $|(x,v) - T(x,v)| \ge \lambda\,\varepsilon(N)$. Since it is true for any $\lambda < c_\phi$, and for any $\tilde{u}$ in a neighborhood of u, it implies that $f_N^0 - \operatorname{esssup}|T - \mathrm{Id}| \ge c_\phi \varepsilon(N)$. That last argument may be adapted if we use an optimal transference plane, rather than a map. This means in particular that the plane π defined above is optimal. But it is not the only one, except if the blobs never intersect. $\square$

Before turning to the proof of our results on the mean field limit, we give some results about the existence and uniqueness of strong solutions to the Vlasov Equation (1.5).

3.2. Uniqueness, stability of solutions to the Vlasov Equation 1.5

The already known results about the well-posedness (in the strong sense) of the Vlasov equation that we are considering are gathered in the following proposition.

PROPOSITION 2. – *For any dimension d, and any $\alpha \le d-1$, and any compactly supported and bounded initial condition f^0 there exists a unique local (in time) strong solution to the Vlasov Equation* (1.5) *that remains bounded and compactly supported. In general, the maximal time of existence T^* of this solution may be finite, but in the two particular cases below we have $T^* = +\infty$:*

- $\alpha < 1$ *(and any d),*
- $d \le 3$*, and* $\alpha \le d-1$.

In the other cases, the maximal time of existence of the strong solution may be bounded from below by some constant depending only on the L^∞ norm and the size of the support of the initial condition f^0. The size of the support at any time t may also be bounded by a constant depending on the same quantities.

The local existence part in Proposition 2 is a consequence of the following lemma which is proved in the appendix and the following Proposition 3

LEMMA 3. – *Let $f \in L^\infty([0,\, T],\, \mathbb{R}^{2d})$ with compact support be a solution to* (1.5) *in the sense of distribution with an F satisfying an (S^α) condition* (1.2) *with $\alpha \le d-1$. Then if we denote by $R(t)$ and $K(t)$ the maximal position and velocity in the support of f:*

$$R_X(t) := \sup\big\{|x| \mid \exists\, v \in \mathbb{R}^d,\ f(t,x,v) > 0\big\},$$
$$R_V(t) := \sup\big\{|v| \mid \exists\, x \in \mathbb{R}^d,\ f(t,x,v) > 0\big\},$$

they satisfy for a numerical constant C

$$R_X(t) \le R_X(0) + \int_0^t R_V(s)\, ds,$$
$$R_V(t) \le R_V(0) + C\, \|f(0)\|_{L^\infty}^{\alpha/d}\, \|f(0)\|_{L^1}^{1-\alpha/d} \int_0^t R_V(s)^\alpha\, ds.$$

The local uniqueness part in Proposition 2 is a consequence of the following stability estimate proved in [43] for $\alpha = d-1$. Its proof may be adapted to less singular case. For instance, the adaptation is done in [32] in the Vortex case.

PROPOSITION 3 (Loeper). – *If f_1 and f_2 are two solutions of Jeans-Vlasov Equation* (1.5) *respectively with different interaction forces F_1 and F_2 both satisfying an (S^α)-condition with $\alpha < d-1$, then*

$$\frac{d}{dt} W_1(f_1(t), f_2(t)) \le C \max(\|\rho_1(t)\|_\infty, \|\rho_2(t)\|_\infty)\, \big[W_1(f_1(t), f_2(t)) + \|F_1 - F_2\|_1\big],$$

where ρ_i denotes the position distribution associated to f_i: $\rho_i(t,x) := \int f_i(t,x,v)\, dv$.

In the case $\alpha = d-1$, Loeper only obtains in [43] a “log-Lipschitz” bound and not a linear one, but it still implies the stability.

Finally, the global character of the solution in Proposition 2 is

- a consequence of Lemma 3 if $\alpha < 1$, since in that case, the estimates obtained in that lemma show that $R(t)$ and $K(t)$ cannot blow up in finite time,
- a much more delicate issue in the case $d \leq 3$, and $\alpha = d - 1$, finally solved in [42], [55] and [51]. Their proofs may also be extended to the less singular case $\alpha < d - 1$.

3.3. A short sketch of the proofs

Here we only present "almost correct" ideas, and refer to the proof for fully correct statements. We put the emphasis on the novelty with respect to our previous work [34]. We concentrate mostly on the proof of Theorem 1: the proof of Theorem 3 is very similar and simpler, and we say only a few words about the propagation of chaos at the end.

We use the notation introduced in Section 3.1 and two quantities defined below. Recalling the real $r \in (0, r^*)$ introduced in assumption iii) of Theorem 1, we define:

$$\tilde{d}_N(t) := \varepsilon^{-(1+r)}\, d_N(t), \qquad \tilde{W}_\infty(t) := \varepsilon^{-1}\, W_\infty(t).$$

The assumption iii) of Theorem 1 implies that $\tilde{d}_N$ is initially larger than 1, and by Proposition 1, $\tilde{W}_\infty$ is initially equal to a constant $c_\phi > 0$ depending only on the cut-off ϕ used in the construction of the blob-approximation.

As mentioned above, the Vlasov Equation (1.5) is satisfied by the empirical distribution μ_N of the interacting particle system provided that $F(0)$ is set to 0. Hence the problem of convergence can be reformulated into a problem of stability of the empirical measures $\mu_N(t)$—seen initially as measure valued perturbations of the smooth profile f^0—but around the smooth solution $f(t)$ of the Vlasov equation. The proof of the two mean-field limit results uses two ingredients to obtain this stability, which is given in term of an explicit control on $W_1\big(\mu_N(t), f(t)\big)$

- A standard stability estimate (See Proposition 3) for solution of the Vlasov-Poisson Equation (1.5), (with the 1 Monge-Kantorovitch-Wasserstein distance W_1):

$$W_1\big(f_N(t), f(t)\big) \leq e^{Ct}\, W_1\big(f_N^0, f^0\big), \quad C := \sup_{s \leq t}\big(\|\rho_f(s)\|_\infty + \|\rho_{f_N}(s)\|_\infty\big).$$

- A control on $W_\infty(t) = W_\infty\big(\mu_N(t), f_N(t)\big)$ (remark that we always have the inequality $W_1\big(\mu_N(t), f_N(t)\big) \leq W_\infty(t)$).

Once this is achieved, we get a quantitative control on the rate of convergence. This is an important improvement with respect to [34], where we used a compactness argument to prove the convergence and did not get any convergence rate. We emphasize that the use of the infinite MKW distance is important. We were not able to perform our calculations with other MKW distances of order $p < +\infty$ as the infinite distance is the only MKW distance with which we can handle a localized singularity in the force and Dirac masses in the empirical distribution.

The control on $W_\infty(t)$ requires to estimate the difference between the force terms acting in the two systems (the particle system and the continuous distribution f_N). Precisely, we need to compare averages on short time intervals of length ε of the forces:

$$\tilde{E}_N(t,i) = \frac{1}{N}\sum_j \int_{t-\varepsilon}^t F\big(X_i(s) - X_j(s)\big)\,ds,$$

$$\tilde{E}_\infty(t,z) = \int_{\mathbb{R}^d}\int_{t-\varepsilon}^t F(x_s - y) f(s,y,w)\,dydw\,ds,$$

when $Z_i = (X_i, V_i)$ and $z = (x, v)$ are close (x_s denotes the position at time s of the point starting at (t, z) when following the characteristics defined by f_N). For this comparison, it is necessary to distinguish the contributions of three domains:

- Contribution of particles j (and point y) far enough from X_i and x in the physical space. This is the simplest case as one does not see the discrete nature of the problem at that level. The estimates need to be adapted to the W_∞ distance used here but are otherwise very similar in spirit to the continuous problem or other previous works for mean field limits.
- Contribution of particles j (and point y) ε-close in the physical space $\mathbb{R}^d$ to X_i and x, but with sufficiently different velocities. It corresponds to a domain of volume of order ε^d, but where the force is singular. Here we start to see the discrete level of the problem and in fact we cannot compare anymore the discrete and continuous forces: Instead we just show that both are small. The continuous force term is handled easily, but the discrete force term requires more work: the short average in time is really required to get rid of possible singularities.

  Precisely, consider a second particle $j \neq i$, and neglect the variation of velocities on $[t-\varepsilon, t]$. Because of (1.2), with $\alpha < 1$, we have

  $$\int_{t-\varepsilon}^t |F(X_i(s) - X_j(s))|\,ds \sim \int_{t-\varepsilon}^t \frac{ds}{|\delta + (s-s_0)(V_i - V_j)|^\alpha} \lesssim \frac{\varepsilon^{1-\alpha}}{|V_i - V_j|^{-\alpha}}$$

  where δ is the minimum distance between the two particles on the time interval $[t-\varepsilon, t]$, which is reached at time s_0. The full contribution is obtained after a careful summation on all the particles j of the domain.

  There is here a major improvement with respect to [34]. In this previous work bounding the number of particles in that domain was straightforward, since we assumed that $\varepsilon \leq d_N$ (that bound was propagated in time) so that particles were mostly equidistributed at scale ε. Instead here, we use the L^∞ bound on f_N and the W_∞ distance to obtain a control of the contribution of all these particles, which is more delicate.
- Contribution of particles ε-close in $\mathbb{R}^{2d}$, i.e., in position and velocity. This a very small domain, of volume of order ε^{2d}, but it contains particles that are close in physical space and are likely to remain close for a rather long time (small relative velocity).

  Again, there is a major improvement with respect to [34], as this case was relatively simple there: under our restrictive assumption on d_N that last domain contained only a bounded number of particle. Here the lower bound on d_N is much smaller, of order ε^{1+r}. It is even surprising that it is possible to control d_N at a scale which is much lower than the natural discrete scale of the problem. The key to this new control is due to the fact that the ODE system is second order so that the trajectories (in position space) can be approximated by straight lines up to second order in time, thanks to a discrete Lipschitz estimate on $\tilde{E}_N$. Using this idea, careful estimates allow to control the influence of one single particle. Then, the number of particles in the domain is bounded, again with the help of $\|f_N\|_\infty$ and W_∞.

All of this leads to the following estimate

$$\frac{\tilde{W}_\infty(t) - \tilde{W}_\infty(t-\varepsilon)}{\varepsilon} \leq C\Big(\tilde{W}_\infty(t) + \varepsilon^{\beta_1}\tilde{W}^d_\infty(t) + \varepsilon^{\beta_2}\tilde{W}^{2d}_\infty(t)\tilde{d}_N(t)^{-\alpha}\Big),$$

where $\beta_1, \beta_2 > 0$ under the assumptions of Theorem 1. The three terms of the r.h.s. come respectively from the three domains mentioned above. We complete the proof with an inequality on $\tilde{d}_N(t)$ obtained in a similar way ($\beta_3, \beta_4 > 0$)

$$\frac{\tilde{d}_N(t) - \tilde{d}_N(t-\varepsilon)}{\varepsilon} \geq -C\Big(\tilde{d}_N(t) + \varepsilon^{\beta_3}\tilde{W}^d_\infty(t) + \varepsilon^{\beta_4}\tilde{W}^{2d}_\infty(t)\tilde{d}_N(t)^{-\alpha}\Big).$$

The two previous inequalities form an (implicit) time discretization of a system of two differential inequalities. As the non-linear terms come with small weight ε^{β_i}, the previous system provides uniform bounds until a critical time T_ε with $T_\varepsilon \to \infty$ as $\varepsilon \to 0$; hence for any fixed T, $T_\varepsilon > T$ for N large enough (depending on T).

About the restriction $\alpha < 1$. – This restriction is clearly manifested when two particles with non vanishing relative velocity become relatively close. The physical explanation is the following: if $\alpha < 1$ the deviation in velocity due to a collision (another particle coming very close) with a sufficiently large relative velocity cannot be too large: for instance, two particles with sufficiently large relative velocity will never bounce back even if they exactly collide at some time. So we do not expect any fast variation in the velocities of the particles and our analysis shows that we can rigorously prove that claim. This is why it is enough to control the distance in $\mathbb{R}^{2d}$ between particles: the only "bad events" are the collisions with very small relative velocities. In contrast when $\alpha > 1$, a particle coming very close to another one can change its velocity over a very short time interval (even if their relative velocity remains of order 1): for instance the two particles can bounce back. Such "collisions" are incompatible with our argument since a control on the distance between particles in position-velocities does not allow to prevent them. When such an event happens, we are not able to control the trajectories of the particles involved after that. Even if it happens only once, we cannot adapt our strategy of proof, which requires a control on W_∞, i.e., a control on all the trajectories.

