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LOCAL *Tb* THEOREM WITH *L*² TESTING CONDITIONS AND GENERAL MEASURES: CALDERÓN-ZYGMUND OPERATORS

BY MICHAEL T. LACEY AND HENRI MARTIKAINEN

ABSTRACT. — Local Tb theorems with L^p type testing conditions have been studied widely in the case of the Lebesgue measure. Such conditions are tied to the scale of the given test function's supporting cube. Until very recently, local Tb theorems in the non-homogeneous case had only been proved assuming scale invariant (L^∞ or BMO) testing conditions. Moving past such strong assumptions in non-homogeneous analysis is a key problem. In a previous paper we overcame this obstacle in the model case of square functions defined using general measures. In this paper we finally tackle the very demanding case of Calderón-Zygmund operators. That is, we prove a non-homogeneous local Tb theorem with L^2 type testing conditions for all Calderón-Zygmund operators. In doing so we prove general twisted martingale transform inequalities which turn out to be subtle in our general framework.

RÉSUMÉ. — Les théorèmes Tb avec conditions de type L^p pour une famille de fonctions de test indexées par les cubes ont été étudiés abondamment dans le cadre de la mesure de Lebesgue. Jusqu'à très récemment, les théorèmes Tb locaux dans les espaces non doublants ont été obtenus sous des conditions invariantes par transformation affine (L^∞ ou BMO). Se dispenser de cette invariance complique la tâche. Dans un article précédent, nous avons développé une méthode permettant de surmonter cette difficulté dans un cas modèle de fonctions carrées définies à l'aide de mesures générales. Dans cet article, on s'attaque au cas des opérateurs de Calderón-Zygmund. Plus précisément, on démontre un théorème Tb local dans le cas non doublant avec des conditions de test L^2 pour tous les opérateurs de Calderón-Zygmund. Un ingrédient essentiel est le contrôle d'une transformation de martingale tordue qui s'avère subtile dans notre cadre.

1. Introduction

In this paper we prove the boundedness of a Calderón-Zygmund operator T on $L^2(\mu)$, where μ can be non-homogeneous, assuming only the existence of certain non-degenerate test functions satisfying local L^2 conditions. For a given test function b_Q , associated with

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a cube $Q \subset \mathbb{R}^n$, the assumptions concern only the scale of Q (unlike, say, with L^∞ or BMO conditions). This is a key difficulty made much harder by the fact that we allow general measures. Indeed, such local Tb theorems with L^p testing functions are known in the homogeneous case, but proving such a result in the non-homogeneous setting is delicate. Here we are able to do this for the first time. The proof requires extensive development and usage of the techniques of non-homogeneous and two-weight dyadic analysis.

Let us begin by introducing the setting and formulate our main theorem. We assume that μ is a measure on \mathbb{R}^n satisfying only the size condition $\mu(B(x,r)) \lesssim r^m$ for some m. We consider Calderón-Zygmund operators T in this setting. First of all, this means that there is a kernel $K \colon \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x=y\} \to \mathbb{C}$ for which there holds for some $C < \infty$ and $\alpha > 0$ that

$$\begin{split} |K(x,y)| &\leq \frac{C}{|x-y|^m}, \qquad x \neq y, \\ |K(x,y) - K(x',y)| &\leq C \frac{|x-x'|^\alpha}{|x-y|^{m+\alpha}}, \qquad |x-y| \geq 2|x-x'|, \end{split}$$

and

$$|K(x,y) - K(x,y')| \le C \frac{|y - y'|^{\alpha}}{|x - y|^{m + \alpha}}, \qquad |x - y| \ge 2|y - y'|.$$

Secondly, we demand that T is a linear operator satisfying the identity

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) d\mu(y), \qquad x \notin \operatorname{spt} f.$$

In this paper we assume a priori that $T: L^2(\mu) \to L^2(\mu)$ boundedly. We are after a new quantitative bound for ||T||, independent of the a priori bound. Such practice is standard, and one can deduce to this situation by, for example, considering suitably truncated operators.

We are ready to state our main theorem—a non-homogeneous local Tb theorem with L^2 type testing conditions for all Calderón-Zygmund operators.

- 1.1. THEOREM. Suppose that $T: L^2(\mu) \to L^2(\mu)$ is a bounded Calderón-Zygmund operator with an adjoint operator T^* . We assume that to every cube $Q \subset \mathbb{R}^n$ there is associated two functions b_Q^T and $b_Q^{T^*}$ satisfying that
 - 1. spt $b_Q^T \subset Q$ and spt $b_Q^{T^*} \subset Q$;
 - 2. $\left| \int_{Q} b_{Q}^{T} d\mu \right| \gtrsim \mu(Q) \text{ and } \left| \int_{Q} b_{Q}^{T^{*}} d\mu \right| \gtrsim \mu(Q);$
 - 3. $\|b_Q^T\|_{L^2(\mu)}^2 \lesssim \mu(Q)$ and $\|b_Q^{T^*}\|_{L^2(\mu)}^2 \lesssim \mu(Q)$;
 - 4. $\|1_Q T b_Q^T\|_{L^2(\mu)}^2 \lesssim \mu(Q)$ and $\|1_Q T^* b_Q^{T^*}\|_{L^2(\mu)}^2 \lesssim \mu(Q)$.

Then we have that $||T|| \lesssim 1$.

Recently in [11] we proved a version of this theorem for square functions defined in the upper half-space. While of independent interest because of the genuinely different context, it is a result with a much simpler proof than the current one. Indeed, the square functions essentially provide a model framework where many technicalities of the Calderón-Zygmund world do not arise. One of them is that the diagonal is completely trivial for square functions while extremely delicate for Calderón-Zygmund operators. Another difference is that the recent Whitney averaging identity over good cubes of Martikainen and Mourgoglou [15]

makes certain probabilistic arguments easy even in the local Tb situation. A critical difference is the fact that the paraproduct operator is much simpler in the square function case.

Before going more to the history and context, we want to discuss the proof of our main theorem, Theorem 1.1, and the references most related to our techniques. The proof is quite simply begun by reducing to the non-homogeneous T1 theorem of Nazarov-Treil-Volberg [17]. More specifically, a local formulation following directly from this is used:

$$\|T\| \leq C_1 + C_2 \sup_{\substack{Q_0 \subset \mathbb{R}^n \\ Q_0 \text{ cube } |f|, |g| \leq 1_{Q_0}}} \frac{|\langle Tf, g \rangle|}{\mu(\lambda Q_0)}.$$

Here $\lambda > 1$ is some fixed large constant. This reduces things to proving that

$$|\langle Tf, g \rangle| \le (C_3 + c||T||)\mu(\lambda Q_0),$$

where c can be taken to be arbitrarily small. Two independent random cubes Q^* and R^* for which $Q_0 \subset Q^* \subset \lambda Q_0$ and $Q_0 \subset R^* \subset \lambda Q_0$ are then used to expand the fixed bounded functions f and g dyadically in to martingale differences adapted to the local test functions.

We now come to the essentials. To handle the complicated paraproducts we require a non-homogeneous version of the twisted martingale difference inequalities of Auscher-Routin [2] or Lacey-Vähäkangas [13]. This is Proposition 2.4 of our current paper—a result of independent interest. Indeed, the proof of Proposition 2.4 turns out to be a demanding task. The key reason lies in the fact that even if we have performed a stopping time argument which gives us that a fixed test function b_F^T behaves nicely on a cube Q i.e., $\int_Q |b_F^T|^2 d\mu \lesssim \mu(Q)$, we cannot say much what happens in the stopping children of Q. That is, in a stopping child Q' of Q we cannot use the simple argument

$$\int_{Q'} |b_F^T|^2 d\mu \le \int_{Q} |b_F^T|^2 d\mu \lesssim \mu(Q) \lesssim \mu(Q')$$

which would only be available if μ would be doubling.

Instead, the proof of Proposition 2.4 becomes about controlling maximal truncations of certain half-twisted martingales $\sum_Q \epsilon_Q D_Q$. Even if we are interested in an L^2 result, we find it convenient to prove a weak type bound for every $p \in (1, \infty)$ and interpolate this (the half-twisted martingales will be L^p bounded for every p unlike the original twisted martingales). But such a weak type bound can be reduced to a testing condition—an idea originally by Sawyer [18], but which can essentially also be found from e.g., [9] by Hytönen et al. The verification of this testing inequality is based crucially on controlling $\sum_Q \epsilon_Q D_Q 1$ in L^p . This control is proved by reducing to the case p=1 using a non-homogeneous John-Nirenberg principle formulated at least by Lacey-Petermichl-Reguera [12] and Hytönen-Pérez-Treil-Volberg [8].

Proposition 2.4 is formulated in such a way that essentially the stopping generation is fixed. For this reason we perform an argument which gives that in the expansion of the pairing $\langle Tf,g\rangle$ we can use only finitely many generations of stopping cubes. This follows from the Carleson property of the stopping cubes by noticing that the large generations provide only an absorbable error. The fact that the functions f and g are bounded plays a role in this reduction, and also later in the proof when we prove the boundedness of a certain paraproduct.

After this, the pairing is split in to standard pieces: separated, nested, diagonal. The goodness is inserted only to the nested sum—an idea already used by Hytönen-Martikainen [7]. The point of adding the goodness like this is to guarantee the collapse of the paraproduct. The crucial thing is that the paraproduct arising from this sum can now be handled using an argument by Lacey-Vähäkangas [14], the non-homogeneous twisted martingale difference inequality proved before, and the reduction to finitely many generations of stopping cubes. The final part of the proof is to deal in this non-homogeneous setting with the extremely delicate surgery of the diagonal using only L^2 test functions.

We have given the technical foundation and references related to these latest techniques. But let us now discuss the history and overall context of the problem. The first local Tb theorem, with L^{∞} control of the test functions and their images, is by Christ [4]. This was proven for doubling measures. Nazarov, Treil and Volberg [16] obtained a nonhomogeneous version of this theorem. The point compared to the global Tb theorems is as follows. The accretivity of a given test function b_Q is only assumed on its supporting cube Q i.e., $|\int_Q b_Q d\mu| \gtrsim \mu(Q)$. While in a global Tb one needs a function which is simultaneously accretive on all the scales. But the remaining conditions are still completely scale invariant: $b_Q \in L^{\infty}(\mu)$ and $Tb_Q \in L^{\infty}(\mu)$ (or Tb_Q belongs to some non-homogeneous BMO space). This scale invariance of the testing conditions is the main thing one wants to get rid of.

The idea of using (in the homogeneous situation) just local L^p type testing conditions was introduced over ten years ago by Auscher, Hofmann, Muscalu, Tao and Thiele [1]. However, their proof works only for the so-called perfect dyadic singular integral operators. The assumptions are of the form $\int_Q |b_Q^1|^p \leq |Q|, \int_Q |b_Q^2|^q \leq |Q|, \int_Q |T^b_Q^1|^{q'} \leq |Q|$ and $\int_Q |T^*b_Q^2|^{p'} \leq |Q|$, where s' denotes the dual exponent of s and $1 < p, q \leq \infty$. Our interest here is solving the long open problem of extending to general measures. That is, we study the p=q=2 case for all Calderón-Zygmund operators in the non-homogeneous setting. Even in the homogeneous setting extending the result of [1] to general Calderón-Zygmund operators is complicated.

Hofmann [6] was able to extend to general Calderón-Zygmund operators but at the price of needing a stronger set of assumptions: $\int_Q |b_Q^1|^s \le |Q|$, $\int_Q |b_Q^2|^s \le |Q|$, $\int_Q |Tb_Q^1|^2 \le |Q|$ and $\int_Q |T^*b_Q^2|^2 \le |Q|$ for some s>2. Auscher and Yang [3] established the theorem for standard Calderón-Zygmund operators in the case $1/p+1/q\le 1$ (and thus in the case p=q=2).

We mention that there is also the question of considering the case 1/p + 1/q > 1. While general exponents are not part of this paper, it has been an extremely active area in the homogeneous world. Hofmann [5] has given a full solution in the case of square functions. In the Calderón-Zygmund world the work of Auscher and Routin [2] continued to shed some light to the general case of exponents, however, not giving a definite answer and involving additional technical conditions. The (almost) full solution is given by Hytönen and Nazarov [10].

1.1. Notation

Consider a dyadic grid \mathcal{D} in \mathbb{R}^n . For $Q \in \mathcal{D}$ we use the following notation:

- $\ell(Q)$ is the side length of Q;
- $\operatorname{ch}(Q) = \{ Q' \in \mathcal{D} : Q' \subset Q, \, \ell(Q') = \ell(Q)/2 \};$
- gen(Q) is determined by $\ell(Q) = 2^{-\text{gen}(Q)}$;
- $Q^{(k)} \in \mathcal{D}$ is the unique cube for which $\ell(Q^{(k)}) = 2^k \ell(Q)$ and $Q \subset Q^{(k)}$.

The average of a function f with respect to the measure μ on a set A is denoted $\langle f \rangle_A = \mu(A)^{-1} \int_A f \, d\mu$. We also use the pairing $\langle f, g \rangle = \int f g \, d\mu$.

The notation $f \lesssim g$ is used to write $f \leq Cg$ for some constant C. This implied constant C is always allowed to depend on the dimension n of the space \mathbb{R}^n , the upper bound m for the dimension of the measure μ , the kernel constants of T and on the implied constants of Theorem 1.1. If we do some L^p estimates we also let the constants depend on p. However, when we need to track the dependence on some parameter s, we usually explicitly write C(s) for a large constant and c(s) for a small constant. In Subsections 2.3 and 2.4 we record the other parameters that we in some cases may absorb to the implied constants.

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2. Beginning of the proof

Let $\lambda > 1$ be a fixed large constant. We begin by noting that by [17] there holds that

$$||T|| \le C_1 + C_2 \sup_{\substack{Q_0 \subset \mathbb{R}^n \\ Q_0 \text{ cube } |f|, |g| \le 1_{Q_0}}} \frac{|\langle Tf, g \rangle|}{\mu(\lambda Q_0)}.$$

Let us fix a cube $Q_0 \subset \mathbb{R}^n$ and functions f, g such that $|f|, |g| \leq 1_{Q_0}$. It suffices to prove that

$$|\langle Tf, q \rangle| < (C_3 + c||T||)\mu(\lambda Q_0),$$

where c is so small that $C_2c \leq 1/2$.