The propagation of chaos results. – To deduce Theorem 2 from Theorem 1, it is enough to show that the conditions i) and iii) under which our mean-field limit theorem is valid, are satisfied with large probability in the limit. This relies on already known results or on rather simple statistical estimates:

– for point i), it relies on a large deviation bound for $\|f_N\|_\infty$, see Proposition 8,
– for point iii) it relies on a simple estimate (not of large deviation type) on $d_N(0)$ proved in [32], see Proposition 5,
– and finally, we use also some large deviation bound on $W_1(\mu^0_N, f^0)$ obtained by Boissard [8], see Proposition 6.

4. Proof of Theorems 1 and 3

4.1. Definition of the transport

We now try to compare the dynamics of μ_N and f_N, which both have a compact support. For that, we choose an optimal transport T^0 (of course depending on N) from f_N^0 to μ_N^0 for the infinite MKW distance. The existence of such a transport is ensured by [15]. T^0 is defined on the support of f_N^0, which is included in $B_{2d}(0, R^0)$ (the size of the support in position and velocity), and Proposition 1 implies that $W_\infty(f_N^0, \mu_N^0) \le \varepsilon$.

Thanks to the assumptions of both theorems, the strong solution f_N to the Vlasov equation is well defined till a time T^*, infinite in the case of Theorem 1, that depends only on C_∞ and R_0 and not on N. Since we are dealing with strong solutions, there exists a well-defined underlying flow, that we will denote by $Z^f = (X^f, V^f) : Z^f(t,s,z)$ being the position-velocity at time t of a particle with position-velocity z at time s.

Moreover, by the assumption of Theorem 1 or because we use a cut-off in Theorem 3, the dynamic of the N particles is well defined, and we can also write in that case a flow $Z^\mu = (X^\mu, V^\mu)$, which is well defined at least at the position and velocity of the particles we are considering. A simple way to get a transport of $f_N(t)$ on $\mu_N(t)$ is to transport along the flows the map T^0, i.e., to define

$$T^t = Z^\mu(t,0) \circ T^0 \circ Z^f(0,t), \qquad \text{and} \quad T^t = (T_x^t, T_v^t).$$

We use the following notation, for a test-"particle" of the continuous system with position-velocity $z_t = (x_t, v_t)$ at time t, $z_s = (x_s, v_s)$ will be its position and velocity at time s for $s \in [t-\tau, t]$. Precisely

$$z_s = Z^f(s,t,z_t).$$

Since f_N is the solution of a transport equation, we have $f_N(t,z_t) = f_N(s,z_s)$. And since the vector-field of that transport equation is divergence free, the flow Z^f is measure-preserving in the sense that for all smooth test functions Φ

$$\int \Phi(z)\, f_N(s,z)\, dz = \int \Phi(Z^f(s,t,z))\, f_N(t,z)\, dz = \int \Phi(z_s)\, f_N(t,z_t)\, dz_t.$$

That property is standard for smooth vector fields but also holds here with a vector-field in $W^{1,1}$, see the lecture notes of Ambrosio [2]. Finally, let us remark that the f_N are solutions to the (continuous) Vlasov equations with an initial L^∞ norm and support that are uniformly bounded in N. Therefore the Proposition 2, and in particular the last assertion in it imply that this remains true uniformly in N for any finite time $T < T^*$. In particular the uniform bound on the whole support $R(T)$ (in position-velocity) implies since $\alpha < d-1$ the existence of a constant C independent of N such that for any $t \in [0,\ T]$

$$(4.1)\qquad \begin{aligned} &\|f_N(t,.,.)\|_\infty \le C, \qquad \|f_N(t,.,.)\|_{L^1} = 1,\\ &\operatorname{supp} f_N(t,.,.) \in B_{2d}\big(0, R(t)\big), \quad \text{with } R(t) \le C,\\ &|E_{f_N}|_\infty(t) := \|E_{f_N}(t,\cdot)\|_\infty \le \sup_x \int |F(x-y)|\, f_N(t,y,w)\, dy\, dw \le C,\\ &|\nabla E_{f_N}(t,x)| \le \int |\nabla F(x-y)|\, f_N(t,y,w)\, dy\, dw \le C, \end{aligned}$$

where E_{f_N} stands for the force-field generated by f_N: $E_{f_N}(t,x) = \int F(x-y) f_N(t,y,w)\, dydw$.

In what follows, the final time T is fixed and independent of N. For simplicity, C will denote a generic universal constant, which may actually depend on T, the size of the initial support, the infinite norms of the $f_N\ldots$ But those constants are always independent of N as in (4.1).

4.2. The quantities to control

We will not be able to control the infinite norm of the field (and its derivative) created by the empirical distribution μ_N, but only a small time average of this norm. For this, we introduce in the case without cut-off a small time step $\tau = \varepsilon^{r'}$ for some $r' > r$ and close to r (the precise condition will appear later). In the case with cut-off where r and r' are useless, the time step will by $\tau = \varepsilon$.

Before going on, we define some important quantities :

– The MKW infinite distance between $\mu_N(t)$ and $f(t)$.

We wish to bound the infinite Wasserstein distance $W_\infty(\mu_N(t), f_N(t))$ between the empirical measure μ_N associated to the N particle system (1.1), and the solution f_N of the Vlasov Equation (1.5) with blobs as initial condition. But for convenience we will work instead with the quantity

$$W_\infty(t) := \sup_{s\le t} \sup_{z\in \mathrm{supp}\, f_N(s)} |T^s(z) - (z)|, \tag{4.2}$$

where the sup on z should be understood precisely as a essential supremum with respect to the measure $f_N(s)$. This is not exactly the infinite Wasserstein distance between $\mu_N(t)$ and $f_N(t)$ (or its supremum in times smaller than $s \le t$). But, since for all s, the transport map T^s sends the measure $f_N(s)$ onto $\mu_N(s)$ by construction, we always have

$$W_\infty(\mu_N(t), f_N(t)) \le \sup_{s\le t} W_\infty(\mu_N(t), f_N(t)) \le W_\infty(t).$$

So that a control on $W_\infty(t)$ implies a control on $W_\infty(\mu_N(t), f_N(t))$. It is in fact a little stronger, since it means that rearrangements in the transport are not necessary to keep the infinite MKW distance bounded. We introduce the supremum in time for technical reasons as it will be simpler to deal with a non-decreasing quantity in the sequel.

– The support of μ_N.

We also need a uniform control on the support in position and velocity of the empirical distributions :

$$R_N(t) := \sup_{s\le t} \max_i |(X_i(t),\ V_i(t))|. \tag{4.3}$$

– The infinite norm $|\nabla^N E|_\infty$ of the time averaged discrete derivative of the force field.

We define a version of the infinite norm of the averaged derivative of the discrete force field E_N

$$|\nabla^N E|_\infty(t) := \sup_{i\ne j} \frac{1}{\tau} \int_{t-\tau}^{t} \frac{|E_N(X_i(s)) - E_N(X_j(s))|\, ds}{|X_i(s) - X_j(s)| + \varepsilon^{(1+r')}}\, ds. \tag{4.4}$$

For $|\nabla^N E|_\infty$, we use the convention that when the interval of integration contains 0 (for $t < \tau$), the integrand is null on the right side for negative times. Remark that the control on $|\nabla^N E|_\infty$ is useless in the cut-off case.

- The minimal distance in $\mathbb{R}^{2d}$, d_N, which was already defined in the Equation (3.1) in the Section 3.
- Two useful integrals $I_\alpha(t,\bar z_t,z_t)$ and $J_{\alpha+1}(t,\bar z_t,z_t)$.

  Finally for any two test trajectories $z_t=(x_t,v_t)$ and $\bar z_t=(\bar x_t,\bar v_t)$, we define

$$I_\alpha(t,\bar z_t,z_t) := \frac{1}{\tau}\int_{t-\tau}^{t} |F(T_x^s(\bar z_s)-T_x^s(z_s)) - F(\bar x_s - x_s)|\,ds, \tag{4.5}$$

  which controls the difference of the two force fields at two points related by the "optimal" transport. We recall that we use here the convention $F(0)=0$, in order to avoid self-interaction. It is important here since we have $T^s(z_s)=T^s(\bar z_s)$ for all $s\in[t-\tau,t]$, for a set of $(z_s,\bar z_s)$ of positive measure (those who are associated to the same particle (X_i,V_i)).

  Defining a second kernel as

$$K_\varepsilon := \min\left(\frac{1}{|x|^{1+\alpha}},\ \frac{1}{\varepsilon^{1+r'}\,|x|^{\alpha}}\right) \quad \text{for } x\neq 0, \quad \text{and } K_\varepsilon(0)=0, \tag{4.6}$$

  we introduce a second useful quantity

$$\begin{aligned} J_{\alpha+1}(t,\bar z_t,z_t) &:= \frac{1}{\tau}\int_{t-\tau}^{t} K_\varepsilon\big(T_x^s(\bar z_s)-T_x^s(z_s)\big)\,ds \\ &= \frac{1}{\tau}\int_{t-\tau}^{t} K_\varepsilon\big(X_i(s)-X_j(s)\big)\,ds, \end{aligned} \tag{4.7}$$

  if i and j are the indices such that $Z_i(t)=T^t(\bar z_t)$ and $Z_j(t)=T^t(z_t)$. $J_{\alpha+1}$ will be useful to control the discrete derivative of the field $|\nabla^N E|_\infty(t)$, and is thus useless in the cut-off case.

All previous quantities are relatively easily bounded by I_α and $J_{\alpha+1}$. Those last two will not be bounded by direct calculation on the discrete system, but we will compare them to similar ones for the continuous system, paying for that in terms of the distance between $\mu_N(t)$ and $f(t)$. That strategy is interesting because the integrals are easier to manipulate than the discrete sums.

REMARK 11. – *Before stating the next proposition, let us mention that we also define for $t<0$, $W(t)=W(0)$ and $d_N(t)=d_N(0)$. This is just a helpful convention. With it the estimates of the next proposition are valid for any $t\geq 0$, and this will be very convenient in the conclusion of the proof of our main theorem. Remark also that $|\nabla^N E|_\infty(0)=0$.*

We summarize the first easy bounds in the following

PROPOSITION 4. – *Under the assumptions of Theorem 1, one has for some constant C uniform in N that, for all $t \geq 0$*

(i) $$R_N(t) \leq W_\infty(t) + R(t) \leq W_\infty(t) + C,$$

(ii) $$W_\infty(t) \leq W_\infty(t-\tau) + \tau W_\infty(t) + C\,\tau \sup_{\bar{z}_t} \int_{|z_t| \leq R(t)} I_\alpha(t, \bar{z}_t, z_t)\, dz_t,$$

(iii) $$|\nabla^N E|_\infty(t) \leq C \sup_{\bar{z}_t} \int_{|z_t| \leq R(t)} J_{\alpha+1}(t, \bar{z}_t, z_t)\, dz_t,$$

(iv) $$d_N(t) + \varepsilon^{1+r'} \geq [d_N(t-\tau) + \varepsilon^{1+r'}]e^{-\tau(1+|\nabla^N E|_\infty(t))}.$$

The points i) *and* ii) *are also satisfied under the assumptions of 3.*

Note that the control on $R_N(t)$ is simple enough that it will actually be used implicitly in the rest many times. In that proposition the crucial estimates are the ii) and iii). Remark also that in the case of very singular interaction force ($\alpha \geq 1$) with cut-off—in short (S^α_m) conditions (3.3)—the control on minimal distance d_N and therefore the control on $|\nabla^N E|_\infty$ are useless, so that the only interesting inequality is the second one.