For small notational convenience we assume that $c_{Q_0}=0$ (that is, Q_0 is centred at the origin). Let $N\in\mathbb{Z}$ be defined by the requirement $2^{N-3}\leq \ell(Q_0)<2^{N-2}$. Consider two independent random squares $Q^*=Q^*(w)=w+[-2^N,2^N)^n$ and $R^*=R^*(w')=w'+[-2^N,2^N)^n$, where $w,w'\in[-2^{N-1},2^{N-1})^n$. The cubes Q^* and Q^* are taken to be the starting cubes of the independent grids Q^* and Q^* (only the cubes inside Q^* and Q^* are included in these grids). The probability measure is the normalized Lebesgue measure on the square $[-2^{N-1},2^{N-1})^n$. Furthermore, note that always spt Q^* 0 and Q^* 1 with some absolute constant Q^* 2. There also holds that $Q^* \cup Q^* \subset \lambda Q_0$ choosing Q^* 3 large enough.

2.1. Martingale difference operators

Let $M_{\mu}h(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h| d\mu$ be the centred maximal function. This is an $L^2(\mu)$ bounded operator. For a small convenience we may assume the normalization $\langle b_Q^T \rangle_Q = 1 = \langle b_Q^{T^*} \rangle_Q$. Let also A denote the constant for which $\|b_Q^T\|_{L^2(\mu)}^2 + \|b_Q^{T^*}\|_{L^2(\mu)}^2 \le A\mu(Q)$ and B the constant for which $||1_Q T b_Q^T||_{L^2(\mu)}^2 + ||1_Q T^* b_Q^{T^*}||_{L^2(\mu)}^2 \le B\mu(Q)$.

Let $\mathcal{F}^1_{Q^*}$ consist of the maximal cubes $Q \in \mathcal{D}^T$, $Q \subset Q^*$, for which at least one of the following three conditions holds:

- $$\begin{split} &1. \ |\langle b_{Q^*}^T \rangle_Q| < 1/2; \\ &2. \ \langle |M_\mu b_{Q*}^T|^2 \rangle_Q > 16A^2 \|M_\mu\|^2; \\ &3. \ \langle |Tb_{Q*}^T|^2 \rangle_Q > 16AB. \end{split}$$

Next, one repeats the previous procedure by replacing Q^* with a fixed $Q \in \mathcal{F}^1_{Q^*}$. The combined collection of stopping cubes resulting from this is called $\mathcal{F}_{Q^*}^2$. This is continued and one sets $\mathcal{F}_{Q^*} = \bigcup_{j=0}^{\infty} \mathcal{F}_{Q^*}^j$. Finally, for every $Q \in \mathcal{D}^T$, $Q \subset Q^*$, we let $Q^a \in \mathcal{F}_{Q^*}$ be the minimal cube $S \in \mathcal{F}_{Q^*}$ for which $Q \subset S$.

Similarly, let $\mathcal{F}_{R^*}^1$ consist of the maximal cubes $R \in \mathcal{D}^{T^*}$, $R \subset R^*$, for which at least one of the following three conditions holds:

- $\begin{array}{l} 1. \ |\langle b_{R^*}^{T^*} \rangle_R| < 1/2; \\ 2. \ \langle |M_\mu b_{R^*}^{T^*}|^2 \rangle_R > 16A^2 \|M_\mu\|^2; \\ 3. \ \langle |T^* b_{R^*}^{T^*}|^2 \rangle_R > 16AB. \end{array}$

We define \mathcal{F}_{R^*} and R^a analogously as above.

The following results are proved essentially in [11].

2.1. Lemma. – For $F \in \mathcal{F}_{O^*}^j$ there holds that

$$\sum_{\substack{S \in \mathcal{F}_{Q^*}^{j+1} \\ S \subset F}} \mu(S) \le \left(1 - \frac{1}{8A}\right) \mu(F) =: \tau \mu(F), \qquad \tau < 1.$$

2.2. COROLLARY. – We have the following Carleson estimate:

$$\sum_{\substack{F \in \mathcal{F}_{Q^*} \\ F \subset Q}} \mu(F) \lesssim \mu(Q), \qquad Q \in \mathcal{D}^T, \, Q \subset Q^*.$$

If $Q \in \mathcal{D}^T$, $Q \subset Q^*$, and $h \in L^1_{\mathrm{loc}}(\mu)$, we define the twisted martingale difference operators

$$\Delta_Q h = \sum_{Q' \in \operatorname{ch}(Q)} \left[\frac{\langle h \rangle_{Q'}}{\langle b_{(Q')^a}^T \rangle_{Q'}} b_{(Q')^a}^T - \frac{\langle h \rangle_Q}{\langle b_{Q^a}^T \rangle_Q} b_{Q^a}^T \right] 1_{Q'}.$$

The operators Δ_R , $R \in \mathcal{D}^{T^*}$, $R \subset R^*$, are analogously defined.

2.2. General twisted martingale transform inequalities

Most of the time we can make do with the following square function estimate.

2.3. Lemma. – Suppose $|h| \le 1$. Then there holds that

$$\sum_{Q \in \mathcal{D}^T} \|\Delta_Q h\|_{L^2(\mu)}^2 \lesssim \mu(Q^*).$$

Proof. – The proof is like that of Proposition 2.5 of [11] - except easier because of the assumption $|h| \le 1$.

But for a few key times (in connection with paraproducts) we absolutely depend on the variant presented in the next proposition. It is much stronger in many sense but requires that we set $Q^a = F$ for a fixed $F \in \mathcal{F}_{Q^*}$. It is not a triviality to then sum over all the generations of stopping cubes F. However, this is an issue that we do not care about since our proof of the local Tb theorem incorporates a reduction to finitely many stopping generations. So we could use the bound that follows in every situation. But just to stress that this stronger estimate and the reduction to finitely many generations is really needed only when dealing with the paraproduct, we mostly use the above bound.

We shall use the following notation. If $F \in \mathcal{F}_{Q^*}$, we let $j \in \mathbb{N}$ be such that $F \in \mathcal{F}_{Q^*}^j$ and define $\mathcal{H} = \mathcal{H}_F = \{H \in \mathcal{F}_{Q^*}^{j+1}: H \subset F\}$.

2.4. PROPOSITION. – Suppose $F \in \mathcal{F}_{Q^*}$ and $h \in L^2(\mu)$. Suppose also that we have constants ϵ_Q , $Q \in \mathcal{D}^T$, which satisfy $|\epsilon_Q| \leq 1$. Then there holds that

$$\left\| \sum_{\substack{Q \in \mathcal{I}^T \\ Q^a = F}} \epsilon_Q \Delta_Q h \right\|_{L^2(\mu)}^2 \lesssim \|h\|_{L^2(\mu)}^2.$$

Proof. – Consider a cube $Q \in \mathcal{D}^T$ for which $Q^a = F$. We define

$$D_Q h := \sum_{Q' \in \operatorname{ch}(Q) \setminus \mathcal{H}} \left[\frac{\langle h \rangle_{Q'}}{\langle b_F^T \rangle_{Q'}} - \frac{\langle h \rangle_Q}{\langle b_F^T \rangle_Q} \right] 1_{Q'}.$$

Our aim is to reduce to these operators. However, for technical reasons certain maximal truncations of them will be needed. Anyway, begin by noticing that

$$\begin{split} \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F}} \epsilon_Q \Delta_Q h \cdot 1_{F \setminus \bigcup \mathcal{H}} \right\|_{L^2(\mu)}^2 &= \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F}} \epsilon_Q D_Q h \cdot b_F^T 1_{F \setminus \bigcup \mathcal{H}} \right\|_{L^2(\mu)}^2 \\ &\lesssim \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F}} \epsilon_Q D_Q h \right\|_{L^2(\mu)}^2, \end{split}$$

since $b_F^T 1_{F \setminus \bigcup \mathcal{H}} \in L^{\infty}(\mu)$.

We are reduced to controlling

$$\sum_{H \in \mathcal{H}} \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ H^{(1)} \subset Q \subset F}} \epsilon_Q \Delta_Q h \cdot 1_H \right\|_{L^2(\mu)}^2.$$

We write

$$\sum_{\substack{Q \in \mathcal{D}^T \\ H^{(1)} \subset Q \subset F}} \epsilon_Q \Delta_Q h \cdot 1_H = \epsilon_{H^{(1)}} \left(\langle h \rangle_H b_H^T - \frac{\langle h \rangle_{H^{(1)}}}{\langle b_F^T \rangle_{H^{(1)}}} b_F^T 1_H \right) + \sum_{\substack{Q \in \mathcal{D}^T \\ H^{(2)} \subset Q \subset F}} \epsilon_Q D_Q h \cdot b_F^T 1_H.$$

We have that

$$\sum_{H \in \mathcal{H}} |\epsilon_{H^{(1)}}|^2 |\langle h \rangle_H|^2 \|b_H^T\|_{L^2(\mu)}^2 \lesssim \sum_{H \in \mathcal{H}} \int_H |h|^2 \, d\mu \leq \|h\|_{L^2(\mu)}^2.$$

Next, notice that

$$\begin{split} \sum_{H \in \mathcal{H}} |\epsilon_{H^{(1)}}|^2 \frac{|\langle h \rangle_{H^{(1)}}|^2}{|\langle b_F^T \rangle_{H^{(1)}}|^2} \int_H |b_F^T|^2 \, d\mu &\lesssim \sum_{H \in \mathcal{H}} |\langle h \rangle_{H^{(1)}}|^2 \int_H |b_F^T|^2 \, d\mu \\ &= \sum_{Q \in \mathcal{Q}^T} |\langle h \rangle_Q|^2 \alpha_Q \lesssim \|h\|_{L^2(\mu)}^2, \end{split}$$

since

$$\alpha_Q := \begin{cases} \sum_{Q' \in \operatorname{ch}(Q) \cap \mathcal{H}} \int_{Q'} |b_F^T|^2 \, d\mu, & \text{if } Q^a = F, \\ 0, & \text{otherwise,} \end{cases}$$

is a Carleson sequence. Let us show this fact now.

To this end, fix a cube $R \in \mathcal{D}^T$. We have that

$$\sum_{Q \subset R} \alpha_Q = \sum_{\substack{Q \subset R \ Q' \in \operatorname{ch}(Q) \cap \mathcal{H}}} \int_{Q'} |b_F^T|^2 d\mu \le \int_R |b_F^T|^2 d\mu.$$

We can assume that $R \subset F$ and that there is a $Q \subset R$ for which $Q^a = F$. But then $R^a = F$ and $\int_R |b_F^T|^2 d\mu \lesssim \mu(R)$ proving the Carleson property.

So we are to deal with

$$\sum_{H \in \mathcal{H}} \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ H^{(2)} \subset Q \subset F}} \epsilon_Q D_Q h \cdot b_F^T 1_H \right\|_{L^2(\mu)}^2.$$

But notice that

$$\sum_{Q:\, H^{(2)}\subset Q\subset F} \epsilon_Q D_Q h$$

is constant on $H^{(1)}$. We can then estimate

$$\begin{split} \sum_{H \in \mathcal{H}} & \left| \left\langle \sum_{Q: \, H^{(2)} \subset Q \subset F} \epsilon_Q D_Q h \right\rangle_{H^{(1)}} \right|^2 \int_H |b_F^T|^2 \, d\mu \\ &= \sum_{H \in \mathcal{H}} \left| \left\langle \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \ell(H^{(1)})}} \epsilon_Q D_Q h \right\rangle_{H^{(1)}} \right|^2 \int_H |b_F^T|^2 \, d\mu \\ &\leq \sum_{H \in \mathcal{H}} \left\langle \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q h \right| \right\rangle_{H^{(1)}}^2 \int_H |b_F^T|^2 \, d\mu \end{split}$$

$$= \sum_{R \in \mathcal{D}^T} \left\langle \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q h \right| \right\rangle_R^2 \alpha_R \lesssim \left\| \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q h \right| \right\|_{L^2(\mu)}^2.$$

The conclusion is that the proposition follows from the $L^2(\mu)$ bound of these maximal truncations. But this bound follows from Proposition 2.12.

To control the maximal truncations we need some heavier tools. For the reader's convenience we formulate and prove the needed principles exactly. The first is a reduction to a testing condition (a dyadic adaptation of Sawyer's idea [18] and also essentially contained in [9]). The second is a non-homogeneous John-Nirenberg principle (essentially found in [12] and [8]).

Suppose that for every $Q \in \mathcal{D}^T$ we are given an operator A_Q satisfying:

- $A_Q h = \sum_{Q' \in \operatorname{ch}(Q)} c_{Q'}(h) 1_{Q'}$ for some constants $c_{Q'}(h)$;
- $|A_Q h| \leq M_{\mu}^{\mathcal{D}^T} h$, where $M_{\mu}^{\mathcal{D}^T} h(x) := \sup_{R \in \mathcal{D}^T} 1_R(x) \mu(R)^{-1} \int_R |h| \, d\mu$.

We set

$$\begin{split} A_{\epsilon}h &:= \sum_{\substack{Q \in \mathcal{D}^T \\ \ell(Q) > \epsilon}} A_Q h, \qquad \epsilon > 0, \\ A_{\#}h &:= \sup_{\epsilon > 0} |A_{\epsilon}h|. \end{split}$$

For $P \in \mathcal{D}^T$, let us define

$$\begin{split} A^P_\epsilon h &= \sum_{\substack{Q \in \mathcal{D}^T \\ Q \subset P \\ \ell(Q) > \epsilon}} A_Q h, \qquad \epsilon > 0, \\ A^P_\# h &:= \sup_{\epsilon > 0} |A^P_\epsilon h|. \end{split}$$

The following lemma states that a certain testing condition for maximal truncations implies an estimate $L^p(\mu) \to L^{p,\infty}(\mu)$ for the maximal truncations.