4.3. Proof of Proposition 4

Step 1. Let us start with (i). Simply write

$$R^N(t) = \sup_{s \leq t} \sup_{z \in \operatorname{supp} f_N(s,\cdot)} |T^s(z)| \leq \sup_{s \leq t} \sup_{z \in \operatorname{supp} f_N(s,\cdot)} |T^s(z) - z| + \sup_{s \leq t} \sup_{z \in \operatorname{supp} f_N(s,\cdot)} |z|,$$

so indeed by the bound (4.1) and the Definition (4.2) of W_∞

$$R^N(t) \leq W_\infty(t) + R(t) \leq W_\infty(t) + C.$$

Step 2. For (ii), for any time $t' \in [t-\tau, t]$ we have

$$\begin{aligned} |T^{t'}_x(\bar{z}_{t'}) - \bar{x}_{t'}| &\leq |T^{t-\tau}_x(\bar{z}_{t-\tau}) - \bar{x}_{t-\tau}| + \int_{t-\tau}^{t'} |T^s_v(\bar{z}_s) - \bar{v}_s|\, ds \\ &\leq |T^{t-\tau}_x(\bar{z}_{t-\tau}) - \bar{x}_{t-\tau}| + \tau W_\infty(t), \end{aligned} \tag{4.8}$$

and for the speeds

$$\begin{aligned} |T^{t'}_v(\bar{z}_{t'}) - \bar{v}_{t'}| &\leq |T^{t-\tau}_v(\bar{z}_{t-\tau}) - \bar{v}_{t-\tau}| \\ &\quad + \int_{t-\tau}^{t'} \int |F(T^s_x(\bar{z}_s) - T^s(z_s)) - F(\bar{x}_s - x_s)| f_N(s, z_s)\, dz_s ds \\ &\leq |T^{t-\tau}_v(\bar{z}_{t-\tau}) - \bar{v}_{t-\tau}| + \int_{t-\tau}^{t} \int |F(T^s_x(\bar{z}_s) - T^s(z_s)) - F(\bar{x}_s - x_s)| f_N(t, z_t)\, dz_t ds. \end{aligned}$$

where we used the fact that the change of variable $z_t \mapsto z_s$ preserves the measure. Since $f_N(t)$ is uniformly bounded in L^∞ and compactly supported in $B_{2d}(0, R(t))$, one gets by the Definition (4.5) of I_α

$$|T^{t'}_v(\bar{z}_{t'}) - \bar{v}_{t'}| \leq |T^{t-\tau}_v(\bar{z}_{t-\tau}) - \bar{v}_{t-\tau}| + C\tau \sup_{\bar{z}_t} \int_{|z_t| \leq R(t)} I_\alpha(t, \bar{z}_t, z_t)\, dz_t. \tag{4.9}$$

Summing the two estimates (4.8) and (4.9), we get for the Euclidean distance on $\mathbb{R}^{2d}$

$$|T^{t'}(\bar z_{t'}) - \bar z_{t'}| \le |T^{t-\tau}(\bar z_{t-\tau}) - \bar z_{t-\tau}| + \tau W_\infty(t) + C\tau \sup_{\bar z_t} \int_{|z_t|\le R(t)} I_\alpha(t,\bar z_t, z_t)\, dz_t.$$

Taking the supremum over all $\bar z_{t'}$ in the support of $f_N(t')$, and then the supremum over all $t' \in [t-\tau, t]$ we get

$$W_\infty(t) \le W_\infty(t-\tau) + \tau W_\infty(t) + C\tau \sup_{\bar z_t} \int_{|z_t|\le R(t)} I_\alpha(t,\bar z_t, z_t)\, dz_t$$

which is exactly (ii).

Step 3. Concerning $|\nabla^N E|_\infty(t)$ in (iii), note that

$$\int_{t-\tau}^t \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}} = \frac{1}{N}\sum_{k\neq i,j} \int_{t-\tau}^t \frac{|F(X_i(s)-X_k(s)) - F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}}\, ds$$
$$+ \frac{1}{N}\int_{t-\tau}^t \frac{|F(X_i(s)-X_j(s)) - F(X_j(s)-X_i(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}}\, ds.$$

By the assumption (1.2), one has that

$$|F(x)-F(y)| \le C\left(\frac{1}{|x|^{\alpha+1}} + \frac{1}{|y|^{\alpha+1}}\right)|x-y|.$$

So

$$\frac{|F(X_i(s)-X_k(s)) - F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}} \le \frac{C}{|X_i(s)-X_k(s)|^{1+\alpha}} + \frac{C}{|X_j(s)-X_k(s)|^{1+\alpha}},$$

and that bound is also true for the remaining term where $k=i$ or j, if we delete the undefined term in the sum. One also obviously has, still by (1.2)

$$\frac{|F(X_i(s)-X_k(s)) - F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}}$$
$$\le \frac{C}{\varepsilon^{1+r'}|X_i(s)-X_k(s)|^{\alpha}} + \frac{C}{\varepsilon^{1+r'}|X_j(s)-X_k(s)|^{\alpha}}.$$

Therefore by the definition of K_ε

$$\frac{|F(X_i(s)-X_k(s)) - F(X_j(s)-X_k(s))|}{|X_i(s)-X_j(s)| + \varepsilon^{1+r'}} \le C\big[K_\varepsilon(X_i(s)-X_k(s)) + K_\varepsilon(X_j(s)-X_k(s))\big].$$

Summing up, this implies that

$$|\nabla^N E|_\infty(t) \le C \max_{i\neq j}\Big(\frac{1}{\tau}\int_{t-\tau}^t \frac{1}{N}\sum_{k\neq i} K_\varepsilon(X_i(s)-X_k(s))\, ds$$
$$+ \frac{1}{\tau}\int_{t-\tau}^t \frac{1}{N}\sum_{k\neq j} K_\varepsilon(X_j(s)-X_k(s)) ds\Big).$$

Transforming the sum into integral thanks to the transport, we get exactly the bound (iii) involving $J_{\alpha+1}$.

Step 4. Finally for $d_N(t)$, consider any $i \neq j$, differentiating the Euclidean distance $|Z_i - Z_j|$, we get

$$\frac{d}{ds}\big|(X_i(s) - X_j(s),\ V_i(s) - V_j(s))\big| \geq -|V_i(s) - V_j(s)| - \big|E_N(X_i(s)) - E_N(X_j(s))\big|.$$

Simply write

$$|E_N(X_i(s)) - E_N(X_j(s))| \leq \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}}\,(|X_i(s) - X_j(s)| + \varepsilon^{1+r'})$$

to obtain that

$$\frac{d}{ds}\big|(X_i(s) - X_j(s),\ V_i(s) - V_j(s))\big| \geq -\left(1 + \frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}}\right)$$
$$\big(|(X_i(s) - X_j(s),\ V_i(s) - V_j(s))| + \varepsilon^{1+r'}\big).$$

Integrating this inequality and taking the minimum, we get

$$\begin{aligned} d_N(t) + \varepsilon^{1+r'} &\geq (d_N(t-\tau) + \varepsilon^{1+r'})\inf_{i\neq j}\exp\left(-\tau - \int_{t-\tau}^{t}\frac{|E_N(X_i(s)) - E_N(X_j(s))|}{|X_i(s) - X_j(s)| + \varepsilon^{1+r'}}\,ds\right) \\ &\geq [d_N(t-\tau) + \varepsilon^{1+r'}]\exp^{-\tau(1+|\nabla^N E|_\infty(t))}. \end{aligned}$$

4.4. The bounds for I_α and $J_{\alpha+1}$

To close the system of inequalities in Proposition 4, it remains to bound the two integrals involving I_α and J_α. It is done with the following lemmas

LEMMA 4. – *Assume that F satisfies an (S^α)-condition* (1.2) *with $\alpha < 1$, and that τ is small enough such that for some constant C (made precise in the proof)*

$$C\,\tau\,(1 + |\nabla^N E|_\infty(t))\,(W_\infty(t) + \tau) \leq d_N(t). \tag{4.10}$$

Then one has the following bounds, uniform in $\bar z_t$

$$\int_{|z_t|\leq R(t)} I_\alpha(t, \bar z_t, z_t)\,dz_t \leq C\left[W_\infty(t) + (W_\infty(t)+\tau)^d\tau^{-\alpha} + (W_\infty(t)+\tau)^{2d}(d_N(t))^{-\alpha}\tau^{-\alpha}\right].$$

$$\begin{aligned} \int_{|z_t \leq R(t)} J_{\alpha+1}(t, \bar z_t, z_t)\,dz_t \leq C\,\big(&1 + (W_\infty(t)+\tau)^d\varepsilon^{-(1+r')}\,\tau^{-\alpha} \\ &+ (W_\infty(t)+\tau)^{2d}\varepsilon^{-(1+r')}\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}\big). \end{aligned}$$

In the cut-off case where the interaction force satisfies an (S_m^α) condition (3.3), we only need to bound the integral of I_α, with the result

LEMMA 5. – *Assume that $1 \leq \alpha < d-1$, and that F satisfies an (S_m^α) condition* (3.3)*. Then one has the following bound, uniform in $\bar z_t$*

$$\begin{aligned} &\int_{|z_t|\leq R(t)} I_\alpha(t, \bar z_t, z_t)\,dz_t \\ &\qquad \leq C\,\big(W_\infty(t) + (W_\infty(t)+\tau)^d\tau^{-1}\varepsilon^{\bar m(1-\alpha)} + (W_\infty(t)+\tau)^{2d}\varepsilon^{-\bar m\alpha}\big). \end{aligned} \tag{4.11}$$

with the convention[(1)] *(if* $\alpha = 1$*) that* $\varepsilon^0 = 1 + |\ln \varepsilon|$.

The proofs with or without cut-off follow the same line and we will prove the above lemmas at the same time. We begin by an explanation of the sketch of the proof, and then perform the technical calculation.

4.4.1. *Rough sketch of the proof.* – The point $\bar z_t = (\bar x_t, \bar v_t)$ is considered fixed through all this subsection (as the integration is carried over $z_t = (x_t, v_t)$). Accordingly we decompose the integration in z_t over several domains. First

$$A_t = \{ z_t \mid |\bar x_t - x_t| \geq 4W_\infty(t) + 2\tau(|\bar v_t - v_t| + \tau |E|_\infty(t)) \}. \tag{4.12}$$

This set consists of points z_t such that x_s and $T^s_x(z_s)$ are sufficiently far away from $\bar x_s$ on the whole interval $[t-\tau, t]$, so that they will not see the singularity of the force. The bound over this domain will be obtained using traditional estimates for convolutions.

Next, one part of the integral can be estimated easily on A^c_t (the part corresponding to the flow of the regular solution f_N to the Vlasov equation). For the other part it is necessary to decompose further. The next domain is

$$B_t = A^c_t \cap \{ z_t \mid |\bar v_t - v_t| \geq 4\, W_\infty(t) + 4\, \tau |E|_\infty |(t) \}. \tag{4.13}$$

This contains all particles z_t that are close to $\bar z_t$ in position (i.e., x_t close to $\bar x_t$), but with enough relative velocity not to interact too much. The small average in time will be useful in that part, as the two particles remain close only a small amount of time.

The last part is of course the remainder

$$C_t = (A_t \cup B_t)^c. \tag{4.14}$$

This is a small set, but where the particles remain close together a relatively long time. Here, we are forced to deal with the corresponding term at the discrete level of the particles. This is the only term which requires the minimal distance in $\mathbb{R}^{2d}$; and the only term for which we need a time step τ small enough as per the assumption in Lemma 4.

4.4.2. *Step 1: Estimate over* A_t. – According to the Definition (4.12), if $z_t \in A_t$, we have for $s \in [t-\tau, t]$

$$|\bar x_s - x_s| \geq |\bar x_t - x_t| - (t-s)|\bar v_t - v_t| - (t-s)^2 |E|_\infty(t) \geq \frac{|\bar x_t - x_t|}{2} \tag{4.15}$$

$$|T^s_x(\bar z_s) - T^s_x(z_s)| \geq |\bar x_s - x_s| - 2W_\infty(s) \geq \frac{|\bar x_t - x_t|}{2}. \tag{4.16}$$

For I_α, we use the direct bound for $z_t \in A_t$

$$\begin{aligned} \left| F(T^s_x(\bar z_s) - T^s_x(z_s)) - F(\bar x_s - x_s) \right| &\leq \frac{C}{|\bar x_t - x_t|^{1+\alpha}} (|T^s_x(\bar z_s) - \bar x_s| + |T^s_x(z_s) - x_s|) \\ &\leq \frac{C}{|\bar x_t - x_t|^{1+\alpha}} W_\infty(s) \leq \frac{C}{|\bar x_t - x_t|^{1+\alpha}} W_\infty(t), \end{aligned}$$

(1) That convention may be justified by the fact that it implies a very simple algebra $(x^{1-\alpha})' \approx x^{-\alpha}$ even if $\alpha = 1$. It allows us to give a unique formula rather than three different cases.

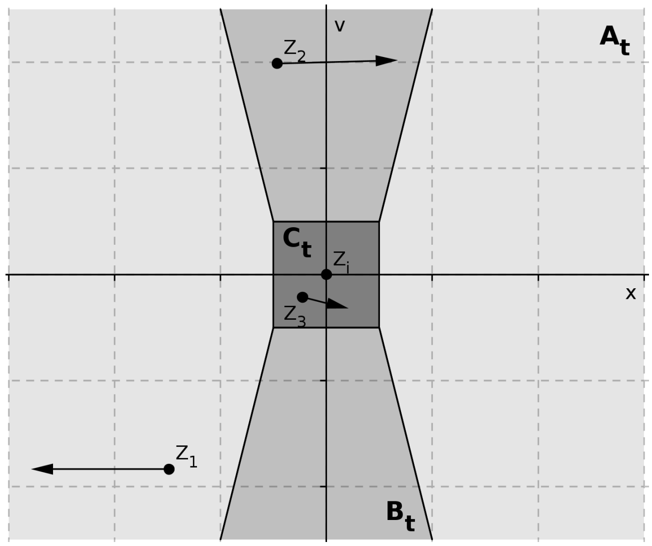

FIGURE 1. The partition of $\mathbb{R}^{2d}$.

and obtain by integration on $[t-\tau, t]$

$$I_\alpha(t, \bar{z}_t, z_t) \le \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}} W_\infty(t).$$

Then integrating in z_t we may get since $\alpha + 1 < d$

$$\begin{aligned}\int_{A_t} I_\alpha(t, \bar{z}_t, z_t)\, dz_t &\le C W_\infty(t) \int_{A_t} \frac{dz_t}{|\bar{x}_t - x_t|^{1+\alpha}} \\ &\le C\, R(t)^{2d-1-\alpha}\, W_\infty(t) \le C\, W_\infty(t).\end{aligned} \tag{4.17}$$

For $J_{\alpha+1}$, we use (4.16) on the set A_t the bound

$$|K_\varepsilon(T^s_x(\bar{z}_s) - T^s_x(z_s))| \le \frac{C}{|\bar{x}_t - x_t|^{1+\alpha}}.$$

Integrating with respect to time and z_t we get since $1 + \alpha < d$.

$$\begin{aligned}\int_{A_t} J_{\alpha+1}(t, \bar{z}_t, z_t)\, dz_t &\le C \int_{A_t} \frac{dz_t}{|\bar{x}_t - x_t|^{1+\alpha}} \\ &\le C\, R(t)^{2d-1-\alpha} \le C.\end{aligned} \tag{4.18}$$

For the cut-off case, the estimation on I_α for this step is unchanged.