2.5. Lemma. – Let $p \in (1, \infty)$. We assume that for every $Q \in \mathcal{D}^T$ and $h \in L^p(\mu)$ there holds that

(2.6)
$$\int_{Q} A_{\#}^{Q} h \, d\mu \lesssim \|h 1_{Q}\|_{L^{p}(\mu)} \mu(Q)^{1/p'}.$$

Then for every $h \in L^p(\mu)$ there holds that $||A_\# h||_{L^{p,\infty}(\mu)} \lesssim ||h||_{L^p(\mu)}$.

Proof. – We assume qualitatively that $A_Q \neq 0$ for only finitely many $Q \in \mathcal{D}^T$. This gives us the a priori information $||A_\# h||_{L^{p,\infty}(\mu)} < \infty$ for $h \in L^p(\mu)$.

Let $\lambda > 0$ and set $\Omega_{\lambda} := \{A_{\#}h > \lambda\}$. Let M_{λ} consist of the maximal $Q \in \mathcal{D}^T$ for which $Q \subset \Omega_{\lambda}$. It is clear that $\Omega_{\lambda} = \bigcup_{Q \in M_{\lambda}} Q$, since for every $x \in \Omega_{\lambda}$ there is a $Q \in \mathcal{D}^T$ such that $x \in Q \subset \Omega_{\lambda}$.

Let us set

$$H_{\lambda}(Q) := Q \cap \{A_{\#}h > 4\lambda, M_{\mu}^{\mathcal{O}^T}h \le \lambda\}, \qquad Q \in M_{\lambda}.$$

We have that

$$\Omega_{4\lambda} \subset \bigcup_{Q \in M_{\lambda}} H_{\lambda}(Q) \cup \{M_{\mu}^{\mathcal{D}^T} h > \lambda\}$$

so that we have for any $\eta > 0$ that

$$\lambda^{p}\mu(\Omega_{4\lambda}) \leq \lambda^{p} \sum_{Q \in M_{\lambda}} \mu(H_{\lambda}(Q)) + \|M_{\mu}^{\mathcal{D}^{T}} h\|_{L^{p,\infty}(\mu)}^{p}$$

$$\leq \lambda^{p} \sum_{\substack{Q \in M_{\lambda} \\ \mu(H_{\lambda}(Q)) > \eta\mu(Q)}} \mu(H_{\lambda}(Q)) + \eta\lambda^{p}\mu(\Omega_{\lambda}) + C\|h\|_{L^{p}(\mu)}^{p}.$$

Let $Q \in M_{\lambda}$ and $x \in Q$. Let $z \in Q^{(1)}$ be such that $A_{\#}h(z) \leq \lambda$. Let $\epsilon > 0$ be arbitrary. If $\ell(Q) > \epsilon$ we have the following identity

$$A_{\epsilon}h(x) = A_{\epsilon}^Q h(x) + \sum_{\substack{R \in \mathcal{D}^T \\ \ell(Q) < \ell(R) \leq 4\ell(Q)}} A_R h(x) + \sum_{\substack{R \in \mathcal{D}^T \\ 4\ell(Q) < \ell(R)}} A_R h(z).$$

We get the bound that

$$|A_{\epsilon}h(x)| \le A_{\#}^{Q}h(x) + 2M_{\mu}^{\mathcal{D}^{T}}h(x) + A_{\#}h(z).$$

It is clear that this bound holds for every $\epsilon > 0$. Therefore, we have that

$$4\lambda < A_{\#}h(x) \le A_{\#}^Q h(x) + 3\lambda, \qquad Q \in M_{\lambda}, \ x \in H_{\lambda}(Q).$$

This yields that if $Q \in M_{\lambda}$ and $\mu(H_{\lambda}(Q)) > \eta \mu(Q)$, then there holds that

$$\int_{Q} A_{\#}^{Q} h \, d\mu \ge \int_{H_{\lambda}(Q)} A_{\#}^{Q} h \, d\mu \ge \lambda \mu(H_{\lambda}(Q)) \ge \eta \lambda \mu(Q).$$

From this we can conclude that

$$\left(\frac{1}{\mu(Q)}\int_{Q}A_{\#}^{Q}h\,d\mu\right)^{p}\geq\eta^{p}\lambda^{p},\qquad Q\in M_{\lambda},\,\mu(H_{\lambda}(Q))>\eta\mu(Q).$$

We can now see using the assumed testing condition that

$$\lambda^{p} \sum_{\substack{Q \in M_{\lambda} \\ \mu(H_{\lambda}(Q)) > n\mu(Q)}} \mu(H_{\lambda}(Q)) \leq C \eta^{-p} \sum_{\substack{Q \in M_{\lambda} \\ \mu(H_{\lambda}(Q)) > n\mu(Q)}} \|h1_{Q}\|_{L^{p}(\mu)}^{p} \leq C \eta^{-p} \|h\|_{L^{p}(\mu)}^{p}.$$

We have shown that

$$4^{-p}(4\lambda)^{p}\mu(\Omega_{4\lambda}) \le C\eta^{-p} \|h\|_{L^{p}(\mu)}^{p} + \eta\lambda^{p}\mu(\Omega_{\lambda}) + C\|h\|_{L^{p}(\mu)}^{p}.$$

This yields that

$$4^{-p}\|A_{\#}h\|_{L^{p,\infty}(\mu)}^{p} \leq C\eta^{-p}\|h\|_{L^{p}(\mu)}^{p} + \eta\|A_{\#}h\|_{L^{p,\infty}(\mu)}^{p} + C\|h\|_{L^{p}(\mu)}^{p}.$$

Taking $\eta = 4^{-p}/2$ and using the fact that $\|A_\# h\|_{L^{p,\infty}(\mu)} < \infty$ we get the claim. \square

The following two lemmata capture our usage of the non-homogeneous John-Nirenberg principle.

2.7. Lemma. – Suppose that for every $P \in \mathcal{D}^T$ there holds that

$$||A_{\#}^{P}1||_{L^{1}(\mu)} \lesssim \mu(P).$$

Then for every $p \in (1, \infty)$ and for every $P \in \mathcal{D}^T$ there holds that

$$||A_{\#}^{P}1||_{L^{p}(\mu)} \lesssim \mu(P)^{1/p}.$$

Proof. – Follows from the next lemma by taking $\varphi_Q = A_Q 1/C$ for a large enough constant C > 1.

- 2.8. Lemma. Assume that for every $Q \in \mathcal{D}^T$ we are given a function φ_Q such that
- $\varphi_Q = \sum_{Q' \in \operatorname{ch}(Q)} c_{Q'} 1_{Q'}$ for some constants $c_{Q'}$;
- $\|\varphi_Q\|_{L^{\infty}(\mu)} \leq 1$.

For every $P \in \mathcal{D}^T$ we set

$$\Phi_P := \sup_{\epsilon > 0} \bigg| \sum_{\substack{Q \in \mathcal{D}^T \\ Q \subset P \\ \ell(Q) > \epsilon}} \varphi_Q \bigg|.$$

Suppose that for every $P \in \mathcal{D}^T$ there holds that

$$\mu(\{x \in P : \Phi_P(x) > 1\}) \le \mu(P)/2.$$

Then for every $P \in \mathcal{D}^T$ and t > 1 there holds that

$$\mu(\{x \in P : \Phi_P(x) > t\}) \le 2^{-(t-1)/2}\mu(P).$$

Proof. – Fix $P_0 \in \mathcal{D}^T$. Let \mathcal{R}_1 consist of the maximal $R \in \mathcal{D}^T$ such that $R \subset P_0$ and

$$\Big| \sum_{Q: R \subseteq Q \subset P_0} \varphi_Q(x) \Big| > 1, \qquad x \in R.$$

The left-hand side is constant on R so this makes sense. Define $S_1 := \bigcup_{R \in \mathcal{R}_1} R$. We have that:

- $\mu(S_1) \le \mu(\{x \in P_0 : \Phi_{P_0}(x) > 1\}) \le \mu(P_0)/2;$
- $\Phi_{P_0} 1_{P_0 \setminus S_1} \leq 1$;
- For $R \in \mathcal{R}_1$ and $x \in R$ we have that

$$\Big| \sum_{Q: R \subsetneq Q \subset P_0} \varphi_Q(x) \Big| \le 2.$$

For $R_0 \in \mathcal{R}_1$ we let $\mathcal{R}_2^{R_0}$ consist of the maximal $R \in \mathcal{D}^T$ such that $R \subset R_0$ and

$$\left| \sum_{Q: R \subseteq Q \subset R_0} \varphi_Q(x) \right| > 1, \quad x \in R.$$

Let $\mathscr{R}_2:=\bigcup_{R_0\in\mathscr{R}_1}\mathscr{R}_2^{R_0}$ and $S_2:=\bigcup_{R\in\mathscr{R}_2}R.$ We have that:

- $\mu(S_2) \le \mu(P_0)/4$;
- $\Phi_{P_0} 1_{P_0 \setminus S_2} \leq 3$;
- For $R \in \mathcal{R}_2$ and $x \in R$ we have that

$$\Big| \sum_{Q: R \subseteq Q \subset P_0} \varphi_Q(x) \Big| \le 4.$$

Continue like this. We establish collections \mathcal{R}_j and sets $S_j = \bigcup_{R \in \mathcal{R}_j} R$ such that there holds $\mu(S_j) \leq 2^{-j} \mu(P_0)$ and $\Phi_{P_0} 1_{P_0 \setminus S_j} \leq 2j-1$. Let t>1 and choose $j_t \in \mathbb{N}$ such that $2j_t-1 \leq t < 2j_t+1$. We have that

$$\mu(\lbrace x \in P_0 : \Phi_{P_0}(x) > t \rbrace) \le \mu(S_{j_t}) \le 2^{-j_t} \mu(P_0) \le 2^{-(t-1)/2} \mu(P_0).$$

An important tool for us is the following standard maximal truncation estimate for martingale differences.

2.9. Lemma. – Suppose we have constants ϵ_Q , $Q \in \mathcal{D}^T$, which satisfy $|\epsilon_Q| \leq 1$. Let $p \in (1, \infty)$. We have for every $h \in L^p(\mu)$ the estimate

$$\left\| \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ \ell(Q) > \epsilon}} \epsilon_Q \sum_{\substack{Q' \in \operatorname{ch}(Q)}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'} \right| \right\|_{L^p(\mu)}^p \lesssim \|h\|_{L^p(\mu)}^p.$$

Proof. – The claim follows by first using Doob's inequality and then using Burkholder's inequality for martingale transforms. \Box

We need a version of this where we have removed the stopping children.

2.10. COROLLARY. – Suppose $F \in \mathcal{F}_{Q^*}$. Suppose also that we have constants ϵ_Q , $Q \in \mathcal{D}^T$, which satisfy $|\epsilon_Q| \leq 1$. Let $p \in (1, \infty)$. We have for every $h \in L^p(\mu)$ the estimate

$$\| \sup_{\epsilon > 0} \Big| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q \sum_{\substack{Q' \in \operatorname{ch}(Q) \setminus \mathcal{H} \\ Q' = 0}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'} \Big| \, \|_{L^p(\mu)}^p \lesssim \|h\|_{L^p(\mu)}^p.$$

Proof. - Notice that

$$\sup_{\epsilon>0} \left| \sum_{\substack{Q \in \mathcal{O}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q \sum_{\substack{Q' \in \operatorname{ch}(Q) \cap \mathcal{H}}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'}(x) \right| \le \sum_{H \in \mathcal{H}} |\langle h \rangle_H - \langle h \rangle_{H^{(1)}} | 1_H(x)$$

$$\leq 2 \sum_{H \in \mathcal{H}} 1_H(x) M_{\mu}^{\mathcal{D}^T} h(x).$$

But then we have that

$$\| \sup_{\epsilon > 0} \Big| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q \sum_{\substack{Q' \in \operatorname{ch}(Q) \cap \mathcal{H}}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'} \Big| \, \Big\|_{L^p(\mu)}^p \lesssim \sum_{H \in \mathcal{H}} \| 1_H M_{\mu}^{\mathcal{D}^T} h \|_{L^p(\mu)}^p$$

$$\leq \|M_{\mu}^{\mathcal{D}^T} h\|_{L^p(\mu)}^p \lesssim \|h\|_{L^p(\mu)}^p.$$

Combining this with the previous lemma we have the result.

The proof of Proposition 2.12 will be based on a reduction to the testing condition (Lemma 2.5). However, to verify the testing condition we still require the following lemma. It is in the proof of this final lemma that the John-Nirenberg type reductions from above are used.

2.11. Lemma. – Suppose $F \in \mathcal{F}_{Q^*}$. Suppose also that we have constants ϵ_Q , $Q \in \mathcal{D}^T$, which satisfy $|\epsilon_Q| \leq 1$. For every $p \in [1, \infty)$ and $P \in \mathcal{D}^T$ there holds that

$$\left\| \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q \subset P \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q 1 \right| \right\|_{L^p(\mu)}^p \lesssim \mu(P).$$

Proof. – By Lemma 2.7 it suffices to prove that for every $P \in \mathcal{D}^T$ there holds that

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q \subset P \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q 1 \right| d\mu \lesssim \mu(P).$$

Let us write

$$\frac{1}{\langle b_F^T \rangle_{Q'}} - \frac{1}{\langle b_F^T \rangle_Q} = \frac{\langle b_F^T \rangle_Q - \langle b_F^T \rangle_{Q'}}{\langle b_F^T \rangle_Q^2} + \frac{[\langle b_F^T \rangle_Q - \langle b_F^T \rangle_{Q'}]^2}{\langle b_F^T \rangle_Q^2 \langle b_F^T \rangle_{Q'}}.$$

Define $\tilde{\epsilon}_Q := \epsilon_Q/\langle b_F^T \rangle_Q^2$, $Q^a = F$. Note that $|\tilde{\epsilon}_Q| \lesssim 1$, and then that

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \ Q \subset P \\ \ell(Q) > \epsilon}} \tilde{\epsilon}_Q \sum_{\substack{Q' \in \operatorname{ch}(Q) \setminus \mathcal{H}}} [\langle b_F^T \rangle_{Q'} - \langle b_F^T \rangle_Q] 1_{Q'} \right| d\mu$$

$$\leq \mu(P)^{1/2} \left(\int_{P} \left[\sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \ Q \subset P \\ \ell(Q) > \epsilon}} \tilde{\epsilon}_Q \sum_{\substack{Q' \in \operatorname{ch}(Q) \setminus \mathcal{H}}} [\langle b_F^T \rangle_{Q'} - \langle b_F^T \rangle_Q] 1_{Q'} \right| \right]^2 d\mu \right)^{1/2}$$

$$\lesssim \mu(P)^{1/2} ||1_P b_F^T||_{L^2(\mu)} \lesssim \mu(P).$$

Here we first appealed to the L^2 bound for maximal truncations of a martingale difference (Corollary 2.10). For the last inequality we have the following explanation. It is trivial if $F \cap P = \emptyset$ or $F \subset P$. Otherwise, we may assume that there is a Q for which $Q^a = F$ and $Q \subset P \subset F$. But then $P^a = F$.

Next, notice that

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^{T} \\ Q^{a} = F, \ Q \subset P \\ \ell(Q) > \epsilon}} \epsilon_{Q} \sum_{\substack{Q' \in \operatorname{ch}(Q) \setminus \mathcal{H}}} \frac{\left[\langle b_{F}^{T} \rangle_{Q} - \langle b_{F}^{T} \rangle_{Q'} \right]^{2}}{\langle b_{F}^{T} \rangle_{Q'}} 1_{Q'} \right| d\mu$$

$$\lesssim \sum_{\substack{Q \in \mathcal{D}^{T} \\ Q^{a} = F, \ Q \subset P}} \int_{Q' \in \operatorname{ch}(Q)} \left(\langle b_{F}^{T} \rangle_{Q'} - \langle b_{F}^{T} \rangle_{Q} \right) 1_{Q'} \right]^{2} d\mu \lesssim \|1_{P} b_{F}^{T}\|_{L^{2}(\mu)}^{2} \lesssim \mu(P). \qquad \Box$$

2.12. PROPOSITION. – Suppose $F \in \mathcal{F}_{Q^*}$. Suppose also that we have constants ϵ_Q , $Q \in \mathcal{D}^T$, which satisfy $|\epsilon_Q| \leq 1$. Then for every $p \in (1, \infty)$ and $h \in L^p(\mu)$ there holds that

$$\left\| \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q h \right| \right\|_{L^p(\mu)}^p \lesssim \|h\|_{L^p(\mu)}^p.$$

Proof. – Fix $1 , <math>h \in L^p(\mu)$ and $P \in \mathcal{D}^T$. By Lemma 2.5 we need to prove that there holds that

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \ Q \subset P \\ \ell(Q) > \epsilon}} \epsilon_Q D_Q h \right| d\mu \lesssim \|h 1_P\|_{L^p(\mu)} \mu(P)^{1/p'}.$$

Indeed, then we have the weak type bound for every p and we can interpolate the sublinear operator to establish the strong type bounds.

We now write

(2.13)
$$\frac{\langle h \rangle_{Q'}}{\langle b_F^T \rangle_{Q'}} - \frac{\langle h \rangle_Q}{\langle b_F^T \rangle_Q} = \left\{ \frac{\langle h \rangle_{Q'}}{\langle b_F^T \rangle_Q} - \frac{\langle h \rangle_Q}{\langle b_F^T \rangle_Q} \right\} + \left\{ \frac{\langle h \rangle_{Q'}}{\langle b_F^T \rangle_{Q'}} - \frac{\langle h \rangle_{Q'}}{\langle b_F^T \rangle_Q} \right\} \\
= \frac{1}{\langle b_F^T \rangle_Q} \left\{ \langle h \rangle_{Q'} - \langle h \rangle_Q \right\}$$

$$+ \left\{ \langle h \rangle_{Q'} - \langle h \rangle_Q \right\} \left\{ \frac{1}{\langle b_F^T \rangle_{Q'}} - \frac{1}{\langle b_F^T \rangle_Q} \right\}$$

$$+ \langle h \rangle_Q \left\{ \frac{1}{\langle b_F^T \rangle_{Q'}} - \frac{1}{\langle b_F^T \rangle_Q} \right\}.$$

This leaves us with three terms to control.

Define $\tilde{\epsilon}_Q := \epsilon_Q/\langle b_F^T \rangle_Q$, $Q^a = F$. Note that $|\tilde{\epsilon}_Q| \lesssim 1$. The control of (2.13) goes as follows:

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q \subset P \\ \ell(Q) > \epsilon}} \tilde{\epsilon}_Q \sum_{\substack{Q' \operatorname{ch}(Q) \setminus \mathcal{H}}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'} \right| d\mu$$

$$\leq \left\| \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q \subset P \\ \ell(Q) > \epsilon}} \tilde{\epsilon}_Q \sum_{\substack{Q' \operatorname{ch}(Q) \setminus \mathcal{H}}} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'} \right| \right\|_{L^p(\mu)} \mu(P)^{1/p'}$$

$$\leq \|h 1_P\|_{L^p(\mu)} \mu(P)^{1/p'}.$$

Here we used Corollary 2.10.

We will then control (2.14). Let us define

$$\Delta_Q^c h = \sum_{Q' \in \operatorname{ch}(Q)} [\langle h \rangle_{Q'} - \langle h \rangle_Q] 1_{Q'},$$

where c stands for classical. Notice that

$$\Delta_Q^c h \cdot D_Q 1 = \sum_{Q' \in \operatorname{ch}(Q) \backslash \mathcal{H}} \left\{ \langle h \rangle_{Q'} - \langle h \rangle_Q \right\} \left\{ \frac{1}{\langle b_F^T \rangle_{Q'}} - \frac{1}{\langle b_F^T \rangle_Q} \right\} 1_{Q'}.$$

The small point we want to make is that the other martingale can in fact be taken classical, since it is multiplied with D_Q which is supported on the children of Q which are not in \mathcal{H} . Now we have that

$$\left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \ Q \subset P \\ \ell(Q) > \epsilon}} \epsilon_Q \Delta_Q^c h \cdot D_Q 1 \right| \le \left(\sum_{\substack{Q \in \mathcal{D}^T \\ Q \in \mathcal{D}^T \\ \ell(Q) > \epsilon}} |\Delta_Q^c (h 1_P)|^2 \right)^{1/2} \left(\sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \ Q \subset P}} |D_Q 1|^2 \right)^{1/2}.$$

It is enough to note that

$$\left\| \left(\sum_{Q \in \mathcal{D}^T} |\Delta_Q^c(h1_P)|^2 \right)^{1/2} \right\|_{L^p(\mu)} \left\| \left(\sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q \subset P}} |D_Q 1|^2 \right)^{1/2} \right\|_{L^{p'}(\mu)} \lesssim \|h1_P\|_{L^p(\mu)} \mu(P)^{1/p'}.$$

To control the last term we used Lemma 2.11. Indeed, this form follows from it by averaging over independent random signs ± 1 .

We are left to control the term with (2.15). To control the averages $\langle h \rangle_Q$ in front, we will perform a standard stopping time. Let $\mathscr{G}_0 = \{P\}$. Let \mathscr{G}_1 consist of the maximal $R \in \mathscr{D}^T$, $R \subset P$, for which $\langle |h| \rangle_R > 4 \langle |h| \rangle_P$. Continuing this in the standard way we get the full stopping tree $\mathscr{G} = \bigcup_{j=0}^\infty \mathscr{G}_j$. For $Q \in \mathscr{D}^T$, $Q \subset P$, we define Q^s to be the minimal $S \in \mathscr{G}$ for which $Q \subset S$. We have that $\langle |h| \rangle_Q \leq 4 \langle |h| \rangle_{Q^s}$.

If $Q^s = S$ we let $\epsilon_Q(S) := \epsilon_Q \frac{\langle h \rangle_Q}{\langle |h| \rangle_S}$. Notice that $|\epsilon_Q(S)| \le 4|\epsilon_Q| \lesssim 1$. We then estimate using the p=1 case of Lemma 2.11:

$$\int_{P} \sup_{\epsilon > 0} \left| \sum_{S \in \emptyset} \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q^s = S \\ \ell(Q) > \epsilon}} \epsilon_Q \langle h \rangle_Q D_Q 1 \right| d\mu$$

$$\leq \sum_{S \in \emptyset} \int_{S} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q^s = S \\ \ell(Q) > \epsilon}} \epsilon_Q \langle h \rangle_Q D_Q 1 \right| d\mu = \sum_{S \in \emptyset} \langle |h| \rangle_S \int_{S} \sup_{\epsilon > 0} \left| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = F, \, Q^s = S \\ \ell(Q) > \epsilon}} \epsilon_Q(S) D_Q 1 \right| d\mu$$

$$\lesssim \sum_{S \in \emptyset} \langle |h| \rangle_S \mu(S) \leq \left(\sum_{S \in \emptyset} \langle |h| \rangle_S^p \mu(S) \right)^{1/p} \left(\sum_{S \in \emptyset} \mu(S) \right)^{1/p'} \lesssim \|h 1_P\|_{L^p(\mu)} \mu(P)^{1/p'}.$$

This completes the proof of the proposition.

2.16. Remark. – We only need the following conclusion of Proposition 2.4. If $|h| \le 1$ and $S \subset \mathbb{R}^n$ is an arbitrary set, then there holds that

$$\left\| \sum_{\substack{Q \in \mathcal{D}^T : Q \subset S \\ Q^a = F}} \epsilon_Q \Delta_Q h \right\|_{L^2(\mu)}^2 = \left\| \sum_{\substack{Q \in \mathcal{D}^T : Q \subset S \\ Q^a = F}} \epsilon_Q \Delta_Q (1_{S \cap F} h) \right\|_{L^2(\mu)}^2 \lesssim \mu(S \cap F).$$

2.3. Further reductions

We now expand (see Proposition 2.8 of [11])

$$f = \sum_{Q \in \mathcal{D}^T} \Delta_Q f + \langle f \rangle_{Q^*} b_{Q^*}^T$$

and

$$g = \sum_{R \in \mathcal{O}^{T^*}} \Delta_R g + \langle g \rangle_{R^*} b_{R^*}^{T^*}.$$

If $Q \in \mathcal{D}^T$ is such that $Q^a \in \mathcal{F}_{Q^*}^j$, we define $\beta(Q) := j$. Let $\beta > 0$ be a large parameter (we shall fix it momentarily). We have

$$|\langle Tf, g \rangle| \leq \Big| \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \langle T(\Delta_Q f), g \rangle \Big| + \Big| \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) > \beta}} \langle T(\Delta_Q f), g \rangle \Big| + |\langle Tb_{Q^*}^T, g \rangle|.$$

Recall that our functions f and g satisfy $|f|, |g| \leq 1_{Q_0}$. Notice that

$$\begin{split} \Big| \sum_{\substack{Q \in \mathcal{I}^T \\ \beta(Q) \ge \beta}} \langle T(\Delta_Q f), g \rangle \Big| &\leq \sum_{j=\beta}^{\infty} \|T\| \Big\| \sum_{F \in \mathcal{I}_{Q^*}^j} \sum_{\substack{Q \in \mathcal{I}^T \\ Q^a = F}} \Delta_Q f \Big\|_{L^2(\mu)} \|g\|_{L^2(\mu)} \\ &\leq \|T\| \mu(Q_0)^{1/2} \sum_{j=\beta}^{\infty} \Big(\sum_{F \in \mathcal{I}_{Q^*}^j} \Big\| \sum_{\substack{Q \in \mathcal{I}^T \\ Q^a = F}} \Delta_Q f \Big\|_{L^2(\mu)}^2 \Big)^{1/2} \\ &\lesssim \|T\| \mu(Q_0)^{1/2} \sum_{j=\beta}^{\infty} \Big(\sum_{F \in \mathcal{I}_{Q^*}^j} \mu(F) \Big)^{1/2} \\ &\leq \|T\| \mu(Q_0)^{1/2} \sum_{j=\beta}^{\infty} \tau^{j/2} \mu(Q^*)^{1/2} \lesssim \tau^{\beta/2} \|T\| \mu(\lambda Q_0)^{1/2} \|T\| \mu(\lambda Q_0)^{1/2}$$

and $|\langle Tb_{Q^*}^T, g \rangle| \le \|1_{Q^*} Tb_{Q^*}^T\|_{L^2(\mu)} \|g\|_{L^2(\mu)} \lesssim \mu(\lambda Q_0).$

Next, we have

$$\begin{split} \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \langle T(\Delta_Q f), g \rangle &= \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(Q) < \beta}} \langle T(\Delta_Q f), \Delta_R g \rangle + \left\langle T\Big(\sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \Delta_Q f\Big), \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) \ge \beta}} \Delta_R g \right\rangle \\ &+ \langle g \rangle_{R^*} \langle f, 1_{R^*} T^* b_{R^*}^{T^*} \rangle - \langle g \rangle_{R^*} \left\langle T\Big(\sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) \ge \beta}} \Delta_Q f\Big), b_{R^*}^{T^*} \right\rangle \\ &- \langle f \rangle_{Q^*} \langle g \rangle_{R^*} \langle T b_{Q^*}^T, b_{R^*}^{T^*} \rangle. \end{split}$$

Again, there holds that

$$\left| \left\langle T \left(\sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \Delta_Q f \right), \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) \ge \beta}} \Delta_R g \right\rangle \right| \lesssim \tau^{\beta/2} \|T\| \mu(\lambda Q_0)$$

and

$$\left| \langle g \rangle_{R^*} \left\langle T \left(\sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) > \beta}} \Delta_Q f \right), b_{R^*}^{T^*} \right\rangle \right| \lesssim \tau^{\beta/2} \|T\| \mu(\lambda Q_0).$$

Also, we have that $|\langle g \rangle_{R^*} \langle f, 1_{R^*} T^* b_{R^*}^{T^*} \rangle| \lesssim \mu(\lambda Q_0)$.