4.4.3. *Step 1′ : Estimate over A_t^c for the "continuous" part of I_α.* – For the remaining term in I_α, we use the rude bound

$$|F(T^s_x(\bar{z}_s) - T^s_x(z_s)) - F(\bar{x}_s - x_s)| \le |F(T^s_x(\bar{z}_s) - T^s_x(z_s))| + |F(\bar{x}_s - x_s)|.$$

The term involving T^s is complicated and requires an additional decomposition. It will be treated in the next sections. The other term is simply bounded by

$$\int_{z_t\in A_t^c}\frac{1}{\tau}\int_{t-\tau}^{t}|F(\bar{x}_s-x_s)|ds\ dz_t \le \frac{1}{\tau}\int_{t-\tau}^{t}\int_{z_t\in A_t^c}\frac{C\,dz_t}{|\bar{x}_s-x_s|^\alpha}\,ds$$
$$\le \frac{1}{\tau}\int_{t-\tau}^{t}\int_{z_s\in Z^f(s,t,A_t^c)}\frac{C\,dz_s}{|\bar{x}_s-x_s|^\alpha}\,ds.$$

From the bounds (4.1), we get that

$$|A_t^c|\le CR(t)^d[W_\infty(t)+\tau(1+|E|_\infty(t))]^d\le C(W_\infty(t)+\tau)^d,$$

where $|\cdot|$ denotes the Lebesgue measure. Since the flow Z_f is measure-preserving, the measure of the set $Z^f(s,t,A_t^c)$ satisfies the same bound. This set is also included in $B_{2d}(0,R)$. We will also need the following lemma

LEMMA 6. – *Let $\Omega\subset B_{2d}(0,R)\subset\mathbb{R}^{2d}$. Then for any $a<d$, there exists a constant C_a depending on a and d such that*

$$\int_\Omega\frac{dz}{|x|^a}\le C_a R^a\,|\Omega|^{1-a/d}.$$

Proof of Lemma 6. – We maximize the integral

$$\int_\omega |x|^{-a}dz$$

over all sets $\omega\subset\mathbb{R}^{2d}$ satisfying $\omega\subset B_{2d}(0,R)$ and $|\omega|=|\Omega|$. It is clear that the maximum is obtained by concentrating as much as possible ω near $x=0$, i.e., with a cylinder of the form $B_d(0,r)\times B_d(0,R)$. Since $|\omega|=|\Omega|$ we have $(c_d)^2r^dR^d=|\Omega|$, where c_d is the volume of the unit ball of dimension d. The integral over this cylinder can now be computed explicitly and gives the lemma. □

Applying Lemma 6, we get

$$\int_{z_t\in A_t^c}\frac{1}{\tau}\int_{t-\tau}^{t}|F(\bar{x}_s-x_s)|\,dz_t ds\le C[W_\infty(t)+\tau]^{d-\alpha}. \tag{4.19}$$

That term does not appear in Lemma 4 since it is strictly smaller than the bound of the remaining term (involving the transport T), as we will see in the next section.

For the cut-off case, the same bound is valid for I_α since $\alpha\le d-1<d$ (the cut-off cannot in fact help to provide a better bound for this term).

At this point, the remaining term to bound in the integral involving I_α is only

$$\int_{z_t\in A_t^c}\frac{1}{\tau}\int_{t-\tau}^{t}|F(T_x^s(\bar{z}_s)-T_x^s(z_s))|\,ds \tag{4.20}$$

and the remainder in $J_{\alpha+1}$ is

$$\int_{A_t^c}J_{\alpha+1}(t,\bar{z}_t,z_t)\,dz_t=\frac{1}{\tau}\int_{A_t^c}\int_{t-\tau}^{t}K_\varepsilon(T_x^s(\bar{z}_s)-T_x^s(z_s))\,dz_t ds. \tag{4.21}$$

Therefore in the next sections we focus on giving a bound for (4.20) and (4.21).

4.4.4. *Step 2: Estimate over B_t.* – We recall the definition of B_t

$$B_t = \left\{ z_t \ \text{ such that } \begin{array}{l} |\bar{x}_t - x_t| \leq 4W_\infty(t) + 2\tau(|\bar{v}_t - v_t| + \tau|E|_\infty(t)) \\ |\bar{v}_t - v_t| \geq 4\,W_\infty(t) + 4\,\tau|E|_\infty|(t) \end{array} \right\}.$$

If $z_t \in B_t$, we have for $s \in [t-\tau, t]$

$$|\bar{v}_s - v_s - \bar{v}_t + v_t| \leq 2\tau|E|_\infty(t) \leq \frac{|\bar{v}_t - v_t|}{2}, \tag{4.22}$$

$$|T_v^s(\bar{z}_s) - T_v^s(z_s) - \bar{v}_t + v_t| \leq |\bar{v}_s - v_s - \bar{v}_t + v_t| + 2W_\infty(s) \leq \frac{|\bar{v}_t - v_t|}{2}. \tag{4.23}$$

This means that the particles involved are close to each others (in the position variables), but with a sufficiently large relative velocity, so that they do not interact a lot on the interval $[t-\tau, t]$.

First we introduce a notation for the term of (4.20)

$$\int_{z_t \in B_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t, \tag{4.24}$$

$$\text{with } I_{bc}(t, \bar{z}_t, z_t) = I_{bc}(t, i, j) := \frac{1}{\tau} \int_{t-\tau}^{t} F(T_x^s(\bar{z}_s) - T_x^s(z_s))\, ds,$$

where (i,j) are such that $T_x^t(\bar{z}_t) = X_i(t)$, $T_x^t(z_t) = X_j(t)$ (and this is also true for any time $s \in [t-\tau, t]$ by definition of T^s. For $z_t \in B_t$, define for $s \in [t-\tau,\ t]$

$$\phi(s) := (T_x^s(\bar{z}_s) - T_x^s(z_s)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|} = (X_i(s) - X_j(s)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|}.$$

Note that $|\phi(s)| \leq |T_x^s(\bar{z}_s) - T_x^s(z_s)|$ and that

$$\begin{aligned} \phi'(s) &= (T_v^s(\bar{z}_s) - T_v^s(z_s)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|} \\ &= |\bar{v}_t - v_t| + (T_v^s(\bar{z}_s) - T_v^s(z_s) - (\bar{v}_t - v_t)) \cdot \frac{\bar{v}_t - v_t}{|\bar{v}_t - v_t|} \geq \frac{|\bar{v}_t - v_t|}{2}, \end{aligned}$$

where we have used (4.23). Therefore ϕ is an increasing function of the time on the interval $[t-\tau, t]$. If it vanishes at some time $s_0 \in [t-\tau, t]$, then the previous bound by below on its derivative implies that

$$|T_x^s(\bar{z}_s) - T_x^s(z_s)| \geq |\phi(s)| \geq |t - s_0|\, \frac{|\bar{v}_t - v_t|}{2}. \tag{4.25}$$

If ϕ is always positive (resp. negative) on $[t-\tau, t]$, then the previous estimate is still true with the choice $s_0 = t - \tau$ (resp. $s_0 = t$). So in any case, estimate (4.25) holds true for some $s_0 \in [t-\tau, t]$. Using this directly gives, as $\alpha < 1$

$$|I_{bc}(t, \bar{z}_t, z_t)| \leq \frac{C}{\tau}\, |\bar{v}_t - v_t|^{-\alpha} \int_{t-\tau}^{t} \frac{ds}{|s - s_0|^\alpha} \leq C\,\tau^{-\alpha}\, |\bar{v}_t - v_t|^{-\alpha}. \tag{4.26}$$

Integrating the above

$$\begin{aligned} \int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)|\, dz_t &\leq C\,\tau^{-\alpha} \int_{A_t^c} \frac{dz_t}{|\bar{v}_t - v_t|^\alpha} \\ &\leq C\,\tau^{-\alpha}\, [W_\infty(t) + \tau]^d\, [R(t)]^{d-\alpha}, \end{aligned}$$

by using the fact that $B_t \subset B_d(0, C[W_\infty(t)+\tau]) \times B_d(0, R(t))$. In conclusion

$$\int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)|\, dz_t \leq C\, \tau^{-\alpha}\, [W_\infty(t)+\tau]^d. \tag{4.27}$$

With the cut-off where $\alpha > 1$, the reasoning follows the same line up to the bound (4.26) which relies on the assumption $\alpha < 1$. (4.26) is replaced by

$$\begin{aligned}
|I_{bc}(t, \bar{z}_t, z_t)| &\leq \frac{C}{\tau} \int_{t-\tau}^{t} \frac{ds}{(|s-s_0||\bar{v}_t - v_t| + 4\varepsilon^{\bar{m}})^\alpha} \\
&\leq \frac{C}{\tau} \int_{t-\tau}^{s_0} \cdots + \frac{C}{\tau} \int_{s_0}^{t} \cdots \leq \frac{2C}{\tau} \int_0^\tau \frac{ds}{(s|\bar{v}_t - v_t| + 4\varepsilon^{\bar{m}})^\alpha} \\
&\leq \frac{C}{\tau |\bar{v}_t - v_t|} \int_0^{\tau|\bar{v}_t - v_t|} \frac{ds}{(s + 4\varepsilon^{\bar{m}})^\alpha} \leq \frac{C\varepsilon^{\bar{m}(1-\alpha)}}{\tau|\bar{v}_t - v_t|}.
\end{aligned}$$

When $\alpha = 1$, the previous calculation leads to

$$|I_{bc}(t, \bar{z}_t, z_t)| \leq \frac{C}{\tau|\bar{v}_t - v_t|}\, \ln\left(1 + C\tau\varepsilon^{-\bar{m}}\right) \leq \frac{C}{\tau|\bar{v}_t - v_t|}\,(1 + |\ln\varepsilon|) =: \frac{C\varepsilon^0}{\tau}$$

where the first bound follows from $\ln(1+x) \leq 1 + \ln(x)$ if $x \geq 1$. In the second one, we use that $\tau = \varepsilon$ in the cut-off case. We also used the convention $\varepsilon^0 = 1 + |\ln(\varepsilon)|$, introduced in Lemma 5, that will be quite convenient since it allows to conclude the proof of the two cases $\alpha \in (1, d-1)$ and $\alpha = 1$ with the same calculation.