The pairing $\langle Tb_{O^*}^T, b_{R^*}^{T^*} \rangle$ is trickier. Let u > 0. We estimate

$$\begin{split} |\langle Tb_{Q^*}^T, b_{R^*}^{T^*} \rangle| &\leq \|1_{Q^*} Tb_{Q^*}^T \|_{L^2(\mu)} \|b_{R^*}^{T^*} \|_{L^2(\mu)} + \|T\| \|b_{Q^*}^T \|_{L^2(\mu)} \|1_{(1+u)Q^* \setminus Q^*} b_{R^*}^{T^*} \|_{L^2(\mu)} \\ &+ \int_{R^* \setminus (1+u)Q^*} \int_{Q^*} \frac{C(u)}{\ell(Q^*)^{m/2} \ell(R^*)^{m/2}} |b_{Q^*}^T(y)| |b_{R^*}^{T^*}(x)| \, d\mu(y) \, d\mu(x) \\ &\leq C(u) \mu(\lambda Q_0) + \|T\| \mu(\lambda Q_0)^{1/2} \|1_{(1+u)Q^* \setminus Q^*} b_{R^*}^{T^*} \|_{L^2(\mu)}. \end{split}$$

Notice that with a fixed w' we have that

$$E_w \|1_{(1+u)Q^* \setminus Q^*} b_{R^*}^{T^*}\|_{L^2(\mu)} \le \left(\int_{R^*} P_w(x \in (1+u)Q^* \setminus Q^*) |b_{R^*}^{T^*}(x)| d\mu(x) \right)^{1/2}$$

$$\le c(u)\mu(\lambda Q_0)^{1/2},$$

where $c(u) \to 0$ when $u \to 0$.

The conclusion of this subsection is that

$$\begin{aligned} |\langle Tf, g \rangle| &\leq E_{w,w'} \bigg| \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \langle T(\Delta_Q f), \Delta_R g \rangle \bigg| \\ &+ C(u)\mu(\lambda Q_0) + c(\beta) \|T\|\mu(\lambda Q_0) + c(u)\|T\|\mu(\lambda Q_0), \end{aligned}$$

where $c(\beta) \to 0$ when $\beta \to 0$ and $c(u) \to 0$ when $u \to 0$. We now fix β and u to be so small that $(c(\beta)+c(u))C_2 \le 1/4$. In the sequel some estimates will depend on the fixed parameter β but this is no longer a concern (and the dependance will not be tracked). We may now focus on proving that

$$E_{w,w'} \bigg| \sum_{\substack{Q \in \mathcal{D}^T \\ \beta(Q) < \beta}} \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \langle T(\Delta_Q f), \Delta_R g \rangle \bigg| \le (C_4 + \tilde{c} ||T||) \mu(\lambda Q_0),$$

where \tilde{c} is so small that $C_2\tilde{c} \leq 1/4$.

2.4. Splitting of the summation

We set $\gamma = \alpha/(2m+2\alpha)$, where $\alpha > 0$ appears in the kernel estimates and m appears in $\mu(B(x,t)) \lesssim t^m$. We also let r > 0 be a large constant that we shall fix later. We will also focus on the part of the summation where $\ell(Q) < \ell(R)$. We will simply split this sum in to three standard pieces:

- $Q: \ell(Q) < \ell(R)$ and $d(Q,R) > \ell(Q)^{\gamma} \ell(R)^{1-\gamma}$;
- $Q: \ell(Q) < 2^{-r}\ell(R)$ and $d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma}$;
- $Q: 2^{-r}\ell(R) < \ell(Q) < \ell(R) \text{ and } d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma}$.

We call the first sum the separated sum, the second sum the nested sum and the last sum the diagonal sum. Here the term nested is the most cryptic, but will be justified using probability by introducing good cubes to the second sum in a specific way (like in [7]).

In the next section we will prove that

$$\sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T : \beta(Q) < \beta \\ \ell(Q) < \ell(R) \\ d(Q,R) > \ell(Q)^{1-\gamma}}} |\langle T(\Delta_Q f), \Delta_R g \rangle| \lesssim \mu(\lambda Q_0).$$

In a section after that we will show that

$$E_{w,w'} \Big| \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T : \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} \langle T(\Delta_Q f), \Delta_R g \rangle \Big| \leq C\mu(\lambda Q_0) + c(r) \|T\|\mu(\lambda Q_0),$$

where $c(r) \to 0$ as $r \to 0$. We may then fix the parameter r at this point of the argument to be so small that $C_2c(r) \leq 1/16$. The estimates of the last sum may depend on r, but this is no longer a concern (and the dependance will not be tracked). In the last section we will prove that

$$E_{w,w'} \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T : \beta(Q) < \beta \\ 2^{-r}\ell(R) < \ell(Q) < \ell(R) \\ d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} |\langle T(\Delta_Q f), \Delta_R g \rangle| \le C\mu(\lambda Q_0) + \hat{c} ||T||\mu(\lambda Q_0),$$

where $\hat{c}C_2 \leq 1/16$. Combining with the symmetric argument for the case $\ell(Q) \geq \ell(R)$ this proves our main theorem.

3. The separated sum

If
$$\ell(Q) < \ell(R)$$
 and $d(Q,R) > \ell(Q)^{\gamma} \ell(R)^{1-\gamma}$, then
$$|\langle T(\Delta_Q f), \Delta_R g \rangle| \lesssim A_{QR} ||\Delta_Q f||_{L^2(\mu)} ||\Delta_R g||_{L^2(\mu)},$$

where

$$\begin{split} A_{QR} &:= \frac{\ell(Q)^{\alpha/2} \ell(R)^{\alpha/2}}{D(Q,R)^{m+\alpha}} \mu(Q)^{1/2} \mu(R)^{1/2}; \\ D(Q,R) &:= \ell(Q) + \ell(R) + d(Q,R). \end{split}$$

Moreover, by [17] this ℓ^2 estimate holds

$$\sum_{Q,R} A_{QR} x_Q y_R \lesssim \left(\sum_Q x_Q^2\right)^{1/2} \left(\sum_R y_R^2\right)^{1/2}.$$

Therefore, we have that

Therefore, we have that
$$\sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T : \beta(Q) < \beta \\ \ell(Q) < \ell(R) \\ d(Q,R) > \ell(Q)^{\gamma} \ell(R)^{1-\gamma}}} |\langle T(\Delta_Q f), \Delta_R g \rangle| \lesssim \left(\sum_Q \|\Delta_Q f\|_{L^2(\mu)}^2 \right)^{1/2} \left(\sum_R \|\Delta_R g\|_{L^2(\mu)}^2 \right)^{1/2}$$

$$\lesssim \mu(Q^*)^{1/2}\mu(R^*)^{1/2} \le \mu(\lambda Q_0).$$

4. The nested sum

We now introduce the good and bad cubes to the argument. This is a key technique of non-homogeneous analysis and is originally by Nazarov-Treil-Volberg [17]. However, due to technical reasons we need to insert them to the argument more in the spirit of [7]. In particular, this trick will give the nested structure.

Define $\mathcal{D}^T_{\mathrm{bad},\,A}$ to be the collection of those cubes $Q\in\mathcal{D}^T$ which are bad with respect to some \mathcal{D}^{T^*} -cube of side length A or larger. We define that this demands that there should exist a cube $S\in\mathcal{D}^{T^*}$ for which $\ell(S)\geq A$ and $d(Q,\operatorname{sk} S)\leq \ell(Q)^{\gamma}\ell(S)^{1-\gamma}$, where $\operatorname{sk} S:=\bigcup_{S'\in\operatorname{ch}(S)}\partial S'$. Let $\mathcal{D}^T_{\operatorname{good},\,A}$ be the collection of those $Q\in\mathcal{D}^T$ which are good with respect to all \mathcal{D}^{T^*} -cubes of side length A and larger. This means that for every $S\in\mathcal{D}^{T^*}$ for which $\ell(S)\geq A$ there holds that $d(Q,\operatorname{sk} S)>\ell(Q)^{\gamma}\ell(S)^{1-\gamma}$.

4.1. Remark. – Notice carefully the usage of the words *some* and *all* above.

Let us write

$$\sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T : \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) \leq \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} \langle T(\Delta_Q f), \Delta_R g \rangle = S_{\text{good}} + S_{\text{bad}},$$

where

$$S_{\text{good}} = \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}_{\text{good}, \ell(R)}^T \colon \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) \leq \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} \langle T(\Delta_Q f), \Delta_R g \rangle$$

and

$$S_{\text{bad}} = \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}_{\text{bad}, \, \ell(R)}^T : \beta(Q) < \beta \\ \ell(Q) \le 2^{-r} \ell(R) \\ d(Q, R) \le \ell(Q)^{\gamma} \ell(R)^{1-\gamma}}} \langle T(\Delta_Q f), \Delta_R g \rangle.$$

4.1. The bad part is small

Notice that for a given $R \in \mathcal{D}^{T^*}$ there holds that

$$\left\| \sum_{\substack{Q \in \mathcal{D}_{\mathrm{bad},\,\ell(R)}^T : \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) \leq \ell(Q)^{\gamma}\ell(R)^{1-\gamma} }} \Delta_Q f \right\|_{L^2(\mu)} \leq \sum_{k=r}^{\infty} \Big(\sum_{\substack{Q \in \mathcal{D}_{\mathrm{bad},\,2^k\ell(Q)}^T \\ \ell(Q) = 2^{-k}\ell(R) \\ d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma} }} \|\Delta_Q f\|_{L^2(\mu)}^2 \Big)^{1/2}$$

so that

$$\begin{split} |S_{\text{bad}}| &\leq \|T\| \sum_{k=r}^{\infty} \sum_{R \in \mathcal{D}^{T^*}} \bigg(\sum_{\substack{Q \in \mathcal{D}_{\text{bad}, 2^k \ell(Q)}^T \\ \ell(Q) = 2^{-k} \ell(R) \\ d(Q, R) \leq \ell(Q)^{\gamma} \ell(R)^{1-\gamma}}} \|\Delta_Q f\|_{L^2(\mu)}^2 \bigg)^{1/2} \|\Delta_R g\|_{L^2(\mu)} \\ &\leq C \|T\| \sum_{k=r}^{\infty} \bigg(\sum_{\substack{Q \in \mathcal{D}_{\text{bad}, 2^k \ell(Q)}^T \\ \text{bad}, 2^k \ell(Q)}} \|\Delta_Q f\|_{L^2(\mu)}^2 \bigg)^{1/2} \bigg(\sum_{\substack{R \in \mathcal{D}^{T^*} \\ R \in \mathcal{D}^{T^*}}} \|\Delta_R g\|_{L^2(\mu)}^2 \bigg)^{1/2} \\ &\leq C \|T\| \mu (\lambda Q_0)^{1/2} \sum_{k=r}^{\infty} \bigg(\sum_{\substack{Q \in \mathcal{D}_{\text{bad}, 2^k \ell(Q)}^T \\ Q \in \mathcal{D}_{\text{bad}, 2^k \ell(Q)}^T}} \|\Delta_Q f\|_{L^2(\mu)}^2 \bigg)^{1/2}, \end{split}$$

where we used that

$$\#\{R \in \mathcal{D}^{T^*}: \ell(R) = 2^k \ell(Q) \text{ and } d(Q,R) \le \ell(Q)^{\gamma} \ell(R)^{1-\gamma}\} \lesssim 1, \qquad k \ge 0.$$

We conclude that

$$\begin{split} E_{w,w'}|S_{\text{bad}}| &\leq C \|T\|\mu(\lambda Q_0)^{1/2} E_w \sum_{k=r}^{\infty} \Big(\sum_{Q \in \mathcal{D}^T} \mathbb{P}_{w'}(Q \in \mathcal{D}_{\text{bad}, \, 2^k \ell(Q)}^T) \|\Delta_Q f\|_{L^2(\mu)}^2 \Big)^{1/2} \\ &\leq C \|T\|\mu(\lambda Q_0) \sum_{k=r}^{\infty} 2^{-\gamma k/2} = c(r) \|T\|\mu(\lambda Q_0), \end{split}$$

where $c(r) \to 0$ as $r \to \infty$. We now fix r so that $c(r)C_2 \le 1/16$.

4.2. The good part

In this sum we will have the nested structure. Indeed, notice that Q is good with respect to R, and so we must have $d(Q,\operatorname{sk} R)>\ell(Q)^\gamma\ell(R)^{1-\gamma}$. The possible scenario $d(Q,R)>\ell(Q)^\gamma\ell(R)^{1-\gamma}$ cannot happen here because of the summing condition $d(Q,R)\leq\ell(Q)^\gamma\ell(R)^{1-\gamma}$. Therefore, for the cube Q to stay away from the skeleton of R (i.e., $\operatorname{sk} R$), it must lie deep inside one of the children of R. That is, there is a child $R_Q\in\operatorname{ch}(R)$ so that $Q\subset R_Q$ and $d(Q,R_Q^c)>\ell(Q)^\gamma\ell(R)^{1-\gamma}$.

Before having to split the argument into a case study, we prove two lemmata.

4.2. Lemma. — If
$$R \in \mathcal{D}^{T^*}$$
, $Q \in \mathcal{D}^T_{good,\,\ell(R)}$ and $Q \subset R$, then there holds that
$$|\langle T(\Delta_Q f), 1_{R_Q^c} b_{R^a}^{T^*} \rangle| \lesssim \left(\frac{\ell(Q)}{\ell(R)}\right)^{\alpha/2} \mu(Q)^{1/2} \|\Delta_Q f\|_{L^2(\mu)}.$$

Proof. – We will first show that

$$\int_{R^a \setminus R_Q} \frac{|b_{R^a}^{T^*}(x)|}{|x - c_Q|^{m+\alpha}} d\mu(x) \lesssim \ell(Q)^{-\alpha/2} \ell(R)^{-\alpha/2}.$$

Let M be such that $R_Q^{(M+1)} = R^a$. Notice that since Q is good with respect to all $S \in \mathcal{D}^{T^*}$ for which $\ell(S) \geq \ell(R)$, there holds that

$$d(Q, \partial R_O^{(j)})^{m+\alpha} \gtrsim \ell(Q)^{\alpha/2} \ell(R_O^{(j)})^{m+\alpha/2} \gtrsim 2^{\alpha j/2} \ell(Q)^{\alpha/2} \ell(R)^{\alpha/2} \mu(R_O^{(j+1)}).$$

Here we used that $\gamma(m+\alpha)=\alpha/2$.