In both cases, the singular part in $1/|\bar{v}_t - v_t|$ is integrable on $\mathbb{R}^d$ and integrating that bound over B_t, we get the estimate

$$\begin{aligned}
\int_{z_t \in B_t} |I_{bc}(t, \bar{z}_t, z_t)|\, dz_t &\leq C\, \tau^{-1} \varepsilon^{\bar{m}(1-\alpha)} \int_{A_t^c} \frac{dz_t}{|\bar{v}_t - v_t|} \\
&\leq C\, \tau^{-1}\, \varepsilon^{\bar{m}(1-\alpha)} [W_\infty(t)+\tau]^d\, [R(t)]^{d-1}, \\
&\leq C\, \tau^{-1}\, \varepsilon^{\bar{m}(1-\alpha)} [W_\infty(t)+\tau]^d.
\end{aligned} \tag{4.28}$$

4.4.5. *Step 3: Estimate over C_t.* – We recall the definition of C_t

$$C_t = \left\{ z_t \;\text{ such that }\; \begin{array}{l} |\bar{x}_t - x_t| \leq 4W_\infty(t) + 2\tau(|\bar{v}_t - v_t| + \tau|E|_\infty(t)) \\ |\bar{v}_t - v_t| \leq 4\, W_\infty(t) + 4\,\tau |E|_\infty|(t) \end{array} \right\}.$$

First remark that $C_t \subset \{|z_t - \bar{z}_t| \leq C(W_\infty(t) + \tau)\}$, so that its volume is bounded by $C(W_\infty(t)+\tau)^{2d}$. From the previous steps, it only remains to bound

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t.$$

We begin by the cut-off case, which is the simpler one. In that case, one simply bounds $I_{bc} \leq C\,\varepsilon^{-\bar{m}\alpha}$ which implies

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t \leq C(W_\infty(t)+\tau)^{2d} \varepsilon^{-\bar{m}\alpha}. \tag{4.29}$$

It remains the case without cut-off. We denote

$$\tilde{C}_t = \{ j \mid \exists z_t \in C_t, \text{ such that } Z_j(t) = T^t(z_t) \},$$

and transform the integral on C_t in a discrete sum

$$\int_{z_t \in C_t} I_{bc}(t, \bar{z}_t, z_t)\, dz_t = \sum_{j \in \tilde{C}_t} a_{ij} I_{Nc}(t, i, j) \text{ with } I_{Nc}(t, i, j) = \frac{1}{\tau} \int_{t-\tau}^{t} \frac{dz_t}{|X_i(s) - X_j(s)|^\alpha}\, ds,$$

where i is the number of the particles associated to $\bar{z}_t$ ($T^t(\bar{z}_t) = Z_i(t)$) and

$$a_{ij} = |\{z_t \in C_t,\ T^t(z_t) = Z_j(t)\}|, \quad \text{so that} \sum_{j \in \tilde{C}_t} a_{ij} = |C_t|.$$

To bound I_{Nc} over $\tilde{C}_t$, we do another decomposition in j. Define

$$JX_t = \left\{ j \in \tilde{C}_t\,,\ |X_j(t) - X_i(t)| \geq \frac{d_N(t)}{2} \right\},$$
$$JV_t = \left\{ j \in \tilde{C}_t\,,\ |V_j(t) - V_i(t)| \geq \frac{d_N(t)}{2} \right\}.$$

By the definition of the minimal distance in $\mathbb{R}^{2d}$, $d_N(t)$, one has that $\tilde{C}_t = JX_t \cup JV_t$. Since

$$|T^t(z_t) - z_t| \leq W_\infty(t),$$

one has by the definition of $\tilde{C}_t$ and C_t that for all $j \in \tilde{C}_t$, $|Z_j(t) - Z_i(t)| \leq C\,(W_\infty(t) + \tau)$.

Let us start with the bound over JX_t. If $j \in JX_t$, one has that

$$|X_j(s) - X_i(s)| \geq |X_j(t) - X_i(t)| - \int_s^t |V_j(u) - V_i(u)|\, du.$$

On the other hand, for $u \in [s, t]$,

$$|V_j(u) - V_i(u)| \leq 2W_\infty(t) + |\bar{v}_u - v_u| \leq 2(W_\infty(t) + \tau|E|_\infty) + |\bar{v}_t - v_t| \leq C(W_\infty(t) + \tau).$$

Therefore assuming that with that constant C

$$C\,\tau(W_\infty(t) + \tau) \leq d_N(t)/4, \tag{4.30}$$

we have that for any $s \in [t-\tau,\ t]$, $|X_j(s) - X_i(s)| \geq d_N(t)/4$. Consequently for any $j \in JX_t$

$$I_{Nc}(t, i, j) \leq C\,[d_N(t)]^{-\alpha}. \tag{4.31}$$

For $j \in JV_t$, we write

$$|(V_j(s) - V_i(s)) - (V_j(t) - V_i(t))| \leq \int_s^t |E_N(X_j(u)) - E_N(X_i(u))|\, du.$$

Note that

$$\begin{aligned} |X_j(s) - X_i(s)| &\leq |X_j(t) - X_i(t)| + \int_s^t |V_j(u) - V_i(u)|\, du \\ &\leq C(W_\infty(t) + \tau) + 2\int_s^t (W_\infty(u) + R(u))\, du \\ &\leq C(W_\infty(t) + \tau). \end{aligned} \tag{4.32}$$

Hence we get for $s \in [t - \tau, t]$, by the Definition (4.4) of $|\nabla^N E|_\infty$

$$\int_s^t |E_N(X_j(u)) - E_N(X_i(u))|\, du \leq C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t) + \tau + \varepsilon^{1+r'}).$$

Note that the constant C still does not depend on $\tau = \varepsilon^{r'}$. Therefore provided that with the previous constant C

$$2C\,\tau\,|\nabla^N E|_\infty\,(W_\infty(t)+\tau) \le d_N(t)/4, \tag{4.33}$$

one has that

$$|V_j(s)-V_i(s)-(V_j(t)-V_i(t))| \le d_N(t)/4 \quad \text{and also} \quad |V_i(s)-V_j(s)| \ge \frac{d_N(t)}{4}.$$

As in the step for B_t (see Equation (4.25)) this implies the dispersion estimate $|X_j(s)-X_i(s)| \ge |s-s_0|\,d_N(t)/4$ for some $s_0 \in [t-\tau,\ t]$. As a consequence for $j \in JV_t$,

$$I_{Nc}(t,i,j) \le \frac{C}{\tau}\,(d_N(t))^{-\alpha}\int_{t-\tau}^{t}\frac{ds}{|s-s_0|^\alpha} \le C\,\tau^{-\alpha}\,(d_N(t))^{-\alpha}. \tag{4.34}$$

Summing (4.31) and (4.34), one gets

$$\sum_{j\in\tilde C_t} a_{ij} I_{Nc}(t,i,j) \le C\,|C_t|\,\left((d_N(t))^{-\alpha}+\tau^{-\alpha}\,(d_N(t))^{-\alpha}\right).$$

Coming back to I_{bc}, using the bound on the volume of $|C_t|$ and keeping only the largest term of the sum

$$\int_{C_t} I_{bc}(t,\bar z_t,z_t)\,dz_t \le C\,(W_\infty(t)+\tau)^{2d}\tau^{-\alpha}\,(d_N(t))^{-\alpha}. \tag{4.35}$$

4.4.6. *Conclusion of the proof of Lemmas 4, 5.* – Assumptions (4.30) and (4.33) are ensured by the assumptions of the lemma. Summing up (4.17) for I_α or (4.18) for $J_{\alpha+1}$, with (4.19), (4.27) and (4.35), we indeed find the conclusion of the first lemma.

In the S^α_m case, no assumption is needed, and summing up the bounds (4.17), (4.19), (4.28), (4.29), we obtain the second lemma.

4.5. A bound on $W_\infty(\mu_N, f_N)$ in the case without cut-off

In this subsection, in order to make the argument clearer, we number explicitly the constants. Let us summarize the important information of Proposition 4 and Lemma 4. We introduce rescaled versions of the important quantities

$$\tilde W_\infty(t) = \varepsilon^{-1}W_\infty(t), \qquad \tilde d_N(t) = \varepsilon^{-(1+r)}\,d_N(t).$$

Remark that by Proposition 1, $\tilde W_\infty(t) = c_\phi > 0$. By assumption (i) in Theorem 1, also note that $\tilde d_N(0) \ge 1$.

Recalling $\tau = \varepsilon^{r'}$ (with $r' > r > 1$), the condition (4.10) of Lemma 4 after rescaling reads

$$C_1\,\varepsilon^{r'-r}\,(1+|\nabla^N E|_\infty(t))\,\tilde W_\infty(t) \le \tilde d_N(t). \tag{4.36}$$

In Lemma 4, we proved that there exist some constants C_0 and C_2 independent of N (and hence ε), such that if (4.36) is satisfied, then for any $t \in [0,\ T]$

$$\begin{aligned}
\tilde W_\infty(t) \quad &\le \tilde W_\infty(t-\tau) + C_0\,\tau\left(\tilde W_\infty(t) + \varepsilon^{\lambda_1}\,\tilde W^d_\infty(t) + \varepsilon^{\lambda_2}\,\tilde W^{2d}_\infty(t)\,\tilde d_N^{-\alpha}(t)\right),\\
|\nabla^N E|_\infty(t) \ &\le C_2\left(1+\varepsilon^{\lambda_3}\,\tilde W^d_\infty(t) + \varepsilon^{\lambda_4}\,\tilde W^{2d}_\infty(t)\,\tilde d_N^{-\alpha}(t))\right)\\
\tilde d_N(t) + \varepsilon^{r'-r} &\ge [\tilde d_N(t-\tau) + \varepsilon^{r'-r}]e^{-\tau(1+|\nabla^N E|_\infty(t))},
\end{aligned}$$

where ε appears four times with four different exponents $\lambda_i, i = 1, \dots, 4$ defined by

$$\lambda_1 = d - 1 - \alpha\, r', \qquad \lambda_2 = 2d - 1 - \alpha(1 + r' + r),$$
$$\lambda_3 = d - 1 - r' - \alpha\, r', \qquad \lambda_4 = 2d - 1 - r' - \alpha(1 + r' + r).$$

To propagate uniform bounds as $\varepsilon \to 0$ and $N \to \infty$, we need all λ_i to be positive. As $r, r' > 0$, it is clear that $\lambda_1 > \lambda_3$ and $\lambda_2 > \lambda_4$. Thus we need only check $\lambda_3 > 0$ and $\lambda_4 > 0$. As $r' > r$, it is sufficient to have

$$r' < \frac{d-1}{1+\alpha}, \quad \text{and} \quad r' < \frac{2d - 1 - \alpha}{1 + 2\,\alpha}.$$

Note that a simple calculation shows that

$$\frac{d-1}{1+\alpha} - \frac{2d-1-\alpha}{1+2\,\alpha} = \frac{\alpha^2 - d}{(1+\alpha)(1+2\alpha)} < 0,$$

so that the first inequality is the stronger one. Thanks to the condition given in Theorem 1, $r < r^* := \frac{d-1}{1+\alpha}$, so that if we choose any $r' \in (r, r^*)$, the corresponding λ_i are all positive. We fix a r' as above and denote $\lambda = \min_i(\lambda_i)$. Then by a rough estimate

$$\begin{aligned} \tilde W_\infty(t) &\le \tilde W_\infty(t-\tau) + C_0\,\tau \left(\tilde W_\infty(t) + 2\,\varepsilon^\lambda\, \tilde W_\infty^{2d}(t)\, d_N^{-\alpha}(t)\right), \\ |\nabla^N E|_\infty(t) &\le C_2 \left(1 \,+ 2\,\varepsilon^\lambda\, \tilde W_\infty^{2d}(t)\, \tilde d_N^{-\alpha}(t)\right), \\ \tilde d_N(t) &\ge [\,\tilde d_N(t-\tau) + \varepsilon^{r'-r}]e^{-(1+|\nabla^N E|_\infty(t))\tau} - \varepsilon^{r'-r}. \end{aligned} \tag{4.37}$$

Next, we define $t_0^N \in [0,\ T]$ as the maximal time such that (4.36) and

$$2\,\varepsilon^\lambda\, \tilde W_\infty^{2d-1}(t)\big[1 + \tilde W_\infty(t)\big]\, \tilde d_N^{-\alpha}(t) \le 1, \tag{4.38}$$

are satisfied on $[0,\ t_0^N]$. This time t_0^N can depend on N and possibly $t_0^N = T$.

We claim that this time t_0^N exists and is strictly positive for N large enough. First as we explain in Remark 11, $|\nabla^N E|_\infty(0) = 0$ and the conditions (4.36) and (4.38) are satisfied at time $t = 0$. In fact, at time 0 they may be rewritten

$$C_1 \varepsilon^{r'-r} \tilde W_\infty(0) \le \tilde d_N(0), \qquad 2\varepsilon^\lambda \tilde W_\infty(0)^{2d-1}\big[1 + \tilde W_\infty(0)\big]\tilde d_N(0)^{-\alpha} \le 1$$

and this is true for N large enough because of our assumption on ε and $d_N(0)$. Next, as we mention in the introduction, we only deal with continuous solutions to the N particle system (1.1). So $\tilde W_\infty(t)$ and $\tilde d_N(t)$ are continuous functions of the time, and $|\nabla^N E|_\infty(t)$ is also continuous in time thanks to the smoothing parameter that appears in its Definition (4.4). Then if the conditions hold at $t = 0$, they also hold on a small neighborhood of 0, and the claim is proved.

Next, as long as $t < t_0^N$, the first equation in (4.37) may be rewritten

$$\tilde W_\infty(t) \le \tilde W_\infty(t-\tau) + 2C_0\tau \tilde W_\infty(t)$$

so that if $2C_0\tau < 1$

$$\begin{aligned} \tilde W_\infty(t) &\le \tilde W_\infty(t-\tau)(1 - 2C_0\tau)^{-1}, \\ |\nabla^N E|_\infty(t) &\le 2\,C_2, \\ \tilde d_N(t) &\ge\ e^{-(1+2C_2)\,t} - \varepsilon^{r'-r} \end{aligned} \tag{4.39}$$

for any $t < t_0^N$. The last inequality implies $\tilde{d}_N(t) \geq \frac{1}{2}\, e^{-(1+2C_2)\, t}$ if $2\varepsilon^{r'-r} e^{(1+2C_2)\, T} < 1$. That condition is fulfilled for ε small enough, i.e., N large enough.

The first inequality in (4.39) iterated gives $W_\infty(t) \leq W_\infty(0)(1-2C_0\tau)^{-\frac{t}{\tau}}$. If $C_0\tau \leq \frac{1}{4}$, then we can use $-\ln(1-x) \leq 2x$ for $x \in [0, \frac{1}{2}]$, and get

$$\tilde{W_\infty}(t) \leq \tilde{W}_\infty(0) e^{4C_0 t}$$

To summarize, under the previous assumption it comes for all $t \in [0, t_0^N]$

$$\begin{aligned} \tilde{W}_\infty(t) &\leq e^{4C_0 t}, \\ |\nabla^N E|_\infty(t) &\leq 2\, C_2, \\ \tilde{d}_N(t) &\geq \frac{1}{2}\, e^{-(1+2C_2)\, t}. \end{aligned} \tag{4.40}$$

Finally, we show that for N large enough, i.e., ε small enough, then one necessarily has $t_0^N = T$. Then we will have (4.40) on $[0,\ T]$ which is the desired result. This is simple enough. By contradiction if $t_0^N < T$ then

$$C_1\, \varepsilon^{(r-r')}\, (1 + |\nabla^N E|_\infty(t_0^N))\, \tilde{W}_\infty(t_0^N) = \tilde{d}_N(t_0^N), \text{ or } 4\, \varepsilon^\lambda\, \tilde{W}_\infty^{2d-1}(t_0^n)\big[1 + \tilde{W}_\infty(t_0^N)\big]\, \tilde{d}_N^{-\alpha}(t_0^N) = 1.$$

But until t_0^N, (4.40) holds. Therefore

$$\varepsilon^\lambda\, \tilde{W}_\infty^{2d-1}(t_0^N)\big[1 + \tilde{W}_\infty(t_0^N)\big]\, \tilde{d}_N^{-\alpha}(t_0^N) \leq \varepsilon^\lambda\, 2^{\alpha+1}\, e^{(\alpha + (8d+2\alpha)\max(C_0,C_2))\, t_0^N} < 1,$$

for ε small enough with respect to T and the C_i. This is the same for (4.36)

$$C_1\, \varepsilon^{(r-r')}\, (1 + |\nabla^N E|_\infty(t_0))\, \tilde{W}_\infty(t_0) \tilde{d}_N^{-1}(t_0) \leq 2\varepsilon^{(r-r')} C_1 (1+2C_2) e^{(1+6\max(C_0,C_2))t_0} < 1.$$

Hence we obtain a contradiction and prove that

$$\forall t \leq T, \quad W_\infty(f_N(t), \mu_N(t)) \leq e^{4C_0 t} W_\infty(f_N^0, \mu_N^0), \tag{4.41}$$

for N large enough.

4.6. A bound on $W_\infty(\mu_N, f_N)$ in the case with cut-off

In the cut-off case, using Lemma 5 together with the inequality ii) of the Proposition 4, we may obtain

$$W_\infty(t) \leq W_\infty(t-\tau) + C_0 W_\infty(t) \left[1 + (W_\infty(t)+\tau)^{d-1} \tau^{-1} \varepsilon^{\bar{m}(1-\alpha)} + (W_\infty(t)+\tau)^{2d-1} \varepsilon^{-\bar{m}\alpha}\right].$$

We again rescale the quantity $W_\infty(t) = \varepsilon \tilde{W_\infty}(t)$. Choosing in that case $\tau = \varepsilon$, it comes for $1 \leq \alpha < d-1$,

$$\tilde{W}_\infty(t) \leq \tilde{W}_\infty(t-\tau) + C_0 \tilde{W}_\infty(t) \tau \left[1 + \varepsilon^{d-2-\bar{m}(\alpha-1)}\, \tilde{W}_\infty^{d-1}(t) + \varepsilon^{2d-1-\bar{m}\alpha}\, \tilde{W}_\infty^{2d-1}(t)\right].$$

As in the previous section, we will get a good bound provided that the powers of ε appearing in parentheses are positive. The two conditions read

$$\bar{m} < \bar{m}^* := \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right).$$

In that case, for N large enough (with respect to e^{Ct}), we get a control of type

$$\frac{d}{dt} \tilde{W}_\infty(t) \leq 4C_0 \tilde{W}_\infty(t),$$

(but discrete in time) which gives that

$$\forall t \leq T, \quad W_\infty(f_N(t), \mu_N(t)) \leq e^{4C_0 t} W_\infty(f_N^0, \mu_N^0), \tag{4.42}$$

for N large enough.

REMARK 12. – *In the cut-off case (and also in the case without cut-off), it seems important to be able to say that the initial configurations Z we choose have a total energy close to the one of f^0. Because, if the empirical distribution μ_N^z is close to f^0, but has a different total energy, we would not expect that they remain close a very long time. Fortunately, such a result is true and under the assumptions of Theorems 1 and 3, the total energy of the empirical distributions is close to the total energy of f^0.*

Unfortunately, the proof is not simple. But, it can be done using the argument presented here for the deterministic theorems. First, the difference between the kinetic energies is easily controlled because our solutions are compactly supported and that there is no singularity there. Next, performing calculations very similar to the ones done in the proofs, we can control the difference between a small average in time of the potential energies, on the small interval of time $[0, \tau]$. Then, we control the average of the total energy, which is constant.

4.7. Estimation of the distance $W_1(f, \mu_N)$

The case without cut-off. – Just apply the stability estimate for solutions of Vlasov equation given by Proposition 3. This is possible since the uniform bound on $\|f_N\|_\infty$ given by point ii) in Theorems 1 and 3, and the uniform bound on the size of the support of Proposition 4, imply a uniform bound on $\|\rho_N\|_\infty \leq \|f_N\|_\infty R(t)^{2d}$. We get

$$\begin{aligned} W_1(f, f_N) &\leq e^{C_0 t}\, W_1(f^0, f_N^0) \\ &\leq e^{C_0 t}\left(W_1(f^0, \mu_N^0) + W_1(\mu_N^0, f_N^0)\right), \\ &\leq e^{C_0 t}\left(W_1(f^0, \mu_N^0) + N^{-\frac{\gamma}{2d}}\right), \end{aligned}$$

where we have used in the last line Proposition 1 with a ϕ such that $c_\phi = 1$. This together with the bound (4.41) concludes the proof since

$$\begin{aligned} W_1(f, \mu_N) &\leq W_1(f, f_N) + W_1(f_N, \mu_N) \\ &\leq W_1(f, f_N) + W_\infty(f_N, \mu_N) \\ &\leq e^{4C_0 t}\Big(W_1(f^0, \mu_N^0) + 2\, N^{-\frac{\gamma}{2d}}\Big). \end{aligned}$$

The case with cut-off. – Proposition 2 implies that the strong solution f_N with initial data f_N^0 has existence times T_N^* bounded from below by some $\bar{T}$ independent of N. And from the condition (S_m^α) restated in (3.3) in term of ε, we get that

$$\|F - F_N\|_1 \leq \varepsilon^{\bar{m}(d-\alpha)} \leq \varepsilon,$$

since $\bar{m} \geq 1$ and $d - \alpha \geq 1$. So we can apply the stability estimate given by Proposition 3 with $F_1 = F$ and $F_2 = F_N$ and get that

$$\begin{aligned} W_1(f(t), f_N(t)) &\leq e^{C_0 t}\big(W_1(f^0, f_N^0) + \varepsilon\big) \\ &\leq e^{C_0 t}\Big(W_1(f^0, \mu_N^0) + W_1(\mu_N^0, f_N^0) + N^{-\frac{\gamma}{2d}}\Big), \\ &\leq e^{C_0 t}\Big(W_1(f^0, \mu_N^0) + 2\, N^{-\frac{\gamma}{2d}}\Big), \end{aligned}$$

for any $t \leq \bar{T}$. With the bound (4.42) it leads to

$$W_1(f, \mu_N) \leq e^{4C_0 t}\Big(W_1(f^0, \mu_N^0) + 3\, N^{-\frac{\gamma}{2d}}\Big),$$

and this concludes the proof in the cut-off case.

5. From deterministic results (Theorems 1 and 3) to propagation of chaos.

The assumptions made in Theorem 1 are in some sense generic, when the initial positions and speeds are chosen with the law $(f^0)^{\otimes N}$. Therefore, to prove Theorem 2 from Theorem 1, we need to

- find a good choice of the parameters γ and r so that there is a small probability that empirical measures, chosen with the law $(f^0)^{\otimes N}$, do not satisfy the conditions i) and ii) of Theorem 1, and are far away from f^0 in W_1 distance;
- apply Theorem 1 on the complementary set that is almost of full measure.

For the first point, we will use results detailed in the next two sections.

5.1. Estimates in probability on the initial distribution

Deviations on the infinite norm of the smoothed empirical distribution f_N. – The precise result we need is given by the Proposition 8 in the appendix. It tells us that if the approximating kernel is $\phi = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]^{2d}}$ and if the Z_i are i.i.d random variables with law f^0, then the random variable f_N satisfies

$$\mathbb{P}\left(\|f_N^0\|_\infty \geq 2^{1+2d}\|f^0\|_\infty\right) \leq C_2 N^\gamma e^{-C_1 N^{1-\gamma}},$$

with $C_2 = (2R^0 + 2)^{2d}$, R^0 the size of the support of f, and $C_1 = (2\ln 2 - 1)\, 2^{2d}\|f\|_\infty$.

We would like to mention that we were first aware of the possibility of getting such estimates in a paper of Bolley, Guillin and Villani [10], where the authors obtain quantitative concentration inequality for $\|f^N - f\|_\infty$ under the additional assumption that f^0 and ϕ are Lipschitz. Unfortunately, they cannot be used in our setting because they would require too large a smoothing parameter. Gao obtains in [26] large (and moderate) deviation principles for $\|f^N - f\|_\infty$. But a large deviation principle is too precise for our purpose, and also less convenient since it provides only an asymptotic estimate, and no quantitative bounds. Finally, we choose to prove a more simple estimate that is well adapted to our problem.

Deviations for the minimal inter-particle distance. – It may be proved with simple arguments that the scale η_m is almost surely larger than $N^{-1/d}$ when $f^0 \in L^\infty$. A precise result is stated in the proposition below, proved in [32, Proposition 4].

PROPOSITION 5. – *There exists a constant c_{2d} depending only on the dimension such that if $f^0 \in L^\infty(\mathbb{R}^{2d})$ and if $Z = (Z_1, \ldots, Z_N)$ has law $(f^0)^{\otimes N}$, then*

$$\mathbb{P}\left(d_N(Z) \geq \frac{l}{N^{1/d}}\right) \geq e^{-c_{2d}\|f^0\|_\infty l^d}.$$

We point out that this is not a large deviation result : the inequalities are in the wrong direction. This is quite natural because d_N is not an average quantity, but an infimum. It is that condition that prevents us from obtaining a "large deviation" type result in Theorem 2, contrarily to the cut-off case of Theorem 4. In fact, the only bound it provides on the "bad" set is

$$\mathbb{P}\left(d_N(Z) \leq \frac{l}{N^{1/d}}\right) \leq 1 - e^{-c_{2d}\|f^0\|_\infty l^d} \leq c_{2d}\|f^0\|_\infty l^d.$$

With the notation of Theorem 1 (recall that $\varepsilon = N^{-\gamma/(2d)}$) it comes that if $s = \gamma\frac{1+r}{2} - 1 > 0$ then

$$\mathbb{P}\left(d_N(Z) \leq \varepsilon^{1+r}\right) = \mathbb{P}\left(d_N(Z) \leq \frac{N^{-s/d}}{N^{1/d}}\right) \leq c_{2d}\|f^0\|_\infty N^{-s}. \tag{5.1}$$

Deviations for the W_1 MKW distance. – It is more or less classical that if the Z_i are independent random variables with identical law f, the empirical measure μ_N^Z goes in probability to f. This theorem is known as the empirical law of large number or Glivenko-Cantelli Theorem and is due in this form to Varadarajan [60]. But, the convergence may be quantified in Wasserstein distance, and recently upper bounds on the large deviations of $W_1(\mu_N^Z, f)$ were obtained by Bolley, Guillin and Villani [10] and Boissard [8]. However the first one concerns only very large deviations, and the last result is more interesting for our purpose. We also mention the very recent work of Fournier and Guillin [21].