We may now estimate

$$\begin{split} \int_{R^a \backslash R_Q} \frac{|b_{R^a}^{T^*}(x)|}{|x - c_Q|^{m + \alpha}} \, d\mu(x) &= \sum_{j=0}^M \int_{R_Q^{(j+1)} \backslash R_Q^{(j)}} \frac{|b_{R^a}^{T^*}(x)|}{|x - c_Q|^{m + \alpha}} \, d\mu(x) \\ &\leq \sum_{j=0}^M \frac{1}{d(Q, \partial R_Q^{(j)})^{m + \alpha}} \int_{R_Q^{(j+1)}} |b_{R^a}^{T^*}(x)| \, d\mu(x) \\ &\lesssim \sum_{j=0}^M \frac{\mu(R_Q^{(j+1)})}{2^{\alpha j/2} \ell(Q)^{\alpha/2} \ell(R)^{\alpha/2} \mu(R_Q^{(j+1)})} \lesssim \ell(Q)^{-\alpha/2} \ell(R)^{-\alpha/2}. \end{split}$$

To end the proof it remains to use the Hölder estimate of K to get that

$$\begin{aligned} |\langle T(\Delta_{Q}f), 1_{R_{Q}^{c}} b_{R^{a}}^{T^{*}} \rangle| &\lesssim \|\Delta_{Q}f\|_{L^{1}(\mu)} \cdot \ell(Q)^{\alpha} \int_{R^{a} \backslash R_{Q}} \frac{|b_{R^{a}}^{T^{*}}(x)|}{|x - c_{Q}|^{m + \alpha}} \, d\mu(x) \\ &\lesssim \left(\frac{\ell(Q)}{\ell(R)}\right)^{\alpha/2} \mu(Q)^{1/2} \|\Delta_{Q}f\|_{L^{2}(\mu)}. \end{aligned}$$

The next lemma is much easier, essentially an argument using only separation (similar arguments appear already in [17], of course).

4.3. Lemma. – Let $R \in \mathcal{D}^{T^*}$, $Q \in \mathcal{D}^T$ be good with respect to R and $Q \subset R$. Then there holds that

$$|\langle T(\Delta_Q f), 1_{R_O^c} \Delta_R g \rangle| \lesssim B_{QR} ||\Delta_Q f||_{L^2(\mu)} ||\Delta_R g||_{L^2(\mu)},$$

where

$$B_{QR} := \Big(\frac{\ell(Q)}{\ell(R)}\Big)^{\alpha/2} \Big(\frac{\mu(Q)}{\mu(R_Q)}\Big)^{1/2}.$$

Proof. – Let $S \in \operatorname{ch}(R)$, $S \neq R_Q$. Then $d(Q, S) \geq d(Q, \partial R_Q) \geq \ell(Q)^{\gamma} \ell(R)^{1-\gamma}$. Using this it is easy to see (like in the separated sum) that there holds that

$$|\langle T(\Delta_Q f), 1_S \Delta_R g \rangle| \lesssim \left(\frac{\ell(Q)}{\ell(R)}\right)^{\alpha/2} \mu(Q)^{1/2} \frac{\mu(S)^{1/2}}{\ell(R)^m} \|\Delta_Q f\|_{L^2(\mu)} \|\Delta_R g\|_{L^2(\mu)}.$$

The claim follows from this since

$$\frac{\mu(S)^{1/2}}{\ell(R)^m} \lesssim \frac{1}{\ell(R)^{m/2}} \lesssim \mu(R_Q)^{-1/2}.$$

Notice that

$$\begin{split} \sum_{R \in \mathcal{D}^{T^*}} \sum_{\substack{Q \in \mathcal{D}^T_{\text{good}, \, \ell(R)} \\ \ell(Q) \leq 2^{-r}\ell(R)}} |\langle T(\Delta_Q f), 1_{R_Q^c} \Delta_R g \rangle| \\ \lesssim \sum_{R \in \mathcal{D}^{T^*}} \sum_{\substack{Q \in \mathcal{D}^T \\ \ell(Q) \leq 2^{-r}\ell(R) \\ Q \subset R}} B_{QR} \|\Delta_Q f\|_{L^2(\mu)} \|\Delta_R g\|_{L^2(\mu)} \\ \lesssim \left(\sum_{Q} \|\Delta_Q f\|_{L^2(\mu)}\right)^{1/2} \left(\sum_{R} \|\Delta_R g\|_{L^2(\mu)}^2\right)^{1/2} \lesssim \mu(Q^*)^{1/2} \mu(R^*)^{1/2} \leq \mu(\lambda Q_0). \end{split}$$

Here we used Lemma 4.3 and the fact that by [17] we have the ℓ^2 estimate

$$\sum_{\substack{R \in \mathcal{D}^{T*} \\ \ell(Q) \leq 2^{-r}\ell(R) \\ Q \subset R_Q \in \operatorname{ch}(R)}} B_{QR} x_Q y_R \lesssim \Big(\sum_Q x_Q^2\Big)^{1/2} \Big(\sum_R y_R^2\Big)^{1/2}.$$

Therefore, we need to only consider

$$S'_{\operatorname{good}} := \sum_{\substack{R \in \mathcal{D}^{T^*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}^T_{\operatorname{good}, \ell(R)} : \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) < \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} \langle T(\Delta_Q f), 1_{R_Q} \Delta_R g \rangle.$$

The case $R_Q^a = R^a$. – Define

$$C_{R_Q} := \frac{\langle g \rangle_{R_Q}}{\langle b_{R^a}^{T^*} \rangle_{R_Q}} - \frac{\langle g \rangle_R}{\langle b_{R^a}^{T^*} \rangle_R}.$$

Writing $1_{R_Q} = 1 - 1_{R_Q^c}$ we see that

$$1_{R_{\mathcal{Q}}}\Delta_{R}g=C_{R_{\mathcal{Q}}}1_{R_{\mathcal{Q}}}b_{R^{a}}^{T^{*}}=C_{R_{\mathcal{Q}}}b_{R^{a}}^{T^{*}}-C_{R_{\mathcal{Q}}}1_{R_{\mathcal{Q}}^{c}}b_{R^{a}}^{T^{*}}.$$

The first part will become part of the paraproduct and we do not touch it further in this subsection.

Notice that now

$$|C_{R_Q}|\mu(R_Q) \lesssim \Big|\int_{R_Q} C_{R_Q} b_{R^a}^{T^*} d\mu\Big| = \Big|\int_{R_Q} \Delta_R g d\mu\Big| \leq \mu(R_Q)^{1/2} \|\Delta_R g\|_{L^2(\mu)}.$$

Therefore, we have using Lemma 4.2 that

$$\begin{split} \sum_{R \in \mathcal{D}^{T^*}} \sum_{Q \in \mathcal{D}^T_{\text{good, }\ell(R)}: \ Q \subset R} |C_{R_Q}| |\langle T(\Delta_Q f), 1_{R_Q^c} b_{R^a}^{T^*} \rangle| \\ \ell(Q) \leq 2^{-r} \ell(R) \\ R_Q^a = R^a \end{split} \lesssim \sum_{R \in \mathcal{D}^{T^*}} \sum_{\substack{Q \in \mathcal{D}^T \\ \ell(Q) \leq 2^{-r} \ell(R) \\ Q \subset R_Q \in \text{ch}(R)}} B_{QR} \|\Delta_Q f\|_{L^2(\mu)} \|\Delta_R g\|_{L^2(\mu)} \lesssim \mu(\lambda Q_0). \end{split}$$

The case $R_Q^a = R_Q$. – We now write

$$1_{R_Q} \Delta_R g = \left(\frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q}^{T^*} \rangle_{R_Q}} b_{R_Q}^{T^*} - \frac{\langle g \rangle_R}{\langle b_{R^a}^{T^*} \rangle_R} b_{R^a}^{T^*} \right) + \frac{\langle g \rangle_R}{\langle b_{R^a}^{T^*} \rangle_R} b_{R^a}^{T^*} 1_{R_Q^c}.$$

The first part is exactly the same thing that we did not touch previously, and we will not do so here either. It will become part of the paraproduct.

But let us notice that Lemma 4.2 again gives that

$$\begin{split} \sum_{R \in \mathcal{D}^{T^*}} \sum_{Q \in \mathcal{D}_{\text{good}, \, \ell(R)}^T \colon Q \subset R} \left| \langle T(\Delta_Q f), \frac{\langle g \rangle_R}{\langle b_{R^a}^{T^*} \rangle_R} b_{R^a}^{T^*} \mathbf{1}_{R_Q^c} \rangle \right| \\ & \qquad \qquad \ell(Q) \leq 2^{-r} \ell(R) \\ & \qquad \qquad R_Q^a = R_Q \end{split} \\ \lesssim \sum_{R \in \mathcal{D}^{T^*}} \sum_{\substack{Q \in \mathcal{D}^T \\ \ell(Q) \leq 2^{-r} \ell(R) \\ Q \subset R_Q \in \text{ch}(R)}} B_{QR} \|\Delta_Q f\|_{L^2(\mu)} \Big(\sum_{\substack{R' \in \text{ch}(R) \\ (R')^a = R'}} \mu(R') \Big)^{1/2} |\langle g \rangle_R| \\ \lesssim \Big(\sum_{Q} \|\Delta_Q f\|_{L^2(\mu)}^2 \Big)^{1/2} \Big(\sum_{R} |\langle g \rangle_R|^2 \sum_{\substack{R' \in \text{ch}(R) \\ (R')^a = R'}} \mu(R') \Big)^{1/2} \\ \lesssim \mu(Q^*) \|g\|_{L^2(\mu)} \leq \mu(\lambda Q_0). \end{split}$$

The paraproduct. – For $Q \in \bigcup_{k \geq r} \mathcal{D}_{\text{good}, 2^k \ell(Q)}^T$ let $\alpha(Q)$ be the smallest k such that $Q \in \mathcal{D}_{\text{good}, 2^k \ell(Q)}^T$. We are reduced to bounding

$$\begin{split} S_{\mathrm{par}} &:= \sum_{\substack{R \in \mathcal{D}^{T*} \\ \beta(R) < \beta}} \sum_{\substack{Q \in \mathcal{D}_{\mathrm{good},\,\ell(R)}^T \colon \beta(Q) < \beta \\ \ell(Q) \leq 2^{-r}\ell(R) \\ d(Q,R) \leq \ell(Q)^{\gamma}\ell(R)^{1-\gamma}}} \left\langle T(\Delta_Q f), \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q}^{T*} \rangle_{R_Q}} b_{R_Q}^{T*} - \frac{\langle g \rangle_R}{\langle b_{R^a}^{T*} \rangle_R} b_{R^a}^{T*} \right\rangle \\ &= \sum_{\substack{Q \in \bigcup_{k \geq r} \mathcal{D}_{\mathrm{good},\,2^k\ell(Q)}^T \colon \beta(Q) < \beta \\ Q \subset R^*}} \sum_{\substack{R \in \mathcal{D}^{T*} \colon \beta(R) < \beta \\ \ell(R) \geq 2^{\alpha(Q)}\ell(Q) \\ Q \subset R}} \left\langle T(\Delta_Q f), \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q}^{T*} \rangle_{R_Q}} b_{R_Q}^{T*} - \frac{\langle g \rangle_R}{\langle b_{R^a}^{T*} \rangle_R} b_{R^a}^{T*} \right\rangle. \end{split}$$

If $Q \in \bigcup_{k \geq r} \mathcal{D}^T_{\operatorname{good}, \, 2^k \ell(Q)}$ we let $H(Q) \in \mathcal{D}^{T^*}$ be the smallest cube satisfying $\beta(H(Q)) < \beta$, $\ell(H(Q)) \geq 2^{\alpha(Q)} \ell(Q)$ and $Q \subset H(Q)$. Let $J(Q) = H(Q)_Q$. We have that

$$S_{\text{par}} = \sum_{Q \in \bigcup_{k \geq r} \mathcal{D}_{\text{good}, 2^k \ell(Q)}^T : \beta(Q) < \beta} \left\langle T(\Delta_Q f), \sum_{\substack{R \in \mathcal{D}^{T^*} \\ H(Q) \subset R \subset R^*}} \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q}^{T^*} \rangle_{R_Q}} b_{R_Q}^{T^*} - \frac{\langle g \rangle_R}{\langle b_{R^*}^{T^*} \rangle_R} b_{R^a}^{T^*} \right\rangle$$

$$= \sum_{\substack{Q \in \bigcup_{k \geq r} \mathcal{D}_{\text{good}, 2^k \ell(Q)}^T : \beta(Q) < \beta}} \left\langle T(\Delta_Q f), \frac{\langle g \rangle_{J(Q)}}{\langle b_{J(Q)^a}^{T^*} \rangle_{J(Q)}} b_{J(Q)^a}^{T^*} - \frac{\langle g \rangle_{R^*}}{\langle b_{R^{*a}}^{T^*} \rangle_{R^*}} b_{R^{*a}}^{T^*} \right\rangle.$$

We may consider the following general situation. We are given a collection $\mathcal{G} \subset \mathcal{D}^T$ so that to every cube $Q \in \mathcal{G}$ there holds $\beta(Q) < \beta$, and there is associated a unique cube $S(Q) \in \mathcal{D}^{T^*}$ satisfying $Q \subset S(Q)$. Our object is to bound

$$P(f,g) := \Big| \sum_{Q \in \mathcal{G}} \Big\langle T(\Delta_Q f), \frac{\langle g \rangle_{S(Q)}}{\langle b_{S(Q)^a}^{T^*} \rangle_{S(Q)}} b_{S(Q)^a}^{T^*} \Big\rangle \Big|.$$

To this end, we first define for $F \in \mathcal{F}_{R^*}$ and $Q \in \mathcal{D}^T$ that

$$\epsilon_Q(F) = \begin{cases} 0 & \text{if } Q \notin \mathcal{G}, \\ 0 & \text{if } Q \in \mathcal{G} \text{ and } S(Q)^a \neq F, \\ \frac{\langle g \rangle_{S(Q)}}{\langle b_F^{1,*} \rangle_{S(Q)}} & \text{if } Q \in \mathcal{G} \text{ and } S(Q)^a = F. \end{cases}$$