PROPOSITION 6 (Boissard [8], Annexe A, Proposition 1.2). – *Assume that f is a non negative measure compactly supported on $B_{2d}(0, R) \subset \mathbb{R}^{2d}$. If $d \geq 2$, and the $Z = (Z_1, \ldots, Z_N)$ are chosen according to the law $(f^0)^{\otimes N}$, then there is an explicit constant $C_1 = 2^{-(2d+1)}R^{-2d}$, such that the associated empirical measures μ_N^Z satisfy*

$$\mathbb{P}\Big(W_1(\mu_N^Z, f) \geq \mathbb{E}[W_1(\mu_N^Z, f)] + L\Big) \leq e^{-C_1 N L^2}.$$

Since it is already known (see [8] *or* [19] *and references therein) that for $d \geq 2$ there exists a numerical constant $C_2(d)$ such that*

$$\mathbb{E}[W_1(\mu_N^Z, f)] \leq C_2 \frac{R}{N^{1/2d}},$$

the previous result with $L = C_2 \frac{R}{N^{1/(2d)}}$ implies that for $C_3(R, d) := C_1(R)C_2(d)^2 R^2$,

$$\mathbb{P}\left(W_1(\mu_N^Z, f) \geq 2\frac{C_2 R}{N^{1/2d}}\right) \leq e^{-C_3 N^{1-1/d}}. \tag{5.2}$$

5.2. From Theorem 1 to Theorem 2

Now take the assumptions of Theorem 2 : F satisfies an (S^α) condition for $\alpha < 1$ and $f^0 \in L^\infty$ with support in some ball $B_{2d}(0, R_0)$ in dimension $d \geq 3$. We choose

$$\gamma \in \left(\gamma^* = \frac{2+2\alpha}{d+\alpha}, 1\right), \text{ and } \quad r \in \left(\frac{2}{\gamma} - 1, r^* = \frac{d-1}{1+\alpha}\right),$$

the condition on γ ensuring that the second interval is non empty. We also define

$$s := \gamma \, \frac{1+r}{2} - 1 > 0, \quad \lambda = 1 - \max\left(\gamma, \frac{1}{d}\right).$$

Denote by ω_1, ω_2 the sets of initial conditions such that respectively (i) and (ii) of Theorem 1 hold and ω_3 s.t. $W_1(\mu_N, f^0) \leq \frac{1}{N^{\gamma/(2d)}}$. Precisely

$$\begin{aligned} \omega_1 &:= \{Z^0 \text{ such that } d_N(Z^0) \geq \varepsilon^{1+r}\}, \\ \omega_2 &:= \{Z^0 \text{ such that } \|f_N^0\|_\infty \leq 2^{1+2d} \|f^0\|_\infty\} \\ \omega_3 &:= \{Z^0 \text{ such that } W_1(\mu_N^0, f^0) \leq \varepsilon\}. \end{aligned}$$

By the results stated in the previous section, one knows that for $N \geq (2C_2 R)^{2d/(1-\gamma)}$

$$\mathbb{P}(\omega_1^c) \leq C\, N^{-s}, \quad \mathbb{P}(\omega_2^c) \leq C N^\gamma e^{-C N^{1-\gamma}}, \quad \mathbb{P}(\omega_3^c) \leq \; e^{-C N^{1-\frac{1}{d}}}. \tag{5.3}$$

Denote $\omega = \omega_1 \cap \omega_2 \cap \omega_3$. Hence $|\omega^c| \leq |\omega_1^c| + |\omega_2^c| + |\omega_3^c|$ and for N large enough

$$\mathbb{P}(\omega^c) \leq C\, N^{-s} + C\, N^\gamma \, e^{-C\, N^{1-\gamma}} + e^{-C N^{1-\frac{1}{d}}} \leq C\, N^{-s}, \tag{5.4}$$

and checking carefully the dependence, we can see that the constant C depends only on $d, R, \|f^0\|_\infty, \gamma$. Since we known that global solutions to the N particles system (1.1) exist for almost all initial conditions (see the discussion on this point in Subsection 6.1), one may apply Theorem 1 to $(f^0)^{\otimes N}$-a.e. initial condition in ω and get on $[0,\ T]$

$$W_1(f, \mu_N) \leq e^{C_0 t} \Big(2\, W_1(f, \mu_N^0) + N^{-\frac{\gamma}{2d}}\Big) \leq 3\, e^{C_0 t}\, N^{-\frac{\gamma}{2d}},$$

which proves that

$$\omega \subset \Big\{\forall t \in [0, T],\ W_1(f, f_N) \leq \frac{3 e^{C_0 t}}{N^{\gamma/(2d)}}\Big\}.$$

The bound 5.4 then gives Theorem 2.

5.3. From Theorem 3 to Theorem 4

In the cut-off case, one can derive Theorem 4 from Theorem 3 in the same manner. As we do not use the minimal distance in that case, the proof is simpler and we get a stronger convergence result.

We only have to consider $\omega = \omega_2 \cap \omega_3$, where ω_2 and ω_3 are defined according to (5.3). Then, the bound (5.4) is replaced for N larger than an explicit constant by

$$\mathbb{P}(\omega^c) \leq C\, N^\gamma \, e^{-C\, N^{1-\gamma}} + e^{-C N^{1-\frac{1}{d}}} \leq C\, N^\gamma \, e^{-C\, N^{-\lambda}}. \tag{5.5}$$

Next, for any $Z^0 \in \omega$, we can apply Theorem 3 and obtain the stability estimate for any $T < T^*$

$$W_1(f, \mu_N) \leq e^{C_0 t} \Big(W_1(f^0, \mu_N^0) + 3\, N^{-\frac{\gamma}{2d}}\Big) \leq 4\, e^{C_0 t}\, N^{-\frac{\gamma}{2d}}.$$

From there, we obtain as before that for N large enough

$$\mathbb{P}\left(\exists\, t\in[0,T],\ W_1(\mu_N(t),f(t))\geq\frac{4e^{C_0t}}{N^{\gamma/(2d)}}\right)\leq CN^{\gamma}e^{-CN^{\lambda}}.$$

Replacing 2^{1+2d} by any $c>1$ in the definition of ω_2, we may also get estimates that are valid till a time T^* as large as possible.

5.4. Proof of Corollary 1

Theorem 2 proves that for fixed time t, $\mu_N(t)$ converges in probability towards $f(t)$ as $N\to+\infty$. This classically implies the convergence in law of $\mu_N(t)$ towards $f(t)$, which is one of the characterizations of $f(t)$-chaotic sequences, given in point iii) of Definition 3. This proves the first part of (2.4).

For the second part, first remark that by the definition of the entropy, $H\big((f^0)^{\otimes N}\big)=NH(f^0)$. Then, the chain rule applied to the Vlasov Equation (1.5) implies that for any smooth β, $\beta(f)$ also satisfies

$$\partial_t\beta(f)+v\cdot\nabla_v\beta(f)+E(t,x)\nabla_v\beta(f)=0.$$

This is classical if f is a smooth solution to the Vlasov equation, but also holds in our case as the vector-field is smooth enough (at least $W^{1,1}$), we refer to [2] for the details (this is indeed an important issue in the theory initiated by DiPerna and Lions [18]). An integration in x and v implies that for any time t,

$$\int_{\mathbb{R}^{2d}}\beta(f(t))\,dz=\int_{\mathbb{R}^{2d}}\beta(f^0)\,dz.$$

Approximating $\beta(x)=x\ln x$ by a monotone sequence of smooth β_n, the monotone convergence theorem (we recall that $f(t)$ and f^0 are bounded and non-negative) allows to conclude that the entropy is preserved by the Vlasov equation: $H(f(t))=H(f^0)$ for all time t.

In addition, the entropy is also preserved by the (linear) Liouville equation, governing the evolution of the distribution of the N particle system. In fact, a similar argument applies since the vector-field associated to the Liouville equation (which is nothing but the one appearing in (1.1)) is smooth enough, at least $W^{1,1}$. If $f^N(t)$ denotes the law at time t of $Z^N(t)$, we also have $H(f^N(t))=H\big((f^0)^{\otimes N}\big)$. All in all, we conclude that the second point of (2.4) holds not only asymptotically but precisely

$$\forall\, t\geq 0,\ \forall N\in\mathbb{N},\quad \frac{1}{N}H\big(f^N(t)\big)=\frac{1}{N}H\big((f^0)^{\otimes N}\big)=H(f^0)=H\big(f(t)\big).$$

On the marginals, the entropy has the two key properties of lower semi-continuity and so-called super additivity: $H(f_k^N)\leq q^{-1}H(f^N)$, with q the divisor in the Euclidean division of N by k: $N=qk+r$. This implies that $H(f_k^N(t))\to H(f(t)^{\otimes k})$. Since the entropy is a strictly convex functional, the weak convergence of the f_k^N and the convergence of their entropy implies the strong convergence in L^1 norm stated in Corollary 1. We refer to [22, Section 8] for a rigorous proof of the last point in a slightly different setting.

6. Related discussions

6.1. The question of existence of solutions to System (1.1)

We have just mentioned till now the most basic question for System (1.1) with a singular force kernel, namely whether one can even expect to have solutions to the system for a fixed number of particles.

Since we only use forces that are singular only at the origin, the usual Cauchy-Lipschitz theory implies that starting from any initial conditions such that $X_i^N \neq X_j^N$ for all $i \neq j$, there exists a unique local solution, defined till the first collision time T^*, when for some couple i, j we have $X_i^N(T^*) = X_j^N(T^*)$. Unfortunately this time T^* depends on the initial configuration (and thus on N) and could be very small.

In the case where the interaction force F derives from a repulsive singular potential ϕ *strong enough*, i.e., if Φ satisfies $\lim_{x\to 0} \Phi(x) = +\infty$, then collisions can never occur and the solutions given by the classical Cauchy-Lipschitz theory are global, i.e., $T^* = +\infty$ for all initial configurations.

In the other cases, it is not possible to extend the local result in such a simple way. One could try to apply the DiPerna-Lions theory [18], that allows to handle vector fields that are locally in $W^{1,1}$. This looks promising since any force satisfying the condition (1.2), with $\alpha < d-1$ has the required local regularity. But unfortunately, the DiPerna-Lions theory also requires a condition on the growth of the vector-field at infinity, which is not satisfied in our case. However if the interaction force F derives from a potential Φ which is bounded at the origin (without any sign condition), the DiPerna-Lions theory still leads to global solutions for almost every initial conditions. This is stated precisely in the following proposition which is a consequence of [31, Theorem 4].

Proposition 7. – *Assume that $F = -\nabla\Phi$ with $\Phi \in W^{2,1}_{\mathrm{loc}}$, and that $\Phi(x) \geq -C(1+|x|^2)$ for some constant $C > 0$. Then for any fixed N, there exists a unique measure preserving and energy preserving flow defined almost everywhere on $\mathbb{R}^{2dN}$ associated to* (1.1). *Such a flow precisely satisfies*

i) *there exists a set $\Omega \subset \mathbb{R}^{2dN}$ with $|\Omega| = 0$ s.t. for any initial data $Z^0 \in \mathbb{R}^{2dN} \setminus \Omega$, we have a trajectory $Z(t)$ solution to* (1.1),

ii) *for a.e. trajectory the energy is conserved: for all time $t \geq 0$,*

$$\frac{1}{2}\sum_{i=1}^{N} |V_i(t)|^2 + \frac{1}{2N}\sum_{i\neq j} \Phi(|X_i(t) - X_j(t)|) = \frac{1}{2}\sum_{i=1}^{N} |V_i^0|^2 + \frac{1}{2N}\sum_{i\neq j} \Phi(|X_i^0 - X_j^0|)$$

iii) *the family of solutions defines a global flow, which preserves the measure on $\mathbb{R}^{2dN}$.*

Remark that if $F = -\nabla\Phi$ then the conditions on Φ are fulfilled whenever F satisfies (1.2). So this proposition implies the global existence of solutions for almost all initial positions and velocities in that case, and this is completely sufficient for our results: Theorem 1 requires only the existence of a solution with given initial data and Theorem 2 requires the existence of solutions for almost all initial data.

In the case of some specific but more singular attractive potentials as the gravitational force ($\alpha = d - 1$) in dimension 2 or 3, and also for some others power law forces, it is known [53] that the set of initial conditions leading to "standard" collisions (possibly multiple and simultaneous), is of zero measure. The 2D case is in fact completely solved in [44]. But, what is unknown even if it seems rather natural, is that the set of initial collisions leading to the so-called “non-collisions” singularities, which do exist [63], is also of zero measure for $N \geq 5$. Up to our knowledge it has only been proved for $N \leq 4$ [54]. In fact, there is a large literature about this N body problem in the physics and mathematics communities. However, that discussion is not really relevant here since in the “strongly” singular case $\alpha \in [1, d-1)$, we use a regularization or cut-off of the force (see the condition (1.3)), thanks to which the question of global existence becomes trivial.