Notice that $|\epsilon_Q(F)| \lesssim |\langle g \rangle_{S(Q)}| \leq 1$. We have that

$$\left\| \sum_{Q \in \mathcal{D}^T} \epsilon_Q(F) \Delta_Q f \right\|_{L^2(\mu)} \le \sum_{j=0}^{\beta-1} \left(\sum_{K \in \mathcal{F}_{Q^*}^j} \left\| \sum_{\substack{Q \in \mathcal{D}^T \\ Q^a = K}} \epsilon_Q(F) \Delta_Q f \right\|_{L^2(\mu)}^2 \right)^{1/2}$$

$$\lesssim \sum_{j=0}^{\beta-1} \left(\sum_{K \in \mathcal{F}_{Q^*}^j} \mu(F \cap K) \right)^{1/2} \le \beta \mu(F)^{1/2}.$$

But as β is already fixed we do not need to mind about this dependence. Using this we now have that

$$\begin{split} P(f,g) &= \Big| \sum_{\substack{F \in \mathcal{D}^{T^*} \\ F^a = F}} \Big\langle \sum_{\substack{Q \in \mathcal{G} \\ S(Q)^a = F}} \frac{\langle g \rangle_{S(Q)}}{\langle b_F^{T^*} \rangle_{S(Q)}} \Delta_Q f, 1_F T^* b_F^{T^*} \Big\rangle \Big| \\ &\leq \sum_{\substack{F \in \mathcal{D}^{T^*} \\ F^a = F}} \Big\| \sum_{\substack{Q \in \mathcal{D}^T \\ Q \in \mathcal{D}^T}} \epsilon_Q(F) \Delta_Q f \Big\|_{L^2(\mu)} \|1_F T^* b_F^{T^*} \|_{L^2(\mu)} \\ &\lesssim \sum_{\substack{F \in \mathcal{D}^{T^*} \\ F^a = F}} \mu(F) \lesssim \mu(R^*) \leq \mu(\lambda Q_0). \end{split}$$

In particular, we have shown that

$$|S_{\text{par}}| \lesssim \mu(\lambda Q_0).$$

This completes our proof of the fact that

$$|S_{\text{good}}| \lesssim \mu(\lambda Q_0).$$

5. The diagonal

We come to the part of the proof which requires a delicate surgery type argument—another key method which originates from Nazarov-Treil-Volberg [17]. The procedure of performing the surgery of the diagonal has evolved quite a lot since, and we need to deal with the general L^2 testing conditions.

For every $Q \in \mathcal{D}^T$ and $R \in \mathcal{D}^{T^*}$ we write $Q = \bigcup_{i=1}^{2^n} Q_i$ and $R = \bigcup_{j=1}^{2^n} R_j$, where $Q_i \in \operatorname{ch}(Q)$ and $R_j \in \operatorname{ch}(R)$. We then fix two indices i and j. We write $Q \sim R$ to mean $\ell(Q) \sim \ell(R)$ and $d(Q,R) \lesssim \min(\ell(Q),\ell(R))$. Notice that $\#\{Q: Q \sim R\} \lesssim 1$. We want to bound a sum of the form

$$S = \sum_{R} \sum_{Q: Q \sim R} A_{Q,i} |\langle T(1_{Q_i} u_{Q,i}), 1_{R_j} v_{R,j} \rangle| B_{R,j},$$

where $A_{Q,i} = A_{Q,i}(f)$, $B_{Q,j} = B_{Q,j}(g) \ge 0$ are constants and $u_{Q,i}$, $v_{R,j}$ are functions such that

(5.1)
$$\sum_{Q} \left[\|1_{Q_i} M_{\mu} u_{Q,i}\|_{L^2(\mu)}^2 + \|1_{Q_i} T u_{Q,i}\|_{L^2(\mu)}^2 \right] A_{Q,i}^2 \lesssim \mu(\lambda Q_0)$$

(5.2)
$$\sum_{R} \left[\|1_{R_{j}} M_{\mu} v_{R,j}\|_{L^{2}(\mu)}^{2} + \|1_{R_{j}} T^{*} v_{R,j}\|_{L^{2}(\mu)}^{2} \right] B_{R,j}^{2} \lesssim \mu(\lambda Q_{0}).$$

Here we recall that M_{μ} is the centred maximal function with respect to the measure μ .

In practice, we shall consider S with the choice that $(A_{Q,i}, u_{Q,i})$ is either

$$A_{Q,i} = \begin{cases} \left| \frac{\langle f \rangle_{Q_i}}{\langle b_{Q_i}^T \rangle_{Q_i}} - \frac{\langle f \rangle_{Q}}{\langle b_{Q^a}^T \rangle_{Q}} \right| \text{ if } Q_i^a = Q^a \\ 0 \qquad \qquad \text{if } Q_i^a = Q_i \end{cases}, \quad u_{Q,i} = b_{Q_i^a}^T$$

or

$$A_{Q,i} = \begin{cases} 0 & \text{if } Q_i^a = Q^a \\ \left| \frac{\langle f \rangle_{Q_i}}{\langle b_{Qa}^T \rangle_{Q_i}} \right| & \text{if } Q_i^a = Q_i \end{cases}, \quad u_{Q,i} = b_{Q_i^a}^T$$

or

$$A_{Q,i} = \begin{cases} 0 & \text{if } Q_i^a = Q^a \\ \left| \frac{\langle f \rangle_Q}{\langle b_{Q^a}^T \rangle_Q} \right| & \text{if } Q_i^a = Q_i \end{cases}, \quad u_{Q,i} = b_{Q^a}^T.$$

Analogous choices are made for $(B_{R,j}, v_{R,j})$. This means that we consider nine different sums S. But to bound a sum of the form S we shall need only the fact that (5.1) and (5.2) hold, which is true with all these choices:

5.3. Lemma. – The inequality (5.1) holds with all the above three choices for $(A_{Q,i}, u_{Q,i})$.

Proof. – This is proved in exactly the same way as the inequality $\sum_{Q} \|\Delta_{Q} f\|_{L^{2}(\mu)}^{2} \lesssim \mu(Q^{*})$. The proof only needs the additional fact that we have also done a stopping time with respect to the properties

$$\int_{Q} |M_{\mu} b_{Q}^{T}|^{2} d\mu \lesssim \mu(Q) \quad \text{and} \quad \int_{Q} |T b_{Q}^{T}|^{2} d\mu \lesssim \mu(Q). \quad \Box$$

5.1. First surgery: the θ -surgery

Suppose for convenience that $\ell(Q_i) \leq \ell(R_j)$. Let θ be a small parameter. We perform surgery on (Q_i, R_j) with the parameter θ . Let $j(\theta) \in \mathbb{Z}$ be such that $2^{-21}\theta \leq 2^{j(\theta)} < 2^{-20}\theta$. Let \mathcal{D}^* be yet another random grid in \mathbb{R}^n , independent of all other grids considered. Let $G := \{g \in \mathcal{D}^* : \ell(g) = 2^{j(\theta)}\ell(Q_i)\}$, and for $x \in \mathbb{R}^n$, let G(x) be the unique cube in G that contains x. We define

$$Q_{i,\partial} := \{x \in Q_i : d(G(x), \partial R_i) < \theta \ell(R_i)/2\} \cup \{x \in Q_i \cap R_i : d(x, \partial G(x)) < \theta \ell(G(x))\}.$$

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Thus points in $Q_{i,\partial}$ belong to Q_i , and are either close to the boundary of R_j , or to the boundary of the grid G. The set $Q_{i,\partial}$ depends on the set R_j as well. However, we have

$$Q_{i,\partial} \subset Q_{i,\text{bad}} := Q_i \cap \left[\bigcup_{\substack{R' \in \mathcal{D}^{T^*} \\ 2^{-r}\ell(Q_i) \leq \ell(R') \leq 2^r \ell(Q_i)}} \{x : d(x,\partial R') < \theta \ell(R')\} \right]$$

$$\cup \bigcup_{\substack{g \in \mathcal{D}^* \\ \ell(g) = 2^{j(\theta)}\ell(Q_i)}} \{x : d(x,\partial g) < \theta \ell(g)\} \right]$$

$$=: Q_i \cap \left[\bigcup_{\substack{a = \text{gen}(Q_i) + r \\ a = \text{gen}(Q_i) - r}} \partial_a^{D^{T^*}}(\theta) \cup \partial_{\text{gen}(Q_i) - j(\theta)}^{D^*}(\theta) \right]$$

$$=: Q_i \cap \left[\partial_{\sim \text{gen}(Q_i)}^{D^{T^*}}(\theta) \cup \partial_{\text{gen}(Q_i) - j(\theta)}^{D^*}(\theta) \right],$$

which depends only on Q_i and the grids \mathcal{D}^{T^*} and \mathcal{D}^* . One should keep in mind that in what follows $Q_{i,\text{bad}} = Q_{i,\text{bad}}(\theta)$.

We set

$$Q_{i,\text{sep}} := Q_i \setminus (Q_{i,\partial} \cup R_j),$$

the part of Q_i strictly separated from R_i . Finally, we have

$$Q_{i,\Delta} := Q_i \setminus (Q_{i,\partial} \cup Q_{i,\text{sep}}) = \bigcup_k L_k,$$

where each L_k is of the form $L_k = (1 - \theta)g \cap Q_i \cap R_j$ for some $g \in G$, and $\#k \lesssim_{\theta} 1$. In fact, L_k is of the form $L_k = (1 - \theta)g$ unless it is close to the boundary of Q_i ; it cannot be close to the boundary of R_j , since such cubes were already subtracted in the $Q_{i,\partial}$ component.

We have the partition

$$Q_i = Q_{i, \text{sep}} \cup Q_{i, \partial} \cup Q_{i, \Delta} = Q_{i, \text{sep}} \cup Q_{i, \partial} \cup \bigcup_k L_k,$$

and in a completely analogous manner also

$$R_j = R_{j, \mathrm{sep}} \cup R_{j, \partial} \cup R_{j, \Delta} = R_{j, \mathrm{sep}} \cup R_{j, \partial} \cup \bigcup_s L_s.$$

A key observation is that all $L_k \subset Q_i \cap R_j$ appearing in the first union are cubes (of the form $(1-\theta)g$ for $g \in G$) unless they are close to ∂Q_i , and they are never close to ∂R_j , while the L_s in the second union are cubes unless they are close to ∂R_j , and they are never close to ∂Q_i . Thus, all $L_k = L_s$ that appear in both unions are cubes and then $5L_k \subset Q_i \cap R_j$.

5.2. Reduction to a deeply diagonal term

Using the above θ -surgery we want to reduce to a term of the form $\langle T(1_H u_{Q,i}), 1_H v_{R,j} \rangle$, where $H = L_k = L_s$ is a cube with $5H \subset Q_i \cap R_j$ and $\#H \leq C(\theta)$. This term will then be split using a different σ -surgery (at the end one will first choose θ small, and then $\sigma = \sigma(\theta)$ small depending on θ). But let us first do the actual reduction.

We write

$$\begin{split} \langle T(1_{Q_i}u_{Q,i}), 1_{R_j}v_{R,j} \rangle &= \sum_{\beta \in \{\text{sep},\,\partial\}} \langle T(1_{Q_i}u_{Q,i}), 1_{R_{j,\beta}}v_{R,j} \rangle + \sum_{\alpha \in \{\text{sep},\,\partial\}} \langle T(1_{Q_{i,\alpha}}u_{Q,i}), 1_{R_{j,\Delta}}v_{R,j} \rangle \\ &+ \sum_{k \neq s} \langle T(1_{L_k}u_{Q,i}), 1_{L_s}v_{R,j} \rangle + \sum_{k = s} \langle T(1_{L_k}u_{Q,i}), 1_{L_k}v_{R,j} \rangle. \end{split}$$

If $\alpha = \text{sep or } \beta = \text{sep or } k \neq s$, then the corresponding pairing is seen to be dominated by

$$C(\theta) \|1_{Q_i} u_{Q,i}\|_{L^2(\mu)} \|1_{R_j} v_{R,j}\|_{L^2(\mu)}$$

using the size estimate of the kernel K together with the fact that the sets are separated by $c(\theta)\ell(Q_i) \sim c(\theta)\ell(R_j)$. In the case $k \neq s$ a further large dependence on θ is gained from the summation $\sum_{k \neq s} 1$.

The sum of the cases $\alpha = \partial$ and $\beta = \partial$ is dominated by

$$\|T\|(\|1_{Q_{i,\mathrm{bad}}}u_{Q,i}\|_{L^2(\mu)}\|1_{R_j}v_{R,j}\|_{L^2(\mu)}+\|1_{Q_i}u_{Q,i}\|_{L^2(\mu)}\|1_{R_{j,\mathrm{bad}}}v_{R,j}\|_{L^2(\mu)}].$$

All in all, we have the estimate

$$\begin{split} |\langle T(1_{Q_i}u_{Q,i}), 1_{R_j}v_{R,j}\rangle| &\leq C(\theta) \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ \|T\| \|1_{Q_{i,\mathrm{bad}}}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ \|T\| \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \|1_{R_{j,\mathrm{bad}}}v_{R,j}\|_{L^2(\mu)} \\ &+ \sum_{H:=L_k=L_s} |\langle T(1_Hu_{Q,i}), 1_Hv_{R,j}\rangle|. \end{split}$$

We will now fix one such H and estimate $|\langle T(1_H u_{Q,i}), 1_H v_{R,j} \rangle|$ with a bound independent of H.