Eventually, the only case in which we are not covered by the existing literature is the case of non potential force satisfying the (S^α)-condition for some $\alpha < 1$, for which we claimed a result without cut-off. In that case, we opt for the following simple strategy. As in the case with larger singularity we use a cut-off or regularization of the interaction force. The existence of global solution is then straightforward. And our results of convergence are valid independently of the size of cut-off (or smoothing parameter) which is used. It can be any positive function of the number of particles N.

Note that this suggests in fact that for a fixed N, the analysis done in this article should imply the existence of solutions for almost all initial conditions. If one checks precisely, our proofs show that trajectories may be extended after a collision where the relative velocity between the two particles goes to a non zero limit. Hence the only collisions that remain problematic are those where the relative velocity of the colliding particles vanishes, but our result controls the probability of this happening. This was mentioned in Remark 6 after Theorem 2.

6.2. The structure of the force term: Potential, repulsive, attractive?

In the particular case where the force derives from a potential $F = -\nabla\Phi$, the system (1.1) is endowed with some important additional structure, for example the conservation of energy

$$\frac{1}{N}\sum_i \frac{|V_i|^2}{2} + \frac{1}{2N^2}\sum_{i\neq j}\Phi(X_i - X_j) = \text{const.}$$

When the forces are repulsive, i.e., $\Phi \geq 0$, this immediately bounds the kinetic energy and separately the potential energy. However this precise structure is never used in this article, which may seem weird at first glance. We present here some arguments that can explain this fact.

First, for the interactions considered in the case without cut-off, again satisfying an (S^α) condition with $\alpha < 1$, the potential Φ is continuous (hence locally bounded). In that case the singularity in the force term is too weak to really see or use a difference between repulsive and attractive interactions. Two particles having a close encounter cannot have a strong influence onto each other, both in the attractive or repulsive case. Similarly the fact that the interaction derives from a potential is not really useful, hence our choice of the slightly more general setting.

It should here be noted that the previous discussion applies to every previous result on the mean-field limit or propagation of chaos in the kinetic case: They all require assumptions (typically $\nabla^2\Phi$ locally bounded) implying that the attractive or repulsive nature of the interaction does not matter; the situation is different for the macroscopic “Euler-like” cases, see the comments in the paragraph devoted to that case. The present contribution shows that mean-field limits and propagation of chaos are essentially valid at least as long as the potential is bounded (instead of at least $W^{2,\infty}_{\rm loc}$ as before). This corresponds to the physical intuition that nothing should go wrong as long as the local interaction between two very close particles is too weak to impact the dynamics.

The exact structure of the interaction kernel should become crucial once this threshold is passed, i.e., for $\Phi(x) \sim C|x|^{1-\alpha}$ at the origin with $\alpha \geq 1$. But here we use in that case a cut-off, which weakens the effect of the interaction between two very close particles. In fact in order to prove the mean-field limit, we are able to show that if the cut-off is large enough, these local interactions may be neglected. So our techniques still do not make any difference between the repulsive or attractive cases.

However in the case where the "strong" singularity is repulsive, the potential energy is bounded, and if we were able to use this fact, we would obtain results depending of the attractive-repulsive character of the interaction. In that respect, we point out that the information contained in a bounded potential energy is actually quite weak and clearly insufficient, at least with our techniques. Assume for instance that $\Phi(x) \sim |x|^{1-\alpha}$ for some $\alpha > 1$ in dimension 3. Then the boundedness of the potential energy implies that the minimal distance in physical space between any two particles is of order $N^{-2/(\alpha-1)}$, which is at best N^{-2} in the Coulomb case, $\alpha = 2$. But it can be checked that the cut-off parameter N^{-m} given in Theorem 4 as a power m which is always much lower than $\frac{2}{\alpha-1}$, i.e., that the cut-off we use is always much larger than the minimal distance provided by the bound on the potential energy. To go further, an interesting idea is to compare the dynamics of the N particles with or without cut-off. But even if the difference between the original force and its mollified version is well localized, it is quite difficult to understand how we can control the difference between the two associated dynamics. We refer to [5] for a first attempt in that direction, in which well-localized and singular perturbations of the free transport are investigated.

Therefore in those singular settings, the repulsive or potential structure of the interaction will only help in a more subtle (and still unidentified) manner. An interesting comparison is the stability in average proved in [4]: This requires repulsive interaction not to control locally the trajectories but in order to use the statistical properties of the flow (through the Gibbs equilibrium).

Appendix

A.1. Large deviation on the infinite norm of f_N

PROPOSITION 8. – *Assume that ρ is a probability on $\mathbb{R}^n$ with support included in $[-R^0, R^0]^n$ and bounded density $f(x)\,dx$. Let ϕ be a bounded cut-off function, with support in $[-\frac{L}{2}, \frac{L}{2}]^n$ and total mass one, and define the usual $\phi_\varepsilon := \frac{1}{\varepsilon^n}\phi(\frac{\cdot}{\varepsilon})$. For any configuration $Z = (Z_i)_{i\leq N}$ we define*

$$f_N^Z := \mu_N^Z * \phi_\varepsilon(N).$$

If $\varepsilon(N) = N^{-\frac{\gamma}{n}}$ *and the* Z_N *are distributed according to* $f^{\otimes N}$*, then we have the explicit "large deviations" bound with* $c_\phi = (2L)^n\|\phi\|_\infty$ *and* $c_0 = (2R^0+2)^n L^{-n}$

$$\forall \beta > 1, \quad \mathbb{P}\left(\|f_N^Z\|_\infty \geq \beta c_\phi \|f\|_\infty\right) \leq c_0 N^\gamma e^{-(\beta\ln\beta-\beta+1)(2L)^n\|f\|_\infty N^{1-\gamma}}. \tag{A.1}$$

In particular, for $\phi = \mathbf{1}_{[-1/2,1/2]^n}$ *and* $\beta = 2$*, we get*

$$\mathbb{P}\left(\|f_N^Z\|_\infty \geq 2^{1+n}\|f\|_\infty\right) \leq (2R^0+2)^n N^\gamma e^{-(2\ln 2-1)2^n\|f\|_\infty N^{1-\gamma}}. \tag{A.2}$$

Proof. – For any $Z \in \mathbb{R}^{nN}$ and $z \in \mathbb{R}^n$, we have

$$\begin{aligned} f_N^Z(z) &= \frac{1}{N}\sum_{i=1}^N \phi_\varepsilon(z-Z_i) = \frac{1}{N\,\varepsilon^n}\sum_{i=1}^N \phi\left(\frac{z-Z_i}{\varepsilon}\right) \\ &\leq \frac{\|\phi\|_\infty}{N\,\varepsilon^n}\#\{i \text{ such that } |z-Z_i|_\infty \leq \tfrac{L\varepsilon}{2}\} \\ \|f_N^Z\|_\infty &\leq \frac{\|\phi\|_\infty}{N\,\varepsilon^n}\sup_{z\in\mathbb{R}^n}\#\{i \text{ such that } |z-Z_i|_\infty \leq \tfrac{L\varepsilon}{2}\}, \end{aligned}$$

where $\#$ stands for the cardinal (of a finite set). It remains to bound the supremum on all the cardinals. The first step will be to replace the sup on all the $z \in \mathbb{R}^n$ by a supremum on a finite number of points. For this, we cover $[-R^0, R^0]^n$ by M cubes C_k of size $L\varepsilon$, centered at the points $(c_k)_{k\leq M}$. The number M of squares needed depends on N via ε, and is bounded by

$$M \leq \left[\frac{2(R^0+1)}{L\varepsilon}\right]^n.$$

Next, for any $z \in \mathbb{R}^d$, there exists a $k \leq M$ such that $|z-c_k| \leq \frac{L\varepsilon}{2}$. This implies that

$$\sup_{z\in\mathbb{R}^n}\#\{i \text{ such that } |z-Z_i|_\infty \leq \tfrac{L\varepsilon}{2}\} \leq \sup_{k\leq M}\#\{i \text{ such that } |c_k-Z_i|_\infty \leq L\varepsilon\}.$$

Now we set $H_k^N := \#\{i \text{ such that } |c_k - Z_i|_\infty \leq L\varepsilon\}$. H_k^N follows a binomial law $B(N, p_k)$ with $p_k = \int_{2C_k} f(z)\,dz$, where $2C_k$ denotes the square with center c_k, but size $2L\varepsilon$. Remark that

$$p_k \leq \bar{p} := (2L\varepsilon)^n\|f\|_\infty.$$

For any λ, the exponential moments of H_k^N are therefore given and bounded by

$$\begin{aligned} \mathbb{E}(e^{\lambda H_k^N}) &= \left[1+(e^\lambda-1)p_k\right]^N \\ &\leq \left[1+(e^\lambda-1)(2L\varepsilon)^n\|f\|_\infty\right]^N \\ &\leq e^{(e^\lambda-1)N(2L\varepsilon)^n\|f\|_\infty}. \end{aligned}$$

Now for the supremum of the H_k^N

$$\begin{aligned} \mathbb{E}(e^{\lambda\sup_k H_k^N}) &\leq \mathbb{E}(e^{\lambda H_1^N}) + \cdots + \mathbb{E}(e^{\lambda H_M^N}) \\ &\leq M e^{(e^\lambda-1)N(2L\varepsilon)^n\|f\|_\infty} \\ &\leq \left[\frac{2(R^0+1)}{L\varepsilon}\right]^n e^{(e^\lambda-1)N(2L\varepsilon)^n\|f\|_\infty}. \end{aligned}$$

Using finally Chebyshev's inequality, we get for any $\beta > 0$

$$\begin{aligned}\mathbb{P}\left(\|f_N^z\|_\infty \geq \beta(2L)^n\|\phi\|_\infty\|f\|_\infty\right) &\leq \mathbb{P}\left(\sup_k H_k^N \geq \beta\|f\|_\infty N(2L\varepsilon)^n\right)\\ &\leq \mathbb{E}(e^{\lambda \sup_k H_k^N})e^{-\lambda\beta\|f\|_\infty N(2L\varepsilon)^n}\\ &\leq \left[\frac{2(R^0+1)}{L\varepsilon}\right]^n e^{\left(e^\lambda-1-\lambda\beta\right)N(2L\varepsilon)^n\|f\|_\infty}.\end{aligned}$$

For $\beta > 1$, the optimal λ is $\ln\beta$ and we get with $c_\phi = (2L)^n\|\phi\|_\infty$

$$\mathbb{P}\left(\|f_N^z\|_\infty \geq \beta c_\phi\|f\|_\infty\right) \leq \left[\frac{2(R^0+1)}{L\varepsilon}\right]^n e^{-(\beta\ln\beta-\beta+1)N(2L\varepsilon)^n\|f\|_\infty}.$$

With the scaling $\varepsilon(N) = N^{-\frac{\gamma}{n}}$, we get

$$\mathbb{P}\left(\|f_N^z\|_\infty \geq \beta c_\phi\|f\|_\infty\right) \leq c_0 N^\gamma e^{-(\beta\ln\beta-\beta+1)(2L)^n\|f\|_\infty N^{1-\gamma}}.$$

Remark finally that the choice of scale $\varepsilon(N) = (\ln N)N^{-\frac{1}{n}}$ is also sufficient to get a probability vanishing faster than any inverse power. □

A.2. Existence of strong solutions to Equation (1.5)

This subsection is devoted to the proof of Lemma 3.

Proof of Lemma 3.. – Given the estimate on f, $\rho = \int f\,dv$ also belongs to L^∞ with the bound

$$\|\rho(t,.)\|_{L^\infty(\mathbb{R}^d)} \leq C\,R_V(t)^d\,\|f(t,.,.)\|_{L^\infty(\mathbb{R}^{2d})}.$$

As we have (1.2) with $\alpha < d-1$, $E = F \star_x \rho$ is Lipschitz. Therefore the solution to (1.5) is given by the characteristics. Namely, we define X and V the unique solutions to

$$\begin{aligned}&\partial_t X(t,s,x,v) = V(t,s,x,v), \qquad && \partial_t V(t,s,x,v) = E(t,X(t,s,x,v)),\\ &X(s,s,x,v) = x, && V(s,s,x,v) = v.\end{aligned}$$

The solution f is now given by

$$f(t,x,v) = f(0,X(0,t,x,v),V(0,t,x,v)),$$

with the consequence that

$$R_X(t) \leq R_X(0) + \int_0^t R_V(s)\,ds, \quad R_V(t) \leq R_V(0) + \int_0^t \|E(s,.)\|_{L^\infty}\,ds.$$

Then, the use of

$$\|E\|_{L^\infty} \leq \|\rho\|_{L^1}^{1-\alpha/d}\,\|\rho\|_{L^\infty}^{\alpha/d},$$

leads to the required inequality. To conclude it is enough to notice that the L^1 and L^∞ norms of f are preserved in this case. This again holds because f is transported along the flow of a sufficiently smooth vector-field, at least $W^{1,1}$. See [2] for the details. □

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