5.3. Second surgery: the σ -surgery

We continue to split

$$\langle T(1_H u_{Q,i}), 1_H v_{R,j} \rangle = \langle T u_{Q,i}, 1_H v_{R,j} \rangle - \langle T(1_{\mathbb{R}^n \setminus 5H} u_{Q,i}), 1_H v_{R,j} \rangle - \langle T(1_{5H \setminus (1+\sigma)H} u_{Q,i}), 1_H v_{R,j} \rangle - \langle T(1_{(1+\sigma)H \setminus H} u_{Q,i}), 1_H v_{R,j} \rangle.$$

We have that

$$|\langle Tu_{Q,i}, 1_H v_{R,j} \rangle| \le ||1_{Q_i} Tu_{Q,i}||_{L^2(\mu)} ||1_{R_i} v_{R,j}||_{L^2(\mu)}.$$

Using separation and the fact that $5H \subset Q_i \cap R_j$ we see that

$$|\langle T(1_{5H\setminus(1+\sigma)H}u_{Q,i}), 1_Hv_{R,j}\rangle| \le C(\sigma)||1_{Q_i}u_{Q,i}||_{L^2(\mu)}||1_{R_i}v_{R,j}||_{L^2(\mu)}.$$

Moreover, there holds that

$$|\langle T(1_{(1+\sigma)H\setminus H}u_{Q,i}), 1_Hv_{R,j}\rangle| \leq ||T|| ||1_{\partial_{\text{gen}(Q,i)-j}(\theta)}^{D^*}(\sigma)1_{Q_i}u_{Q,i}||_{L^2(\mu)} ||1_{R_j}v_{R,j}||_{L^2(\mu)}.$$

The term $\langle T(1_{\mathbb{R}^n\setminus 5H}u_{Q,i}), 1_Hv_{R,j}\rangle$ still requires further splitting. We write

$$\begin{split} \langle T(1_{\mathbb{R}^n\backslash 5H}u_{Q,i}), 1_Hv_{R,j}\rangle &= \langle T(1_{\mathbb{R}^n\backslash 5H}u_{Q,i}) - \langle b_H^{T^*}/\mu(H), T(1_{\mathbb{R}^n\backslash 5H}u_{Q,i})\rangle, 1_Hv_{R,j}\rangle \\ &+ \langle b_H^{T^*}, T(1_{\mathbb{R}^n\backslash 5H}u_{Q,i})\rangle \frac{1}{\mu(H)} \int_H v_{R,j} \, d\mu. \end{split}$$

5.5. Lemma. – There holds that

$$\begin{aligned} |\langle T(1_{\mathbb{R}^{n}\setminus 5H}u_{Q,i}) - \langle b_{H}^{T^{*}}/\mu(H), T(1_{\mathbb{R}^{n}\setminus 5H}u_{Q,i}) \rangle, 1_{H}v_{R,j} \rangle| \\ &\lesssim ||1_{Q_{i}}M_{\mu}u_{Q,i}||_{L^{2}(\mu)} ||1_{R_{i}}v_{R,j}||_{L^{2}(\mu)}. \end{aligned}$$

Proof. – Let $\Phi := T(1_{\mathbb{R}^n \setminus 5H} u_{Q,i})$. We need to bound

$$\int |1_H(x)v_{R,j}(x)||\Phi(x) - \langle b_H^{T^*}/\mu(H), \Phi \rangle| d\mu(x).$$

For $x \in H$ we have that

$$|\Phi(x) - \langle b_H^{T^*}/\mu(H), \Phi \rangle| \le \frac{1}{\mu(H)} \int_H |b_H^{T^*}(y)| |\Psi(x) - \Psi(y)| \, d\mu(y).$$

But for $x, y \in H$ there holds that

$$|\Psi(x) - \Psi(y)| \lesssim \ell(H)^{\alpha} \int_{|x-z| > c\ell(H)} \frac{|u_{Q,i}(z)|}{|x-z|^{m+\alpha}} d\mu(z) \lesssim M_{\mu} u_{Q,i}(x).$$

Therefore, for $x \in H$ we have that

$$|\Phi(x) - \langle b_H^{T^*}/\mu(H), \Phi \rangle| \lesssim M_{\mu} u_{Q,i}(x)$$

using which we see that

$$\int |1_{H}(x)v_{R,j}(x)| |\Phi(x) - \langle b_{H}^{T^{*}}/\mu(H), \Phi \rangle | d\mu(x) \lesssim \int |1_{H}(x)v_{R,j}(x)| |1_{H}(x)M_{\mu}u_{Q,i}(x)| d\mu(x)$$

$$\lesssim \|1_{Q_{i}}M_{\mu}u_{Q,i}\|_{L^{2}(\mu)} \|1_{R_{i}}v_{R,j}\|_{L^{2}(\mu)}. \quad \Box$$

We are left to deal with

$$\left| \langle b_H^{T^*}, T(1_{\mathbb{R}^n \setminus 5H} u_{Q,i}) \rangle \frac{1}{\mu(H)} \int_H v_{R,j} \, d\mu \right| \leq |\langle b_H^{T^*}, T(1_{\mathbb{R}^n \setminus 5H} u_{Q,i}) \rangle |\mu(H)^{-1/2} ||1_{R_j} v_{R,j}||_{L^2(\mu)}.$$

Our final splitting is as follows:

$$\langle b_H^{T^*}, T(1_{\mathbb{R}^n \setminus 5H} u_{Q,i}) \rangle = \langle b_H^{T^*}, Tu_{Q,i} \rangle - \langle b_H^{T^*}, T(1_{5H \setminus (1+\sigma)H} u_{Q,i}) \rangle - \langle b_H^{T^*}, T(1_{(1+\sigma)H \setminus H} u_{Q,i}) \rangle - \langle b_H^{T^*}, T(1_{H} u_{Q,i}) \rangle.$$

We have that

$$|\langle b_H^{T^*}, Tu_{Q,i}\rangle| \leq \|b_H^{T^*}\|_{L^2(\mu)} \|1_{Q_i} Tu_{Q,i}\|_{L^2(\mu)} \lesssim \mu(H)^{1/2} \|1_{Q_i} Tu_{Q,i}\|_{L^2(\mu)}.$$

Using again separation and the fact that $5H \subset Q_i \cap R_j$ we see that

$$\begin{aligned} |\langle b_H^{T^*}, T(1_{5H\setminus(1+\sigma)H}u_{Q,i})\rangle| &\leq C(\sigma) \|b_H^{T^*}\|_{L^2(\mu)} \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \\ &\lesssim C(\sigma)\mu(H)^{1/2} \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)}. \end{aligned}$$

Next, notice that

$$\begin{split} |\langle b_H^{T^*}, T(1_{(1+\sigma)H\setminus H}u_{Q,i})\rangle| &\leq \|T\| \|b_H^{T^*}\|_{L^2(\mu)} \|1_{\partial_{\mathrm{gen}(Q_i)-j(\theta)}^{D^*}(\sigma)} 1_{Q_i} u_{Q,i}\|_{L^2(\mu)} \\ &\lesssim \|T\| \mu(H)^{1/2} \|1_{\partial_{\mathrm{gen}(Q_i)-j(\theta)}^{D^*}(\sigma)} 1_{Q_i} u_{Q,i}\|_{L^2(\mu)}. \end{split}$$

Finally, we have that

$$\begin{aligned} |\langle b_H^{T^*}, T(1_H u_{Q,i}) \rangle| &= |\langle T^* b_H^{T^*}, 1_H u_{Q,i} \rangle| \le \|1_H T^* b_H^{T^*}\|_{L^2(\mu)} \|1_{Q_i} u_{Q,i}\|_{L^2(\mu)} \\ &\lesssim \mu(H)^{1/2} \|1_{Q_i} u_{Q,i}\|_{L^2(\mu)}. \end{aligned}$$

Collecting the estimates we see that our σ -surgery yields the final bound

$$\begin{split} |\langle T(1_H u_{Q,i}), 1_H v_{R,j} \rangle| & \leq C(\sigma) \|1_{Q_i} u_{Q,i}\|_{L^2(\mu)} \|1_{R_j} v_{R,j}\|_{L^2(\mu)} \\ & + C \|1_{Q_i} M_\mu u_{Q,i}\|_{L^2(\mu)} \|1_{R_j} v_{R,j}\|_{L^2(\mu)} \\ & + C \|1_{Q_i} T u_{Q,i}\|_{L^2(\mu)} \|1_{R_j} v_{R,j}\|_{L^2(\mu)} \\ & + C \|T\| \|1_{\partial^{D^*}_{\mathrm{gen}(Q_i) - j(\theta)}(\sigma)} 1_{Q_i} u_{Q,i}\|_{L^2(\mu)} \|1_{R_j} v_{R,j}\|_{L^2(\mu)}. \end{split}$$

5.4. The final estimate through averaging

Combining the different surgeries we see that

$$\begin{split} |\langle T(1_{Q_i}u_{Q,i}), 1_{R_j}v_{R,j}\rangle| &\leq C(\theta,\sigma) \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ \|T\| \|1_{Q_{i,\text{bad}}}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ \|T\| \|1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \|1_{R_{j,\text{bad}}}v_{R,j}\|_{L^2(\mu)} \\ &+ C(\theta) \|1_{Q_i}M_{\mu}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ C(\theta) \|1_{Q_i}Tu_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)} \\ &+ C(\theta) \|T\| \|1_{\partial^{D^*}_{\text{gen}(Q_i)-j(\theta)}(\sigma)} 1_{Q_i}u_{Q,i}\|_{L^2(\mu)} \|1_{R_j}v_{R,j}\|_{L^2(\mu)}. \end{split}$$

Using the Cauchy-Schwarz inequality, the property $\#\{Q: Q \sim R\} \lesssim 1$, the inequalities (5.1) and (5.2), the independence of the different dyadic grids \mathcal{D}^T , \mathcal{D}^{T^*} and \mathcal{D}^* , and the fact that e.g., $\mathbb{P}_{\mathcal{D}^*}(x \in \partial_k^{D^*}(\sigma)) \leq c(\sigma)$ for every point x and every generation $k \in \mathbb{Z}$, we see that

$$\begin{split} E_{\mathcal{D}^T} E_{\mathcal{D}^{T*}} E_{\mathcal{D}^*} \sum_{R} \sum_{Q: Q \sim R} A_{Q,i} |\langle T(1_{Q_i} u_{Q,i}), 1_{R_j} v_{R,j} \rangle| B_{R,j} \\ \leq [C(\theta, \sigma) + c(\theta) ||T|| + C(\theta) c(\sigma) ||T||] \mu(\lambda Q_0). \end{split}$$

Here $\lim_{p\to 0} c(p) = 0$. Let v > 0 be small enough. First choose θ so small that $c(\theta) \le v$. Then choose σ so small that $C(\theta)c(\sigma) \le v$. We have proved the diagonal bound

$$E_{\mathcal{D}^T} E_{\mathcal{D}^{T^*}} \sum_{R} \sum_{Q: Q \sim R} A_{Q,i} |\langle T(1_{Q_i} u_{Q,i}), 1_{R_j} v_{R,j} \rangle| B_{R,j} \leq [C + 2v \|T\|] \mu(\lambda Q_0).$$

We have completed the proof of our main theorem, Theorem 1.1.

BIBLIOGRAPHY

- [1] P. AUSCHER, S. HOFMANN, C. MUSCALU, T. TAO, C. THIELE, Carleson measures, trees, extrapolation, and *T*(*b*) theorems, *Publ. Mat.* **46** (2002), 257–325.
- [2] P. Auscher, E. Routin, Local *Tb* theorems and Hardy inequalities, *J. Geom. Anal.* **23** (2013), 303–374.
- [3] P. Auscher, Q. X. Yang, BCR algorithm and the T(b) theorem, *Publ. Mat.* **53** (2009), 179–196.
- [4] M. Christ, A T(b) theorem with remarks on analytic capacity and the Cauchy integral, *Collog. Math.* **60/61** (1990), 601–628.

- [5] S. HOFMANN, A local *Tb* theorem for square functions, in *Perspectives in partial differential equations, harmonic analysis and applications*, Proc. Sympos. Pure Math. **79**, Amer. Math. Soc., Providence, RI, 2008, 175–185.
- [6] S. HOFMANN, A proof of the local *Tb* theorem for standard Calderón-Zygmund operators, preprint arXiv:0705.0840.
- [7] T. HYTÖNEN, H. MARTIKAINEN, On general local *Tb* theorems, *Trans. Amer. Math. Soc.* **364** (2012), 4819–4846.
- [8] T. HYTÖNEN, C. PÉREZ, S. TREIL, A. VOLBERG, Sharp weighted estimates for dyadic shifts and the A_2 conjecture, *J. reine angew. Math.* **687** (2014), 43–86.
- [9] T. P. HYTÖNEN, M. T. LACEY, H. MARTIKAINEN, T. ORPONEN, M. C. REGUERA, E. T. SAWYER, I. URIARTE-TUERO, Weak and strong type estimates for maximal truncations of Calderón-Zygmund operators on A_p weighted spaces, *J. Anal. Math.* 118 (2012), 177–220.
- [10] T. P. HYTÖNEN, F. NAZAROV, The local *Tb* theorem with rough test functions, preprint arXiv:1206.0907.
- [11] M. T. LACEY, H. MARTIKAINEN, Local Tb theorem with L^2 testing conditions and general measures: Square functions, to appear in J. Anal. Math.
- [12] M. T. LACEY, S. PETERMICHL, M. C. REGUERA, Sharp A_2 inequality for Haar shift operators, *Math. Ann.* **348** (2010), 127–141.
- [13] M. T. LACEY, A. V. VÄHÄKANGAS, The perfect local *Tb* theorem and twisted martingale transforms, *Proc. Amer. Math. Soc.* **142** (2014), 1689–1700.
- [14] M. T. LACEY, A. V. VÄHÄKANGAS, On the local *Tb* theorem: a direct proof under duality assumption, to appear in *Proc. Edinb. Math. Soc.*
- [15] H. Martikainen, M. Mourgoglou, Square functions with general measures, *Proc. Amer. Math. Soc.* **142** (2014), 3923–3931.
- [16] F. NAZAROV, S. TREIL, A. VOLBERG, Accretive system *Tb*-theorems on nonhomogeneous spaces, *Duke Math. J.* **113** (2002), 259–312.
- [17] F. NAZAROV, S. TREIL, A. VOLBERG, The *Tb*-theorem on non-homogeneous spaces, *Acta Math.* **190** (2003), 151–239.
- [18] E. T. SAWYER, A characterization of two weight norm inequalities for fractional and Poisson integrals, *Trans. Amer. Math. Soc.* **308** (1988), 533–545.

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